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by

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# Hildenbrand Distribution Economies as Limiting Empirical Distributions of Random Economies *

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**Abstract:** The present note shows that the concept of a *distribution economy* (Hildenbrand (1974)) is closely related to a framework of an exchange economy in which the agents' individual characteristics (i.e. preferences and endowments) are random (Hildenbrand (1971), Bhattacharya and Majumdar (1973), Föllmer (1974)). A *random exchange economy* is fully specified by the distribution of the family of random variables representing the agents' individual characteristics. This distribution is a probability measure $\mu$ on $(S^A, \mathcal{B}(S)^A)$ with $S$ denoting the space of individual characteristics, $\mathcal{B}(S)$ the Borel $\sigma$-algebra generated by an appropriate topology on $S$ and $A$ a denumerable set of agents. The linkage between a Hildenbrand distribution economy and an ergodic random exchange economy with a countably infinite set of agents endowed with the graph topology of the integer lattice $\mathbb{Z}^d$ is established in this paper by a convergence result for the empirical distribution of the latter. For any increasing sequence of finite subsets of $A$ exhausting $A$, the associated sequence of empirical distributions converges almost everywhere on the underlying probability space to some distribution $\nu$ on $(S, \mathcal{B}(S))$. As far as aggregate variables of the economy, such as the mean demand or the equilibrium price system are concerned, any infinite random exchange economy with converging limiting empirical distribution $\nu$ is equivalent to a Hildenbrand distribution economy characterized by the same distribution $\nu$. This relationship suggests an approach to endogenous modelling of distributions of individual characteristics in General Equilibrium Theory. Thereby, specific distributions of characteristics can be obtained from a specific stochastic microstructure of local interaction between agents affecting their individual characteristics.

JEL classification: D50

Key words: General Equilibrium Theory, Arrow-Debreu economy, Random Economy, Distribution Economy, Empirical Distribution

# 1 Introduction

The present note shows that the concept of a *distribution economy* (Hildenbrand (1974)) is closely related to a framework of an economy in which the agents' individual characteristics (i.e. preferences and endowments) are random. In this framework, called *random exchange economy*, to each agent $a$ from a denumerable set $A$ there is assigned a random variable $X_a : \Omega \to \mathcal{P}_{mo,sco} \times T$ rather than an element from $\mathcal{P}_{mo,sco} \times T$ as in the traditional Arrow-Debreu framework. Here $\mathcal{P}_{mo,sco}$ denotes the space of monotone, strictly convex preference relations[1] and $T$ the space of initial endowments with $T \subset \mathbb{R}^L_{++}$ compact. The product space of individual characteristics will be denoted by $S$ throughout this paper.

A random exchange economy is fully specified by the distribution of the family of random variables $(X_a)_{a\in A}$ being a probability measure $\mu$ on $(S^A, \mathcal{B}(S)^A)$ with $\mathcal{B}(S)$ denoting the Borel $\sigma$-algebra generated by an appropriate topology on $S$ (Debreu (1969), Hildenbrand (1974)). Random exchange economies, both with and without the assumption of stochastic independence of agents, have been investigated, for example, by Hildenbrand (1971), Bhattacharya and Majumdar (1973), Föllmer (1974), Karmann (1976) and Majumdar and Rotar (2001).

The linkage between a distribution economy and an ergodic random exchange economy with a countably infinite set of agents is provided by a convergence result for the empirical distribution of the latter. The empirical distribution associated with some finite set $A' \subset A$ can be interpreted as the (random) relative frequency of the appearance of individual characteristics from some Borel subset of $S$ within the subset $A'$ of agents of the random economy. For any increasing sequence of finite subsets from $A$ exhausting $A$, the sequence of associated empirical distributions converges almost everywhere on the underlying probability space $\Omega$ to some distribution $\nu$ on $(S, \mathcal{B}(S))$ (see Proposition 2 of this paper). Although the un-

[1] The notion of a preference relation and its various properties are defined in Appendix C.

derlying mathematical statement is well-known (see Varadarajan (1958) for the case of independent random variables), the relationship between a Hildenbrand distribution economy and an ergodic random exchange economy has found little address in economics so far. The result implies that any countably infinite random exchange economy with empirical distribution $\nu$ is equivalent to a Hildenbrand distribution economy characterized by the same distribution $\nu$, as far as aggregate variables of the economy like mean demand or equilibrium price system are concerned.

Moreover, the result suggests an approach to endogenous modelling of distributions of individual characteristics in General Equilibrium Theory. By this approach, specific distributions of individual characteristics can be obtained from a specific stochastic microstructure of local interaction of agents. To the author's knowledge, such an approach has been first proposed by Grandmont (1992):

> An important issue to investigate would then be how such macroeconomic distributions might arise endogenously from specific socioeconomic interactive processes involving for instance imitation and/or differenciation effects at the micro-level. One could for instance envision a more "adaptive" viewpoint, in the spirit of [Hildenbrand (1971), Föllmer (1974)], in which the decision rule (here the demand function) or the preferences of an individual are influenced in a stochastic (Markovian) fashion by those of his immediate neighbor(s), and generate endogenously a macroeconomic distribution by looking for invariant distributions. The properties of these invariant distributions might in turn generate enough strong macroeconomic structure to allow us to proceed on secure grounds. These avenues, which might eventually lead to some kind of "Statistical Economics" (in the sense we talk of "Statistical Mechanics"), may not sound quite orthodox after so much emphasis put for so long on "individual rationality" as the main structuring language in our profession. Yet they are presumably worth exploring.

