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The Impact of Resale on 2-Bidder First-Price Auctions where One Bidder's Value is Commonly Known

by

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The Impact of Resale on 2-Bidder First-Price Auctions where One Bidder's Value is Commonly Known

By Thomas Tröger,[1]

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Abstract

We consider 2-bidder first-price auctions where one bidder's value is commonly known. Such auctions induce an inefficient allocation. We show that a resale opportunity, where the auction winner can make a take-it-or-leave-it offer to the loser, increases (reduces) the inefficiency of the market when the buyer with the commonly known value is weak (strong). Resale always reduces all bidders' payoffs and increases the initial seller's revenue.

Keywords: asymmetric first-price auctions, resale, efficiency *JEL*: D44

1 Introduction

A worrisome property of sealed-bid first-price auctions with asymmetric bidders is that they can lead to an inefficient allocation (Vickrey, 1961, Griesmer et al, 1967, Plum 1992, Maskin and Riley, 2000). In such a situation, the winning bidder might have an incentive to try to resell the good to a bidder with a higher value. Rational bidders will anticipate the possibility of resale and their bidding incentives will be different from the situation without a resale opportunity. What then is the impact of a resale opportunity on efficiency, seller revenue, and bidder payoffs? In particular, if the seller can prohibit resale, should she do so?

This paper analyzes the impact of resale on first-price auctions with asymmetric bidders. We consider a class of 2-bidder environments that

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generalizes an example of Vickrey (1961). Buyer 1 has a private value for the good on sale. Buyer 2 has a commonly known, positive, value. A typical example would be the sale of an asset in the context of a market where buyer 2 is the incumbent firm who's value is known from previous interactions, and buyer 1 is a potential entrant. We will put particular emphasis on two special cases: a market where one buyer is known to be weak (i.e., where buyer 2's value is close to 0), and a market where one buyer is known to be strong (i.e., where buyer 2's value is close to the highest possible value of buyer 1). As we will show, resale has different implications in markets with a weak buyer compared to markets with a strong buyer.

The interaction is modelled as follows. A first-price auction without reserve price takes place. The winner either consumes the good right away or offers it for resale via a take-it-or-leave-it offer to the auction loser. Because the winner's resale offer is based on her posterior belief about the loser's value, it makes a difference whether or not the auction winner observes the loser's bid. We assume that the loser's bid remains private, like in a Dutch auction. We construct a (perfect Bayesian) equilibrium in this environment, as well as an equilibrium when no resale opportunity exists.[1] Neither with nor without a resale opportunity, the equilibrium allocation is fully efficient, so the question whether resale is good or bad for efficiency is nontrivial.

Conventional wisdom would suggest that resale can only be good for efficiency because any resale transaction in itself realizes additional gains from trade. Our results show, however, that if buyer 2 is weak the anticipation of resale distorts bidding in the auction so strongly that the post-resale allocation is more inefficient than the auction allocation when resale is not possible. The reason resale is bad for efficiency is that it strengthens the weak buyer by giving her monopoly power in the resale market.[2]

When buyer 2 is strong, the effect of a resale opportunity on the efficiency of the allocation is quite different. Without a resale opportunity, the allocation remains far from efficient because the strong buyer lets her opponent win too often in order to get the good cheaper. But if there is a resale opportunity, the private-value buyer will successfully resell to the strong buyer because she knows her value. Hence, the resale market corrects most inefficiencies that arise from the initial auction allocation when a strong buyer is in the market.

We complete our analysis by showing that resale reduces the buyers'

[1]Both with and without resale, the equilibrium is essentially unique. We defer the details concerning equilibrium uniqueness to a different paper, Tröger (2004).

[2]If the buyers have, say, equal bargaining power in the resale market, the resale opportunity can still be detrimental to efficiency, but the effect is weaker.

payoffs and increases the seller's revenue. These results are driven by our assumption of full bargaining power to the resale seller. The resale opportunity strengthens the bidders' incentives to try to win the initial auction. The intensified bidding competition in the auction shifts rents from the buyers to the seller. Interestingly, the seller gains even when the resale opportunity reduces the efficiency of the allocation.

In summary, our results have the following implications. When one buyer in the market is known to be weak and the seller is efficiency-minded, resale is a pure waste: it reduces all buyers' payoffs and at the same time makes the market even more inefficient than it is without resale. On the other hand, resale increases the seller's revenue. When one buyer in the market is known to be strong, resale still reduces the bidders' payoffs, but also makes the market more efficient. In markets with a strong buyer, the seller should allow resale both on efficiency and on revenue grounds.

Previous studies of first-price auctions with resale have mostly focussed on the impact of a resale opportunity on seller revenue. Gupta and Lebrun (1999) consider two asymmetric bidders with initially private values, but assume common knowledge of values and efficient trade in the resale market. Haile (1999, 2003) analyzes resale in symmetric environments where at the time of the initial auction each bidder is uncertain about her own use value.[3] In both Haile's and Gupta and Lebrun's models, the resale opportunity increases the initial seller's revenue if the resale seller has sufficient bargaining power in the resale market. Hence, our result on the revenue of the initial seller is consistent with their's.

A first result on the impact of resale on the efficiency of first-price auctions is derived in Krishna (2002, Ch. 4.4). He does not ask whether a resale opportunity can be detrimental to efficiency, but shows that a resale market may fail to achieve an efficient allocation. Krishna considers two asymmetric private-value bidders and a resale market where, as in our model, the auction winner makes a take-it-or-leave-it offer to the loser. In contrast to our model, he assumes that both the winner's and the loser's bid become public after the auction. Therefore, resale would restore efficiency if the bidders used strictly increasing bid functions. Krishna shows that the anticipation of resale prevents bidders from using strictly increasing bid functions. Thus, bidders retain some private information when they enter the resale market, and this causes some inefficiencies to prevail.

[3] In Haile's terminlogy, the term "use value" is used in the same way as our term "value." His term "valuation" refers to the opportunity cost of not winning the initial auction.

A natural candidate for generalization of our results is an environment with multiple symmetric private-value buyers and a buyer with a commonly known value. Martinez (2002) constructs an equilibrium in such an environment without resale. Garratt and Tröger (2004) allow resale and construct an equilibrium in the case where the commonly known value equals 0; the respective buyer is then called a speculator because she buys only in order to sell. The main qualitative difference between Garratt and Tröger's equilibrium and the resale equilibrium constructed here is that with multiple private-value buyers, for many distributions of values the speculator will not submit a positive bid at all. In these latter cases, the allocation will be efficient, whether or not resale is possible. It is thus unclear whether a resale opportunity is always detrimental to efficiency in markets with multiple private-value buyers and a commonly-known weak buyer. On the other hand, the intuition behind our result that resale is good for efficiency when a commonly-known strong buyer is in the market suggests that the result generalizes to the case of multiple private-value buyers.

Zheng (2002) constructs a seller-revenue maximizing mechanism with asymmetric private-value buyers where the initial seller cannot prevent resale. His results do not apply directly to our case of a private-value buyer and a buyer with a commonly known value, but an optimal mechanism with resale is easy to find when buyer 1's value distribution has an increasing hazard rate. The seller makes a take-it-or-leave-it offer to buyer 2. The amount of the offer is equal to the payoff buyer 2 obtains from making an optimal take-it-or-leave-it offer to buyer 1. This way, the revenue-maximizing Myerson (1981) allocation is implemented. Hence, being unable to prevent resale does not harm a mechanism-designing seller if buyer 1's value distribution has an increasing hazard rate.[4]

In Section 2 we establish the equilibrium conditions with and without resale. In Section 3 we construct the equilibrium in the market without resale. In Section 4 the equilibrium with resale is constructed. Section 5 analyzes the impact of resale on efficiency, and Section 6 contains the results about the impact of resale on buyer payoffs and seller revenue. The Appendix contains proofs.

