Bonn Econ Discussion Papers

Discussion Paper 28/2003

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December 2003

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Optimal Dynamic Choice of Durable and Perishable Goods †

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Abstract

We analyze the life cycle consumption choice model for multiple goods, focusing on the distinction between durables and perishables. As an approximation of the fact that rather high transaction costs and market imperfections prevail in markets for used durables, we assume that investment in durables is irreversible. In contrast to the additive model with one perishable good, the optimal consumption plan is not myopic. Instead, it depends on past as well as on (expected) future prices. The optimal stock level of the durable good is obtained by tracking a certain *shadow level*: The household purchases just enough durables to keep the stock always above this shadow level. It is shown that this shadow level is given by a backward integral equation that replaces the Euler equation. For the perishable good, the 'usual' Euler equation determines the optimal choice in terms of the optimal stock of durables. Since the optimal stock level aggregates past as well as future prices, the consumption of perishables ceases to be myopic as well. The solutions show that durables play an important part in intertemporal consumption decisions. In fact, major purchases of durables are being made early in life, whereas no durables are bought in the retirement years. Through substitution and complementarity effects, this has a significant impact on the consumption of perishable goods. On the technical side, the paper provides a new approach to singular control problems that might be widely applicable in other contexts like irreversible investment, price rigidities etc. We present a numerical algorithm that allows one to calculate the shadow level for arbitrary period utility functions and time horizons. Explicit solutions are given for the case of a homogeneous Markov setup with infinite time horizon and Cobb–Douglas type period utilities. This setup includes prices driven by Brownian motion and/or Poisson processes.

JEL subject classification. D91, E21/2

Key words and phrases. Intertemporal Consumption Choice, Durable Goods, Irreversible Investment, Singular Control

† We thank Peter Hammond, Chloé Koos, Dirk Krüger, and seminar participants at Stanford University, UCLA, and Humboldt University for comments. Special thanks go to Michael Magill who asked Frank Riedel to solve this problem on the 2000 Boston Econometric Society conference. Support of Deutsche Forschungsgemeinschaft through SFB 373, "Quantification and Simulation of Economic Processes" and grant Ri–1128–1–1 is gratefully acknowledged. Part of the paper was being written while Riedel was visiting UC Berkeley with a grant by the German Research Foundation DFG (Ri–1128–1–1) and later Stanford University with a Feodor Lynen grant by the Alexander von Humboldt foundation. The hospitality of both institutions is gratefully acknowledged.

Introduction

Households face a complex intertemporal allocation problem. Uncertain of their future and confronted with a wide array of commodities, they are to choose a contingent consumption and investment plan for a long time horizon of, typically, about 30 years. A challenging part of the problem is the choice between perishable and durable goods. While perishables usually form the basis of daily survival and are thus purchased day by day, durables provide service over extended time periods and are renewed or repaired only when necessary. So, with current prices for perishable and durable goods and future expectations about those prices in mind, households balance the long term effects of durables and their immediate needs for perishable goods.

This interesting trade–off is rather rarely addressed in the literature. In fact, due to the considerable complexity created already by time and uncertainty, most *models* of intertemporal consumption choice do not distinguish between durable and perishable goods and assume instead that there is only one (aggregate) good for consumption at each point in time. This makes it impossible to analyze the important tradeoff between consumption decisions across goods. However, as documented by, e.g., the Survey of Consumer Expenditures (provided by the US Dept of Labor, http://www.bls.gov/cex/csxreprt.htm), up to 30% of expenditures can be attributed to purchases of durable goods. It thus becomes an issue to analyze the utility maximization problem of a household who can consume both perishable goods as well as durables at the same time. This is the aim of the present paper.

Markets for used durables often suffer from rather high transaction costs. These costs are due to prevalent market imperfections, ranging from asymmetry of information and adverse selection, as illustrated nicely by Akerlof's (1970) celebrated lemons' model, to search and opportunity costs due to inefficient market organization. For a large number of durable goods such as apparel, footwear, furniture, and appliances, these transaction costs are so high as compared to the achievable resale price that owners refrain from trying to sell the good and prefer disposing of it altogether. Therefore, it is important to have transaction costs for durable goods in the model. In the spirit of a simplifying approach started by Arrow (1968), we assume that investments in such durables are irreversible. Note that this rules out durables with well established secondary markets such as houses for which an approach as proposed by Grossman and Laroque (1990) is more compelling. Taking a different point of view, one may also argue that in fact many commodities do not vanish immediately after purchase. Instead, even as perishable groceries as milk do provide utility over some period of time, leaving electricity as a notable exception. In this interpretation, the good's service flow is to be seen as the level of satisfaction provided by the recent past consumption of the good — and such a satisfaction level is not traded, of course. The latter interpretation corresponds to the Hindy, Huang, Kreps (1992) approach.

