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A Characterization of Melioration in Game Theoretic Frameworks

by

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A Characterization of Melioration in Game Theoretic Frameworks

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Abstract

The theory of melioration, originally formulated for individual decision-making contexts, is formally extended to a game theoretic framework. This paper proves the existence of a distribution that is a melioration strategy on the part of all players. Further, the question of stability of such a distribution is studied.

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Keywords: Bounded Rationality; Melioration; Game Theory; Existence of Equilibrium; Stability.

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1 Introduction

Melioration is a simple myopic dynamic-adjustment theory of boundedly rational individual behavior, formulated for decision problems where the decision-maker has to repeatedly select one alternative from a set of alternatives (see Herrnstein 1990, 1991, 1997; Herrnstein and Vaughan 1980; Herrnstein and Prelec 1988, 1991, 1992). Melioration states that individuals consider the value obtained from a single choice of each alternative, and then shift choice to alternatives that provide a higher value. That is, melioration is a local, myopic maximization, where alternatives are treated as competing among themselves. On the contrary, overall value is maximized when the joint influence of all alternatives on value is scrutinized. Obviously, melioration may imply a significant loss in terms of overall value.

From a bounded rationality point of view, it is important to stress that the theory of melioration was developed on the basis of an intense experimental research that involves hundreds of experiments with animals and human-beings (see the above references and Davison and McCarthy 1988; Williams 1988; Kudadjie-Gyamfi and Rachlin 1996; Warry, Remington, and Barke 1999; see also Arthur 1991; de Palma, Myers, and Papageorgiou 1994; Bartolome 1995; Fehr and Zych 1998; Rabin 1998; Read, Loewenstein, and Rabin 1999).

It is conjectured here that the type of behavior described in melioration is not restricted to individual decision-making contexts, but applies to interdependent environments. Therefore, in this paper, the theory of melioration is formally characterized in a game theoretic context. To this end, the original modeling developed by Herrnstein and Prelec (1988, 1991, 1992) for individual decision-making contexts will be followed. In doing so, the notions of *Matching Distribution* and *Stable Matching Equilibrium* will be extended to interdependent environments.

The outline of the formalization of the theory of melioration in a game theoretic environment is as follows. First, an individual binary relation will be formulated on the basis of the principles of melioration. It will be illustrated that the melioration binary relation does not necessarily satisfy either asymmetry, or negative transitivity. This binary relation will be used to define the individual Stable Matching-reply correspondences. A Stable Matching-reply correspondence singles out, for every strategy of the opponents, those i-th distributions that might be an end result according to melioration. It will be shown that some properties of the melioration binary relation guarantees that the individual Stable Matching reply-correspondences are continuous and single-valued. Hence, Brouwer's fixed point theorem can be applied which shows that there is a distribution that is a melioration strategy on the part of all players. Stability of such a distribution will be analyzed. After showing that not all fixed points are stable, known results on the stability of dynamic systems will be used to present some results on

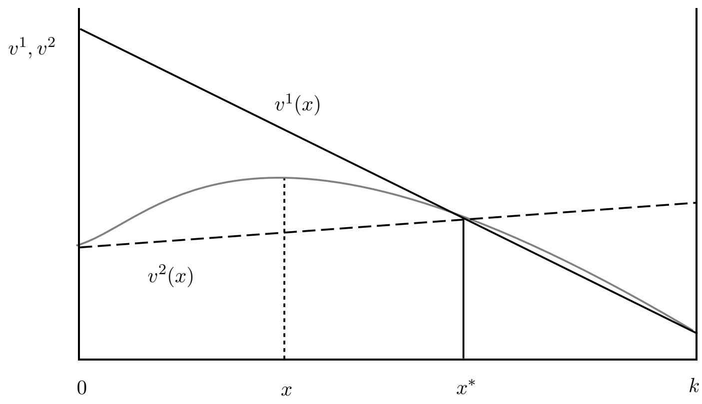

Figure 1: Individual Decision-Making Contexts

stable fixed points.

The structure of the paper is as follows. Section 2 presents an example of a decision-problem relevant to the theory of melioration in independent decision problems, and motivates its extension to a game theoretic context. Section 3 introduces the notation. In Section 4 the modeling of melioration in a game theoretic framework is developed. Section 5 concludes the paper.

2 An Example

The following abstract example will help to clarify the nature of the decision-problems relevant to melioration in individual decision-making contexts, and will introduce its extension to a game theoretic framework.

An individual has to choose repeatedly between two alternatives, say 1 and 2. These alternatives may represent two economic activities, two investment opportunities, two types of food, two hobbies, two different routes to work, whether to work or spend time with the family, to smoke or not to smoke, to use public transport or the car, etc. Let it be that, the more an alternative is chosen, the less value is obtained from a single choice of that alternative. Figure 1 represents these hypothetical value functions. The abscissa axis represents the choice rate at which alternative 1 is chosen over an observation interval, for example, over the last ten choices ($k = 10$). The observation interval fulfills the function of providing a "sufficient statistic for consumption history" (Herrnstein and Prelec 1991, p. 139). Reading the abscissa axis from right to left shows the choice rate of alternative 2. Note that although there are two alternatives, the choice distribution can be represented in the one dimensional space (in general, when L alternatives, the choice distribution is represented in the $L-1$ dimensional space). Therefore, when $L = 2$, superscripts can be omitted, and hence x represents the choice rate of alternative 1 and $(k-x)$ the choice rate of alternative 2. The continuous line in Figure 1 represents alternative 1's value function, the broken line alternative 2's value function. The value functions of alternatives 1 and 2 show the value obtained from a single choice of alternative 1 or 2, respectively, depending on the distribution of choice in the observation interval. The dotted curve in Figure 1 indicates the sum of the values of both alternatives weighted by their relative choice rate. It shows the average value obtained from a single choice, given the choice distribution. Note then that, if the choice rate of alternative 1 is low, a single choice of alternative 1 produces a higher value than a choice of alternative 2, and hence alternative 1 would be chosen according to melioration. That is, if the value attached to a single choice of one alternative is higher than that of the other alternative, melioration calls for the choice of the former. This process will eventually lead to a distribution by which the value derived from a single choice of one of the alternatives is equalized with that derived from a single choice of the other alternative (distribution x^* in Figure 1). On the other hand, overall value is maximized when the dotted curve is maximized, that is, at x.