[4]Why do we assume that the seller uses a standard first-price auction instead of designing an optimal mechanism? Wilson (1987, p. 36-37) suggests that standard auction mechanisms continue to be used because they are detail-free—the rules are independent of the fine structure of the environment, like the probability distribution for a buyer's value. But the mechanism-designing seller's optimal take-it-or-leave-it offer depends on buyer 1's value distribution as well as buyer 2's value.

2 Model

We consider two risk-neutral buyers who are interested in consuming a single indivisible good. The good is initially owned by a seller who has no value for it. Buyer 1 has the random value $\tilde{\theta}_1 \in [0, 1]$ for the good. Buyer 2 has the commonly known value $\theta_2 \in (0, 1)$. Let F denote the distribution function for $\tilde{\theta}_1$. We assume that F is continuous, $F(0) = 0$, $F(1) = 1$, and F has a positive and continuous density on $[0, 1]$.

We consider a 2-period interaction. Before period 1, buyer 1 privately learns the realization of her value, $\tilde{\theta}_1 = \theta_1$. In period 1, the good is offered via a sealed-bid first-price auction without reserve price. The highest bidding agent becomes the new owner of the good. To simplify the presentation we assume that buyer 1 wins all ties. The agent who wins in period 1 either consumes the good in period 1 or makes a take-it-or-leave-it offer in period 2; if she fails to resell the good she consumes it in period 2. The buyers discount payoffs that are obtained in period 2, according to a factor $\delta \in (0, 1)$.[5]

Actions taken in period 2 may depend on information that is revealed during period 1. We assume that after period 1, the winner's bid becomes public; the loser's bid remains private. This implies that that we consider a first-price auction as implemented via a Dutch auction. Such a descending auction stops at the moment the highest bid is revealed, such that the losers' stopping bids remain private. This assumption is needed for tractability, but it is also common for real sellers to keep the losers' bids private in sealed-bid first-price auctions.

Buyer 1's bid function is denoted b_1. A bidding strategy for buyer 2 is given by a random bid $\tilde{b}_2$ with distribution function H (we allow for randomization because otherwise there is no equilibrium).

If buyer 1 wins in period 1, she offers the good for resale at price θ_2 if $\theta_1 \leq \delta\theta_2$, and otherwise consumes the good. In both cases, buyer 2's payoff equals 0.

Now consider any bid $b_2 > 0$ of buyer 2 that wins with positive probability. Upon winning with bid b_2, buyer 2's posterior distribution $\Pi(\cdot \mid b_2)$ for buyer 1's value is given, for all $\theta_1 \in [0, 1]$, by

$$\Pi(\theta_1 \mid b_2) = \Pr[\tilde{\theta}_1 \leq \theta_1 \mid b_1(\tilde{\theta}_1) < b_2]. \tag{1}$$

The posterior distribution $\Pi(\cdot \mid b_2)$ is arbitrary if bid b_2 wins with probability 0. The optimal take-it-or-leave-it offer of buyer 2 if she wins with bid b_2 is

[5] The fact that we consider a 2-period interaction does not mean that the world is over after two periods. We only assume that the resale offer includes a credible commitment of the resale seller to keep the good if she fails to resell it.

denoted

$$T(b_2) \in \arg\max_{p \geq \theta_2} (1 - \Pi_-(p \mid b_2))p + \Pi_-(p \mid b_2)\theta_2, \tag{2}$$

where $\Pi_-(p \mid b_2) = \lim_{p' \nearrow p} \Pi(p' \mid b_2)$ denotes the probability that the offer p is not accepted. Buyer 2's expected resale payoff equals

$$R(b_2) = \max_{p \geq \theta_2} (1 - \Pi_-(p \mid b_2))p + \Pi_-(p \mid b_2)\theta_2.$$

If buyer 2 wins with a bid b_2 such that $\delta R(b_2) \geq \theta_2$, she offers the good for resale; if $\delta R(b_2) < \theta_2$, she consumes the good in the first period.

Given buyer 2's bid distribution H, the expected payoff of buyer 1 with value θ_1 and bid b equals

$$\begin{aligned} u_1(b, \theta_1) &= H(b)(\max\{\theta_1, \delta\theta_2\} - b) \\ &\quad + \delta \int_{(b,\infty)} \mathbf{1}_{\theta_1 \geq T(b'), \delta R(b') \geq \theta_2} (\theta_1 - T(b'))\mathrm{d}H(b'), \end{aligned} \tag{3}$$

Given buyer 1's bid function b_1, the expected payoff of buyer 2 with bid $b_2 \geq 0$ equals

$$u_2(b_2) = \Pr[b_1(\tilde{\theta}_1) < b_2](\max\{\theta_2, \delta R(b_2)\} - b_2). \tag{4}$$

A *(perfect Bayesian) equilibrium outcome of the first-price auction with resale* is a tuple (b_1, H, T) such that (2), (5), and (6) hold.

$$\forall \theta_1 \in [0, 1]: \; b_1(\theta_1) \in \arg\max_{b \geq 0} u_1(b, \theta_1), \tag{5}$$

$$\Pr[\tilde{b}_2 \in \arg\max_{b \geq 0} u_2(b)] = 1. \tag{6}$$

We will compare the auction with resale with the auction where resale is not possible. Without resale, given buyer 2's bid distribution H, the expected payoff of buyer 1 with value θ_1 and bid b equals

$$v_1(b, \theta_1) = H(b)(\theta_1 - b). \tag{7}$$

Moreover, given buyer 1's bid function b_1, the expected payoff of buyer 2 with bid b_2 equals

$$v_2(b_2) = \Pr[b_1(\tilde{\theta}_1) < b_2](\theta_2 - b_2). \tag{8}$$

A *(Bayesian Nash) equilibrium of the first-price auction without resale* is a pair (b_1, H) such that

$$\forall \theta_1 \in [0, 1]: \; b_1(\theta_1) \in \arg\max_{b \geq 0} v_1(b, \theta_1), \tag{9}$$

$$\Pr[\tilde{b}_2 \in \arg\max_{b \geq 0} v_2(b)] = 1. \tag{10}$$

3 Equilibrium when resale is not possible

The first step towards evaluating the impact of resale is to analyze the auction market where resale is not possible. The proposition below constructs an equilibrium.[6] Because there is no resale, the equilibrium bids are determined by the trade-off between payment amount and winning probability. Buyer 2, who's value is commonly known, randomizes her bid over a certain interval.

Proposition 1 *Let*

$$V_2 = \max_{b\in[0,\theta_2]} F(b)(\theta_2 - b), \tag{11}$$

$$\underline{b} = \max\ \arg\max_{b\in[0,\theta_2]} F(b)(\theta_2 - b),\ \textit{and}\ \overline{b} = \theta_2 - V_2. \tag{12}$$

The first-price auction without resale has an equilibrium (b_1, H) *with the following properties. Buyer 1's bid function* b_1 *is given by*

$$b_1(\theta_1) = \begin{cases} \theta_1, & \textit{if } \theta_1 \leq \underline{b}, \\ \theta_2 - \frac{V_2}{F(\theta_1)}, & \textit{if } \theta_1 \geq \underline{b}. \end{cases} \tag{13}$$

Buyer 2's bid distribution H *has the support* $[\underline{b}, \overline{b}]$*, and is given by*

$$H(b_2) = e^{-\int_{b_2}^{\overline{b}} \frac{1}{\phi(b)-b} db} \quad (b_2 \in [\underline{b}, \overline{b}]), \tag{14}$$

where ϕ *denotes the inverse of* b_1*.*

The proof is straightforward (for details see the Appendix). Buyer 1's bid function b_1 is such that buyer 2 obtains the payoff V_2 with any bid in the range $[\underline{b}, \overline{b}]$. By (11), it is not profitable for buyer 2 to deviate to a bid below $\underline{b}$. Because $\overline{b} = b_1(1)$, a deviation to a bid above $\overline{b}$ is not profitable either. Buyer 2's bid distribution H is such that buyer 1's bid function is optimal. Observe that H may have an atom at $\underline{b}$.

