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# Regularity of Digits and Significant Digits of Random Variables

by

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# Regularity of Digits and Significant Digits of Random Variables

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**Abstract**

A random variable $X$ is digit-regular (respectively, significant-digit-regular) if the probability that every block of $k$ given consecutive digits (significant digits) appears in the $b$-adic expansion of $X$ approaches $b^{-k}$ as the block moves to the right, for all integers $b > 1$ and $k \geq 1$. Necessary and sufficient conditions are established, in terms of convergence of Fourier coefficients, and in terms of convergence in distribution modulo 1, for a random variable to be digit-regular (significant-digit regular), and basic relationships between digit-regularity and various classical classes of probability measures and normal numbers are given. These results provide a theoretical basis for analyses of roundoff errors in numerical algorithms which use floating-point arithmetic, and for detection of fraud in numerical data via using goodness-of-fit of the least significant digits to uniform, complementing recent tests for leading significant digits based on Benford's law.

*Key words:* normal numbers, significant digits, Benford's law, digit-regular random variable, significant-digit-regular random variable, law of least significant digits, floating-point numbers, nonleading digits, trailing digits
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## 1 Introduction

For each positive integer $n$, each integer $b > 1$ and each real number $r \geq 0$, let $D_n^{(b)}(r)$ denote the $n^{\text{th}}$ *digit (base $b$)* of $r$; that is,

$$r = \sum_{n=-\infty}^{\infty} b^{-n} D_n^{(b)}(r), \quad \text{where } D_n^{(b)}(r) \in \{0, 1, \ldots, b-1\},$$

and if $r$ has two $b$-adic expansions, then the terminating one, i.e., the one with $\lim_{n\to\infty} D_n^{(b)}(r) = 0$, is chosen. (For example, if $b = 10$ and $r = .02 = .019999\cdots$, then $D_2^{(10)}(r) = 2$ and $D_n^{(10)}(r) = 0$ for all $n \neq 2$.)

Similarly, for each $n \in \mathbb{N}$, $b \in \mathbb{N}\backslash\{1\}$ and $r > 0$, $S_n^{(b)}(r)$ will denote the $n^{\text{th}}$ *significant digit (base $b$) of* $r$, that is,

$$S_n^{(b)}(X) = D_{n+m-1}^{(b)}(X) \text{ for all } n \in \mathbb{N} \text{ on the set } \{b^{-m} \leq X < b^{-m+1}\}. \quad (1.1)$$

(So, e.g., $S_1^{(10)}(\pi/100) = S_1^{(10)}(\pi/10) = 3$, and $S_1^{(10)}(.01999) = S_1^{(10)}(.02) = D_2^{(10)}(.02) = 2$.) Also, for convenience of notation, set $S_n^{(b)}(0) = 0$ for all $n, b$.

The main goal of this article is to study the limiting behavior of the $n$-th digits and $n$-th significant digits, that is, the behavior of the trailing or least significant digits, for various classes of random variables. Non-leading significant digits play an important role in the analysis of roundoff errors in numerical algorithms using floating-point arithmetic (cf. [6]), and in statistical tests for fraud or human error in numerical data (e.g., [13], [14]). "Digit-regular" and "significant-digit-regular" random variables are defined and basic relationships are established between digit-regularity and various related classical notions including normal numbers, convergence of Fourier coefficients, and convergence in distribution.

The organization is as follows: §2 defines *digit-regular* random variables, and establishes necessary and sufficient conditions for a random variable to be digit-regular in terms of convergence of Fourier coefficients and in terms of convergence in distribution; §3 is the analog for *significant-digit-regular* random variables, with examples to show that neither digit-regularity nor significant-digit-regularity imply the other; and §4 defines *strongly digit-regular* distributions, establishes basic properties including equivalence of strong digit-regularity and strong significant-digit-regularity, and derives rates of convergence for digit-regularity of absolutely continuous distributions.

## 2 Digit-regular Random Variables

In the sequel, $X$ will denote a nonnegative random variable defined on some probability space $(\Omega, \mathcal{F}, P)$.

**Definition 2.1** $X$ is *digit-regular* (d.r.) *base* $b$ if

$$P(D_{n+j}^{(b)}(X) = d_j,\ 1 \le j \le k) \to b^{-k} \text{ as } n \to \infty \text{ for all } k \in \mathbb{N} \text{ and all } d_j \in \{0, 1, \dots, b-1\};$$

and is *digit regular* if it is d.r. base $b$ for all integers $b > 1$.

In particular, a random variable is digit-regular base 2 if, in the binary expansion of $X$, the probability that the $n$-th digit of $X$ is 0 approaches $1/2$ as $n$ goes to infinity, and, more generally, the probability that any given string of $k$ consecutive digits starting at the $n$-th place in the binary expansion approaches $2^{-k}$ as $n$ goes to infinity.

**Proposition 2.2** *If $X$ is d.r. base $b$ for some integer $b > 1$, then $X$ is continuous, i.e., $P(X = r) = 0$ for all $r \ge 0$.*

**Proof.** Suppose, by way of contradiction, that $P(X = x^*) > 0$ for some $x^* \ge 0$, and let $c_n = D_n^{(b)}(x^*)$, $n \in \mathbb{N}$. Fix $m \in \mathbb{N}$ such that $P(X = x^*) > b^{-m}$. It is clear that there exist digits $d_j \in \{0, 1, \dots, b-1\}$, $j = 1, \dots, m$, such that $(c_{n+1}, \dots, c_{n+m}) = (d_1, \dots, d_m)$ for infinitely many $n \in \mathbb{N}$. Then

$$\limsup_{n\to\infty} P(D_{n+j}^{(b)}(X) = d_j,\ 1 \le j \le m) \ge P(X = x^*) > b^{-m},$$

a contradiction. $\square$

The next example shows that a random variable $X$ may be continuous and a.s. completely normal, but not digit-regular. (Recall that a real number $x$ is normal base $b$ if the limiting frequency of the occurrence of every $k$-tuple of $\{0, 1, \dots, b-1\}$ in the $b$-adic expansion of $x$ is $b^{-k}$, and $x$ is (completely) normal if it is normal base $b$ for all $b$ [1,12].)

**Example 2.3** Let $x^* \in (0, 1)$ be completely normal, with binary expansion $x^* = \sum_{k=1}^{\infty} D_k^{(2)}(x^*) 2^{-k}$. Define $X$, via its binary expansion, by $D_n^{(2)}(X) \equiv D_n^{(2)}(x^*)$ if $n \ne k^k$ for any $k \in \mathbb{N}$, and let $\{D_{k^k}^{(2)}(X)\}$ be i.i.d uniform on $\{0, 1\}$. Clearly $X$ is continuous, and it is easy to see that since $x^*$ is completely normal, for every base $b$ and every $j \in \{0, 1, \dots, b-1\}$,

$$\lim_{n\to\infty} \frac{\#\{i \le n : D_i^{(b)}(X) = j\}}{n} = \lim_{n\to\infty} \frac{\#\{i \le n : D_i^{(b)}(x^*) = j\}}{n} = b^{-1}.$$

The argument for longer blocks is similar, which shows that $X(\omega)$ is completely normal for all $\omega$. Clearly $X$ is not d.r. base 2.

Conversely, digit-regularity base $b$ does not imply almost sure normality.

**Example 2.4** Let $\{X_n\}$ be i.i.d. Bernoulli random variables with $P(X_n = 0) = P(X_n = 1) = 1/2$, and let $\{Z_n\}$ be independent Bernoulli random variables, independent of the $\{X_n\}$, with $P(Z_n = 1) = 1 - P(Z_n = 0) = 1 - \frac{1}{n}$. Define the random variable $X$, via its binary representation, as follows: for any $m \in \mathbb{N}$, let

$$D_n^{(2)}(X) = Z_m X_n \quad \text{for all } n \in B_m := \{m^m, m^m + 1, \ldots, (m+1)^{m+1} - 1\}.$$

To see that $X$ is d.r. base 2, let $k \in \mathbb{N}$ and $(d_1, \ldots, d_k) \in \{0, 1, \ldots, b-1\}^k$. Fix $n > k$; then $\{n+1, \ldots, n+k\} \subset B_m \cup B_{m+1}$ for some $m = m(n,k)$. By definition of $X$, $\{X_n\}$, $\{Z_n\}$ and $m$,

$$\begin{aligned}
&P\left(D_{n+j}^{(2)}(X) = d_j, 1 \le j \le k\right) = \\
&P\left(D_{n+j}^{(2)}(X) = d_j, 1 \le j \le k \text{ and } Z_m = Z_{m+1} = 1\right) \\
&\qquad + P\left(D_{n+j}^{(2)}(X) = d_j, 1 \le j \le k \text{ and } Z_m Z_{m+1} = 0\right) \\
&= 2^{-k}\left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{m+1}\right) \\
&\qquad + P\left(D_{n+j}^{(2)}(X) = d_j, 1 \le j \le k \text{ and } Z_m Z_{m+1} = 0\right).
\end{aligned}$$

Since $m \to \infty$ as $n \to \infty$, $\lim_{n\to\infty} P\left(D_{n+j}^{(2)}(X) = d_j, 1 \le j \le k\right) = 2^{-k}$.