In the same spirit, Blume and Durlauf (2001) comment:

The interactions approach can contribute to the development of such an understanding by identifying how certain aggregate behaviors emerge from particular classes of individual characteristics and particular specifications of how individuals interact. One does not, however, get something for nothing by employing this approach to aggregate dynamics. Particular emergent phenomena depend upon particular sets of individual specifications.

The structure of the paper is as follows: Section 2 explains the notion of a random exchange economy and, in particular, introduces the framework of a Gibbsian random exchange economy. Section 3 contains the convergence result for empirical distributions of ergodic random exchange economies and explains the relationship with Hildenbrand distribution economies. Section 4 concludes with a discussion of the results. The appendices contain a short statement of some mathematical and economic definitions and results on which the paper is based.

# 2 The notion of a random exchange economy

In economic literature, the argument has been frequently made that individual choice involves a certain degree of randomness.[2] The question why randomness appears, and, more specifically, whether it is an intrinsic randomness or the artifact of some hidden variables, which, if observed, would again determine individual choice, is certainly a deep one. Although analogous questions have been posed and discussed in the context of theories and modelling approaches in several disciplines of science, and some conclusions from that analysis might well apply to economics, no attempt will be made in this paper to explain the roots of randomness in individual economic behaviour . Instead, the paper follows a pragmatic path to use randomness in the description of large economic systems as a way to circumvent

[2] See, for example Quand (1956), Davidson and Marschak (1959), Block and Marschak (1960) and Mossin (1968).

the complexity of individual behaviour, e.g. with regard to the formation of preferences. By modelling individual characteristics as random variables, the modeller can embody in a model empirical knowledge, e.g. that, on average, some percentage of a population behaves in a certain way without explaining *why* a particular individual does or does not.

In the same spirit, Hildenbrand (1971) introduced randomness into General Equilibrium Theory by specifying an exchange economy with random individual characteristics. In this framework, there is assigned to each agent $a$ from a finite set $A$ a random variable $X_a$ assuming values in $\mathcal{P}_{mo,sco} \times T$ (where $T$ denotes a compact set $T \subset \mathbb{R}^L_+$) rather than a fixed preference relation and a fixed endowment as in the Arrow-Debreu framework. Consequently, for a given price $p \in \mathbb{R}^L_{++}$, individual demand and aggregate demand are random. As Hildenbrand (1971) assumes that individual characteristics of agents are stochastically independent, it follows, in principle from the Law of Large Numbers, that per-capita excess demand converges to zero (in an appropriate sense) on the underlying probability space in the limit of countably infinitely many agents.

Bhattacharya and Majumdar (1973) modify Hildenbrand's framework in two respects. First, equilibrium price itself is modelled as a random variable equilibrating supply and demand almost everywhere on the probability space even for finitely many agents rather than on average as in Hildenbrand's (1971) framework.[3] Second, the convergence of the empirical distribution for dependent sequences of random agents is shown for the case of exchangeable agents and strong mixing, thus relaxing the assumption of independent agents.

Indeed, the assumption of independence of agents' individual characteristics might, in many cases, neglect an important aspect of socioeconomic phenomena.

[3]Bhattacharya and Majumdar (1973) also show the convergence of sample distributions of sequences of finite random economies for certain types of stochastic dependence between agents. These results are similar to Proposition 2 in section 3 of this note.

Interaction between individuals affecting their tastes, attitudes and expectations is likely to be fundamental in explaining a broad range of socioeconomic facts (see Blume und Durlauf (2001) and Kapteyn et al. (1997)).

A stochastic description of an exchange economy with locally interdependent characteristics has been proposed by Föllmer (1974) extending the Hildenbrand (1971) framework. Föllmer considers from the outset a countably infinite random economy indexed by the $d-$dimensional integer lattice $\mathbb{Z}^d$ (i.e. elements of $A$ are identified with lattice sites). The $d-$dimensional integer lattice $\mathbb{Z}^d$ induces a simple graph topology on $A$ with each agent having $2d$ next neighbors.[4] In this paper, we follow Föllmer (1974) in assuming the lattice $\mathbb{Z}^d$ as a simple model of social influence structures. Following the aproach to infinite random fields introduced by Dobrushin (1968) and Lanford and Ruelle, the stochastic dependence of agents resulting from a given microstructure of local interaction is characterized by a specification $\gamma$ being an appropriate family of probability kernels.[5]

The following very general definition characterizes the class of random exchange economies considered in this paper:

**Definition 1.** A *Gibbsian random exchange economy* with specification $\gamma$ is a family of random variables $(X_a)_{a\in A}$ with values in $(S, \mathcal{B}(S))$, the distribution of which is a Gibbs measure with respect to a specification $\gamma$ (see Appendix A for the definitions of the mathematical concepts involved).

[4]The integer lattice has been often used in the context of random-field models in statistical physics. A related concept of a random economy (including production) on more general graph structures has been formulated and analysed by Evstigneev and Taksar (1994,1995).

[5]For a detailed mathematical treatment of random fields see Georgii (1988). Basic notions are summarized in Appendix A of this paper.

## 3 Distribution economies as limiting empirical distributions of random economies

The notion of a *distribution economy*, introduced by Hildenbrand (1974), is a general framework for the study of large economies in which any individual agent has little influence on the market outcome. The latter condition, known as *perfect competition*, is fundamental for the Walrasian General Equilibrium Theory. Historically, there emerged two approaches to the modelling of large economies. In the first, due to Debreu and Scarf (1964), appropriate sequences of finite economies with an increasing number of agents, so-called replica economies, are considered. The second, due to Auman (1964), is based upon the notion of an idealized economy with a continuum of agents.