Consider now that there is a second player, say player 2, whose behavior has an influence on the value functions of our initial individual, player 1. Denote player 1's value functions for alternatives 1 and 2 by

$$v_1^1(x_1, x_2) = 10 - 0.9x_1 - 0.1x_2$$

$$v_1^2(x_1, x_2) = 4 - 0.1(10 - x_1) + x_2$$

where x_1 and x_2 are the choice rates of alternative 1 by players 1 and 2, respectively. When $x_2 = 0$, then player 1's value functions can be represented as in Figure 1. There, the choice rate that maximizes player 1's overall value is to choose alternative-1 40% of the time, while the melioration strategy is to choose alternative-1 70% of the time. Figure 2 represents player 1's value functions when player 2 always choose alternative 1. Note now that the optimal and the meliorating strategies are a choice rate in alternative 1 equal to 0. This is an example of the types of interdependencies to be studied below. First, for every choice distribution of the opponents, a characterization of those distributions that are melioration strategies on the part of player i is developed. Second, this paper analyzes the question of existence of a distribution that is a melioration strategy on the part of all players. Finally, the issue of stability of such distributions is studied.

3 Notation

There are L alternatives subject to repeated choice indexed by $\ell = 1, \ldots, L$, and N players indexed by $i = 1, \ldots, N$. Denote by $x_i = (x_i^1, \ldots, x_i^L)$ player i's choice distribution over an observation interval, where x_i^ℓ denotes the choice rate at which alternative ℓ is chosen by player i. Player

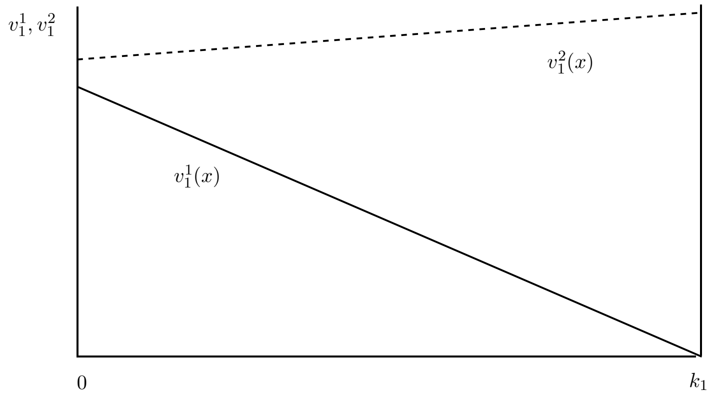

Figure 2: Player 1's value functions

i's total choice rate (i.e., player i's observation interval) is represented by $k_i \in \mathbb{R}_{++}$. Thus, $\sum_{\ell=1}^{\ell=L} x_i^\ell = k_i$. Let $X_i = \{x_i \in \mathbb{R}_+^L : \sum_{\ell=1}^{\ell=L} x_i^\ell = k_i\}$ be player i's choice set. $x = (x_1, \ldots, x_N)$ denotes the vector of the players' choice distribution, which will often be represented as (x_i, x_{-i}), where $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_N)$. Therefore, $X = X_1 \times \cdots \times X_N \subset \mathbb{R}^{N \times L}$ is the choice space of the game, and $X_{-i} = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \cdots \times X_N$. Function $v_i^\ell(x)$, $v_i^\ell : X \to \mathbb{R}$, is the i-th value function associated to alternative ℓ, that represents the value obtained by player i from a single choice of ℓ when $x \in X$. Individual total welfare, denoted by $u_i(x)$, $u_i : X \to \mathbb{R}$, is the weighted sum of all the value functions,

$$u_i(x) = \sum_{\ell=1}^{\ell=L} x_i^\ell v_i^\ell(x) \tag{1}$$

It only remains to introduce function $E_i : X_i \times X_i \times X_{-i} \to \mathbb{R}$,

$$E_i(y_i; x_i, x_{-i}) = \sum_{\ell=1}^{\ell=L} (y_i^\ell / k_i) v_i^\ell(x_i, x_{-i}) \tag{2}$$

that represents the i-th average value of distribution $x_i \in X_i$ weighted by distribution $y_i \in X_i$, when the opponents play $x_{-i} \in X_{-i}$. The i-th average value of distribution $x_i \in X_i$ when $x_{-i} \in X_{-i}$ is represented by

$$E_i(x_i; x_i, x_{-i}) = u_i(x)/k_i = \sum_{\ell=1}^{\ell=L} (x_i^\ell / k_i) v_i^\ell(x_i, x_{-i}) \tag{3}$$

4 Stable Matching Equilibrium in Games

Consider the following assumption on the value functions.