To see that $X$ is not normal base 2, note that by the Borel-Cantelli Lemma, $P(Z_n = 0 \text{ infinitely often}) = 1$, so $P$-almost surely there are infinitely many blocks $B_m$ where $D_n^{(2)}(X) = 0$ for all $n \in B_m$. But this implies that $\limsup_{n\to\infty} \frac{1}{n}\#\{i \le n : D_i^{(2)}(X) = 0\} = 1$, so $X$ is a.s. not normal base 2, (and hence not normal).

For each real Borel probability measure $\mu$, and each integer $n$, let $\phi_\mu(n)$ denote the $n^{\text{th}}$ Fourier coefficient of $\mu$, that is

$$\phi_\mu(n) = E(\exp(2\pi i n X)), \text{ where } X \text{ is a random variable with law } \mathcal{L}(X) = \mu.$$

**Theorem 2.5** *Let $X$ be a nonnegative random variable with distribution $\mu$. Then for each integer $b > 1$, the following are equivalent:*

(i) *$X$ is d.r. base $b$;*

(ii) *$X_n^{(b)} := b^n X (\text{mod } 1)$ converges in distribution as $n \to \infty$ to the uniform distribution on $[0,1)$;*

(iii) $\phi_\mu(mb^n) \to 0$ *as* $n \to \infty$ *for each integer* $m \neq 0$.

**Proof.** Fix $b \in \mathbb{N}\backslash\{1\}$. For integers $m \geq 1$ and $d_i \in \{0, 1, \ldots, b-1\}$, $i = 1, \ldots, m$, let

$$A^{(b)}(d_1, \ldots, d_m) = \{r \in [0,1) : D_i^{(b)}(r) = d_i,\ 1 \leq i \leq m\}.$$

Since only terminating expansions are considered,

$$A^{(b)}(d_1, \ldots, d_m) = \left[\sum_{i=1}^m d_i b^{-i},\ \sum_{i=1}^m d_i b^{-i} + b^{-m}\right), \tag{2.1}$$

and for any digits $(d_1, \ldots, d_m) \neq (0, \ldots, 0)$,

$$\bigcup\Bigg\{A^{(b)}(\tilde{d}_1, \ldots, \tilde{d}_m) : 0 \leq \tilde{d}_i \leq b-1, i = 1, \ldots, m; \\ \sum_{i=1}^m \tilde{d}_i b^{-i} \leq \sum_{i=1}^m d_i b^{-i} - b^{-m}\Bigg\} = \left[0, \sum_{i=1}^m d_i b^{-i}\right). \tag{2.2}$$

Note that

$$P\left(X_n^{(b)} \in A^{(b)}(d_1, \ldots, d_m)\right) = P\left(D_{n+i}^{(b)}(X) = d_i,\ 1 \leq i \leq m\right). \tag{2.3}$$

"(i) $\Rightarrow$ (ii)" If $X$ is d.r. base $b$, it follows from (2.1)–(2.3) that

$$P\left(X_n^{(b)} < \sum_{i=1}^m d_i b^{-i}\right) \to \sum_{i=1}^m d_i b^{-i} \quad \text{as } n \to \infty,$$

for every integer $m \geq 1$ and digits $0 \leq d_i \leq b-1$, which implies (ii) since the set $\{\sum_{i=1}^m d_i b^{-i} : m \geq 1,\ d_i \in \{0, 1, \ldots, b-1\}\}$ is dense in $[0,1]$.

"(ii) $\Rightarrow$ (i)" If (ii) holds, then by (2.2) and (2.3),

$$P\left(X_n^{(b)} \in A^{(b)}(d_1, \ldots, d_m)\right) \to b^{-m} \quad \text{as } n \to \infty.$$

By the definition of d.r., this implies (i).

"(ii) $\Leftrightarrow$ (iii)" Let $\lambda$ denote Lebesgue measure on $[0,1]$, and for each $n \in \mathbb{N}$, let $\mu_n = \mathcal{L}(X_n^{(b)})$. For each $n \in \mathbb{N}$, the Fourier coefficients $\{\phi_{\mu_n}(m)\}_{m=-\infty}^{\infty}$ uniquely determine $\mu_n$ [1, p. 361], and for each integer $m \neq 0$, $\phi_\lambda(m) = 0$ [1, Ex. 26.3]. Hence, by Lévy's Continuity Theorem [1, Th. 26.3]

$$X_n^{(b)} \xrightarrow{\mathcal{D}} U(0,1) \Leftrightarrow \phi_{\mu_n}(m) \to 0 \quad \text{for all integers } m \neq 0. \tag{2.4}$$

Fix an integer $m \neq 0$, and note that

$$\phi_{\mu_n}(m) = E[\exp(2\pi i m b^n X(\text{mod}1))] = E[\exp(2\pi i m b^n X)] = \phi_\mu(mb^n),$$

where the first equality follows by the definition of $X_n^{(b)}$, the second since $m \neq 0$, $b > 1$ and $n \geq 1$ are integers, and the last by definition of $\phi_\mu$. With (2.4), this completes the proof. $\square$

**Corollary 2.6** *If $X$ is a random variable with distribution $\mu$, and if $\phi_\mu(n) \to 0$ as $|n| \to \infty$, then $X$ is digit-regular.*

**Proof.** Immediate, since $\phi_\mu(n) \to 0$ as $|n| \to \infty$ implies $\phi_\mu(mb^n) \to 0$ as $n \to \infty$, since $b > 1$ and $m \neq 0$ are integers. $\square$

The next proposition shows that a random variable which is continuous and digit-regular base $b$ need not be digit-regular for other bases, nor be almost surely normal.

**Proposition 2.7** *Let $X$ have the classical middle-thirds Cantor-Lebesgue distribution on $(0,1)$, that is, letting $\{X_k\}_{k=1}^{\infty}$ be i.i.d. with $P(X_1 = 0) = P(X_1 = 2) = 1/2$,*

$$X = \sum_{k=1}^{\infty} X_k 3^{-k}.$$

*Then $X$ is digit-regular and normal base 2, but is neither digit-regular nor normal base 3.*

**Proof.** Since the ternary expansion of $X$ contains no 1's, clearly $X$ is neither d.r. nor normal base 3.

By a theorem of Feldman and Smorodinsky [5, p. 707] (see also [11]), since $\log 2/\log 3$ is irrational, and the ternary digit process for $X$ is non-degenerate and i.i.d., $X$ is a.s. normal base 2.

To see that $X$ is d.r. base 2, let $\nu$ denote the distribution of $Y = \frac{1}{2}X$, so $Y$ has the "right-thirds" Cantor-Lebesgue distribution on $(0,1)$. The measure $\nu$ satisfies the hypotheses of Theorem 5 of [9] with $p = 3$, $q = 2$ and $\mu = \nu$ since: 2 and 3 are multiplicatively independent; $\nu$ is continuous; $\nu$ is invariant under the map $T_3(x) = 3x (\text{mod } 1)$; $\nu$ is $T_3$-exact (i.e., satisfies (6) of [8]), since $\nu$ is a Bernoulli convolution [8, (12)] with g.c.d. $\{i_0 : p_{i_0} > 0\} = 1$ [8, p. 602]; $\nu$ satisfies (5) of [8], since $\nu$ is a Bernoulli convolution [8, p. 602]; and $\mu$ is trivially absolutely continuous with respect to some measure of the form $\delta(t) * T_r\nu$, since taking $t = 0$ and $r = 1$ yields $\nu$. Thus by [9, Theorem 5], $2^n X (\text{mod } 1)$ converges in distribution to the uniform distribution on $(0,1)$, so by Theorem 2.5, $Y$ is d.r. base 2. But $X = 2Y$ is d.r. base 2 if and only if $Y$ is d.r. base 2 by definition of digit-regularity, since $D_n^{(2)}(X) = D_n^{(2)}(2Y) = D_{n+1}^{(2)}(Y)$. $\square$

The converse of Corollary 2.6 is false, as the next proposition shows. By Proposition 2.2, digit-regularity implies continuity of a distribution, so by the

Riemann-Lebesgue Lemma [1, Theorem 26.1], the next proposition will also show that digit-regularity does not imply absolute continuity of the distribution. In order to establish the existence of a d.r. random variable whose Fourier coefficients do not vanish at infinity, the following number-theoretic lemma is needed. Recall that a subset $S$ of $\mathbb{N}$ has density zero in $\mathbb{N}$ if $\lim_{n\to\infty} \frac{1}{n} \#\{k \le n : k \in S\} = 0$.