We obtain the following results. The household consumes perishables in such a way that marginal period utility in the perishable good is proportional to its price. Stated differently, an Euler equation holds true, similar to the case without durables. However, marginal utility in the perishable good directly depends on the current stock of the durables. The slope of indifference curves between consumption of the perishable today and tomorrow thus depends on the stock level of the durable, and so considerable cross effects occur. For example, in a deterministic world, if the subjective discount factor is equal to the interest rate, consumption of the perishable good is constant over time when durables are not taken into account. With durables, however, hump–shaped patterns of consumption of the perishable good can emerge due to the cross effects.

Purchases of the durable involve more complex considerations by the household since it provides service flow over extended periods of time. The decision whether to buy some amount of the durable good depends on the current stock level and on future expectations about prices and wealth. In order to describe the optimal purchasing behavior, we introduce an auxiliary concept, the *minimal shadow stock level* which gives an optimal lower bound for the stock of durables to be held at each point in time. The minimal shadow stock level solves a generalized Euler equation. In a Markov framework with Cobb–Douglas period utility, the minimal shadow level solves indeed the 'usual' Euler equation one obtains in the frictionless model without any transaction costs. Hence, the minimal shadow level is, in this case, the stock level a household (with a suitably adjusted wealth level) would entertain with a perfect resale market. The optimal stock can be derived from the minimal shadow level in a straightforward manner: The household purchases just enough to keep the stock level at or above the minimal shadow level.

This type of purchasing behavior has important implications for comparative statics. Although the shadow level reacts directly when there are exogenous price shocks or new information is released, this does not necessarily imply a change in purchasing behavior. Indeed, if the current stock level is way above the optimal shadow level then small changes in prices do not affect the purchasing behavior at all: We thus have an explanation for 'sticky' behavior by households. If the current stock level is directly *at* the optimal shadow level, changes in prices do have an impact on purchasing behavior. A higher price of the durable, e.g., leads to an immediate stop of purchases—we have some kind of infinite downward elasticity here. Good news, like a lower price for the durable, lead to more purchases in the 'standard' way— the upward impact is less drastic, therefore.

The same kind of stickiness carries over to the cross effects. We illustrate this for the case when durables and perishables are complements. In that case, purchases of the durable are always accompanied by purchases of the perishable good. However, purchases of the perishable, due to, e.g., lower perishable prices, do not entail purchases of the durable if the current stock of durables is sufficiently high.

For perishable goods and time–separable utility functions, the first order conditions imply that properly discounted marginal period utility is a martingale. If it is suitable to approximate marginal period utility in a linear way, one concludes that discounted consumption is also a martingale. This is Hall's (1978) *Random Walk Theorem*. Mankiw (1982) extends Hall's analysis to durable goods, and concludes that durable good expenditure should follow an ARMA(1,1) process. The data reject this hypothesis. Our model predicts that properly discounted marginal period utility in the perishable good is a martingale, whereas the stock of durables is governed by the running maximum of a stochastic process. Under some parametric assumptions, that stochastic process is a semimartingale. Therefore, purchases of durable goods should be related to all time highs in some index process. This could be a weighted average of perishable and durable prices, or a stock index, e.g. It might be tempting to test this hypothesis empirically.

In the remaining part of this introduction, we discuss the more technical contributions of our approach in more detail.

To start with, let us describe the mathematical model. At each point in time, utility is obtained both from the consumption of perishables and the service flow generated by the stock of durables. We assume that the service flow from durables is proportional to the stock held. This stock decays over time with some exponential rate. We consider general concave period utilities which are not necessarily separable and thus allow us to account for cross effects between durable and perishable goods. These cross effects are illustrated by the marginal utilities with respect to additional perishable and durable consumption. In fact, at a given point in time, marginal utility in the perishable good is a function both of current consumption of the perishable good and of the current stock of durables. The marginal utility with respect to durable consumption is more complex. Additional purchases of the durable affect future marginal utilities since they not only yield a higher service flow today, but also in any future point in time. As a consequence, the marginal utility in the durable aggregates the properly discounted future marginal period utilities. In particular, the marginal rate of substitution for durables between two periods is directly affected by the consumption decision in other periods.