However, if the idealized model of an Auman economy is to be considered as a meaningful formalization of a real economy, one has to show that it is, in an appropriate sense, the limit of a sequence of finite economies. To establish this "convergence", Hildenbrand (1974) has proposed a framework in which the demand side of a large economy is characterized by a distribution $\nu$ on the Polish space $(S, \mathcal{B}(S))$.[6]

Consider first an Auman economy $\mathcal{E}$, i.e. a measurable map $\mathcal{E} : [0,1] \to S$. Let $[0,1]$ be equipped with the Borel $\sigma$-algebra and the normalized Lebesgue measure $\lambda$. The distribution $\nu_{\mathcal{E}}$ of $\mathcal{E}$ is defined as the image measure induced by $\mathcal{E}$:

$$\mu_{\mathcal{E}}(B) = \lambda(\{\mathcal{E}^{-1}(B)\}) \quad \forall\ B \in \mathcal{B}(S).$$

Vice versa, due to Skorohod's theorem (Skorokhod (1964)), with any given probability measure $\nu$ on the Polish space $(S, \mathcal{B}(S))$ one can associate an Auman economy such that $\mu$ is its distribution (in the sense of the last equation).

[6] The fact that, among others, the space of monotone, strictly convex preferences on $\mathbb{R}^L_+$ is a Polish space has been shown by Grodal (1974).

The crucial fact, established in Theorem 1 of Hildenbrand (1970), is, that any non-atomic probability measure $\nu$ on the space $(S, \mathcal{B}(S))$ (arising as the distribution of an Auman economy) is the limit (with respect to weak convergence) of a purely competitive sequence of distributions $\nu_n$ on $(S, \mathcal{B}(S))$ arising from a purely competitive sequence of finite economies[7][8] $\mathcal{E}_n : A_n \to S$.

Now we turn to the relationship between random exchange economies and Hildenbrand distribution economies. This will be illustrated by the following simple example: Consider a random exchange economy with a countable set of agents represented by $\mathbb{N}$ and a finite space of characteristics $S_0 = \{s_1, ..., s_N\}$. To each agent there is associated a random variable $X_a : \Omega \to S_0$ defined on some probability space $(\Omega, \mathcal{F}, P)$. For simplicity, assume that the family of random agents $(X_a)_{a \in \mathbb{N}}$ is identically independently distributed. Let $\nu$ denote the law of each $X_a$.

The *empirical distribution* $Y_n$ of the finite family of random agents $(X_a)_{a \in \{1,...,n\}}$ is a *random* distribution,[9] i.e. a disribution on $S_0$ depending on $\omega \in \Omega$. In the finite case, one can proceed as follows. For any $s_i \in S_0$ define a random variable $Y_n(s_i, \cdot) : \Omega \to [0, 1]$ with

$$Y_n(s_i, \omega) := \frac{1}{n} |\{a \in \{1, ..., n\} | X_a(\omega) = s_i\}|, \tag{1}$$

wherein $| \cdot |$ denotes the cardinality of a set. The random variable $Y_n(s_i, \cdot)$ is the

[7] A sequence of finite economies $\mathcal{E}_n : A_n \to S$ with finite $A_n$ is called *purely competitive* if the following conditions hold:

(i) the number $|A_n|$ of agents tends to infinity

(ii) the sequence $\mu_{\mathcal{E}_n}$ of preference-endowment distributions converges weakly to a limit $\nu$

[8] For a finite economy $\mathcal{E}_n$, one obtains the distribution $\mu_{\mathcal{E}_n}$ via

$$\mu_{\mathcal{E}_n}(B) = \frac{|\mathcal{E}_n^{-1}(B)|}{|A_n|} \qquad \forall B \in \mathcal{B}(S)$$

[9] A random distribution is a random variable assuming values in the space of probability measures on some measurable space.

(random) relative frequency of the realisation $s_i$ from the finite set of individual characteristics in a sample consisting of the first $n$ agents.

By the strong law of large numbers we have weak convergence of the sequence almost everywhere on $\Omega$:

$$\lim_{n\to\infty} Y_n(s_i,\cdot) \to \nu(s_i) \;\; P-a.s.$$

The result vindicates the intuition that the relative frequency of $s_i$ in the infinite sequence $(X_a)_{a\in\mathbb{N}}$ is equal to $\nu(s_i)$. By a similar argument, we can obtain convergence of the empirical distribution without the assumption of stochastic independence of $(X_a)_{a\in\mathbb{N}}$, if we consider states which are ergodic w.r.t. an appropriate measure-preserving transformation on $\Omega$.

As a result, in the simple setup described above the limiting empirical distribution specifies a Hildenbrand distribution economy. The main conceptual point of this paper is that any Hildenbrand distribution economy described by a probability measure on the full space of individual characteristics can be derived as the empirical distribution from some random economy $(X_a)_{a\in\mathbb{Z}^d}$. Vice versa, every ergodic random economy gives rise to some Hildenbrand distribution economy given by the empirical distribution of the random economy.

It is important to emphasize that the empirical distribution contains much less information about the economy than does the law of $(X_a)_{a\in\mathbb{Z}^d}$. There will be a multiplicity of quite different random economies sharing the same limiting empirical distribution. Nevertheless, the information contained in the empirical distribution suffices for the computation of any macroscopic variable considered by the classical Arrow-Debreu general equilibrium model.

The remainder of this section extends the argument presented above to general spaces of individual characteristics considered in General Equilibrium Theory. In doing so, we first consider the case of independent agents (Proposition 1), then the general case of dependent agents (Gibbsian random economies) will be tackled

(Proposition 2).