**Lemma 2.8** *The set $S := \{mb^n : m, b, n \in \mathbb{N}, m \ge 1, b \ge 2, n \ge 2, b^n > m\}$ has density zero in $\mathbb{N}$.*

**Proof.** It suffices to show that $\sum_{s\in S} \frac{1}{s} < \infty$. Note that

$$
\begin{aligned}
\sum_{n\ge 2}\sum_{m\ge 1}\frac{1}{m}\sum_{b>m^{1/n}}\frac{1}{b^n} &= \sum_{n\ge 2}\sum_{b\ge 2}\frac{1}{b^n}\sum_{1\le m\le b^n-1}\frac{1}{m} \\
&\le \sum_{n\ge 2}\sum_{b\ge 2}\frac{1}{b^n}\left(1+\int_1^{b^n}\frac{dx}{x}\right) \le \sum_{n\ge 2}\sum_{b\ge 2}\frac{1}{b^n}(1+\ln b^n) \\
&\le 2\ln b\sum_{n\ge 2}\sum_{b\ge 2}\frac{n}{b^n} = 2\ln b\left(\sum_{n\ge 2}\frac{n}{2^n}+\sum_{n\ge 2}\sum_{b\ge 3}\frac{n}{b^n}\right) \\
&\le 2\ln b\left(\sum_{n\ge 2}\frac{n}{2^n}+\sum_{n\ge 2}\int_2^{\infty}\frac{n}{x^n}\,dx\right) \\
&= 2\ln b\left(\sum_{n\ge 2}\frac{n}{2^n}+\sum_{n\ge 2}\frac{n}{n-1}\frac{1}{2^{n-1}}\right) < \infty. \qquad \square
\end{aligned}
$$

**Proposition 2.9** *There exist random variables which are digit-regular whose Fourier coefficients do not vanish at infinity.*

**Proof.** Let $n_1, n_2 \ldots$ be a strictly increasing sequence of positive integers such that there is no solution to $mb^n = (n_{i_1} + \cdots + n_{i_k}) - (n_{j_1} + \cdots + n_{j_{\hat{k}}})$ for any integers $m, b, n$ with $m \ge 1$, $b \ge 2$, $n \ge 2$ and $b^n > m$, where the $k + \hat{k}$ summands are all distinct. Also, assume that the $n_i$'s are such that $0$ cannot be so represented. Such a sequence is easy to construct since by Lemma 2.8 the powers $\{mb^n : b \in \mathbb{N}\backslash\{1\}, n \in \mathbb{N}, b^n > m\}$ have density zero in $\mathbb{N}$, so there exist positive integers $y_1 < y_2 < \cdots$ such that the interval $[y_i - i, y_i + i]$ contains no members of $S$. Define $\{n_i\}$ inductively by $n_1 = y_1$, and $n_{k+1} = y_{n_1+\cdots+n_k}$. If $mb^n = n_{k+1} + \sum_{1\le i\le k} \delta_i n_i$, where $\delta_i \in \{0, \pm 1\}$, then $mb^n \in [n_{k+1} - (n_1 + \cdots + n_k), n_{k+1} + (n_1 + \cdots + n_k)]$, which contradicts the definition of the $\{y_k\}$.

Next, define the Riesz products (cf. [15, §V.7])

$$
p_k(t) = \prod_{j=1}^{k}(1 + \cos 2\pi n_j t), \qquad k = 1, 2, \ldots, \quad t \in [0, 1].
$$

It is easy to check that the $mb^n$-th Fourier coefficients of $p_k(t)$ are all 0 if $n \geq 2$, $m \geq 1$, and $b^n > m$. For example, if $k = 2$,

$$p_2(t) =$$
$$\Big(1 + (\exp(2\pi i n_1 t) + \exp(-2\pi i n_1 t))/2\Big)\Big(1 + (\exp(2\pi i n_2 t) + \exp(-2\pi i n_2 t))/2\Big)$$
$$= 1 + (\exp(2\pi i n_1 t))/2 + (\exp(2\pi i n_2 t))/2 + \cdots + (\exp 2\pi i((n_1 + n_2)t))/4.$$

(There are 9 terms in all.) None of these terms can be of the form $c \cdot \exp(2\pi i m b^k t)$ unless $c = 0$, or $k = 0$ or 1, since $mb^k$ cannot be a sum or difference of $0, n_1, n_2$. Thus, the $mb^n$-th Fourier coefficients, $n \geq 2$, $b^n > m$, are all 0. Note that $p_k(t) \geq 0$ for all $t \in [0,1]$, and that $\int_0^1 p_k(t)dt = 1$, since the constant term in the Fourier expansion of $p_k(t)$ is always 1, which follows from the assumption that 0 cannot be represented as $0 = \sum_{i=1}^k \delta_i n_i$, where $\delta_i \in \{-1, 1\}$. Thus for each $k \geq 2$, $p_k(t)$ is the density function of a Borel probability $P_k$ on $[0,1]$. By Prokhorov's theorem, there is a subsequence $(P_{k_j})$ of $(P_k)$ such that $P_{k_j}$ converges weakly to a probability measure $\mu$ on $[0,1]$. Since weak convergence implies convergence of integrals of bounded continuous functions, and since for $b^n > |m|$, the $mb^n$-th Fourier coefficients of $P_k$ are 0 for all $k$, the same is true for the limiting measure $\mu$. It remains to show that $\limsup_{n\to\infty} \phi_\mu(n) > 0$. Let $\{\hat{p}_k(n)\}$ denote the Fourier coefficients of $\{p_k\}$, so $p_k(t) = \sum_{n \in \mathbb{Z}} \hat{p}_k(n) e^{2\pi i n t}$. The key observation is that

$$\hat{p}_k(n_m) \geq \frac{1}{2} \qquad \text{for all } k \geq m \geq 1. \tag{2.5}$$

To see(2.5), write

$$\hat{p}_k(n_m) = \int_0^1 (\exp 2\pi i n_m t) \prod_{j=1}^k \left(1 + \frac{1}{2}\exp(2\pi i n_j t) + \frac{1}{2}\exp(-2\pi i n_j t)\right) dt.$$

Since $k \geq m$, the product in the last equality is a linear combination of exponential terms, amongst them $\frac{1}{2}\exp(-2\pi i n_m t)$, whose contribution to $\hat{p}_k(n_m)$ is $\frac{1}{2}$. Since the contribution of any exponential term is either zero or positive, this establishes (2.5).

Since $P_{k_j}$ converges weakly to $\mu$, (2.5) implies that $\lim_{j\to\infty} \hat{p}_{k_j}(n_m) = \phi_\mu(\mu_m) \geq \frac{1}{2}$ for all $m \geq 1$, so since $n_m \to \infty$ as $n \to \infty$, $\limsup_{n\to\infty} \phi_\mu(n) \geq \frac{1}{2}$. □

(Note that the $mb^n$-th Fourier coefficient of $P_k$ is zero for all $n \geq 2$, $b \geq 2$ and $m \neq 0$ such that $b^n > |m|$, which follows from the properties of the $(n_j)$. Hence $\phi_\mu(mb^n) = 0$ for all $n \geq 2$, $b \geq 2$ and $m \neq 0$ such that $b^n > |m|$.)

**Proposition 2.10** *Every random variable with a density is digit-regular and a.s. completely normal.*

**Proof.** Let $X$ be a random variable with a.c. distribution $\mu$; and, without loss of generality, $0 \le X < 1$. By the Riemann-Lebesgue Lemma, $\phi_\mu(n) \to 0$ as $n \to \infty$, so $X$ is d.r. by Corollary 2.6. As is well known [2, Prob. 8, p. 107], every random variable with a.c. distribution is a.s. completely normal. $\square$

## 3 Significant-digit-regular Random Variables

**Definition 3.1** $X$ is *significant-digit-regular* (s.d.r.) *base* $b$ if

$$P(S^{(b)}_{n+j}(X) = d_j, 1 \le j \le k) \to b^{-k} \text{ as } n \to \infty \text{ for all } k \in \mathbb{N}$$
$$\text{and all } d_j \in \{0, 1, \dots, b-1\};$$

and is *significant-digit-regular* if it is s.d.r. base $b$ for all $b$.

If $X$ is a random variable with values in $(0,1)$, and $\hat{X} = X + 1$, then it follows from (1.1) that

$$S^{(b)}_{n+1}(\hat{X}) = D^{(b)}_n(X) = D^{(b)}_n(\hat{X}) \quad \text{for all } b \ge 2 \text{ and } n \ge 1, \tag{3.1}$$

so $X$ is d.r. base $b$ if and only if $\hat{X}$ is d.r. base $b$ if and only if $\hat{X}$ is s.d.r. base $b$. Since $\hat{X}$ is a.s. normal base $b$ if and only if $X$ is a.s. normal base $b$, and $\hat{X}$ is absolutely continuous if and only if $X$ is absolutely continuous, the analog of Example 2.3 obtained by replacing $X$ by $X + 1$ yields a random variable which is continuous and a.s. completely normal, but is not s.d.r. Similarly, the analogs of Example 2.4 and Proposition 2.7, respectively, show that significant-digit-regularity base 2 does not imply a.s. normality, and that significant-digit-regularity base 2 does not imply significant-digit-regularity base 3. The analog of Proposition 2.9, that significant-digit-regularity does not imply absolute continuity of a random variable, is an immediate consequence of the Riemann-Lebesgue Lemma and Proposition 4.5 below.

Let $I_B$ denote the indicator function of the set $B$ and let $\lfloor a \rfloor$ denote the integer part of $a$.