We will assume that the equilibrium constructed in Proposition 1 represents the market outcome when no resale opportunity exists. It is therefore important to know whether the equilibrium is unique. Tröger (2004) proves uniqueness under an assumption similar to, but not identical to, the restriction to weakly undominated strategies: we show that any equilibrium

[6] The equilibrium construction is similar to Vickrey (1961), who assumes that buyer 1's value is uniformly distributed. A similar equilibrium construction also appears in Kaplan and Zamir (2000), who construct an equilibrium for 2-bidder first-price auctions where the *maximum* value across bidders is commonly known.

where buyer 1 does not bid strictly *above* her value is—up to bids of buyer-1 types that win with probability 0—identical to the equilibrium constructed in Proposition 1.[7] Without the assumption that buyer 1 does not bid above her value, equilibrium uniqueness cannot be obtained: Tröger (2004) shows that there exist equilibria where the losing buyer-1 types bid above their values and as a consequence all buyer-1 types as well as buyer 2 bid more aggressively than in the equilibrium of Proposition 1.

Let us sketch the arguments leading to the result that the equilibrium is unique if no buyer-1 type bids above her value. Let us assume for simplicity that buyer 2's bid distribution is Lipschitz continuous, with a possible exception at the lower end of its support, and that buyer 1's bid function is strictly increasing and continuous (Tröger, 2004, presents a proof without simplifying assumptions). Let (b_1, H) denote an arbitrary equilibrium. Let ϕ denote the inverse of b_1. Let $\underline{b}$ and $\overline{b}$ denote the lower and upper end of the support of H, and let V_2 denote buyer 2's equilibrium payoff. Because nobody bids higher than necessary in order to win for sure, $\overline{b} = b_1(1)$. This implies the formula for $\overline{b}$ in (12). For all $b_2 \in [0, \overline{b}]$, buyer 2's payoff

$$v_2(b_2) = F(\phi(b_2))(\theta_2 - b_2). \tag{15}$$

Because no buyer-1 type bids above her value,

$$\forall b \in [0, \overline{b}] : \ \phi(b) \geq b. \tag{16}$$

Moreover,

$$\phi(\underline{b}) = \underline{b}. \tag{17}$$

(Suppose that $\phi(\underline{b}) > \underline{b}$. Then all types $\theta_1 \in (\underline{b}, \phi(\underline{b}))$ bid less than $\underline{b}$; these types obtain payoff 0, and any bid in $(\underline{b}, \theta_1)$ is a profitable deviation.)

From (15) and (17),

$$V_2 = F(\underline{b})(\theta_2 - \underline{b}).$$

Hence, for all $b \leq \overline{b}$,

$$F(\underline{b})(\theta_2 - \underline{b}) = V_2 \geq v_2(b) \overset{(16)}{\geq} F(b)(\theta_2 - b).$$

This implies (11).

[7] Only bids strictly *below* value are weakly undominated for any strictly positive type. However, there exist examples of parameter constellations such that the unique equilibrium has the property that a positive mass of buyer-1 types bid exactly their values (Tröger, 2004).

Observe that $\phi(b) > b$ for all $b \in (\underline{b}, \bar{b}]$ (if not, type $\theta_1 = \phi(b)$ bids her value, which results in 0 payoff, but she can do better by bidding below her value). Therefore,

$$V_2 \geq v_2(b) > F(b)(\theta_2 - b),$$

which confirms the formula for $\underline{b}$ in (12).

All buyer-2 bids in the range $[\underline{b}, \bar{b}]$ yield the payoff V_2 (otherwise buyer 2's bid distribution would have a gap, which would lead to a contradiction). Hence, (15) implies for all $\theta_1 \geq \underline{b}$,

$$F(\theta_1)(\theta_2 - b_1(\theta_1)) = V_2,$$

This implies (13) in the cases where $\theta_1 \geq \underline{b}$. Types $\theta_1 < \underline{b}$ win with probability 0, so their bids are not uniquely determined—they only have to be aggressive enough so that buyer 2 has no incentive to submit a bid below $\underline{b}$.

The bid distribution H is such that the bid function b_1 becomes optimal for buyer 1. Because H is differentiable for Lebesgue-almost every $b \in (\underline{b}, \bar{b})$, the first order condition for type $\theta_1 = \phi(b)$ yields

$$0 = \frac{\partial v_1}{\partial b}(b, \theta_1)\Big|_{\theta_1 = \phi(b)} = H'(b)(\phi(b) - b) - H(b) \text{ a.e. } b \in (\underline{b}, \bar{b}).$$

This differential equation, with boundary condition $H(\bar{b}) = 1$, has the unique solution (14). This completes the equilibrium uniqueness proof.

The equilibrium allocation is inefficient. Buyer 2 sometimes outbids buyer-1 types above θ_2, and buyer 2 is sometimes outbid by a buyer-1 type below θ_2. Therefore, if a resale opportunity arises, the buyers may try to realize additional gains of trade. Rational bidders will, of course, anticipate a resale opportunity. In the next section we investigate how a resale opportunity changes the bidding incentives in the auction.

4 Equilibrium when resale is possible

We now consider the auction market with a resale opportunity. The main purpose of this section is to construct an equilibrium (Proposition 2). In the subsequent sections we will compare the efficiency and revenue properties of this equilibrium to the no-resale equilibrium outcome established in Proposition 1.

In equilibrium, like in the no-resale equilibrium, buyer 2 randomizes her bid over a certain interval, and buyer 1's bid function is strictly increasing.

The bids are, however, different from the no-resale case because of the anticipated resale payoffs. When buyer 2 wins with a bid equal to $b_1(\theta_1)$ for some $\theta_1 \in [0,1]$, she faces the buyer-1 type pool $[0,\theta_1)$ in the resale market, which yields the resale payoff

$$M(\theta_1) = \max_{p\in[\theta_2,\theta_1]} (1 - \frac{F(p)}{F(\theta_1)})p + \frac{F(p)}{F(\theta_1)}\theta_2 \tag{18}$$

if $\theta_1 \geq \theta_2$, and $M(\theta_1) = \theta_2$ otherwise. Buyer 2 offers the good for resale if and only if $\delta M(\theta_1) \geq \theta_2$. To simplify the presentation of the equilibrium, we assume that $\delta M(1) \geq \theta_2$; i.e., we exclude discount factors so small that buyer 2 never wants to offer the good for resale. Define $\hat{\theta}$ by $\delta M(\hat{\theta}) = \theta_2$. Using this notation, we can now present the equilibrium.

Proposition 2 *Assume $\delta M(1) \geq \theta_2$. Let*

$$U_2 = \max_{b\in[\delta\theta_2,\theta_2]} F(b)(\theta_2 - b), \tag{19}$$

$$\underline{b} = \max \ \arg \max_{b\in[\delta\theta_2,\theta_2]} F(b)(\theta_2 - b), \ \text{and } \overline{b} = \delta M(1) - U_2. \tag{20}$$

Then the first-price auction with resale has an equilibrium outcome (b_1, H, T) with the following properties. Buyer 1's bid function b_1 is given by

$$b_1(\theta_1) = \begin{cases} K(\theta_1 - \delta\theta_2) + \delta\theta_2 & \text{if } \theta_1 \leq \delta\theta_2, \\ \theta_1 & \text{if } \theta_1 \in (\delta\theta_2, \underline{b}], \\ \max\{\theta_2, \delta M(\theta_1)\} - \frac{U_2}{F(\theta_1)} & \text{if } \theta_1 \in (\underline{b}, 1], \end{cases} \tag{21}$$

where $K > 0$ is any sufficiently small constant.

For all $b \in [\underline{b}, \overline{b}]$, let

$$m(b) = \phi(b) - b - \mathbf{1}_{b\geq b_1(\hat{\theta})}\delta(\phi(b) - T(b)). \tag{22}$$

where ϕ denotes the inverse of b_1.