Assumption 1: *For all $i = 1, \ldots, N$, the value functions are continuous everywhere in X.*

First, in order to make possible to compare different choice distributions on the basis of the principles of melioration, a binary relation is introduced. Consider that Figure 1 above represents the two value functions of a player i when x_{-i}, and let x_i and y_i represent two choice distributions at the left (alternatively, at the right) of the intersection of the two value functions.[1] Let $x_i > y_i$ ($x_i < y_i$), and note that $v_i^1(x_i, x_{-i}) > v_i^2(x_i, x_{-i})$, ($v_i^1(x_i, x_{-i}) < v_i^2(x_i, x_{-i})$). According to melioration, since at x_i $v_i^1(x_i, x_{-i}) > v_i^2(x_i, x_{-i})$ ($v_i^1(x_i, x_{-i}) < v_i^2(x_i, x_{-i})$), the choice rate of alternative 1 will be increased (decreased). Therefore, if x_i is compared with y_i, and since $x_i > y_i$ ($x_i < y_i$), x_i is more attractive than y_i from the perspective of x_i, when x_{-i}. Note that the comparison of two distributions is done from the perspective of one of the two. That is, melioration incorporates the role of the *status quo*. Definition 1 below is a generalization of this argument to the case of L alternatives.

Definition 1: *For $x_i, y_i \in X_i$ and $x_{-i} \in X_{-i}$, it is said that $(x_i, x_{-i}) \succ_i (y_i, x_{-i})$ whenever $E_i(x_i; x_i, x_{-i}) > E_i(y_i; x_i, x_{-i})$.*

Then, $(x_i, x_{-i}) \succeq_i (y_i, x_{-i})$ whenever $E_i(x_i; y_i, x_{-i}) \geq E_i(y_i; y_i, x_{-i})$. The interpretation of the $\succ_i$ binary relation is as follows. $(x_i, x_{-i}) \succ_i (y_i, x_{-i})$ means that, given the strategy of the opponents, x_{-i}, to player i from the perspective of distribution x_i, distribution y_i is inferior. Or in other terms, since $(x_i, x_{-i}) \succ_i (y_i, x_{-i})$ whenever $E_i(x_i; x_i, x_{-i}) > E_i(y_i; x_i, x_{-i})$, $(x_i, x_{-i}) \succ_i (y_i, x_{-i})$ means that, given x_{-i}, the average value of distribution x_i weighted by distribution y_i is lower than the average value of distribution x_i weighted by distribution x_i itself. Below it is shown that $\succ_i$ is irreflexive, but is neither necessarily asymmetric, nor necessarily negatively transitive. Proposition 1 states this, but first consider the following standard definition of negative transitivity.

Definition 2: *The binary relation $\succ_i$ is negatively transitive if when $(x_i, x_{-i}) \succ_i (y_i, x_{-i})$, then for any third element z_i, either $(x_i, x_{-i}) \succ_i (z_i, x_{-i})$, or $(z_i, x_{-i}) \succ_i (y_i, x_{-i})$, or both.*

Proposition 1: *The $\succ_i$ binary relation is irreflexive, but is neither necessarily asymmetric, nor necessarily negatively transitive.*

Proof (i) That $\succ_i$ is irreflexive is obvious from its definition. (ii) Consider the example in Figure 3. Note that $(x_i - y_i)(v_i^1(x_i, x_{-i}) - v_i^2(x_i, x_{-i})) > 0$ and $(y_i - x_i)(v_i^1(y_i, x_{-i}) - v_i^2(y_i, x_{-i})) > 0$, which means that $(x_i, x_{-i}) \succ_i (y_i, x_{-i})$ and $(y_i, x_{-i}) \succ_i (x_i, x_{-i})$, and therefore $\succ_i$ is non-asymmetric. (iii) Note

[1] Recall that when $L = 2$, the i-th choice distribution may be represented by the i-th choice rate of alternative 1.

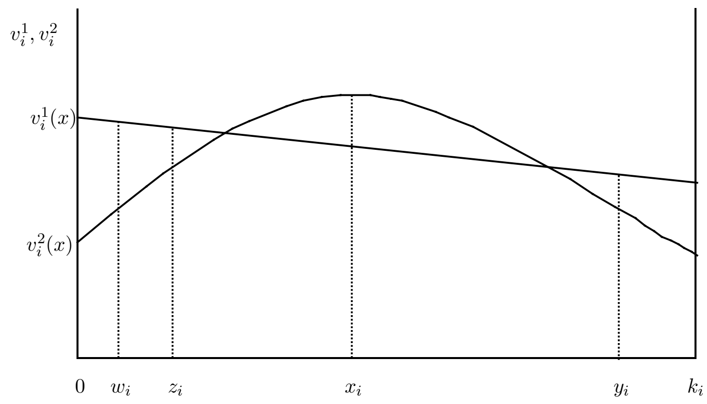

Figure 3: Properties of $\succ_i$

that in Figure 3 while $(z_i, x_{-i}) \succ_i (w_i, x_{-i})$ (that is, $(z_i - w_i)(v_i^1(z_i, x_{-i}) - v_i^2(z_i, x_{-i})) > 0$), it is not true either that $(z_i, x_{-i}) \succ_i (x_i, x_{-i})$ ($(z_i - x_i)(v_i^1(z_i, x_{-i}) - v_i^2(z_i, x_{-i})) < 0$), or that $(x_i, x_{-i}) \succ_i (w_i, x_{-i})$ ($(x_i - w_i)(v_i^1(x_i, x_{-i}) - v_i^2(x_i, x_{-i})) < 0$). Therefore, $\succ_i$ violates negative transitivity. ■

Before defining those distributions on the part of player i that might be an end result of i-th meliorating behavior when the strategy of the opponents is taken as given, consider the following broader definition.