**Theorem 3.2** *For all nonnegative random variables $X$ and all $b \in \mathbb{N} \backslash \{1\}$, the following are equivalent:*

(i) *$X$ is s.d.r. base $b$;*

(ii) *$b^{-\lfloor \log_b X \rfloor} X$ is d.r. base $b$;*

(iii) $\sum_{j \in \mathbb{Z}} E\left[I_{\{b^{-j} \le X < b^{-j+1}\}} \exp(2\pi i m b^{n+j} X)\right] \to 0$ *as $n \to \infty$ for each integer $m \ne 0$.*

**Proof.** Fix $b \in \mathbb{N}\backslash\{1\}$. Then

$$
\begin{aligned}
P & \left(S_{n+j}^{(b)}(X)=d_j, 1 \leq j \leq k\right) \\
& =\sum_{m \in \mathbb{Z}} P\left(S_{n+j}^{(b)}(X)=d_j, \quad 1 \leq j \leq k ; \quad b^{-m} \leq X<b^{-m+1}\right) \\
& =\sum_{m \in \mathbb{Z}} P\left(D_{n+j+m-1}^{(b)}(X)=d_j, \quad 1 \leq j \leq k ; \quad b^{-m} \leq X<b^{-m+1}\right) \\
& =\sum_{m \in \mathbb{Z}} P\left(D_{n+j-1}^{(b)}(b^m X)=d_j, \quad 1 \leq j \leq k ; \quad b^{-m} \leq X<b^{-m+1}\right) \\
& =\sum_{m \in \mathbb{Z}} P\left(D_{n+j-1}^{(b)}(b^{-\lfloor\log_b X\rfloor} X)=d_j, \quad 1 \leq j \leq k ; \quad b^{-m} \leq X<b^{-m+1}\right) \\
& =P\left(D_{n+j-1}^{(b)}(b^{-\lfloor\log_b X\rfloor} X)=d_j, \quad 1 \leq j \leq k\right), \qquad (3.2)
\end{aligned}
$$

where the second equality follows from (1.1); the third equality since $D_i^{(b)}(b^j r) = D_{i+j}^{(b)}(r)$ for $i, j \in \mathbb{Z}$, $r > 0$; and the fourth inequality since $b^{-m} \leq X < b^{-m+1} \Leftrightarrow -m \leq \log_b X < -m+1 \Leftrightarrow \lfloor \log_b X \rfloor = -m$. This establishes the equivalence of (i) and (ii).

Let $\mu$ denote the distribution of $b^{-\lfloor\log_b X\rfloor}X$. By Theorem 2.5, (ii) is equivalent to $\phi_\mu(mb^n) \to 0$ as $n \to \infty$ for each $m \neq 0$. But $\phi_\mu(mb^n) = E[\exp(2\pi i m b^n \cdot b^{-\lfloor\log_b X\rfloor}X)]$, and by dominated convergence, $\phi_\mu(mb^n) = \sum_{j\in\mathbb{Z}} E[I_{\{b^{-j}\leq X<b^{-j+1}\}} \exp(2\pi i m b^{n+j}X)]$, which establishes the equivalence of (ii) and (iii). $\square$

The next two results are the s.d.r. analogs of d.r. Proposition 2.2 and Proposition 2.10, respectively.

**Proposition 3.3** *If $X$ is s.d.r. base $b$ for some integer $b > 1$, then $X$ is continuous.*

**Proof.** Analogous to proof of Proposition 2.2.

**Proposition 3.4** *Every random variable with a density is significant-digit-regular and a.s. completely normal.*

**Proof.** Let $X$ be any r.v. with density, and fix base $b \geq 2$. Let $Y = b^{-\lfloor\log_b X\rfloor}X$ be the r.v. in Theorem 3.2(ii), so $Y$ also has a density, and by Proposition 2.10, $Y$ is d.r. base $b$ (in fact, for all bases). Theorem 3.2 then implies that $X$ is s.d.r. base $b$. $\square$

The next two examples show that digit-regularity base $b$ does not imply significant-digit-regularity base $b$, nor conversely.

**Example 3.5** The special case base $b = 2$ will be shown; the argument for general $b$ is analogous. Let $\{X_n\}_{n=1}^{\infty}$ be Bernoulli random variables defined as follows: $X_1$ is uniform on $\{0,1\}$, i.e., $P(X_1 = 0) = P(X_2 = 1) = 1/2$; $X_2 = 1 - X_1$; $X_{2^k} = X_1$ for all $k > 1$; and $\{X_1, X_n : n \neq 2^k \text{ for any } k\}$ are i.i.d., uniform on $\{0,1\}$. Let $X = \sum_{n=1}^{\infty} X_n 2^{-n}$, so $D_n^{(2)}(X) = X_n$ for all $n \geq 1$. Note that for each $m \in \mathbb{N}$ there exists $N = N(m)$ such that for all $n \geq N$, $D_{n+1}^{(2)}(X), \ldots, D_{n+m}^{(2)}(X)$ are i.i.d. uniform on $\{0,1\}$, which clearly implies that $X$ is d.r. (base 2).

To see that $X$ is not s.d.r. (base 2), note that $X_1 = 0 \Leftrightarrow X_2 = 1$, so on $\{X_1 = 1\}$, $D_n^{(2)}(X) = S_n^{(2)}(X)$ for all $n \geq 1$. Similarly, on $\{X_1 = 0\}$, $D_{n+1}^{(2)}(X) = S_n^{(2)}(X)$ for all $n \geq 1$.

Thus for $n = 2^k$ for some $k \geq 2$, $P(S_n^{(2)}(X) = 1) = P(S_n^{(2)}(X) = 1 \mid X_1 = 1)P(X_1 = 1) + P(S_n^{(2)}(X) = 1 \mid X_1 = 0)P(X_1 = 0) = P(X_n = 1 \mid X_1 = 1) \cdot \frac{1}{2} + P(X_{n+1} = 1 \mid X_1 = 0)\frac{1}{2} = \frac{3}{4} \neq \frac{1}{2}$, so $X$ is not s.d.r. (base 2).

**Example 3.6** Let $\{X_n\}_{n=1}^{\infty}$ be as in Example 3.5, and let $X = \sum_{n=1}^{\infty} \hat{X}_n 2^{-n}$, where $\{\hat{X}_n\}_{n=1}^{\infty}$ are Bernoulli random variables defined as follows: on $\{X_1 = 1\}$, $\hat{X}_n = X_n$ for all $n \geq 1$; on $\{X_1 = 0\} = \{X_2 = 1\}$, $\hat{X}_1 = \hat{X}_2 = 0$, $\hat{X}_3 = 1$, and $\hat{X}_n = X_{n-2}$ for $n \geq 4$. Since $D_n^{(2)}(X) = \hat{X}_n$ for all $n \geq 1$, the definition of $X$ implies that on $\{X_1 = 1\}$, $S_n^{(2)}(X) = D_n^{(2)}(X) = \hat{X}_n = X_n$ for all $n \geq 1$, and on $\{X_1 = 0\} = \{X_2 = 1\}$, $S_n^{(2)}(X) = D_{n+2}^{(2)}(X) = \hat{X}_{n+2} = X_n$ for all $n \geq 2$. In particular, $S_n^{(2)}(X) = X_n$ for all $n \geq 2$. Since $P(S_{n+j}^{(2)}(X) = d_j, 1 \leq j \leq m) = P(X_{n+j} = d_j, 1 \leq j \leq m)$, it follows as in Example 3.5 that for each $m \geq 1$ there exists $N = N(m)$ such that for all $n \geq N$, $X_{n+1}, \ldots, X_{n+m}$ i.i.d. uniform on $\{0,1\}$, so $P(X_{n+j} = d_j, 1 \leq j \leq m) = \left(\frac{1}{2}\right)^m$ for all $n \geq N(m)$, which shows that $X$ is s.d.r. (base 2).

To see that $X$ is not d.r. (base 2), let $n = 2^k$ for some $k \geq 3$, so $n \geq 4$ and $X_{n-2}$ is independent of $X_1$. Then $P(D_n^{(2)}(X) = 1) = P(\hat{X}_n = 1) = P(\hat{X}_n = 1 \mid X_1 = 1) \cdot \frac{1}{2} + P(\hat{X}_n = 1 \mid X_1 = 0) \cdot \frac{1}{2} = P(X_n = 1 \mid X_1 = 1) \cdot \frac{1}{2} + P(X_{n-2} = 1 \mid X_1 = 0) \cdot \frac{1}{2} = P(X_1 = 1 \mid X_1 = 1) \cdot \frac{1}{2} + P(X_{n-2} = 1) \cdot \frac{1}{2} = \frac{3}{4}$, so $X$ is not d.r. base 2.

For any base $b > 1$ and $n \in \mathbb{N}$ put

$$I_b(n) = \{(d_1, \ldots, d_n) : 1 \leq d_1 \leq b-1; 0 \leq d_i \leq b-1 \text{ for all } i = 2, \ldots, n\}$$

and

$$J_b(n) = \{(d_1, \ldots, d_n) : 0 \leq d_i \leq b-1 \text{ for all } i = 1, \ldots, n\}.$$

The following theorem, whose proof uses an elementary argument, shows that the significant digits of a random variable satisfying Benford's law converge to uniformity exponentially fast; the bound improves that in [6, Theorem 4] which only proves $O(b^{-n})$.

**Definition 3.7** Let $b$ be any integer $> 1$. A positive random variable $X$ is said to satisfy *Benford's law base b* (BL($b$)) if for all $(d_1, \ldots, d_k) \in I_b(k)$

$$P(S_j^{(b)}(X) = d_j, 1 \leq j \leq k) = \log_b \left[ 1 + \left( \sum_{i=1}^{k} d_i b^{k-i} \right)^{-1} \right] \tag{3.3}$$

(see [7]).