First, the notion of an empirical distribution has to be extended to processes with a general state space. For a finite sequence of random variables we have

**Definition 2** Let $X = (X_1, \ldots, X_n)$ be a finite sequence of random variables defined on some probability space $(\Omega, \mathcal{F}, P)$ with values in a measurable space $(E, \mathcal{E})$. The *empirical distribution* of $X$, denoted by $Y_n(\cdot, \cdot)$, is a random distribution on $(E, \mathcal{E})$, i.e. a map $Y_n : \mathcal{E} \times \Omega \to [0, 1]$, which is specified setwise by

$$Y_n(B, \cdot) = \frac{1}{n} \sum_{k=1}^{n} I_B \circ X_k(\cdot), \quad B \in \mathcal{E} \tag{2}$$

where $I_B$ denotes the indicator function of $B$.

$Y_n(B, \cdot)$ can be interpreted as the "relative frequency" of the appearance of a value from $B$ in the finite sequence. This "relative frequency" is itself a random variable on $\Omega$.[10]

For infinite sequences of random variables, the empirical distribution is defined as the limit of the sequence $(Y_n)_{n \in \mathbb{N}}$ for $n \to \infty$ with respect to weak convergence, if this limit exists.

**Definition 3** Let $X = (X_1, X_2, \ldots)$ be an infinite sequence of random variables defined on some probability space $(\Omega, \mathcal{F}, P)$ with values in a measurable space $(E, \mathcal{E})$. If there exists a probability measure $Y$ on $(E, \mathcal{E})$ such that the sequence of empirical distributions $(Y_n)_{n \in \mathbb{N}}$, where $Y_k$ is the empirical distribution of the finite

[10] Note that the empirical distribution can also be written as

$$Y_n(B, \cdot) = \frac{1}{n} \sum_{k=1}^{n} \delta_{X_k(\cdot)}(B) \quad \forall B \in \mathcal{E} \tag{3}$$

with the Dirac measure $\delta_x(\cdot)$ on $(E, \mathcal{E})$, with $x \in E$, defined as

$$\delta_x(B) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{else} \end{cases} \qquad \forall B \in \mathcal{E}. \tag{4}$$

sequence $(X_1, \ldots, X_k)$, for $P-$almost every $\omega \in \Omega$ weakly converges to $Y$ with respect to weak convergence of probability measures, i.e.

$$P\{\omega \in \Omega \mid Y_n(\cdot, \omega) \to Y(\cdot)\} = 1$$

then $Y$ is called *limiting empirical distribution* of $X$ (or simply empirical distribution of $X$).

The following proposition asserts that the empirical distribution of a random economy with identically independently distributed agents converges on $\Omega$ ($P-$a.s.). The result can be found already in Hildenbrand (1971). We provide a slightly more explicit proof based on Dudley (1989).

**Proposition 1 (Hildenbrand)** Let $\mathcal{E}$ denote a countably infinite random economy with independently identically distributed random agents, i.e. an i.i.d. family of random variables $(X_a)_{a \in \mathbb{N}}$ on some probability space $(\Omega, \mathcal{F}, P)$ taking values in the Polish space $S$ of individual characteristics. Let $\nu$ denote the distribution of each $X_a$ on $(S, \mathcal{B}(S))$. Then the limiting empirical distribution of $\mathcal{E}$ exists and is equal to $\nu$ $P-$a.s. on $\Omega$.

PROOF: Let $A_n$ denote the finite subset of agents consisting of agents $1, \ldots, n$ constituting the finite random economy $\mathcal{E}_n$ and let $Y_{\mathcal{E}_n}$ denote the empirical distribution of the finite random economy $\mathcal{E}_n$. We have to show that for $P-$almost any $\omega \in \Omega$ the sequence of probability measures $Y_{\mathcal{E}_n}(\omega)$ converges to $\nu$ for $n \to \infty$. We remind that a sequence of probability measures $M_n$ on a topological space $(G, \mathcal{T})$ converges weakly to a probability measure $M$ if for any continuous bounded function $f : G \to \mathbb{R}$ we have $\int_G f dM_n \longrightarrow \int_G f dM$ for $n \to \infty$.

By equation (3) one can see that for any bounded continuous function $f : S \to \mathbb{R}$ we have

$$\int_S f dY_{\mathcal{E}_n} = \frac{1}{n}(f(X_1) + \cdots + f(X_n)).$$

This expression converges for $n \to \infty$ due to the strong law of large numbers to

$\int_S f d\nu$ $P-$almost surely on $S$.

Because the set of measure 0 (w.r.t. $P$) for which convergence does not occur might depend on $f$, it remains to be shown that the measure of the set on which the empirical distribution converges has measure 1. As any separable metric space has a totally bounded metrization, we can assume that $(S, d)$ is compact. By Theorem 11.3.3 from Dudley (1989), it suffices to show that we have convergence for all $f \in BL(S, d)$ where $BL(S, d)$ denotes the set of bounded Lipschitz functions on $(S, d)$.[11]

Because $BL(S, d)$ is itself separable with respect to the supremum norm $\|\cdot\|_\infty$ (cf. Dudley (1989), p.308), there exists a countable dense subset $(f_m)_{m\in\mathbb{N}}$ of $BL(S, d)$. Take any $f \in BL(S, d)$. For any $\varepsilon > 0$ there is a $m(\varepsilon)$ such that: $\|f - f_{m(\varepsilon)}\|_\infty < \frac{\varepsilon}{4}$. Then for any $f \in BL(S, d)$ we can estimate

$$|\int f dY_{\mathcal{E}_n} - \int f d\nu| = |\int (f - f_{m(\varepsilon)} + f_{m(\varepsilon)}) dY_{\mathcal{E}_n} - \int (f - f_{m(\varepsilon)} + f_{m(\varepsilon)}) d\nu|$$

$$\leq |\int (f - f_{m(\varepsilon)}) dY_{\mathcal{E}_n}| + |\int (f - f_{m(\varepsilon)}) d\nu| + |\int f_{m(\varepsilon)} dY_{\mathcal{E}_n} - \int f_{m(\varepsilon)} d\nu|$$