We use a Lagrangian first order approach to address the household's utility maximization problem. This allows for essentially arbitrary price processes; in particular, we do not need the usual Markovian assumptions which are necessary for an approach by dynamic programming. The non-separability of our utility function entails that the first order conditions for optimal consumption of perishable and durable goods are interdependent. Yet, the purely local form of the utility gradient for consumption of perishables allows us to disentangle these two conditions: We can describe explicitly the optimal plan for perishables in terms of the plan for durables. Plugging this description into the first order condition for durable consumption, we are left with a transformed first order condition only involving the durable consumption plan. The main complication arises then from the fact that the first order condition for the durable good need not (and typically will not) be binding because durables cannot be sold. As a consequence, determining the optimal consumption plan for durables is rather involved and relies on a new key concept which we call the minimal shadow level of durable stock. This shadow level describes the evolution of the minimal stock of durables the agent would feel comfortable with at each point in time and in each scenario. The optimal consumption plan for durables can be recovered from this process following the simple rule to always purchase just enough of the durable good to keep the stock of durables above the minimal shadow level. We provide a stochastic backward equation which characterizes this minimal level process. This equation may be viewed as a substitute for the Hamilton–Jacobi–Bellman equation in our non–Markovian setting, and it explicitly relates the level process with the prices of durable and perishable goods, with the agent's preference structure, and with the economy's information flow. This kind of backward equations is studied in Bank and ElKaroui (2002) and, using some of their results, we characterize the solution of the minimal level equation in terms of an non–standard optimal stopping problem.

With an infinite time horizon and a Cobb-Douglas-utility, the method yields closed–form solutions for a wide range of price dynamics, including Brownian motion, Poisson processes and, in fact, general Lévy processes as driving factors. As mentioned above, it turns out that in such a setting the minimal shadow stock level is proportional to the optimal level which would be chosen in a world where the durable good trades without frictions. This explicit solution reveals a remarkable asymmetry in consumption behavior during booms and crashes in the durable good market. In fact, at times when prices for durables soar, our model predicts that this leaves the consumption rate for perishables essentially unaffected. By contrast, when prices for durables plummet this not only triggers additional consumption of durables but has also a direct impact on the agent's purchases of perishables.

We also discuss some algorithmic aspects of the choice problem. In a first step, we show how our approach has a natural counterpart in discrete time models. Therefore, we solve both the continuous–time as well as the discrete time model. Moreover, we show that the discrete time solutions converge to the continuous time solution. Of course, this is an important robustness property of the model. For the minimal level equation in discrete time, we provide an efficient algorithm that allows to calculate the minimal shadow level for all utility functions and time horizons.

We conclude the introduction by reviewing some related literature. Hindy and Huang (1993) study a model with one durable good in a Brownian framework with constant coefficients. They provide explicit solutions for power utility functions. Their model is complemented by our previous work (Bank and Riedel (2001)), where we develop the level equation approach and extend Hindy and Huang's explicit solutions to the class of Lévy processes. Cuoco and Liu (2000) study a model with one durable good which can be resold with transaction costs. They work in a Brownian framework with power utilities, and show that the optimal policy keeps the fraction of stock and wealth in between two constant bounds. Grossman and Laroque (1990) develop a model with one durable good which can be resold only as a whole, incurring transaction costs. They show that a so–called (s, S)–policy is optimal. Damgaard, Fuglsbjerg, and Munk (2003) generalize their model to a two good model including a perishable good. Detemple and Giannikos (1996) analyze a two good model in which one good is perishable and the other good provides 'status' as well as 'service'. Status is interpreted as being a perishable good, whereas service is modelled as durable, as in our model. Moreover, status and service are perfect substitutes. The assumption that status is perishable ensures that the (quasi–)durable good is purchased at every point in time (under appropriate conditions). Technically, this avoids the problems created by non-binding first order conditions. The optimal solution is found by solving a backward stochastic differential equation. Dunn and Singleton (1986) derive testable restrictions for bond returns from the first order conditions of a Cobb–Douglas separable utility function over perishable and durable goods; they assume that the durable good can be resold without transaction costs. Mamaysky (2001) discusses implications for the term structure of interest rates. Last not least we note that the present paper bears relation to the literature on irreversible investment as started by Arrow (1968); see Pindyck (1991) for an overview. In fact, the methodology presented here is bound to have applications for these problems, as future work will hopefully show.

The paper is organized as follows. The next section formulates the problem. Section 2 characterizes the optimal consumption choices, and Section 3 provides the numerical algorithm as well as explicit solutions.

1 Statement of the Problem

The consumer chooses a contingent consumption plan for one perishable and one durable good over his time horizon $T \in (0, \infty]$. Uncertainty is described by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$, where Ω is the set of states of nature, $\mathcal{F}$ a σ–field on Ω containing all possible events, $\mathcal{F}_t$ the information available at time t, and $\mathbb{P}$ a probability measure on the measurable space $(\Omega, \mathcal{F})$. The information flow $(\mathcal{F}_t)_{0 \le t \le T}$ is assumed to satisfy the usual conditions of right continuity and completeness.