Buyer 2's bid distribution H has the support $[\underline{b}, \overline{b}]$ and is given by

$$H(b_2) = e^{-\int_{b_2}^{\overline{b}} \frac{1}{m(b)} db} \quad (b_2 \in [\underline{b}, \overline{b}]). \tag{23}$$

If buyer 2 wins with a bid $b \in [\underline{b}, b_1(\hat{\theta}))$, she consumes the good in the first period. If buyer 2 wins with a bid $b \in [b_1(\hat{\theta}), \overline{b}]$, she makes a resale offer

$$T(b) \in \tau(b) \stackrel{def}{=} \arg \max_{p\in[\theta_2,\phi(b)]} (F(\phi(b)) - F(p))p + F(p)\theta_2. \tag{24}$$

The details of the proof are in the Appendix. Because M is strictly increasing and continuous, buyer 1's bid function b_1 has the same properties and its inverse ϕ is well-defined. The bid function b_1 is such that buyer 2 obtains the payoff U_2 with any bid in the range $[\underline{b}, \overline{b}]$. By (19), it is not profitable for buyer 2 to deviate to a bid in $[\delta\theta_2, \underline{b})$. When K is sufficiently small, a buyer-2 deviation to a bid below $\delta\theta_2$ is not profitable either. Buyer 2's bid distribution H is such that buyer 1's bid function is optimal (observe that H may have an atom at $\underline{b}$). Formula (24) for the resale offer function T reflects buyer 2's posterior belief that with bid b she wins against the buyer-1 type pool $[0, \phi(b))$.

In Tröger (2004) we show that the equilibrium constructed in Proposition 2 is essentially unique under the assumption that no buyer-1 type bids higher than the maximum of her value and the discounted buyer-2 value $\delta\theta_2$. This assumption is in the spirit of iterated weak dominance. Buyer 2 will not accept any resale offer above her value θ_2. Hence, when buyer 1 wins the auction she cannot make more than $\delta\theta_2$ by reselling the good. Moreover, the resale offer buyer 1 obtains when she loses is independent of her own bid (recall that losing bids remain private). Therefore, no buyer-1 type $\theta_1 \in [0, 1]$ has a positive reason to bid higher than $\max\{\theta_1, \delta\theta_2\}$.[8] Like in the no-resale case, multiple equilibria exist if no restricting assumption about buyer 1's bid function is made (see Tröger, 2004).

Let us sketch the arguments leading to the result that the equilibrium is essentially unique if no buyer-1 type bids higher than the maximum of her value and $\delta\theta_2$. Let us assume for simplicity that buyer 2's bid distribution is Lipschitz continuous, with a possible exception at the lower end of its support, and that buyer 1's bid function is strictly increasing and continuous (the proof in Tröger, 2004, does not make these simplifying assumptions). Let (b_1, H, T) denote an arbitrary equilibrium outcome. Let $\underline{b}$ and $\overline{b}$ denote the lower and upper end of the support of H, let U_2 denote buyer 2's equilibrium payoff, and let ϕ denote the inverse of b_1.

Because nobody bids higher than necessary in order to win for sure, $\overline{b} = b_1(1)$. This, together with the assumption $\delta M(1) \geq \theta_2$, implies the formula for $\overline{b}$ in (20). Also by assumption, a buyer-2 bid $b \in (\delta\theta_2, \theta_2]$ wins at least against the buyer-1 types in $[0, b)$. This yields a lower bound for buyer 2's equilibrium payoff,

$$U_2 \geq \max_{b \in [\delta\theta_2, \theta_2]} F(b)(\theta_2 - b) > 0. \qquad (25)$$

[8]This iterated-weak-dominance argument also suggests that buyer 1 bids strictly *below* $\max\{\theta_1, \delta\theta_2\}$ if $\theta_1 > 0$, but for equilibrium-existence reasons analogous to the no-resale case (see footnote 7) we do not adopt this restriction.

Buyer 2's infimum bid, $\underline{b}$, is not lower than the discounted buyer-2 value $\delta\theta_2$ (if it were, all buyer-1 types would overbid the lowest buyer-2 bid to get a positive payoff, but then buyer 2's equilibrium payoff $U_2 = 0$),

$$\underline{b} \geq \delta\theta_2. \tag{26}$$

Like in the no-resale case, one shows that type $\underline{b}$ bids her value,

$$\phi(\underline{b}) = \underline{b}. \tag{27}$$

For all b_2, buyer 2's payoff

$$u_2(b_2) = F(\phi(b_2))(\max\{\theta_2, \delta M(\phi(b_2))\} - b_2). \tag{28}$$

The support of buyer 2's bid distribution must begin below buyer 2's value,

$$\underline{b} < \theta_2. \tag{29}$$

Suppose this were not so. Then buyer 2's lowest equilibrium bids are not smaller than θ_2. Because $U_2 > 0$, buyer 2 must be offering the good for resale after winning with any of these bids. Because her resale payoff must recover her auction bid, her infimum equilibrium resale price must be higher than $\underline{b}$, and some buyer-1 types must be accepting this resale price. Hence, there exists a buyer-1 type above the infimum equilibrium resale price who never wins the auction but always waits for resale. For this type it is profitable to deviate by slightly overbidding buyer 2's infimum bid $\underline{b}$, in order to get the good cheaper—contradiction.[9]

Formula (27) together with continuity of ϕ implies that any given buyer-1 type above $\underline{b}$ wins against all buyer-2 bids sufficiently close to $\underline{b}$. Hence, (29) implies that all buyer-2 bids sufficiently close to $\underline{b}$ lose against all buyer-1 types above θ_2. Therefore, buyer 2 consumes the good after winning with any bid close to $\underline{b}$. Thus, (27) together with (28) implies

$$U_2 = F(\underline{b})(\theta_2 - \underline{b}).$$

This, together with (25), (26), and (29), shows (19). The proof of (20) is analogous to the proof of (12) in the no-resale case (see p. 8).

All buyer-2 bids in the range $[\underline{b}, \overline{b}]$ yield the payoff U_2 (otherwise buyer 2's bid distribution would have a gap, which would lead to a contradiction). Hence, (28) implies for all $\theta_1 \geq \underline{b}$,

$$F(\theta_1)(\max\{\theta_2, \delta M(\theta_1)\} - b_1(\theta_1)) = U_2.$$

[9] The thrust of the argument is similar to the argument in Garratt and Tröger (2004) that shows that speculators make arbitrarily small bids, and thus cannot obtain positive profits, in first-price auctions with resale.

This shows (21) in the cases where $\theta_1 \geq \underline{b}$. Types $\theta_1 < \underline{b}$ win with probability 0, so their bids are not uniquely determined.

The bid distribution H is such that the bid function b_1 is optimal for buyer 1. For all $b \in (\underline{b}, b_1(\hat{\theta}))$, type $\theta_1 = \phi(b)$ does not get a resale offer if buyer 2 wins with a bid just above b; i.e., on the margin, type θ_1 does not lose a resale opportunity. Therefore, for all $b \in (\underline{b}, b_1(\hat{\theta}))$ where H is differentiable, type θ_1's first order condition is like in the case without resale,

$$0 = \frac{\partial u_1}{\partial b}(b, \theta_1)\Big|_{\theta_1=\phi(b)} = H'(b)(\phi(b) - b) - H(b) \text{ a.e. } b \in (\underline{b}, b_1(\hat{\theta})). \quad (30)$$

For all $b \in (b_1(\hat{\theta}), \bar{b})$, type $\theta_1 = \phi(b)$ loses the resale payoff $\theta_1 - T(b)$ on the margin, with marginal probability $H'(b)$. Therefore, her first-order condition is augmented by a resale term,

$$\begin{aligned} 0 &= \frac{\partial u_1}{\partial b}(b, \theta_1)\Big|_{\theta_1=\phi(b)} \qquad \text{a.e. } b \in (b_1(\hat{\theta}), \bar{b}) \\ &= H'(b)(\phi(b) - b) - H(b) - \delta H'(b)(\phi(b) - T(b)). \end{aligned} \quad (31)$$

Rearranging and combining (30) and (31) yields the differential equation

$$H'(b) = \frac{H(b)}{m(b)} \text{ a.e. } b \in (\underline{b}, \bar{b}).$$

This differential equation, with boundary condition $H(\bar{b}) = 1$, has the unique solution (23).