Therefore we have convergence for all $f \in BL(S, d)$ provided it is the case for $(f_m)_{m\in\mathbb{N}}$. But for functions from $(f_m)_{m\in\mathbb{N}}$ the set of possible non-convergence points is a denumerable union of sets of measure zero and therefore also has measure zero. □

Now we turn to the case of a Gibbsian random exchange economy introduced in Definition 1. Such an economy is a random field $(X_a)_{a\in\mathbb{Z}^d}$ with each random variable $X_a$ taking values in $S$. The distribution of a Gibbsian random economy

[11] Let $(S, d)$ be a metric space. For a real-valued function $f$ on $S$, let $\|f\|_L$ denote the Lipschitz seminorm defined by

$$\|f\|_L = sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x, y)}$$

Let $\|f\|_\infty$ denote the supremum norm $\|f\|_\infty = sup_x |f(x)|$. We obtain a norm by setting $\|f\|_{BL} = \|f\|_L + \|f\|_\infty$. Functions from $BL(S, d) = \{f : S \to \mathbb{R} \mid \|f\|_{BL} < \infty\}$ are called bounded Lipschitz functions on $S$.

is a probability measure on the product space $(S^{\mathbb{Z}^d}, \mathcal{B}(S)^{\mathbb{Z}^d})$. We confine our consideration to shift-invariant Gibbs measures $\mu$, which are ergodic with respect to lattice shift (See Appendix B for details). Whereas for our present argument we take ergodicity as an assumption, it is argued elsewhere using a dynamical framework (Hohnisch and Kondratiev (2003)) that ergodic Gibbs measures are the only appropriate measures within the class of shift-invariant Gibbs measures to represent equilibrium states of real economic systems.

The limiting empirical distribution for a Gibbsian random economy on the integer lattice $\mathbb{Z}^d$ can be defined as follows: A finite subset $\Lambda \subset \mathbb{Z}^d$ is called *finite volume*. For a sequence of finite volumes $(\Lambda_n)_{n \in \mathbb{N}}$ let $(\Lambda_n) \nearrow \mathbb{Z}^d$ denote the situation that $\Lambda_n \subset \Lambda_{n+1}$ and $\bigcup_{n \in \mathbb{N}} \Lambda_n = \mathbb{Z}^d$. For any finite volume $\Lambda_n \subset \mathbb{Z}^d$ and any set $B \in \mathcal{B}(S)$ the empirical distribution can be defined as

$$Y_{\Lambda_n}(B) = \frac{1}{|\Lambda_n|} \sum_{k \in \Lambda_n} 1_B \circ X_k$$

for any $B \in \mathcal{B}(S)$. The following result is the generalization of Proposition 1 to the case of Gibbsian random economies.

**Proposition 2** Let $(X_a)_{a \in \mathbb{Z}^d}$ denote an ergodic (with respect to lattice shift) Gibbsian random economy with distribution $\mu$ on $(S^{\mathbb{Z}^d}, \mathcal{B}(S)^{\mathbb{Z}^d})$. Let $\nu$ denote the marginal distribution of each $X_a$ on $(S, \mathcal{B}(S))$. Then for any sequence of finite volumes with $(\Lambda_n) \nearrow \mathbb{Z}^d$ $\lim_{n \to \infty} Y_{\Lambda_n}$ exists and is equal to $\nu$ $\mu$-a.s. on $\Omega$.

PROOF: The proof is almost identical to that of Proposition 1 except that a generalized law of large numbers has to be used. For any bounded continuous function $f : S \to \mathbb{R}$ we have

$$\int_S f dY_{\Lambda_n} = \frac{1}{|\Lambda_n|} \sum_{a \in \Lambda_n} f(X_a)$$

This expression converges for $n \to \infty$ by the multidimensional version of Birkhoff's ergodic theorem to $\int_S f d\nu$ $\mu-$almost surely on $\Omega$ (See Appendix B). The same argument as in Proposition 1 can be used to show that the measure of the set on which convergence does not occur is zero. Therefore $\nu$ is the limiting empirical distribution of $\mathcal{E}$. $\square$

Proposition 2 has the following implication: The convergence of the empirical distribution of an ergodic Gibbsian random exchange economy allows to associate to any such economy a Hildenbrand distribution economy specified by a probability measure $\nu$ on $(S, \mathcal{B}(S))$ in the sense that properties of the latter, e.g. equilibrium prices, per-capita aggregate demand and per-capita aggregate excess demand, could be as well obtained from a random economy with the same distribution $\nu$ as limiting empirical distribution. Therefore, given a market outcome we cannot distinguish whether the underlying framework is a Gibbsian random economy or an Arrow-Debreu economy. This result might be surprising in that it shows that, if one is interested in global variables, the Arrow-Debreu framework is equivalent to an economy with random characteristics, provided we have reason to admit only ergodic states to represent real economic systems.[12]

A comment on the interplay of the two forms of interaction in the model of a Gibbsian random economy, namely the interaction of agents' characteristics and market interaction (the coordination via the price system), seems appropriate. An equilibrium price exists once we are in the case of proposition 2, i.e. the distribution of the individual characteristics is an ergodic probability measure or, intuitively, once the system has "settled down" into equilibrium. In that case we have a complete separation of the two types of interaction, the former endogenizing the distribution of individual characteristics, the latter the price system.