**Theorem 3.8** *Let $X$ satisfy BL(b) for some base $b > 1$. Then for all $k \in \mathbb{N}$, $(d_1, \ldots, d_k) \in J_b(k)$ and $n \geq 2$,*

$$\left| P\left( S_{n+j}^{(b)}(X) = d_j, 1 \leq j \leq k \right) - b^{-k} \right| \leq \frac{3}{b^{k+n-1} \ln b} . \tag{3.4}$$

**Proof.** Denoting the probability in (3.4) by $p_n(d_1, \ldots, d_k)$, (3.3) implies that

$$\begin{aligned} p_n(d_1, \ldots, d_k) = & \sum_{(\tilde{d}_1, \ldots, \tilde{d}_n) \in I_b(n)} P\Big( \left\{ S_i^{(b)}(X) = \tilde{d}_i, 1 \leq i \leq n \right\} \\ & \cap \left\{ S_{n+j}^{(b)}(X) = d_j, 1 \leq j \leq k \right\} \Big) \\ = & \sum_{m=b^{n-1}}^{b^n - 1} \log_b \left[ 1 + \left( b^k m + \sum_{j=1}^{k} d_j b^{k-j} \right)^{-1} \right] . \end{aligned}$$

Let $d_j \in \{0, 1, \ldots, b-1\}$, $\tilde{d}_j \in \{0, 1, \ldots, b-1\}$ be digits such that

$$\sum_{j=1}^{k} \tilde{d}_j b^{k-j} = 1 + \sum_{j=1}^{k} d_j b^{k-j}. \tag{3.5}$$

Putting $a_m = b^k m + \sum_{j=1}^{k} d_j b^{k-j}$ it follows that for all $n = 2, 3, \ldots$,

$$\begin{aligned} p_n(d_1, \ldots, d_k) - p_n(\tilde{d}_1, \ldots, \tilde{d}_k) &= \sum_{m=b^{n-1}}^{b^n - 1} \left( \log_b \left[ 1 + \frac{1}{a_m} \right] - \log_b \left[ 1 + \frac{1}{1 + a_m} \right] \right) \\ &= \sum_{m=b^{n-1}}^{b^n - 1} \log_b \left[ 1 + \frac{1}{a_m^2 + 2a_m} \right] \\ &\leq \frac{1}{\ln b} \sum_{m=b^{n-1}}^{b^n - 1} \frac{1}{(b^k m)^2 + 2b^k m} \\ &\leq \frac{1}{b^{2k} \ln b} \sum_{m=b^{n-1}}^{\infty} \frac{1}{m^2} \leq \frac{1}{b^{2k}(b^{n-1} - 1) \ln b} \\ &\leq \frac{2}{b^{2k+n-1} \ln b} . \end{aligned}$$

Let $p_{n,1}, p_{n,2}, \ldots, p_{n,b^k}$ denote the probabilities $p_n(d_1, \ldots, d_k)$ in lexicographic order starting with $(0, \ldots, 0, 0, 0)$, $(0, \ldots, 0, 0, 1), \ldots, (0, \ldots, 0, 0, b-1)$,

$(0,\ldots,0,1,0),\ldots,(0,\ldots,0,1,b-1),\ldots$ and ending with $(b-1,\ldots,b-1)$. As shown above

$$|p_{n,i} - p_{n,i+1}| \le \frac{2}{b^{2k+n-1}\ln b}\,, \qquad 1 \le i < b^k. \tag{3.6}$$

Since $1 - b^k p_{n,b^k} = \sum_{i=1}^{b^k-1} i(p_{n,i} - p_{n,i+1})$, (3.6) implies that

$$|p_{n,b^k} - b^{-k}| \le \frac{1}{b^k}\sum_{i=1}^{b^k-1} \frac{2i}{b^{2k+n-1}\ln b} \le \frac{1}{b^{k+n-1}\ln b}\,.$$

Using induction and (3.6) yields

$$|p_{n,b^k-m} - b^{-k}| \le \frac{b^k+2m}{b^{2k+n-1}\ln b}\,, \qquad 0 \le m \le b^k - 1. \qquad \square$$

## 4 Strongly Digit-Regular Distributions

**Definition 4.1** $X$ is called *strongly digit-regular* (strongly d.r.) *base* $b$ if for all Borel sets $B \subset [0,\infty)$ with $P(X\in B) > 0$, and for all $k \in \mathbb{N}$ and $d_j \in \{0,1,\ldots,b-1\}$,

$$P(D^{(b)}_{n+j}(X) = d_j, 1 \le j \le k \mid X \in B) \to b^{-k} \quad \text{as } n \to \infty, \tag{4.1}$$

and is *strongly digit-regular* if it is strongly d.r. base $b$ for all $b$.

Similarly, $X$ is *strongly significant-digit-regular* (strongly s.d.r.) *base* $b$ if (4.1) holds with $D^{(b)}_{n+j}(X)$ replaced by $S^{(b)}_{n+j}(X)$, and is strongly s.d.r. if it is strongly s.d.r. base $b$ for all $b$.

In contrast to the fact that neither digit-regularity base $b$ nor significant-digit-regularity base $b$ imply the other (Examples 3.5 and 3.6), in the context of conditional regularity (strong d.r. and s.d.r.), these concepts are equivalent. Note that the basic idea behind Examples 3.5 and 3.6 was exactly that of constructing digit-regular variables which were *not* conditionally digit-regular.

**Theorem 4.2** *Let $b \in \mathbb{N}\backslash\{1\}$. The following are equivalent:*

(i) *$X$ is strongly d.r. base $b$;*

(ii) *$X$ is strongly s.d.r. base $b$;*

(iii) *for each bounded Borel measurable function $f : [0,\infty) \to \mathbb{R}$, and each integer $m \neq 0$,*

$$E[f(X)\exp(2\pi i m b^n X)] \to 0 \quad \textit{as } n \to \infty; \tag{4.2}$$

(iv) $E[I_{[c,d]}(X)\exp(2\pi imb^nX)] \to 0$ *as* $n \to \infty$ *for all real numbers* $0 \le c \le d$ *and integers* $m \neq 0$.

**Proof.** Fix $b \in \mathbb{N}\backslash\{1\}$.

"(i) $\Leftrightarrow$ (ii)" This follows from (1.1) combined with a simple conditioning argument.

"(i) $\Rightarrow$ (iii)" Assume $X$ is strongly d.r. base $b$ and let $B \subset [0,\infty)$ be a Borel set such that $P(X \in B) > 0$. Applying Theorem 2.5 to the probability measure $P(\cdot \mid X \in B)$ shows that (4.2) holds for $f = I_B$, the indicator function of $B$. Thus, (4.2) holds for all Borel measurable simple functions $f : [0,\infty) \to \mathbb{R}$. If $f : [0,\infty) \to \mathbb{R}$ is bounded and Borel measurable, for every $\epsilon > 0$ there exist a Borel measurable simple function $f_\epsilon : [0,\infty) \to \mathbb{R}$ such that $|f(t) - f_\epsilon(t)| \le \epsilon$ for all $t \ge 0$, which proves (iii).

"(iii) $\Rightarrow$ (i)" is an immediate consequence of Theorem 2.5, (iii) $\Rightarrow$ (iv) trivially, and (iv) $\Rightarrow$ (iii) follows from a classical approximation result (see [1, Theorem 17.1]). $\square$

**Remarks.** Note that (iv) implies that $X$ is continuous. In light of Theorem 4.2, the random variable in Example 3.5 is d.r. but not strongly d.r., and that in Example 3.6 is s.d.r., but not strongly s.d.r.

By a standard approximation argument it is easy to see that Theorem 4.2(iv) is equivalent to $E[I_{[c,d[}(X)\exp(2\pi imb^nX)] \to 0$ as $n \to \infty$ for all real numbers $0 \le c < d$ and all integers $m \neq 0$, so letting $c = b^{-j}$ and $d = b^{-j+1}$ yields

$$E[I_{\{b^{-j} \le X < b^{-j+1}\}}\exp(2\pi imb^{n+j}X)] \to 0 \text{ as } n \to \infty, \text{ for each } m \neq 0 \text{ and } j \in \mathbb{Z}.$$

Since $X > 0$, for each $\epsilon > 0$ there exists $N = N(\epsilon)$ such that $P\left(\bigcup_{|j|>N}\{b^{-j} \le X < b^{-j+1}\}\right) < \epsilon$, which implies that the lim sup in Theorem 3.2(iii) is $< \epsilon$ as $n \to \infty$ for all $m \neq 0$; this yields a direct proof that the condition in Theorem 4.2(iv) implies that $X$ is s.d.r. base $b$.

**Theorem 4.3** *If $X$ has a density, then $X$ is strongly d.r. and strongly s.d.r.*

**Proof.** If $g$ is a density of $X$ and $B \subset [0,\infty)$ is a Borel set such that $P(X \in B) > 0$, then $\frac{1}{P(X\in B)} I_B g$ is a density of $X$ with respect to the conditional probability measure $P(\cdot \mid X \in B)$, and the conclusions follow by Propositions 2.10 and 3.4. $\square$

Certain statistical tests for detection of fraud or human error in numerical data are based on goodness-of-fit of least significant (or final) digits to uniform, the idea being that in true data the least significant digits are uniform, but in fabricated data, which may reflect individual preferences for particular digits or strings of digits, the least significant digits are not uniform. In classical tests of this type, the underlying true distribution of least significant digits of data is simply *assumed* to be uniform (e.g., [14, p. 572], [13, p. 66]); the next corollary gives a theoretical basis for the assumption of uniformity of final digits in true data.