Definition 3: *$x_i^* \in X_i$ is a Matching Strategy in Games (MSG) on the part of player i if, given $x_{-i} \in X_{-i}$, all active alternatives of player i enjoy equal value, that is*

$$\textit{for all } \ell = 1, \ldots, L \textit{ such that } x_i^{*\ell} > 0, v_i^\ell(x_i^*, x_{-i}) = E_i(x_i^*; x_i^*, x_{-i}) \tag{4}$$

Note that an interior MSG is a distribution where the values attached to a single choice of each one of the alternatives are equalized. Corner MSG are distributions where only the values of the active alternatives (i.e., those $x_i^{*\ell} > 0$) are equalized. Those distributions where all choice is absorbed by one alternative are also MSG.

Definition 4: *$x_i^* \in X_i$ is a Stable Matching Strategy in Games (SMSG) on the part of player i if, given $x_{-i} \in X_{-i}$, for every sufficiently small $\varepsilon > 0$ and for any other distribution $y_i \in X_i$, $(x_i', x_{-i}) \succ_i (y_i, x_{-i})$, where $x_i' = (1-\varepsilon)x_i^* + \varepsilon y_i$.*

Remark 1 below shows that every distribution that is an SMSG on the part of player i is a MSG. Note that $(x_i', x_{-i}) \succ_i (y_i, x_{-i})$ means $E_i(x_i'; x_i', x_{-i}) > E_i(y_i; x_i', x_{-i})$, which in turn, since

$$(x_i' - y_i) = (1-\varepsilon)(x_i^* - y_i) = \frac{(1-\varepsilon)}{\varepsilon}(x_i^* - x_i') \tag{5}$$

is equivalent to $E_i(x_i^*; x_i', x_{-i}) > E_i(x_i'; x_i', x_{-i})$. The interpretation of an SMSG is as follows. Given the strategy of the opponents, an SMSG on the part of player i, x_i^*, is a strategy that is immune to experimentation with alternative distributions. Consider the following example where $L = 2$. Given the strategy of the opponents, if player i deviates from x_i^* to x_i', Definition 4 implies that if the choice rate of alternative 1 has been increased (decreased), then the value function of alternative 1 at x_i' is lower (higher) than the value function of alternative 2, which makes player i to decreased (increased) alternative 1's choice rate, and hence to return to x_i^*. In other words, Definition 4 states that, given the strategy of the opponents, if player i being at x_i^* tries out any other distribution y_i, he/she will move to distribution x_i' from which perspective distribution y_i appears inferior, hence making player i to return to x_i^*.

Remark 1: *Every SMSG is a MSG.*

Proof Let, by way of contradiction, x_i^* be an SMSG on the part of player i, but not a MSG, when $x_{-i} \in X_{-i}$. Then, given x_{-i}, there are at least two alternatives, say ℓ and h, such that $x_i^{*\ell}, x_i^{*h} > 0$, and $v_i^\ell(x_i^*, x_{-i}) > v_i^h(x_i^*, x_{-i})$. Denote by y_i a distribution identical to distribution x_i^* in all its components, except that $y_i^\ell = x_i^{*\ell} + x_i^{*h}$ and $y_i^h = 0$. Let $x_i' = (1-\varepsilon)x_i^* + \varepsilon y_i$, for any sufficiently small $\varepsilon > 0$, and consider

$$\begin{aligned} & E_i(x_i'; x_i', x_{-i}) - E_i(y_i; x_i', x_{-i}) \\ & = \sum_{\ell=1}^{\ell=L} (x_i'^\ell - y_i^\ell) v_i^\ell(x_i', x_{-i}) \\ & = (1-\varepsilon) x_i^{*h} (v_i^h(x_i', x_{-i}) - v_i^\ell(x_i', x_{-i})) \end{aligned} \tag{6}$$

Continuity of the value functions (Assumption 1) implies that, for any sufficiently small $\varepsilon > 0$, (6) is strictly lower than zero, which contradicts x_i^* being an SMSG when x_{-i}. Therefore, every SMSG is a MSG. ■

Define the neighborhood of a distribution $x_i \in X_i$ as $V(x_i, \delta) = \{z_i \in X_i : \|x_i - z_i\| < \delta\}$, for some scalar $\delta > 0$. The following proposition shows that every SMSG is locally unique.

Proposition 2: *For any given $x_{-i} \in X_{-i}$, let x_i^* be a Stable Matching Strategy in Games on the part of player i, then there is a neighborhood of x_i^*, say $V(x_i^*, \delta)$, such that no $x_i \in V(x_i^*, \delta)$, $x_i \neq x_i^*$, is a Stable Matching Strategy in Games on the part of player i.*

Proof Let x_i and x_i' be two different distributions that are SMSGs on the part of player i when $x_{-i} \in X_{-i}$, and where $x_i' \in V(x_i, \delta)$. Take $x_i'' = (1-\varepsilon)x_i + \varepsilon x_i'$, being $\varepsilon > 0$ any sufficiently small scalar. Since x_i is an SMSG, $E_i(x_i''; x_i'', x_{-i}) > E_i(x_i'; x_i'', x_{-i})$, but this contradicts x_i' being an SMSG. ■

The following definition singles out for any $x_{-i} \in X_{-i}$ those x_i that are SMSGs on the part of player i.