[12] See Hohnisch and Kondratiev (2003)

# 4 Discussion

Having established the consistency of the notion of a Gibbsian random exchange economy within the Walrasian equilibrium concept, we argue that the notion of a Gibbsian random exchange economy provides a framework to study structural properties of an economic system which are beyond the scope of the Arrow-Debreu model with fixed individual characteristics. The distribution of individual characteristics is endogenous in the Gibbsian framework. The local structure of a spatial process indexed by the integer lattice $\mathbb{Z}^d$ is given by its specification, a family of appropriate probability kernels. In the case that one considers a finite state space of individual characteristics and finite range interaction, these kernels reduce to a particularly simple form, namely to the conditional probabilities of a single agent to be in any of the possible states given a fixed configuration of states in his neighborhood. Also, the set of Gibbs measures can be easily constructed in that case. Moreover, these conditional probabilities can be investigated both experimentally and empirically. Thus it is possible, at least in simple cases, to model a distribution of characteristics in a population of economic entities from direct observation.

# Appendices

## A Spatial stochastic processes (Random fields)

The mathematical concepts discussed in this Appendix are widely applied in different areas of science to model random systems consisting of many interacting entities, each capable of being in different states. Depending on the particular area of application, the entities are economic agents, internet servers, neurons or molecules. The denumerable set of these entities will be denoted by $A$. For each entity, the set of possible states will be assumed the same, for simplicity. This set is called *state space*. The general notions can be defined with the state space being any measurable space $(E, \mathcal{E})$. The stochastic description of such systems starts with the following definition:

**Definition A.1** A *spatial stochastic process* (or *random field* or *stochastic field*) is a family of random variables $(X_a)_{a \in A}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ with values in a measurable space $(E, \mathcal{E})$ and with the index set $A$ countably infinite.

The mathematical problem is how to derive global properties of a random field from the local structure of stochastic dependence between individual variables, in the same sense as the law of a random chain can be constructed immediately from its transition probabilities. The appropriate framework to study this problem for random fields has been suggested by Dobrushin (1968) and Lanford and Ruelle (1969) and is referred to as the DLR-approach. Initially, the DLR-approach considered a particular spatial structure of the index set $A$, namely the d-dimensional integer lattice, i.e. $A = \mathbb{Z}^d$, but an extension to more general structures of the index set has been soon provided by Preston (1974). In Preston's general approach, the set of entities $A$ is equipped with a neighborhood structure assigning to each

$a \in A$ the finite set $\mathcal{N}(a) \subset A$ of *next neighbors* of $a$. Any such neigborhood structure can be represented by a graph. The elements of $A$ are then identified with the nodes of the graph.[13] This review will be confined to the case of $\mathbb{Z}^d$ as index set for simplicity.

The surprising fact resulting from the analysis of random fields is that a local dependence structure of the random field, represented by an appropriate family of probability kernels discussed later in this section, does not, in general, determine uniquely the global law of the field if $d \geq 2$. This phenomenon is known as *phase multiplicity*. It is extensively discussed in several monographs, e.g. (Georgii (1988)), and the reader is referred to them for a detailed treatment.

In the following, some of the technicalities underlying the DLR-framework will be explained. We start with the notion of a *probability kernel*.

**Definition A.2** Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be measurable spaces. A function $\pi : \mathcal{X} \times Y \to [0, \infty]$ is called a *probability kernel from* $(Y, \mathcal{Y})$ *to* $(X, \mathcal{X})$ if:
(i) $\pi(\cdot|y)$ is a measure on $(X, \mathcal{X})$ for all $y \in Y$
(ii) $\pi(A|\cdot)$ is $\mathcal{Y}$-measurable for each $A \in \mathcal{X}$
(iii) $\pi(X|\cdot) = 1$
Further, let $\mathcal{X}_0$ be a sub-$\sigma$-algebra of $\mathcal{X}$. A probability kernel from $(X, \mathcal{X}_0)$ to $(X, \mathcal{X})$ is said to be *proper* if $\pi(A \cap B|\cdot) = \pi(A|\cdot) \circ 1_B$ for every $A \in \mathcal{X}$ and $B \in \mathcal{X}_0$.

**Proposition A.1** Let $\mathcal{P}(X, \mathcal{X})$ and $\mathcal{P}(Y, \mathcal{Y})$ denote the spaces of probability measures on $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$, respectively.

[13] The extension of the DLR-approach from the integer lattice $\mathbb{Z}^d$ to more general spatial structures for the index set $A$ is especially important for applications in economics, as the integer lattice $\mathbb{Z}^d$ provides only a crude model for socioeconomic communication and influence networks, created e.g. by peer relationship within a social group. A random economy framework with an underlying graph structure has been proposed by Evstigneev and Taksar (1994)(1995).

(i) Let $\mu \in \mathcal{P}(Y,\mathcal{Y})$. A probability kernel $\pi$ from $(Y,\mathcal{Y})$ to $(X,\mathcal{X})$ defines a mapping from $\mathcal{P}(Y,\mathcal{Y})$ to $\mathcal{P}(X,\mathcal{X})$ via

$$(\mu\pi)(A) := \int d\mu\pi(A|.) \qquad \forall\ A \in \mathcal{X}.$$

(ii) For each measurable $f : X \to \mathbb{R}$ a probability kernel $\pi$ induces a measurable function $\pi f := \int \pi(dx|\cdot)f(x)$.