**Corollary 4.4 (Least-significant-digit law)** *If $X$ has a density, then the significant digits base $b$ of $X$, $S_n^{(b)}(X)$, are asymptotically independent and uniformly distributed on $\{0, 1, \ldots, b-1\}$ for all integers $b > 1$.*

The next proposition generalizes the conclusion of Proposition 2.9 to strongly d.r. distributions.

**Proposition 4.5** *There exist random variables which are strongly digit-regular (equivalently strongly significant-digit-regular) whose Fourier coefficients do not vanish at infinity.*

**Proof.** Refine the construction in Proposition 2.9 as follows. Let $S$ be the set of integers $\{mb^n : m, b, n \in \mathbb{N}, m \geq 1, b \geq 2, n \geq 2, b^n > m\}$ in Lemma 2.8. First, it will be shown that there exist positive integers $12 \leq n_1 < n_2 < \cdots$ satisfying

$$n_t - 2(n_1 + n_2 + \cdots + n_{t-1}) \geq 4^t, \qquad t \in \mathbb{N} \tag{4.3a}$$

and

$$\Big[n_t - 2(n_1 + \cdots + n_{t-1}), n_t + 2(n_1 + \cdots + n_{t-1})\Big] \cap (S \cup \{0\}) = \emptyset, \quad t \in \mathbb{N}, \tag{4.3b}$$

(where void sums are taken to be zero). To see (4.3a)–(4.3b), first note that by Lemma 2.8, $S$ has density zero, so for each $t \in \mathbb{N}$ there exists a sequence of integers $12 \leq y_{t,1} < y_{t,2} < \cdots$ satisfying

$$[y_{t,j} - 2t, y_{t,j} + 2t] \cap (S \cup \{0\}) = \emptyset \quad \text{for all } j \in \mathbb{N}. \tag{4.4}$$

Define the sequence $(n_t)$ recursively as follows. Let $n_1 = y_{1,1}$, and note that, by (4.4), (4.3b) holds for $t = 1$. For each $t \in \mathbb{N}$, choose $k_t \in \mathbb{N}$ so large that $n_{t+1} := y_{n_1+\cdots+n_t, k_t}$ satisfies

$$n_{t+1} \geq 4n_t \quad \text{and} \quad n_{t+1} \geq 3 \cdot 4^{t+1}.$$

(Note that $n_1 \geq 12$.) Then (4.4) implies (4.3b), and for each $t \in \mathbb{N}$, $n_t - 2(n_1 + \cdots + n_{t-1}) = n_t\left(1 - 2\left(\frac{n_1}{n_t} + \cdots + \frac{n_{t-1}}{n_t}\right)\right) \geq n_t\left(1 - 2\left(\left(\frac{1}{4}\right)^{t-1} + \cdots + \frac{1}{4}\right)\right) \geq n_t\left(1 - 2\sum_{j=1}^{\infty}\left(\frac{1}{4}\right)^j\right) = \frac{1}{3}\, n_t \geq 4^t$, which proves (4.3a).

Define the Riesz products $(p_k)$ and $\mu$ as in Proposition 2.9, with the $(n_j)$ as defined above. Since

$$\begin{aligned}\left|\int_c^d \exp(2\pi i\alpha t)dt\right| &= \left|\frac{1}{2\pi i\alpha}\left(\exp(2\pi i\alpha d)-\exp(2\pi i\alpha c)\right)\right| \\ &< \frac{1}{|\alpha|} \quad \text{for all } 0 \le c \le d, \quad |\alpha| > 0,\end{aligned}$$

it follows from the definition of the $(p_k)$ that

$$\left|\int_c^d \exp(2\pi i m b^n t)p_k(t)dt\right| \le \frac{1}{mb^n} + \sum_{j=1}^k \left(\Sigma_{k,j}^{(+)} + \Sigma_{k,j}^{(-)}\right), \tag{4.5}$$

where $\Sigma_{k,j}^{(+)}$ is a sum of $2^{j-1}\binom{k}{j}$ terms of the form

$$\frac{1}{2^j}\frac{1}{|mb^n + n_{i_j} \pm n_{i_{j-1}} \pm \cdots \pm n_{i_1}|}$$

and $\Sigma_{k,j}^{(-)}$ is a sum of $2^{j-1}\binom{k}{j}$ terms of the form

$$\frac{1}{2^j}\frac{1}{|mb^n - n_{i_j} \pm n_{i_{j-1}} \pm \cdots \pm n_{i_1}|},$$

where $1 \le i_1 < i_2 < \cdots < i_j \le k$. (Note that $\Sigma_{k,j}^{(+)}$ and $\Sigma_{k,j}^{(-)}$ also depend on $m$, $b$, and $n$.)

For the rest of the proof fix $m \ge 1$ and $b \ge 2$. Let $n \ge 2$ be such that $b^n > m$ and $mb^n \ge n_2$, and let $u = u(m,b,n)$ be given by $n_u \le mb^n < n_{u+1}$. Since $mb^n \in S$, it follows from (4.3b) that

$$n_u + 2(n_1 + \cdots + n_{u-1}) < mb^n < n_{u+1} - 2(n_1 + \cdots + n_u). \tag{4.6}$$

Letting $i_j = t$, it follows that $j \le t \le k$, and by (4.3a) and (4.6),

$$\begin{aligned} mb^n + n_{i_j} \pm n_{i_{j-1}} \pm \cdots \pm n_{i_1} &\ge mb^n + n_t - (n_1 + \cdots + n_{t-1}) \\ &> n_u + 2(n_1 + \cdots + n_{u-1}) + n_t - (n_1 + \cdots + n_{t-1}) \\ &\ge 4^u + 4^t. \end{aligned}$$

Therefore,

$$\Sigma_{k,j}^{(+)} \le \frac{1}{2^j}\sum_{t=j}^k \binom{t-1}{j-1}\frac{2^{j-1}}{4^u + 4^t}.$$

[Note that given $i_j = t$ there are $\binom{t-1}{j-1}$ sequences of the form $1 \le i_1 < i_2 < \cdots < i_{j-1} \le t-1$, and each integer $n_{i_1}, \ldots, n_{i_{j-1}}$ can have the coefficient $\pm 1$

($2^{j-1}$ possibilities).] This implies

$$\sum_{j=1}^{k} \Sigma_{k,j}^{(+)} \le \sum_{j=1}^{k} \frac{1}{2^j} \sum_{t=j}^{k} \binom{t-1}{j-1} 2^{j-1} \frac{1}{4^u+4^t}$$
$$= \frac{1}{2} \sum_{t=1}^{k} \frac{1}{4^u+4^t} \sum_{j=1}^{t} \binom{t-1}{j-1}$$

which implies

$$\sum_{j=1}^{k} \Sigma_{k,j}^{(+)} < \sum_{t=1}^{k} \frac{2^t}{4^u+4^t}\,. \tag{4.7}$$

Furthermore, for $1 \le t \le u$, by (4.6) and (4.3a),

$$\begin{aligned} mb^n - n_{i_j} \pm n_{i_{j-1}} \pm \cdots \pm n_{i_1} &\ge mb^n - n_t - (n_1 + \cdots + n_{t-1}) \\ &> n_u + 2(n_1 + \cdots + n_{u-1}) - (n_1 + \cdots + n_t) \\ &\ge n_1 + \cdots + n_{u-1} \ge 4^{u-1} \ge \frac{1}{8}\,(4^u + 4^t). \end{aligned}$$

For $t = u+1$, (4.6) and (4.3a) imply that

$$\begin{aligned} n_{i_j} \pm n_{i_{j-1}} \pm \cdots \pm n_{i_1} - mb^n &\ge n_{u+1} - (n_1 + \cdots + n_u) - n_{u+1} + 2(n_1 + \cdots n_u) \\ &= n_1 + \cdots + n_u \ge 4^u > \frac{1}{8}\,(4^u + 4^t). \end{aligned}$$

Finally, for $u+2 \le t \le k$, by (4.6) and (4.3a),

$$\begin{aligned} n_{i_j} \pm n_{i_{j-1}} \pm \cdots \pm n_{i_1} - mb^n &\ge n_t - (n_1 + \cdots + n_{t-1}) - n_{u+1} + 2(n_1 + \cdots + n_u) \\ &\ge n_t - (n_1 + \cdots + n_{t-1}) - n_{t-1} > n_t - 2(n_1 + \cdots + n_{t-1}) \\ &\ge 4^t > \frac{1}{2}\,(4^u + 4^t). \end{aligned}$$

This implies, for $k \ge u+2$,

$$\begin{aligned} \sum_{j=1}^{k} \Sigma_{k,j}^{(-)} &\le 4 \sum_{j=1}^{k} \sum_{t=j}^{k} \binom{t-1}{j-1} \frac{1}{4^u+4^t} \\ &= 4 \sum_{t=1}^{k} \frac{1}{4^u+4^t} \sum_{j=1}^{t} \binom{t-1}{j-1} = 4 \sum_{t=1}^{k} \frac{2^{t-1}}{4^u+4^t} \\ &= 2 \sum_{t=1}^{k} \frac{2^t}{4^u+4^t}\,. \end{aligned}$$