Definition 5: *The Stable Matching-reply correspondence of player i, $\mu_i(x_{-i})$, is a correspondence from X_{-i} to a subset of X_i according to the following rule,*

$$\begin{aligned}\mu_i(x_{-i}) \quad = \quad & \{x_i \in X_i : ((1-\varepsilon)x_i + \varepsilon y_i, x_{-i}) \succ_i (y_i, x_{-i}) \textit{ for every} \\ & \textit{sufficiently small } \varepsilon > 0, \textit{ and for any } y_i \in X_i, y_i \neq x_i\} \quad (7)\end{aligned}$$

By Equation (5), the i-th Stable Matching-reply correspondence μ_i, can also be written as

$$\begin{aligned}\mu_i(x_{-i}) \quad = \quad & \{x_i \in X_i : E_i(x_i; x'_i, x_{-i}) > E_i(x'_i; x'_i, x_{-i}) \\ & \text{for all } x'_i \in V(x_i, \delta), x'_i \neq x_i\} \qquad (8)\end{aligned}$$

Note that, since SMSGs are locally unique, if for any $x_{-i} \in X_{-i}$, μ_i is composed of more than one element, μ_i is not convex-valued. Before introducing the definition of local insatiability, consider the following remark on SMSGs.

Remark 2: *Given any $x_{-i} \in X_{-i}$, for any x_i that is an SMSG on the part of player i, there is no other distribution $y_i \in V(x_i, \delta)$ such that $(y_i, x_{-i}) \succ_i (x_i, x_{-i})$.*

Proof (i) $(y_i, x_{-i}) \succ_i (x_i, x_{-i})$ means that $E_i(y_i; y_i, x_{-i}) > E_i(x_i; y_i, x_{-i})$, which contradicts that x_i is an SMSG on the part of player i when x_{-i}. ■

Definition 6: *$\succ_i$ is defined as locally insatiable whenever for all $x_{-i} \in X_{-i}$, and all $x_i \in X_i$, either x_i is an SMSG, or there is an $y_i \in V(x_i, \delta)$ such that $(y_i, x_{-i}) \succ_i (x_i, x_{-i})$.*

It is worth noting that Definition 6 excludes only the possibility that for any distribution of the opponents, $x_{-i} \in X_{-i}$, there is a distribution $x_i \in X_i$ and a neighborhood of x_i such that for all distributions in this neighborhood distinct from x_i, say y_i, $E_i(y_i; y_i, x_{-i}) = E_i(x_i; y_i, x_{-i})$. Consider the following remark.

Remark 3: *Given any $x_{-i} \in X_{-i}$ and any distribution $x_i \in X_i$, the impossibility of finding a distribution $y_i \in V(x_i, \delta)$ such that $(y_i, x_{-i}) \succ_i (x_i, x_{-i})$, does not mean that x_i is an SMSG.*

Proof Consider, for example, that $L = 2$ and for some $x_{-i} \in X_{-i}$, $v_i^1(x_i, x_{-i}) = v_i^2(x_i, x_{-i})$ for all $x_i \in X_i$. Clearly, for any $x_i \in X_i$ and all $y_i \in X_i$, it is not true either that $(y_i, x_{-i}) \succ_i (x_i, x_{-i})$, or that $(x'_i, x_{-i}) \succ_i (y_i, x_{-i})$, $x'_i = (1-\varepsilon)x_i + \varepsilon y_i$, $\varepsilon > 0$. ■

Take the following correspondences P_i and R_i from X to X_i

$$P_i(x) = \{y_i \in X_i : (y_i, x_{-i}) \succ_i (x_i, x_{-i})\}$$

$$R_i(x) = \{y_i \in X_i : (y_i, x_{-i}) \succeq_i (x_i, x_{-i})\}$$

For any given $x_{-i} \in X_{-i}$, $P_i(x)$ is the set of all distributions y_i in X_i from the perspective of which, distribution $x_i \in X_i$ is inferior. $R_i(x)$ is the set of all distributions y_i in X_i that, given $x_{-i} \in X_{-i}$, are not inferior from the perspective of distribution $x_i \in X_i$. It will be shown now that convexity of the set $P_i(x)$ and local insatiability of $\succ_i$ are sufficient conditions to ensure that, for any given $x_{-i} \in X_{-i}$, there is a distribution that is an SMSG on the part of player i. In fact, the imposition of these properties imply that $\mu_i(x_{-i})$ is nonempty, single-valued, and continuous.