(iii) Let $(Z,\mathcal{Z})$ denote a third measurable space. The composition $\pi_1\pi_2$ of a probability kernel $\pi_1$ from $(Z,\mathcal{Z})$ to $(Y,\mathcal{Y})$ and $\pi_2$ from $(Y,\mathcal{Y})$ to $(X,\mathcal{X})$ defined by the formula

$$\pi_1\pi_2(A|z) = \int \pi_1(dy|z)\pi_2(A|y) \quad \forall\ A \in \mathcal{X}, z \in Z.$$

is a probability kernel from $(Z,\mathcal{Z})$ to $(X,\mathcal{X})$.

PROOF: Preston (1982)

The relationship between a proper probability kernel and conditional probabilities is characterized by the following proposition:

**Proposition A.2** Let $(X,\mathcal{X})$ be a measurable space, $\mathcal{B}$ a sub-$\sigma$-algebra of $\mathcal{X}$, $\pi$ a proper probability kernel from $(X,\mathcal{B})$ to $(X,\mathcal{X})$ and $\mu \in \mathcal{P}(X,\mathcal{X})$. Then

$$\mu(A|\mathcal{B}) = \pi(A|\cdot) \qquad \mu - a.s. \quad \forall\ A \in \mathcal{X}$$

if and only if $\mu\pi = \mu$.

PROOF: Georgii (1988), p.15

Following the standard approach in the theory of stochastic processes, we set $\Omega = E^{\mathbb{Z}^d}$ with $E^{\mathbb{Z}^d} = \{\omega = (\omega_a)_{a\in\mathbb{Z}^d} : \omega_a \in E\}$. The associated $\sigma$-algebra on $\Omega$ is the product $\sigma$-algebra $\mathcal{E}^{\mathbb{Z}^d}$. It will be denoted by $\mathcal{F}$. The variable $X_a$ is then a projection map $X_a : \Omega \to E$ taking $\omega \to \omega_a$ for each $\omega \in \Omega$. Further, for each $\Lambda \subset \mathbb{Z}^d$ let $X_\Lambda : \Omega \to E^\Lambda$ denote the projection onto the coordinates in $\Lambda$. For any $\Delta \subset \mathbb{Z}^d$, $\mathcal{F}_\Delta$ denotes the $\sigma$-algebra of events involving knowledge only about variables inside $\Delta$, i.e. $\mathcal{F}_\Delta$ is generated by the events $\{\sigma_\Lambda \in A\}$ with $\Lambda \in \mathcal{A}, \Lambda \subset \Delta; A \in E^\Lambda$ and $\mathcal{A} := \{\Lambda \subset \mathbb{Z}^d | 0 < |\Lambda| < \infty\}$.

It turns out that the appropriate local structure characterizing the random field $(X_a)_{a\in\mathbb{Z}^d}$ is provided by a family of probability kernels $(\gamma_\Lambda)_{\Lambda\in\mathcal{A}}$ required to be versions of conditional distributions of the random field relative to certain sub-$\sigma$-algebras of $\mathcal{F}$. More precisely, for each finite non-empty set $\Lambda \subset \mathbb{Z}^d$, the probability kernel $\gamma_\Lambda(A|\cdot)$ from $(\Omega, \mathcal{F}_{\mathbb{Z}^d\setminus\Lambda})$ to $(\Omega, \mathcal{F})$ is required to be equal almost everywhere on $\Omega$ to the conditional distribution $\mu(A|\mathcal{F}_{\mathbb{Z}^d\setminus\Lambda})$ relative to the $\sigma$-algebra $\mathcal{F}_{\mathbb{Z}^d\setminus\Lambda}$ of events outside $\Lambda$.

Due to this requirement, the probability kernels $(\gamma_\Lambda)_{\Lambda\in\mathcal{A}}$ have to fulfill certain a-priori consistency conditions:

**Definition A.3** A family of proper probability kernels $\gamma = (\gamma_\Lambda)_{\Lambda\in\mathcal{A}}$ which satisfy the consistency condition $\gamma_\Lambda\gamma_{\Lambda_0} = \gamma_\Lambda$ for any $\Lambda_0 \subset \Lambda$ is called a *specification* for the random field.

The next definition introduces the notion of a Gibbs measure:

**Definition A.4** A probability measure $\mu$ is said to be *admitted* by a specification $\gamma$ if the following condition holds:

$$\mu(A|\mathcal{F}_{\mathbb{Z}^d\setminus\Lambda}) = \gamma_\Lambda(A|\cdot) \quad \mu - a.s. \qquad \forall\ A \in \mathcal{F}.$$

The set of all probability measures admitted by the specification $\gamma$ will be denoted by $\mathcal{G}(\gamma)$. Measures from $\mathcal{G}(\gamma)$ are called *Gibbs measures* (with respect to $\gamma$).[14]

For a comprehensive discussion of existence and uniqueness results for Gibbs

[14]The abstract notion of a specification would be of little practical relevance without a constructive approach to obtain specifications for concrete systems. The great success of the DLR-approach lies in the fact that it allows to explicitely derive specifications from *interaction potentials* (see Georgii (1988), chapter 2). In economics, interaction potentials can be obtained e.g. in appropriate experiments.

measures the reader is referred to Georgii (1988). The basic surprising result is that $\mathcal{G}(\gamma)$ might not be a singleton if $d \geq 2$.

# B Ergodicity of random fields on the integer lattice $\mathbb{Z}^d$

In this section, basic notions related to ergodicity of spatial stochastic processes (random fields) are summarized. In particular, the generalized Birkhoff ergodic theorem is used in Section 3 to derive the main result of this paper. For a detailed treatment see Georgii (1988).