The optimal-resale-price correspondence τ defined in (24) is strictly increasing (see the strict monotone comparative statics results of Edlin and Shannon, 1998). Hence its set of points of multiple-valuedness is countable, and the probability that buyer 2 makes a bid in this set equals 0. Therefore, in the equilibrium range $[b_1(\hat{\theta}), \bar{b}]$ the resale offer function T is uniquely determined with probability 1. This completes the equilibrium uniqueness proof.

The fact that resale payoffs are discounted is crucial for the equilibrium uniqueness result. Without discounting, there exist equilibria where some high buyer-1 types make lower bids than some intermediate buyer-1 types (Tröger, 2004). Discounting induces strictly increasing payoff differences for buyer 1, which leads to a weakly increasing equilibrium bid function, which is the basis of the equilibrium uniqueness proof. By strictly increasing payoff differences we mean that buyer 1's payoff difference from switching to a larger bid that wins the original auction with higher probability is strictly increasing in her type. Without discounting, a higher type does not care more than a lower type whether she obtains the good today or tomorrow. Thus, payoff differences are only weakly increasing, which makes a nonmonotonous equilibrium bid function possible.

Like in the no-resale case, the equilibrium allocation of Proposition 2 is inefficient: it happens with positive probability that a buyer-1 type above θ_2 is outbid by buyer 2 and buyer 2's resale offer is so high that buyer 1 rejects it. An important question now is whether the resale opportunity increases or reduces the inefficiency of the allocation.

5 The impact of resale on efficiency

In this section we evaluate the impact of resale on the efficiency of the allocation. Apart from its general interest, such an evaluation is of direct importance to government agencies who (i) are legally obligated to strive for an efficient allocation when they act as sellers and (ii) are able to prevent resale. In our model, preferences are quasi-linear. Hence, efficiency only depends on who consumes the good and when. The appropriate efficiency measure is the expectation of the discounted realized consumption value of the good; this expectation is called the equilibrium surplus. The no-resale equilibrium surplus is

$$S_{\text{no resale}}(\theta_2) = E[\mathbf{1}_{b_1(\tilde{\theta}_1)\geq\tilde{b}_2}\tilde{\theta}_1 + \mathbf{1}_{b_1(\tilde{\theta}_1)<\tilde{b}_2}\theta_2],$$

where (b_1, H) denotes the equilibrium from Proposition 1, and $\tilde{b}_2$ is an independent random variable with distribution H. Observe that the equilibrium surplus $S_{\text{no resale}}(\theta_2)$ also depends on the value distribution F. We suppress this dependence because we will present comparative statics with respect to θ_2 that are valid for all F.

With a resale opportunity, the equilibrium surplus is

$$\begin{aligned} S_{\text{resale}}(\theta_2,\delta) \quad = \quad & \Pr[b_1(\tilde{\theta}_1)<\tilde{b}_2,\ \delta R(\tilde{b}_2)<\theta_2]\theta_2 \\ & +\delta\Pr[b_1(\tilde{\theta}_1)<\tilde{b}_2,\ \delta R(\tilde{b}_2)\geq\theta_2,\ \tilde{\theta}_1<T(\tilde{b}_2)]\theta_2 \\ & +\delta E[\mathbf{1}_{b_1(\tilde{\theta}_1)<\tilde{b}_2,\ \delta R(\tilde{b}_2)\geq\theta_2,\ \tilde{\theta}_1\geq T(\tilde{b}_2)}\tilde{\theta}_1] \\ & +\delta\Pr[b_1(\tilde{\theta}_1)\geq\tilde{b}_2,\ \tilde{\theta}_1<\delta\theta_2]\theta_2 + E[\mathbf{1}_{b_1(\tilde{\theta}_1)\geq\tilde{b}_2,\tilde{\theta}_1\geq\delta\theta_2}\tilde{\theta}_1], \end{aligned}$$

where (b_1, H, T) denotes the equilibrium outcome from Proposition 2, and $\tilde{b}_2$ is an independent random variable with distribution H.

In Proposition 3 we show that a resale opportunity can increase or reduce the equilibrium surplus, depending on the parameters; i.e, resale can be good or bad for efficiency. If buyer 2's value is large (i.e., close to the maximum possible value of buyer 1) and the discount factor is close to 1, resale is good; if buyer 2's value is small resale is bad.

Proposition 3 *If δ and θ_2 are sufficiently close to 1,*

$$S_{resale}(\theta_2, \delta) > S_{no\ resale}(\theta_2). \tag{32}$$

If θ_2 is sufficiently close to 0,

$$S_{resale}(\theta_2, \delta) < S_{no\ resale}(\theta_2). \tag{33}$$

The basic ideas behind the proof—which is in the Appendix—are as follows. Suppose that the discount factor is close to 1 and buyer 2's value is close to 1. If a buyer-1 type below the discounted buyer-2 value wins the auction, she will make a take-it-or-leave-it offer equal to buyer 2's value if resale is possible. Hence, whoever wins the auction the good will be eventually allocated to buyer 2 with a high probability. Thus, the final allocation is approximately efficient if resale is possible. In the no-resale equilibrium, buyer 2 lets buyer 1 obtains the good with a probability that stays bounded away from 0 even if buyer 2's value approaches 1. Hence, without resale the allocation is not approximately efficient if buyer 2's value is close to 1.

Now suppose buyer 2's value is small. Without resale, the auction is approximately efficient for the following reason. Buyer 1 can win for sure by bidding buyer 2's value; i.e., by submitting a small bid. Hence, the equilibrium payoff of every not-too-small buyer-1 type is close to her value, and this means in equilibrium she must be winning the auction with high probability. With a resale opportunity, buyer 2's auction winning probability stays bounded away from 0 even if her value is arbitrarily close to 0 (if her winning probability tended to 0, high buyer-1 types would make very small bids, but then it would be profitable for buyer 2 to overbid these high types and offer the good for resale). But if buyer 2's auction winning probability stays bounded away from 0, the same is true for the probability that she eventually keeps the good because buyer 2 sets a monopoly price in the resale market. Hence, with resale the allocation is not approximately efficient if buyer 2's value is close to 0.

Proposition 3 provides a guideline for an efficiency-minded seller. If she knows there is a strong buyer in the market, she should not try to establish a no-resale regime. But if she knows that a weak buyer is present, allowing resale is harmful. Of course, we have obtained these results only in a specific 2-bidder model. In general, it may not be commonly known whether a weak or a strong buyer is present, and matters can get more complicated.

6 The impact of resale on payoffs and revenue

This section completes our analysis with an investigation of the distributional impact of a resale opportunity.

In Proposition 4 we show that resale reduces both bidders' payoffs. For buyer 1 this holds for any possible type, unless the type's no-resale payoff already equals 0, in which case her resale-equilibrium payoff equals 0 as well. For these results, we assume that the discount factor is sufficiently close to 1.

Proposition 4 *For all $\theta_1 \in [0,1]$, let $U_1(\theta_1)$ and $V_1(\theta_1)$ denote the equilibrium payoffs with and without resale of buyer 1 with type θ_1. Let U_2 and V_2 denote buyer 2's respective equilibrium payoffs. Then, if δ is sufficiently close to 1,*

$$\forall \theta_1 \in [0,1]: \text{ if } U_1(\theta_1) > 0 \text{ then } V_1(\theta_1) > U_1(\theta_1), \tag{34}$$

$$V_2 > U_2. \tag{35}$$

The proof is in the Appendix and is based on the following ideas. In the resale equilibrium only buyer-1 types above $\delta\theta_2$ can get a positive payoff. No such type will resell the good, and if she obtains the good, she pays at least $\delta\theta_2$, whether she buys in the auction or in the resale market. But without a resale opportunity she can win for sure with a bid $\overline{b} < \delta\theta_2$ if δ is sufficiently close to 1. Hence, her payoff is smaller in the resale equilibrium compared to the no-resale equilibrium. As for buyer 2, without resale her payoff is positive because she can win by bidding below her value θ_2. With resale, buyer 2 submits, with positive probability, bids arbitrarily close to $\underline{b} \in [\delta\theta_2, \theta_2)$. Such bids win against no buyer-1 type above θ_2. Hence, with resale buyer 2's payoff tends to 0 as the discount factor tends to 1.