Theorem 1: *If $\succ_i$ satisfies local insatiability and $P_i(x)$ is convex for all $x \in X$, then the Stable Matching-reply correspondence of player i, $\mu_i(x_{-i})$, is nonempty, single-valued, and continuous everywhere in X_{-i}.*

PROOF It is firstly shown that when $\succ_i$ is locally insatiable and for all $x \in X$ $P_i(x)$ is convex, $\mu_i(x_{-i})$ is nonempty and single-valued for all $x_{-i} \in X_{-i}$. To this end, Sonnenschein's (1971) Theorem 4 is used. The first step is to show that for any $x_{-i} \in X_{-i}$ there exists an $x_i^* \in X_i$ such that for all $x_i \in X_i$, $(x_i^*, x_{-i}) \succeq_i (x_i, x_{-i})$. That is, for any $x_{-i} \in X_{-i}$, it must be shown that for all finite set of distributions $x_i^0, x_i^1, \ldots, x_i^s, \ldots, x_i^m \in X_i$

$$\bigcap_{s=0}^{s=m} R_i(x_i^s, x_{-i}) \neq \emptyset \tag{9}$$

Consider the set $M = \{0, 1, \ldots, m\}$ and $T \subseteq M$. First, it is shown that

$$H_{s \in T}(\{x_i^s\}) \subseteq \bigcup_{s \in T} R_i(x_i^s, x_{-i}) \tag{10}$$

where $H_{s\in T}(\{x_i^s\})$ denotes the convex hull of the collection of points $\{x_i^s\}$, $s \in T$. Let, by way of contradiction, that there is some $T \subseteq M$ for which (10) does not hold. Then there exists a $z_i \in H_{s\in T}(\{x_i^s\})$ and $z_i \notin \bigcup_{s\in T} R_i(x_i^s, x_{-i})$. Note that when $(z_i, x_{-i}) \succeq_i (x_i^s, x_{-i})$ does not hold, $(x_i^s, x_{-i}) \succ_i (z_i, x_{-i})$ holds. Hence, for all $s \in T$, $x_i^s \in P_i(z_i, x_{-i})$. By convexity of $P_i(z_i, x_{-i})$, it is held that $H_{s\in T}(\{x_i^s\}) \subset P_i(z_i, x_{-i})$, which implies that $z_i \in P_i(z_i, x_{-i})$, a contradiction. Therefore, (10) must hold. Since (10) holds, then by Sonnenshein's (1971, Corollary 5) extension of the Knaster-Kuratowski-Mazurkiewicz Lemma to arbitrary collections of points, (9) holds. Since, (9) holds, there is at least one $x_i^* \in X_i$ such that $P_i(x_i^*, x_{-i}) = \emptyset$. According to local insatiability, if x_i^* is not an SMSG, there is another $x_i' \in V_i(x_i^*, \delta)$ such that $(x_i', x_{-i}) \succ_i (x_i^*, x_{-i})$, which violates $P_i(x_i^*, x_{-i}) = \emptyset$. Hence, x_i^* is an SMSG on the part of player i when

x_{-i}. It is shown now that in fact x_i^* is the unique SMSG on the part of player i, when x_{-i}. Let, by way of contradiction, $x_i' \in X_i$ be another SMSG on the part of player i when x_{-i}. Take $x_i'' = (1-\varepsilon)x_i' + \varepsilon x_i^*$, being $\varepsilon > 0$ any sufficiently small scalar. Then, since x_i' is an SMSG, $E_i(x_i''; x_i'', x_{-i}) > E_i(x_i^*; x_i'', x_{-i})$, which implies that $x_i'' \in P_i(x_i^*, x_{-i})$, a contradiction. Then, when $\succ_i$ is locally insatiable and for all $x \in X$ $P_i(x)$ is convex, $\mu_i(x_{-i})$ is nonempty and single-valued. Now it is shown that, in fact, $\mu_i(x_{-i})$ is a continuous function. Denote by $\{x^q\}$ the sequence of vectors $x^q \in X$ where $q = 1, 2, \ldots$. Suppose that $\{x_{-i}^q\} \subseteq X_{-i}$, $x_{-i}^q \to x_{-i}$, $\{x_i^q\} \subseteq X_i$, $x_i^q \to x_i$, $x_i^q = \mu_i(x_{-i}^q)$, and $x_i \neq \mu_i(x_{-i})$. Then, if $x_i \neq \mu_i(x_{-i})$, local insatiability of $\succ_i$ implies that there exists another distribution $y_i \in V_i(x_i, \delta)$ such that $(y_i, x_{-i}) \succ_i (x_i, x_{-i})$. Since the value functions are continuous everywhere in X by assumption, there exist neighborhoods $V(y_i, \delta_1)$, $V(x_i, \delta_2)$, and $V(x_{-i}, \delta_3)$, such that for all distributions $y_i' \in V(y_i, \delta_1)$, $x_i' \in V(x_i, \delta_2)$, and $x_{-i}' \in V(x_{-i}, \delta_3)$, $(y_i', x_{-i}') \succ_i (x_i', x_{-i}')$ holds. There are positive integers M' such that for $m' > M'$ it is held that $x_i^{m'} \in V(x_i, \delta_2)$, and M'' such that for $m'' > M''$ it is held that $x_{-i}^{m''} \in V(x_{-i}, \delta_3)$. Take $M = max\{M', M''\}$ and $m = max\{m', m''\}$, then for $m > M$, $(y_i', x_{-i}^m) \succ_i (x_i^m, x_{-i}^m)$, $y_i' \in V(y_i, \delta_1)$, which implies that $x_i^m \neq \mu_i(x_{-i}^m)$, a contradiction. Hence, $\mu_i(x_{-i})$ must be a continuous function everywhere in X_{-i}. ∎

Hence, by the application of Brouwer's fixed point theorem, it can be shown that there is a fixed point of $\mu(x) = \mu_1(x_{-1}) \times \cdots \times \mu_N(x_{-N})$, $\mu : X \to X$. That is, there is a distribution $x^* \in X$ that is an SMSG on the part of all players $i = 1, \ldots, N$.