The starting point of ergodic theory is the notion of a transformation that preserves the structure of the measurable space, as defined below.

**Definition B.1** Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and $T$ a measurable transformation on $(\Omega, \mathcal{F}, \mu)$. The transformation $T$ is said to be *measure preserving* if

$$\mu(T^{-1}A) = \mu(A) \quad \forall\ A \in \mathcal{F}.$$

To formulate ergodicity for random fields, one has to keep in mind that the linearly ordered index set referring to time is replaced by a more general index set. In this appendix, the $d-$dimensional integer lattice $\mathbb{Z}^d$ is taken as index set. Let $(E, \mathcal{E})$ denote a measurable space and let $(X_i)_{i\in\mathbb{Z}^d}$ be a random field with state space $(E, \mathcal{E})$. The distribution of $(X_i)_{i\in\mathbb{Z}^d}$ is a probability measure on $(\Omega, \mathcal{F})$ with $(\Omega, \mathcal{F}) = (E^{\mathbb{Z}^d}, \mathcal{E}^{\mathbb{Z}^d})$.

Stationarity of a random process corresponds in this situation to invariance of its distribution with respect to *lattice shift*.

**Definition B.2** For any $j \in \mathbb{Z}^d$ the map $\theta_j : \Omega \to \Omega$ given by

$$\theta_j(\omega) = (\omega_{i-j})_{i\in\mathbb{Z}^d}$$

is called a *lattice shift*.

With this definition one can define a property analogous to stationarity.

**Definition B.3** A measure $\mu$ on $\Omega$ is called *shift-invariant* if $\mu(A) = \mu(\theta_j(A))$ for any $A \in \mathcal{F}$ and any $j \in \mathbb{Z}^d$. A random field is called shift invariant, if its distribution is shift invariant.

In defining the notion of ergodicity for random fields on the integer lattice, the role of the $\sigma$-algebra of (time-) shift invariant events is taken by the set $\mathcal{J}$ of events invariant with respect to lattice shift.

**Definition B.4** The sub-$\sigma$-algebra of $\mathcal{F}$

$$\mathcal{J} = \{A \in \mathcal{F} : \theta_j(A) = A \quad \forall\, j \in \mathbb{Z}^d\}$$

is called $\sigma$-algebra of lattice-shift-invariant events.

**Definition B.5** A random field with a shift invariant law $\mu$ on $(E^{\mathbb{Z}^d}, \mathcal{F})$ is called *ergodic* (with respect to lattice-shift) if it is trivial on $\mathcal{J}$, e.g. for any $A \in \mathcal{J}$ we have $\mu(A) = 1$ or $\mu(A) = 0$.

The following statement generalizes Birkhoff's ergodic theorem.

**Theorem B.1** Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $(\theta_j)_{j \in \mathbb{Z}^d}$ measure-preserving transformations on $\Omega$. Then for each increasing sequence of finite volumes $(\Lambda_n)_{n \in \mathbb{N}}$ exhausting $\mathbb{Z}^d$ and any $f \in L^2(\Omega, \mathcal{F}, \mu)$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j \in \Lambda_n} f \circ \theta j = \mu(f|\mathcal{J}))\ \ \mu - a.s.$$

PROOF: Georgii (1988)

# C Properties of individual preferences

The individual taste of each agent is formalized by a binary relation on $X_a$, i.e. a subset of $X_a \times X_a$. It is customary to denote such a binary relation by the symbol $\succsim_a$ and for any pair of consumption bundles $(x_a, y_a) \in \succsim_a$ to write $x_a \succsim_a y_a$ with the interpretation that *agent a considers the bundle $x_a$ as at least as good as $y_a$.*

The neoclassical notion of rationality implies in particular that a binary relation formalizing individual taste possesses the following three properties:

(P1) $x \succsim y$ or $y \succsim x$ (or both) $\forall\ x, y \in \mathbb{R}_+^L$ (completeness)

(P2) if $x \succsim y$ and $y \succsim z$ then $x \succsim z\ \forall\ x, y, z \in \mathbb{R}_+^L$ (transitivity)

(P3) the sets $\{x | y \succsim x\}$ and $\{x | x \succsim y\}$ are closed sets (continuity)

A binary relation on $\mathbb{R}_+^L$ possessing these three properties is called a *preference relation.* Let $\mathcal{P}$ denote the set of all preference relations on $X_a = \mathbb{R}_+^L$. From a preference relation $\succsim$ one can derive two related binary relations on $X_a = \mathbb{R}_+^L$:

(i) The indifference relation $\sim$, defined by: $x \sim y :\Leftrightarrow x \succsim y$ and $y \succsim x$

(ii) The strict preference relation $\succ$, defined by: $x \succ y :\Leftrightarrow x \succsim y$ but not $y \succsim x$.

There are two further standard assumptions on a preference relation:

(MO) A preference relation $\succsim \in \mathcal{P}$ is called *monotonic* if $x \succ y$ for every $x, y \in \mathbb{R}_+^L$ with $x_j > y_j$ for at least one component and $x_j \geq y_j$ for all other components.

(SCO) A preference relation $\succsim \in \mathcal{P}$ is called *strongly convex* if $x, y \in \mathbb{R}_+^L$ with $x \sim y$ and $x \neq y$ implies that $\lambda x + (1 - \lambda) y \succ y$ for every $\lambda,\ 0 < \lambda < 1$.

The set of all preference relations that satisfy the properties (MO) and (SCO) will be denoted by $\mathcal{P}_{mo,sco}$.

For details, see Debreu (1959).

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