Finally, we evaluate the impact of a resale opportunity on the revenue of the initial seller.

Proposition 5 *If the discount factor is sufficiently close to 1, the seller's expected revenue is larger in the resale equilibrium than in the no-resale equilibrium.*

To prove this, it suffices to observe that the seller's no-resale revenue is smaller than θ_2, while her revenue in the resale equilibrium is larger than

$\delta\theta_2$. These lower and upper bounds are immediate from the equilibrium constructions.[10]

Provided the discount factor is sufficiently close to 1, we can conclude that the seller should certainly allow resale when one buyer is known to be strong (i.e., θ_2 large) because then resale increases her revenue as well as the efficiency of the market. If one buyer is known to be weak (i.e., θ_2 small), a seller who considers prohibiting resale has to weigh her revenue losses against the gains in efficiency.

The seller could also try to increase her revenue by setting a reserve price, but—in contrast to simply allowing resale—setting an effective reserve price requires knowledge of the value distribution F and of θ_2. It is easy to see that any given reserve price is revenue-*de*creasing for some (F, θ_2) pair, but allowing resale increases revenue for all (F, θ_2) pairs. In the spirit of Wilson (1987), allowing resale is (in the class of environments considered here) a *robust* way of increasing revenue, while setting a revenue-increasing reserve price requires knowledge of the market parameters.

Appendix

Proof of Proposition 1. Observe that $b_1(\theta_1) < \theta_1$ for all $\theta_1 > \underline{b}$, by definition of $\underline{b}$ in (12). Hence, $\phi(b) > b$ for all $b \in (\underline{b}, \overline{b}]$. This implies that H is continuous and strictly increasing on $(\underline{b}, \overline{b}]$. Moreover, the limit $H(\underline{b}) = \lim_{b \searrow \underline{b}} H(b)$ exists. Hence, H is a distribution function.

Let us verify (10). We have to show that any bid $b_2 \in [\underline{b}, \overline{b}]$ is optimal. By (13),

$$\forall \theta_1 \geq \underline{b}: \; F(\theta_1)(\theta_2 - b_1(\theta_1)) = V_2.$$

Hence,

$$\forall b_2 \in [\underline{b}, \overline{b}]: \; v_2(b_2) = F(\phi(b_2))(\theta_2 - b_2) = V_2. \tag{36}$$

For $b_2 \geq \overline{b}$,

$$v_2(b_2) \leq (\theta_2 - \overline{b}) \overset{(36)}{=} V_2.$$

For $b_2 \leq \underline{b}$,

$$v_2(b_2) \overset{(13)}{=} F(b_2)(\theta_2 - b_2) \overset{(11)}{\leq} V_2.$$

[10] Observe that a seller who's utility is increasing in her realized revenue should allow resale independently of her risk attitudes if the discount factor is sufficiently close to 1. Her realized revenue is less than θ_2 without resale, but larger than or equal to $\delta\theta$ in the resale equilibrium.

Let us now verify (9). For all types $\theta_1 \leq \underline{b}$, no bid $b \leq \underline{b}$ yields a positive payoff, and any bid $b > \underline{b}$ leads to a negative payoff.

Now consider types $\theta_1 > \underline{b}$. For all $b \in (\underline{b}, \overline{b})$, (14) implies

$$\frac{H'(b)}{H(b)} = \frac{1}{\phi(b) - b}.$$

Hence,

$$\frac{\partial v_1}{\partial b}(b, \theta_1) = H(b) \left(\frac{\theta_1 - b}{\phi(b) - b} - 1 \right).$$

One sees that $\partial v_1/\partial b < 0$ if $\theta_1 < \phi(b)$, $=$ if $=$, and $>$ if $<$. Therefore, the bid $b_1(\theta_1)$ is optimal for type θ_1 among all bids in the range $(\underline{b}, \overline{b})$. Because v_1 is continuous from the right at $\underline{b}$, a deviation to the bid $b = \underline{b}$ is not profitable; similarly, a deviation to $b = \overline{b}$ is not profitable. Any bid $b < \underline{b}$ leads to a 0 payoff, and any bid $b > \overline{b}$ leads to the payoff $\theta_1 - b < \theta_1 - \overline{b} = v_1(\overline{b}, \theta_1)$. *QED*

Proof of Proposition 2. Step 1: b_1 is strictly increasing and continuous on $[0, 1]$, so that it has a continuous inverse ϕ.

Clearly, b_1 is strictly increasing and continuous on $[0, \underline{b}]$. By definition of U_2 and $\underline{b}$, we have $\underline{b} = \theta_2 - U_2/F(\underline{b})$, which shows continuity of b_1 at $\underline{b}$. To prove that b_1 is strictly increasing and continuous on $[\underline{b}, 1]$, it is sufficient that M has these properties on $[\theta_2, 1]$. Continuity of M follows from Berge's Theorem of the Maximum. Strict monotonicity of M at θ_2 is clear because $M(\theta_2) = \theta_2 < M(\theta_1)$ for any $\theta_1 > \theta_2$. To complete the proof of strict monotonicity, consider any $\theta_1'' > \theta_1' > \theta_2$. Define for all $\theta_1 > \theta_2$ and $p \in [\theta_1, \theta_2]$,

$$g(p, \theta_1) = (1 - \frac{F(p)}{F(\theta_1)})p + \frac{F(p)}{F(\theta_1)}\theta_2.$$

Let p' be a maximizer in (18) if $\theta_1 = \theta_1'$, and let p'' be a maximizer in (18) if $\theta_1 = \theta_1''$. Then we have $p' > \theta_2$ and thus

$$M(\theta_1') = g(p', \theta_1') < g(p', \theta_1'') \leq g(p'', \theta_1'') = M(\theta_1'').$$

Step 2: buyer 2's posterior beliefs and resale decision.

For all $b_2 \leq \overline{b}$ such that buyer 2 wins with positive probability, and all $\theta_1 \in [0, \phi(b_2)]$, let

$$\Pi(\theta_1 \mid b_2) = \frac{F(\theta_1)}{F(\phi(b_2))}. \tag{37}$$

Then condition (1) is satisfied, buyer 2's resale offer satisfies (24), and buyer 2's resale payoff when she wins with bid b_2 equals $\delta M(\phi(b_2))$. Hence, buyer 2 offers the good for resale if and only if she wins with a bid $b_2 \geq b_1(\hat{\theta})$.

Step 3:

$$\forall \theta_1 > \underline{b}: \; b_1(\theta_1) < \theta_1. \tag{38}$$

If $\theta_1 \leq \theta_2$, the definitions of U_2 and $\underline{b}$ imply $F(\theta_1)(\theta_2 - \theta_1) < U_2$, which implies (38) by definition of b_1. If $\theta_1 \in (\theta_2, \hat{\theta}]$, inequality (38) is immediate. If $\theta_1 > \hat{\theta}$, inequality (38) follows from $M(\theta_1) < \theta_1$.

Step 4:

$$\forall b' > \underline{b} \; \exists \epsilon > 0 \forall b \in (b', \overline{b}] : \; m(b) \geq \epsilon. \tag{39}$$

For all $b \in [b_1(\hat{\theta}), \overline{b}]$,

$$\delta T(b) \geq \delta M(\phi(b)) \overset{\underline{b}<\theta_2<\hat{\theta},\ (21)}{>} b.$$

Hence, $m(b) \geq \phi(b)(1-\delta) \geq \hat{\theta}(1-\delta) \overset{\text{def}}{=} \underline{m}^1 > 0$.