By (4.5) and (4.7) this yields, for $0 \le c \le d \le 1$,

$$\left| \int_c^d \exp(2\pi i m b^n s) p_k(s) ds \right| \le \frac{1}{n_u} + 3 \sum_{j=1}^{k} \frac{2^j}{4^u+4^j}\,, \qquad k \ge u+2. \tag{4.8}$$

By symmetry, (4.8) also holds for integers $m \leq -1$, $b \geq 2$ and $n \geq 2$ such that $b^n > |m|$, $|m|b^n \geq n_2$ and $u \geq 2$ satisfying $n_u \leq |m|b^n < n_{u+1}$. Since (as shown in the proof of Proposition 2.9) the limiting measure $\mu$ satisfies $\phi_\mu(mb^n) = 0$ for $m \neq 0$, and $n \geq 2$ such that $b^n > |m|$, by Theorem 2.5 and Proposition 2.2 this implies that $\mu$ is continuous. Letting the random variable $X_k$ have distribution $P_k$, and $X_\infty$ have distribution $\mu$, since the set of discontinuities of the function $t \mapsto I_{[c,d]}(t) \cdot \exp(2\pi i m b^n t)$ has $\mu$-measure zero for all $0 \leq c \leq d \leq 1$, it follows (cf. [1, Theorem 25.7]) that

$$I_{[c,d]}(X_{k_j}) \exp(2\pi i m b^n X_{k_j}) \to I_{[c,d]}(X_\infty) \exp(2\pi i m b^n X_\infty) \quad \text{weakly as } j \to \infty.$$

Since those functions are uniformly bounded, this implies that

$$E[I_{[c,d]}(X_{k_j}) \exp(2\pi i m b^n X_{k_j}] \to E[I_{[c,d]}(X_\infty) \exp(2\pi i m b^n X_\infty)] \quad \text{as } j \to \infty,$$

so by (4.8),

$$\left| \int_0^1 I_{[c,d]}(s) \exp(2\pi i m b^n s) d\mu(s) \right| \leq \frac{1}{n_u} + 3 \sum_{j=1}^{\infty} \frac{2^j}{4^u + 4^j}.$$

(Note that $n \to \infty$ implies $u \to \infty$.) Therefore $\lim_{n\to\infty} \int_0^1 I_{[c,d]}(s) \exp(2\pi i m b^n s) d\mu(s) = 0$ for all $m \neq 0$ and $b \geq 2$. Hence a random variable $X_\infty$ with law $\mu$ is strongly d.r. (and also strongly s.d.r), but as in Proposition 2.9, it is easily seen that $\limsup_{n\to\infty} |\phi_\mu(n)| \geq \frac{1}{2}$. □

For fixed integers $b > 1$, $m \geq 1$, and $(d_1, \ldots, d_m) \in J_b(m)$, set

$$\langle d_1, \ldots, d_m \rangle_b := \sum_{k=1}^{m} d_k b^{-k}.$$

**Lemma 4.6** *Let $X$ be a random variable with density $f$ such that $0 \leq X < 1$. Then for all $b \in \mathbb{N}$, $b \geq 2$ and $(d_1, \ldots, d_k) \in J_b(k)$,*

(i) $P(D_j^{(b)}(X) = d_j, 1 \leq j \leq k) = \int_{\langle d_1,\ldots,d_k\rangle_b}^{\langle d_1,\ldots,d_k\rangle_b + b^{-k}} f(x)dx$

*and for all $n \in \mathbb{N}$,*

(ii) $P(D_{n+j}^{(b)}(X) = d_j, 1 \leq j \leq k) = \sum_{(a_1,\ldots,a_n)\in J_b(n)} \int_{\langle a_1,\ldots,a_n,d_1,\ldots,d_k\rangle_b}^{\langle a_1,\ldots,a_n,d_1,\ldots,d_k\rangle_b + b^{-(n+k)}} f(x)dx.$

**Proof.** Immediate from the definitions of $D_n^{(b)}$, $\langle d_1, \ldots, d_k\rangle_b$, and $\langle a_1, \ldots, a_n, d_1, \ldots, d_k\rangle_b$. □

**Theorem 4.7** *Let $X$ be a random variable such that $0 \leq X < 1$.*

(a) *Suppose that $X$ has density $f \in C^1$, and $|f'(t)| \leq L$ for all $t \in [0, 1]$. Then for all $j, k, b \in \mathbb{N}$, $b \geq 2$, $n \geq 0$ and all $d_j$, $\tilde{d}_j \in \{0, 1, \ldots, b-1\}$ satisfying (3.5),*

(i) $|P(D^{(b)}_{n+j}(X) = d_j, 1 \le j \le k) - P(D^{(b)}_{n+j}(X) = \tilde{d}_j, 1 \le j \le k)| \le Lb^{-(n+2k)}$;

(ii) $|P(D^{(b)}_{n+j}(X) = d_j, 1 \le j \le k) - b^{-k}| \le \frac{3L}{2}\, b^{-(n+k)}$; *and*

(iii) $|P(D^{(b)}_{k}(X) = d_1, D^{(b)}_{i}(X) = d_2) - b^{-2}| \le \frac{3L}{2}\, b^{-(k+1)}$, $1 \le k < i$.

(b) *Conversely, suppose that there exists some base* $b \ge 2$ *and a constant* $K$ *such that for all integers* $j \ge 1$, $k \ge 1$ *and* $n \ge 0$,

(iv) $|P(D^{(b)}_{n+j}(X) = d_j, 1 \le j \le k) - b^{-k}| \le Kb^{-(n+k)}$.

*Then* $X$ *is absolutely continuous with bounded density* $f$.

**Proof.** (a) "(i)" Note that in case $n \in \mathbb{N}$

$$\begin{aligned}
&|P\Big(D^{(b)}_{n+j}(X) = d_j, 1 \le j \le k\Big) - P\Big(D^{(b)}_{n+j}(X) = \tilde{d}_j, 1 \le j \le k\Big)| \\
&\qquad\le \sum_{(a_1,\ldots,a_n)\in J_b(n)} \int_{\langle a_1,\ldots,a_n,d_1,\ldots,d_k\rangle_b}^{\langle a_1,\ldots,a_n,d_1,\ldots,d_k\rangle_b + b^{-(n+k)}} |f(x) - f(x + b^{-(n+k)})|dx \\
&\qquad\le Lb^{-(n+2k)},
\end{aligned}$$

where the first inequality follows from Lemma 4.6(ii), and the second since $|f'(t)| \le L$.

"(ii)" Fix any integer $n \ge 0$ and let $\pi_{n,1}, \ldots, \pi_{n,b^k}$ denote the probabilities $P(D^{(b)}_{n+j}(X) = d_j, 1 \le j \le k)$ in lexicographic order on $(d_1, \ldots, d_k)$; i.e., $\pi_{n,1} = P(D^{(b)}_{n+j}(X) = 0, 1 \le j \le k)$, $\pi_{n,2} = P\Big((D^{(b)}_{n+1}(X), D^{(b)}_{n+2}(X), \ldots, D^{(b)}_{n+k}(X)) = (0, \ldots, 0, 1)\Big)$, etc. Then (ii) is equivalent to

$$|\pi_{n,i} - b^{-k}| \le \frac{3L}{2}\, b^{-(n+k)} \quad \text{for all } i = 1, \ldots, b^k.$$

In fact, starting with the identity

$$1 - b^k \pi_{n,b^k} = \sum_{i=1}^{b^k-1} i(\pi_{n,i} - \pi_{n,i+1})$$

(note that $\pi_{n,1} + \cdots + \pi_{n,b^k} = 1$) it follows from (i) that

$$\begin{aligned}
|b^{-k} - \pi_{n,b^k}| &\le b^{-k} \sum_{i=1}^{b^k-1} i|\pi_{n,i} - \pi_{n,i+1}| \\
&\le b^{-k} \sum_{i=1}^{b^k-1} iLb^{-(n+2k)} = \frac{L}{2}\, b^{-(n+3k)} b^k (b^k - 1) \\
&= \frac{L}{2}\,(b^k - 1) b^{-(n+2k)}.
\end{aligned}$$

By (i), this implies

$$\begin{aligned}
|b^{-k} - \pi_{n,b^k-1}| &\le |b^{-k} - \pi_{n,b^k}| + |\pi_{n,b^k} - \pi_{n,b^k-1}| \\
&\le \frac{L}{2}\,(b^k - 1) b^{-(n+2k)} + Lb^{-(n+2k)} = \frac{L}{2}\, b^{-(n+2k)}(b^k - 1 + 2),
\end{aligned}$$

and it follows by induction that for $0 \le j \le b^k - 1$,

$$|\pi_{n,b^k-j} - b^{-k}| \le \frac{L}{2}\,(b^k - 1 + 2j) b^{-(n+2k)} < \frac{3L}{2}\, b^{-(n+k)}.$$