Theorem 2: *If for all $i = 1, \ldots, N$, $\succ_i$ satisfies local insatiability and $P_i(x)$ is convex for all $x \in X$, then there is an $x^* \in X$ such that is an SMSG on the part of all players $i = 1, \ldots, N$.*

Proof The set X is nonempty, compact, and convex. Also, Theorem 1 shows that when for all $i = 1, \ldots, N$, $\succ_i$ satisfies local insatiability and $P_i(x)$ is convex for all $x \in X$, the correspondences $\mu_i(x_{-i})$ are nonempty, continuous, and single-valued. Therefore, Brouwer's fixed point theorem can be applied, which ensures that μ has a fixed point. That is, there is $x^* \in X$ such that $x^* = \mu(x^*)$, and therefore $x_i^* = \mu(x_{-i}^*)$ for all $i = 1, \ldots, N$. ∎

Consider the following example where the stability of a fixed point of μ is analyzed.

Example 1: There are two players that have to choose repeatedly between two alternatives with the following value functions

$$\begin{array}{ll} v_1^1(x) = 0.5 - x_1 & v_1^2(x) = 0.6 - 1.5x_2 \\ v_2^1(x) = 0.5 - x_2 & v_2^2(x) = 0.6 - 1.5x_1 \end{array}$$

where $x_1, x_2 \in [0,1]$ denote the relative choice rates of alternative 1 over an observation interval by players 1 and 2 respectively,[2] v_1^1 and v_1^2 are the value functions of player 1 for alternatives 1 and 2 respectively, and v_2^1 and v_2^2 are the value functions of player 2 for alternatives 1 and 2 respectively. It is easy to see that for every strategy of the opponent, there is one and only distribution that is an SMSG and that it can be found by the tendency to equalize the value functions. Then,

$$\mu_i(x_{-i}) = \{x_i \in X_i : v_i^1(x) = v_i^2(x)\}$$

Hence,

$$\mu_1(x_2) = -0.1 + 1.5x_2$$
$$\mu_2(x_1) = -0.1 + 1.5x_1$$

There is one and only one interior fixed point, $x^* = (0.2, 0.2)$, such that given x_1^*, x_2^* is an SMSG on the part of player 2 and, given x_2^*, x_1^* is an SMSG on the part of player 1 (note that corner distributions $(0,0)$ and $(1,1)$ are also fixed points of μ). However, for any $x' = (0.2+\varepsilon, 0.2+\varepsilon)$, $\varepsilon > 0$, $v_i^1(x') > v_i^2(x')$, $i = 1,2$, and hence players do not return to x^*, but tend to the corner distribution $x = (1,1)$ (alternatively if $x' = (0.2-\varepsilon, 0.2-\varepsilon)$, then players tend to the corner distribution $(0,0)$). Hence, there are points close to x^* where the meliorating behavior of players describe a tendency to diverge from it.

Example 1 clearly shows the necessity of differentiating between stable and unstable fixed points of μ. To this end consider the following definition of a melioration process where the strategies of the opponents are not assumed to remain fixed. Let t denote time, $\dot{x}$ the time derivative of x, and $x(t)$ the choice distribution at time t.

Definition 7: *The meliorating process is defined according to the following dynamic system*

$$\begin{aligned} \dot{x}_1 &= \gamma_1(v_1(x) - \bar{v}_1(x)) \\ &\vdots \\ \dot{x}_N &= \gamma_N(v_N(x) - \bar{v}(x)_N) \end{aligned} \tag{11}$$

where for all $i = 1, \ldots, N$, $\bar{v}_i(x) = \frac{\sum_{\ell=1}^{\ell=L} v_i^\ell(x)}{L}$, $\gamma_i > 0$, *being for all* $t \geq 0$ $x_i(t) \in X_i$.

Definition 7 is the simplest way of specifying a meliorating process. Note that when there are two alternatives, Definition 7 states that for all $i = 1, \ldots, N$, at any $x \in X$ if the i-th value functions are not equal, then the i-th choice rate of alternative 1 will, if possible, follow the direction indicated

[2] Of course, it does not make any difference to take x_i as alternative 1's choice rate by player i in the observation interval ($x_i \in [0, k_i]$), than to take it as the relative choice rate in the observation interval ($x_i \in [0,1]$).

by the sign of the difference between the i-th value functions. The following definition singles out those fixed points of μ that are stable according to the meliorating process.

Definition 8: *Let $x^* \in X$ be a fixed point of μ, then x^* is a* Stable Matching Equilibrium in Games *(SMEG) if there is a set $V(x^*, \delta)$ such that, for any starting point $x^0 \in V(x^*, \delta)$, the meliorating process asymptotically converges to x^*.*

If a fixed point of μ is not stable, it is said to be an *Unstable Matching Equilibrium in Games* (UMEG). In the following, based on known results on the asymptotic stability of dynamic systems in differential equations (see, e.g., Hirsch and Smale 1974), some results on Stable Matching Equilibrium in Games are presented. Denote $f(x) = (f_1(x), \ldots, f_N(x))$, where $f_i(x) = (f_i^1(x), \ldots, f_i^L(x))$, being $f_i^\ell(x) = \gamma_i(v_i^\ell(x) - \bar{v}_i(x))$.