For all $b \in [b', b_1(\hat{\theta}))$, the continuity of ϕ together with (38) implies that

$$m(b) \geq \min_{\hat{b} \in [b', b_1(\hat{\theta})]} \phi(\hat{b}) - \hat{b} \overset{\text{def}}{=} \underline{m}^2 > 0.$$

Thus, (39) holds with $\epsilon = \min\{\underline{m}^1, \underline{m}^2\}$ if $b' < b_1(\hat{\theta})$, and $\epsilon = \underline{m}^1$ if $b' \geq b_1(\hat{\theta})$.

Step 5: H is a probability distribution function.

By (39), H is strictly increasing and continuous on $(\underline{b}, \overline{b}]$, and the limit $H(\underline{b}) = \lim_{b \searrow \underline{b}} H(b)$ exists.

Step 6: proof of (6).

It is sufficient to show the following:

$$\begin{aligned} \forall b_2 \in [\underline{b}, \overline{b}] : \quad & u_2(b_2) = U_2, \\ \forall b_2 \notin [\underline{b}, \overline{b}] : \quad & u_2(b_2) \leq U_2, \end{aligned}$$

If $b_2 \in [\underline{b}, b_1(\hat{\theta}))$ then $\phi(b_2) < \hat{\theta}$, hence $\delta M(\phi(b_2)) < \theta_2$, hence

$$u_2(b_2) = F(\phi(b_2))(\theta_2 - b_2) \overset{(21)}{=} U_2.$$

If $b_2 \in [\delta\theta_2, \underline{b}]$ then $\phi(b_2) = b_2 < \hat{\theta}$ by (21). Hence,

$$u_2(b_2) = F(b_2)(\theta_2 - b_2) \overset{(19)}{\leq} U_2.$$

If $K > 0$ is chosen small enough,

$$\forall b_2 < \delta\theta_2 : \ u_2(b_2) \overset{(21)}{<} u_2(\delta\theta_2) = U_2.$$

If $b_2 \in [b_1(\hat{\theta}), \bar{b}]$ then $\phi(b_2) \geq \hat{\theta}$, hence

$$u_2(b_2) = F(\phi(b_2))(\delta M(\phi(b_2)) - b_2) \overset{(21)}{=} U_2.$$

Step 7: proof of (5).

Consider first buyer-1 types $\theta_1 \leq \underline{b}$. For all $b_2 > 0$, the resale offer $T(b_2) \geq \theta_2 > \underline{b} \geq \theta_1$. Hence, type θ_1 never gets an acceptable resale offer. Hence, for all $b \geq 0$,

$$u_1(b, \theta_1) = H(b)(\underbrace{\max\{\theta_1, \delta\theta_2\}}_{\leq \underline{b}} - b).$$

Because $H(b) = 0$ for all $b < \underline{b}$, any bid $b < \underline{b}$ is optimal for every type $\theta_1 < \underline{b}$, and the bid $b = \underline{b}$ is optimal for type $\theta_1 = \underline{b}$.

To show (5) for buyer-1 types $\theta_1 > \underline{b}$, observe first that $u_1(b, \theta_1) < u_1(\bar{b}, \theta_1)$ if $b > \bar{b}$, and $u_1(b, \theta_1) = 0$ if $b < \underline{b}$. Therefore, it is sufficient to focus on the bidding range $b \in [\underline{b}, \bar{b}]$.

An application of strict monotone comparative statics (see Edlin and Shannon, 1998) shows that T is strictly increasing on $[b_1(\hat{\theta}), \bar{b}]$. Let $\underline{T} = T(b_1(\hat{\theta}))$ denote the minimum equilibrium resale price. The supremum winning bid of buyer 2 that leads to an acceptable resale offer for type $\theta_1 \geq \underline{T}$ is denoted

$$T^{-1}(\theta_1) = \sup\{b' \leq \bar{b} \mid T(b') \leq \theta_1\}.$$

For all buyer-1 types $\theta_1 \leq \underline{T}$,

$$u_1(b, \theta_1) = H(b)(\theta_1 - b).$$

For all buyer-1 types $\theta_1 \geq \underline{T}$,

$$u_1(b, \theta_1) = H(b)(\theta_1 - b) + \delta \int_{(\max\{b, b_1(\hat{\theta})\}, T^{-1}(\theta_1))} (\theta_1 - T(b'))\mathrm{d}H(b').$$

Let D denote the set of discontinuities of T in $(b_1(\hat{\theta}), \bar{b})$. Because T is strictly increasing, D is countable. By the definitions of m and H, the function H is differentiable at all $b \in (\underline{b}, \bar{b}) \setminus (D \cup \{b_1(\hat{\theta})\})$, and

$$H'(b) = \frac{H(b)}{m(b)}. \tag{40}$$

Now consider $\theta_1 \in [\underline{T}, \hat{\theta}]$. For all $b \in (T^{-1}(\theta_1), \bar{b}) \setminus D$,

$$\begin{aligned}
\frac{\partial u_1}{\partial b}(b,\theta_1) &= H'(b)(\theta_1 - b) - H(b) \\
&\overset{(40)}{=} H'(b)(\theta_1 - b - m(b)) \\
&\overset{b>b_1(\hat{\theta})}{=} H'(b)(\theta_1 - \phi(b) - \delta(T(b) - \phi(b))) \\
&\overset{T(b)>\theta_1}{\leq} H'(b)(1-\delta)(\theta_1 - \phi(b)) \\
&\overset{\phi(b)>\theta_1}{\leq} 0.
\end{aligned}$$

For all $b \in (b_1(\hat{\theta}), T^{-1}(\theta_1)) \setminus D$,

$$\begin{aligned}
\frac{\partial u_1}{\partial b}(b,\theta_1) &= H'(b)(\theta_1 - b) - H(b) - \delta(\theta_1 - T(b))H'(b) \\
&\overset{(40)}{=} H'(b)(\theta_1 - b - \delta(\theta_1 - T(b)) - m(b)) \\
&= H'(b)(1-\delta)(\theta_1 - \phi(b)) \ \leq 0.
\end{aligned}$$

For all $b \in (\underline{b}, b_1(\hat{\theta}))$,

$$\begin{aligned}
\frac{\partial u_1}{\partial b}(b,\theta_1) &\overset{(40)}{=} H'(b)(\theta_1 - b - m(b)) \\
&= H'(b)(\theta_1 - \phi(b)) \\
&\quad \geq 0 \quad \text{if } b < b_1(\theta_1), \\
&\quad \leq 0 \quad \text{if } b > b_1(\theta_1).
\end{aligned}$$

In summary, for all $\theta_1 \in [\underline{T}, \hat{\theta}]$ and all $b \in (\underline{b}, \bar{b}) \setminus D$,

$$\text{if } b < b_1(\theta_1): \quad \frac{\partial u_1}{\partial b}(b,\theta_1) \geq 0, \tag{41}$$

$$\text{if } b > b_1(\theta_1): \quad \frac{\partial u_1}{\partial b}(b,\theta_1) \leq 0. \tag{42}$$

For every $b' > \underline{b}$, (39) implies that the function H is Lipschitz continuous on $[b', \bar{b}]$. Hence, $u_1(\cdot, \theta_1)$ is Lipschitz on $[b', \bar{b}]$. Hence, on the domain $[b', \bar{b}]$ the function $u_1(\cdot, \theta_1)$ can be written as the integral over its derivative. Hence, the inequalities (41) and (42) show that for type θ_1 the bid $b = b_1(\theta_1)$ is optimal in the range $[b', \bar{b}]$. A deviation to the bid $b = \underline{b}$ is not profitable because b' is arbitrary and $u_1(\cdot, \theta_1)$ is continuous from the right at $\underline{b}$.