"(iii)" If $i = k+1$, (iii) follows immediately from (ii). If $i \ge k+2$, then (writing $d_{i-k+1}$ instead of $d_2$),

$$\begin{aligned}
&|P(D_k^{(b)}(X) = d_1, D_i^{(b)}(X) = d_{i-k+1}) - b^{-2}| \\
&= |\sum_{(d_2,\ldots,d_{i-k}) \in J_b(i-k-1)} P(D_k^{(b)}(X) = d_1, D_{k+1}^{(b)}(X) = d_2, \ldots, D_i^{(b)}(X) = d_{i-k+1}) \\
&\qquad - b^{-2}| \\
&\le \sum_{(d_2,\ldots,d_{i-k}) \in J_b(i-k-1)} |P(D_k^{(b)}(X) = d_1, D_{k+1}^{(b)}(X) = d_2, \ldots, D_i^{(b)}(X) = d_{i-k+1}) \\
&\qquad - b^{-(i-k+1)}| \\
&\le \frac{3L}{2}\, b^{i-k-1} b^{-i} = \frac{3L}{2}\, b^{-(k+1)}. \qquad \text{This proves (a).}
\end{aligned}$$

(b) Fix the base $b$ as in (iv). For $n \in \mathbb{N}$, let $\mathcal{P}_n$ denote the partition of $[0,1)$ consisting of the $b^n$ sets $\{x \in [0,1) : D_j^{(b)}(x) = d_j, 1 \le j \le n\}$ for all $(d_1, \ldots, d_n) \in J_b(n)$, and let $\mathcal{F}_n$ denote the $\sigma$-algebra $\sigma(\mathcal{P}_n)$ generated by $\mathcal{P}_n$. Note that

$$\sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right) = \mathbb{B}([0,1)), \tag{4.9}$$

the $\sigma$-algebra of Borel sets on $[0,1)$. Let $\mu$ denote the distribution of $X$, and let $\lambda$ denote Lebesgue measure on $[0,1)$, so $\lambda(A) = b^{-n}$ for all $A \in \mathcal{P}_n$.

Let $(Y_n)_{n\in\mathbb{N}}$ be random variables on $[0,1)$ defined by

$$\begin{aligned} Y_n &= \sum_{A\in\mathcal{P}_n} \frac{\mu(A)}{\lambda(A)} I_A \\ &= b^n \sum_{(d_1,\dots,d_n)\in J_b(n)} P(D_j^{(b)}(X) = d_j, 1 \le j \le n) I_{\{D_j^{(b)}=d_j, 1\le j\le n\}}. \end{aligned}$$

It is easily seen that $(Y_n)$ is an $(\mathcal{F}_n)$-martingale satisfying $\int_0^1 Y_n d\lambda = 1$ for all $n \in \mathbb{N}$ (cf. [4, Chapter V, No. 6]). By (iv), $0 \le Y_n \le K+1$ for all $n \in \mathbb{N}$, so the martingale convergence theorem implies the existence of a random variable $Y_\infty \in L^1[0,1)$ such that $Y_n \to Y_\infty$ $\lambda$-almost surely. Since the $(Y_n)$ are uniformly bounded, the bounded convergence theorem implies that $\int Y_\infty d\lambda = 1$. Finally, it follows from (4.9) that $Y_\infty$ is a bounded density of $X$ (cf. [4, Chapter V, No. 56]). □

For $i \in \mathbb{N}$, let $I_i = I_i(b,d,X) = I_{\{D_i^{(b)}(X)=d\}}$, and $\widetilde{I}_i = I_i - E(I_i)$.

**Corollary 4.8** *Suppose $X$ has density $f \in C^1$. If $0 \le X < 1$ and $|f'(t)| \le L$ for all $t \in [0,1]$, then for any integer $d$, $0 \le d < b$,*

(i) $|E(I_i) - b^{-1}| \le \frac{3L}{2} b^{-i}$, $i \ge 1$;

(ii) $|E(I_iI_j) - b^{-2}| \le \frac{3L}{2} b^{-(i+1)}$, $1 \le i < j$;
*and*

(iii) $|E(\widetilde{I}_i\widetilde{I}_j)| \le \frac{9L(L+2)}{4} b^{-(i+1)}$, $1 \le i \le j$.

**Proof.** Conclusions (i) and (ii) follow immediately from Theorem 4.7 (ii) and (iii), respectively. For (iii), note that

$$\begin{aligned} |E(\widetilde{I}_i\widetilde{I}_j)| &= |E(I_iI_j) - E(I_i)E(I_j)| \\ &\le |E(I_iI_j) - b^{-2}| + |b^{-2} - (E(I_i) - b^{-1} + b^{-1})(E(I_j) - b^{-1} + b^{-1})| \\ &\le \frac{3L}{2} b^{-(i+1)} + |E(I_i) - b^{-1}||E(I_j) - b^{-1}| + b^{-1}|E(I_i) - b^{-1}| \\ &\quad + b^{-1}|E(I_j) - b^{-1}| \\ &\le \frac{3L}{2} b^{-(i+1)} + \frac{3L}{2} b^{-i} \cdot \frac{3L}{2} b^{-j} + b^{-1}\frac{3L}{2} b^{-i} + b^{-1} \cdot \frac{3L}{2} b^{-j} \\ &\le \left(\frac{3L}{2} + \frac{9L^2}{4} + \frac{3L}{2} + \frac{3L}{2}\right) b^{-(i+1)}. \end{aligned}$$

□

**Theorem 4.9** *Fix $b \in \mathbb{N}\backslash\{1\}$, and let $X$ be a random variable with $0 \le X < 1$ such that, for any integer $0 \le d < b$, $E(I_n) \to b^{-1}$ as $n \to \infty$, and $|E(\widetilde{I}_i\widetilde{I}_j)| = O(b^{-(i+1)})$, $1 \le i \le j$. Then $X$ is a.s. simply normal base $b$.*

**Proof.** First note that
$$\frac{1}{n}\sum_{i=1}^{n} I_i \to \frac{1}{b} \text{ a.s.}$$
is equivalent to
$$\frac{1}{m^2}\sum_{i=1}^{m^2} I_i \to \frac{1}{b} \text{ a.s.,} \tag{4.10}$$
[since $I_i \geq 0$ implies that for $m^2 \leq k < (m+1)^2$,
$$\frac{m^2}{(m+1)^2}\frac{1}{m^2}\sum_{i=1}^{m^2} I_i \leq \frac{1}{k}\sum_{i=1}^{k} I_i \leq \frac{(m+1)^2}{m^2}\frac{1}{(m+1)^2}\sum_{i=1}^{(m+1)^2} I_i,$$
and $\frac{m^2}{(m+1)^2} \to 1$, $\frac{(m+1)^2}{m^2} \to 1$ as $m \to \infty$].

Since $E(I_n) \to b^{-1}$, (4.10) is equivalent to
$$\frac{1}{n^2}\sum_{i=1}^{n^2} \widetilde{I}_i \to 0 \quad \text{a.s.} \tag{4.11}$$
By the Borel-Cantelli Lemma, to show (4.3) it suffices to show that for all $\epsilon > 0$,
$$\sum_{n=1}^{\infty} P\left(n^{-2}\left|\sum_{i=1}^{n^2} \widetilde{I}_i\right| > \epsilon\right) < \infty. \tag{4.12}$$
By Tschebyschev's inequality, the left hand side in (4.12) is
$$\sum_{n=1}^{\infty} P\left(n^{-2}\left|\sum_{i=1}^{n^2} \widetilde{I}_i\right| > \epsilon\right) \leq \sum_{n=1}^{\infty} \epsilon^{-2} n^{-4} \mathrm{Var}\left[\sum_{i=1}^{n^2} \widetilde{I}_i\right] = \sum_{n=1}^{\infty} \epsilon^{-2} n^{-4} E\left[\sum_{i=1}^{n^2}\sum_{j=1}^{n^2} \widetilde{I}_i\widetilde{I}_j\right].$$
Hence, it suffices to show that
$$\sum_{n=1}^{\infty} n^{-4}\left(\sum_{i=1}^{n^2}\sum_{j=1}^{n^2} E(\widetilde{I}_i\widetilde{I}_j)\right) < \infty. \tag{4.13}$$
Since $|\widetilde{I}_i| = |I_i - E(I_i)| \leq 2$,
$$\sum_{n=1}^{\infty} n^{-4}\sum_{i=1}^{n^2} E(\widetilde{I}_i^2) \leq \sum_{n=1}^{\infty} 4n^2 n^{-4} < \infty.$$
But
$$\begin{aligned}
\sum_{1\leq i<j\leq n^2} |E(\widetilde{I}_i\widetilde{I}_j)| &= \sum_{i=1}^{n^2-1}\sum_{j=i+1}^{n^2} |E(\widetilde{I}_i\widetilde{I}_j)| \\
&\leq \sum_{i=1}^{n^2-1} c(n^2-i)b^{-(i+1)} \leq cn^2\sum_{i=1}^{\infty} b^{-(i+1)} \\
&= cn^2 b^{-2}(1-b^{-1})^{-1} = cn^2 b^{-1}(b-1)^{-1} \leq cn^2
\end{aligned}$$

for some $c > 0$, where the first inequality follows by the hypothesis that $|E(\widetilde{I}_i \widetilde{I}_j)| = O(b^{-(i+1)})$. This establishes (4.11). $\square$

**Remark.** It follows from Corollary 4.8 and Theorem 4.9 that if $0 \leq X < 1$ has density $f \in C^1$, then $X$ is a.s. simply normal base $b$ for all $b > 1$; this is a very special case of the fact [2] that every random variable with density is a.s. normal.

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