Theorem 3: *If*

(i) $x^* \in int(X)$ is a fixed point of $\mu : X \subset \mathbb{R}^{N\times L} \to X \subset \mathbb{R}^{N\times L}$,

(ii) for all $i = 1, \ldots, N$ and $\ell = 1, \ldots, N$ value functions v_i^ℓ are continuously differentiable, and

(iii) all the eigenvalues of the Jacobian matrix of $f(x^*)$ have negative real parts,

then x^* is a Stable Matching Equilibrium in Games.

Proof When the value functions are not linear, since they are assumed to be continuously differentiable, the meliorating process can be linearized around the fixed point x^*. The linearization of the meliorating process around x^* can be written in matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(x^*)}{\partial x_1} & \cdots & \frac{\partial f_1(x^*)}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N(x^*)}{\partial x_1} & \cdots & \frac{\partial f_N(x^*)}{\partial x_N} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \tag{12}$$

Since, by assumption, all the eigenvalues of the squared matrix of (12) have negative real parts, it is guaranteed that for every $V(x^*, \delta)$ there is a $V_1(x^*, \delta)$ in $V(x^*, \delta)$ such that if $x^0 \in V_1(x^*, \delta)$ $x(t)$ are in $V(x^*, \delta)$ and $lim_{t\to\infty} x(t) = x^*$. If the meliorating process is a linear system, then (iii) guarantees that x^* is an SMEG such that starting from any $x \in X$ the meliorating process converges to x^*. ∎

It is easy to show that in the case of two players and two alternatives, x^* will be a Stable Matching Equilibrium in Games if

$$\frac{\partial f_1(x^*)}{\partial x_1}\frac{\partial f_2(x^*)}{\partial x_2} > \frac{\partial f_1(x^*)}{\partial x_2}\frac{\partial f_2(x^*)}{\partial x_1}$$

That is, x^* will be a Stable Matching Equilibrium in Games if the product of the crossed effects at x^* is lower than the product of the direct effects.

Theorem 4: *If*

(i) $x^* \in int(X)$ *is a fixed point of* $\mu : X \subset \mathbb{R} \to X \subset \mathbb{R}$,

(ii) for $i = 1, 2$ *and* $\ell = 1, 2$ *value functions* v_i^ℓ *are continuously differentiable, and*

(iii) $\frac{\partial f_1(x^*)}{\partial x_1}\frac{\partial f_2(x^*)}{\partial x_2} > \frac{\partial f_1(x^*)}{\partial x_2}\frac{\partial f_2(x^*)}{\partial x_1}$

then x^* *is a Stable Matching Equilibrium in Games.*

Proof Note that, since x_1^* is an SMSG when x_2^* and vice versa, by definition of SMSG

$$lim_{t\to\infty} x_1(t; x_2^*) = x_1^* \text{ for all } x_1^0 \in V(x_1^*, \delta_1)$$
$$lim_{t\to\infty} x_2(t; x_1^*) = x_2^* \text{ for all } x_2^0 \in V(x_2^*, \delta_2)$$

That is, the fact that x_i^* is an SMSG when x_j^* means that, if x_j^* remains fixed, for all x_i close to x_i^* the meliorating process converges to x_i^*. This implies that $\frac{\partial f_1(x^*)}{\partial x_1}$ and $\frac{\partial f_2(x^*)}{\partial x_2}$ must be negative. Denote by r the eigenvalues of the Jacobian matrix of $f(x)$, and, for convenience, take

$$\begin{bmatrix} \frac{\partial f_1(x^*)}{\partial x_1} & \frac{\partial f_1(x^*)}{\partial x_2} \\ \frac{\partial f_2(x^*)}{\partial x_1} & \frac{\partial f_2(x^*)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{13}$$

Then,

$$r = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}$$

Given that $\frac{\partial f_1(x^*)}{\partial x_1}$ and $\frac{\partial f_2(x^*)}{\partial x_2}$ are negative, to guarantee that the eigenvalues have negative real parts, it is sufficient to impose that

$$\frac{\partial f_1(x^*)}{\partial x_1}\frac{\partial f_2(x^*)}{\partial x_2} > \frac{\partial f_1(x^*)}{\partial x_2}\frac{\partial f_2(x^*)}{\partial x_1}$$

Hence, (i), (ii), and (iii) above guarantee that x^* is a Stable Matching Equilibrium in Games. ∎

5 Conclusions

In this paper the theory of melioration, originally intended for application in independent decision-problems, has been formally extended to a game theoretic framework.

A melioration binary relation has been defined. It has been shown that, to guarantee the existence of a distribution that is a Stable Matching Strategy in Games on the part of all players, it is sufficient to assume that the melioration binary relation is locally insatiable and that the set $P_i(x) = \{y_i \in X_i : (y_i, x_{-i}) \succ_i (x_i, x_{-i})\}$ is convex for all $x \in X$. However, not every such a distribution is stable when no restriction on the behavior of the opponents is imposed. A meliorating process where the behavior of the opponents is not assumed to be given has been defined. This has been used to differentiate between stable and unstable fixed points of the Stable Matching-reply correspondence, μ. The former are referred to as Stable Matching Equilibria in Games.

Finally, experimental research should be conducted to evaluate the predictive value of melioration in game theoretic contexts. To date, some clear initial support to the consideration of melioration in interdependent environments has been found in a two-person, two-alternative, time-dependent game (see Apesteguia 2001).

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