Now consider types $\theta_1 \in (\underline{b}, \underline{T})$. Arguments similar to those above show that for type θ_1 the bid $b = b_1(\theta_1)$ is optimal in the range $[\underline{b}, b_1(\hat{\theta})]$. For

$b \in (b_1(\hat{\theta}), \bar{b}) \setminus D$,

$$\begin{aligned} \frac{\partial u_1}{\partial b}(b,\theta_1) &= H'(b)(\theta_1 - b) - H(b) \\ &\overset{\theta' \overset{\text{def}}{=} T(b) > \theta_1}{\leq} H'(b)(\theta' - b) - H(b) - \delta(\theta' - T(b))H'(b) \\ &\overset{\theta' \geq \underline{T}}{=} \frac{\partial u_1}{\partial b}(b,\theta') \\ &\overset{b > b_1(\theta')}{\leq} 0, \end{aligned}$$

which completes the optimality proof for types $\theta_1 \in (\underline{b}, \underline{T})$.

Types $\theta_1 \in (b_1(\hat{\theta}), 1]$ are treated similarly to the cases that were already discussed. *QED*

Proof of Proposition 3. It is sufficient to show the following four things,

$$\limsup_{\theta_2 \to 1} S_{\text{no resale}}(\theta_2) < 1, \tag{43}$$

$$\lim_{\theta_2 \to 0} S_{\text{no resale}}(\theta_2) = E[\tilde{\theta}_1], \tag{44}$$

$$\lim_{\theta_2 \to 1,\ \delta \to 1} S_{\text{resale}}(\theta_2, \delta) = 1, \tag{45}$$

$$\limsup_{\theta_2 \to 0} S_{\text{resale}}(\theta_2, \delta) < E[\tilde{\theta}_1]. \tag{46}$$

Proof of (43). Observe that V_2, $\underline{b}$, $\bar{b}$, ϕ, and H, as defined in Proposition 1, are functions of θ_2 and δ. Berge's Theorem of the Maximum implies

$$\begin{aligned} \lim_{\theta_2 \to 1} V_2 &= V_2^1 \overset{\text{def}}{=} \max_{b \in [0,1]} F(b)(1-b), \\ \lim_{\theta_2 \to 1} \underline{b} &\leq \underline{b}^1 \overset{\text{def}}{=} \max\ \arg\max_{b \in [0,1]} F(b)(1-b), \\ \lim_{\theta_2 \to 1} \bar{b} &= \bar{b}^1 \overset{\text{def}}{=} 1 - V_2^1. \end{aligned}$$

Hence, $\bar{b}^1 = 1 - F(\underline{b}^1)(1 - \underline{b}^1)$ and thus $\underline{b}^1 < \bar{b}^1$. Moreover, defining $\hat{b} = (\bar{b}^1 + \underline{b}^1)/2$, there exists $\underline{\theta} < 1$ such that

$$\forall \theta_2 \geq \underline{\theta}: \ \hat{b} \in (\underline{b}, \bar{b}) \tag{47}$$

In the compact range

$$\{(b, \theta_2) \mid \theta_2 \in [\underline{\theta}, 1],\ b \in [\hat{b}, \bar{b}]\},$$

the continuous mapping $(b, \theta_2) \mapsto \phi(b) - b$ is strictly positive. Hence, there exists a lower bound $\epsilon > 0$. Therefore,

$$\liminf_{\theta_2 \to 1} H(\hat{b}) \geq e^{-(\bar{b}^1 - \hat{b})/\epsilon} > 0.$$

Moreover,

$$\lim_{\theta_2 \to 1} \phi(\hat{b}) < 1.$$

Therefore,

$$\liminf_{\theta_2 \to 1} \Pr[b_1(\tilde{\theta}_1) \geq \tilde{b}_2] \geq \liminf_{\theta_2 \to 1} H(\hat{b})(1 - F(\phi(\hat{b}))) > 0.$$

This implies (43).

To prove (44), it is sufficient to show that

$$\lim_{\theta_2 \to 0} \Pr[b_1(\tilde{\theta}_1) \geq \tilde{b}_2] = 1. \tag{48}$$

Consider any $\epsilon > 0$. Fix any $\theta_1 > 0$ with $F(\theta_1) < \epsilon$. Note that

$$H(b_1(\theta_1))\theta_1 \geq v_1(b_1(\theta_1), \theta_1) \geq v_1(\bar{b}, \theta_1) = \theta_1 - \bar{b}.$$

Hence,

$$H(b_1(\theta_1)) \geq 1 - \frac{\bar{b}}{\theta_1} \to_{\theta_2 \to 0} 1.$$

If θ_2 is so small that $H(b_1(\theta_1)) > 1 - \epsilon$,

$$\Pr[b_1(\tilde{\theta}_1) \geq \tilde{b}_2] \geq \Pr[\tilde{\theta}_1 \geq \theta_1] \Pr[b_1(\theta_1) \geq \tilde{b}_2] \geq (1 - \epsilon)^2.$$

This proves (48) because ϵ is arbitrary.

As for (45), observe that $T(\tilde{b}_2) \geq \theta_2$. Hence, $S_{\text{resale}}(\theta_2, \delta) \geq \delta\theta_2$.

To prove (46), observe that $\delta > \theta_2 / M(1)$ if θ_2 is sufficiently small. Hence, Proposition 2 applies. By Berge's Theorem

$$\lim_{\theta_2 \to 0} \bar{b} = \bar{b}^0 \stackrel{\text{def}}{=} \delta \max_{p \in [0,1]} (1 - F(p))p > 0.$$

Define $\hat{b} = \bar{b}^0 / 2$. Then, because $m(b) \leq 1$,

$$H(\hat{b}) = e^{-\int_{\hat{b}}^{\bar{b}} \frac{1}{m(b)} \mathrm{d}b} \leq e^{\hat{b} - \bar{b}}.$$

Hence,

$$\limsup_{\theta_2 \to 0} H(\hat{b}) \leq e^{-\hat{b}} < 1.$$

Hence,

$$\liminf_{\theta_2 \to 0} \Pr[\tilde{b}_2 > \hat{b},\ \tilde{\theta}_1 \in (\hat{b}/2, \hat{b})] \geq (1 - e^{-\hat{b}})(F(\hat{b}) - F(\hat{b}/2)) > 0.$$

Consider any $\theta_2 < \hat{b}/2$. Suppose that $\tilde{b}_2 > \hat{b}$ and $\tilde{\theta}_1 \in (\hat{b}/2, \hat{b})$. Then buyer 2 wins the auction because $b_1(\tilde{\theta}_1) \leq \tilde{\theta}_1 < \tilde{b}_2$. Moreover, $T(\tilde{b}_2) > \tilde{b}_2 > \tilde{\theta}_1$ if $\delta R(\tilde{b}_2) \geq \theta_2$. Hence, buyer 2 consumes the good, which yields the value θ_2 or the discounted value $\delta\theta_2$, whereas the efficient allocation, to buyer 1, would yield at least the value $\hat{b}/2$. This completes the proof. *QED*

Proof of Proposition 4. To prove (34), consider any θ_1 such that $U_1(\theta_1) > 0$. Then $\theta_1 > \underline{b}^r \geq \delta\theta_2$, where $\underline{b}^r$ denotes the $\underline{b}$ defined in (20). By (21), $b_1(\theta_1) > \underline{b}^r \geq \delta\theta_2$. Also, $T(\tilde{b}_2) \geq \theta_2$. Hence,

$$U_1(\theta_1) \leq \theta_1 - \delta\theta_2, \tag{49}$$

On the other hand,

$$V_1(\theta_1) \geq v_1(\overline{b}^n, \theta_1) = \theta_1 - \overline{b}^n, \tag{50}$$

where $\overline{b}^n$ denotes the $\overline{b}$ defined in (12). If δ is sufficiently close to 1, $\overline{b}^n < \delta\theta_2$, hence (50) implies

$$V_1(\theta_1) > \theta_1 - \delta\theta_2 \stackrel{(49)}{\geq} U_1(\theta_1).$$

To prove (35), observe that V_2 is given by (11), and U_2 is given by (19). In particular, $V_2 > 0$ and $\lim_{\delta \to 1} U_2 = 0$. Hence, $V_2 > U_2$ if δ is sufficiently close to 1. *QED*

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