Consumption of the perishable good occurs in rates, and we let $c_t(\omega)$ describe the period consumption rate of the perishable good at time t in state ω. As usual, we will omit the state variable ω in the following. Of course, the consumption choice c_t cannot depend on future information. Therefore, the process $c = (c_t)_{0 \le t \le T}$ is taken to be a nonnegative, optional process.

In contrast to the perishable good, the durable provides utility over longer periods of time. We let D_t denote the cumulative purchases of the durable good up to time t. Resale of the durable is not possible in our model, and the choice of durable consumption also has to be based on the available information. Therefore, D is a nondecreasing, right continuous, optional process. We set $D_{0-} \stackrel{\Delta}{=} 0$ which means nothing has been bought before time 0. Note that we allow for jumps as well as for singular increases in D. This allows for purchases of durables in bulks (houses, cars, etc.), as well as in a singular way which may obtain when purchases are related to singular events such as 'the Dow Jones reaches a new all–time–high'. Of course, the more 'standard' way of purchases in rates is allowed for as well. Formally, the consumption set is thus given by

$$\begin{aligned}\mathcal{X}_+ \triangleq \{(c, D) :&\text{optional stochastic processes}\\ &c \text{ nonnegative}\\ &D \text{ nondecreasing, right continuous with } D_{0-} = 0\}\end{aligned}$$

Markets are complete in our model. At time 0, the consumer can buy one unit of the perishable good for contingent delivery at time t and state ω at (forward) price $\phi_t(\omega)$. Similarly, the contingent state–price for the durable good is given by $\psi_t(\omega)$. Both processes ϕ and ψ are assumed to be strictly positive and optional. The (forward) price of the consumption plan (c, D) is therefore

$$\Pi(c, D) \triangleq \mathbb{E}\int_0^T \phi_t c_t\, dt + \mathbb{E}\int_0^T \psi_t\, dD_t\,.$$

The consumer has initial wealth $w > 0$, and so his budget set is

$$\mathcal{B}(w) \triangleq \{(c, D) \in \mathcal{X}_+ : \Pi(c, D) \leq w\}\ .$$

The stock y^D of the durable good depreciates at rate $\beta > 0$ as described by the evolution equation

$$dy_t^D = dD_t - \beta y_t^D dt,\ t \geq 0, \quad , y_{0-}^D = \eta\,, \tag{1}$$

where the nonnegative constant η accounts for the possibility that the consumer starts with a positive stock at time 0. Hence,

$$y_t^D = e^{-\beta t}\eta + \int_0^t e^{-\beta(t-s)}\, dD_s \quad (t \geq 0)\,.$$

Consumer's preferences are given by the von Neumann–Morgenstern utility functional

$$U(c, D) \triangleq \mathbb{E}\int_0^T u\left(t, c_t, y_t^D\right) dt\,.$$

The felicity function $u = u(t, c, y)$ is assumed to be jointly continuous in (t, c, y) and strictly increasing, strictly concave in (c, y) for fixed t. Moreover, we assume that u is continuously differentiable on the open cone $\{(c, y) : c, y > 0\}$ and satisfies the Inada conditions

$$\begin{aligned}\partial_c u(t, 0+, y) &= \partial_y u(t, i(t, \phi, 0+), 0+) = +\infty\\ \partial_c u(t, +\infty, y) &= \partial_y u(t, i(t, \phi, +\infty), +\infty) = 0\end{aligned} \tag{2}$$

for all $t, \phi, y > 0$, where $i(t, ., y)$ denotes the inverse of $\partial_c u(t, ., y)$.[1]

In the sequel, we study the utility maximization problem:

$$\text{Maximize } U(c, D) \text{ subject to } (c, D) \in \mathcal{B}(w) \tag{3}$$

under the standing assumption of well–posedness

$$\sup_{(c,D)\in\mathcal{B}(w)} U(c, D) < +\infty\,.$$

In order to ensure existence of a solution, we furthermore assume the (weak) integrability conditions $\mathbb{E}\sup_{0\le t\le T}\psi_t < +\infty$ and $\mathbb{E}\int_0^T \partial_y u(t, i(t, \phi, y), y)e^{-\beta t}\,dt < +\infty$, as well as the regularity condition that ψ is lower-semicontinuous in expectation, i.e., that $\liminf_n \mathbb{E}\psi_{T^n} \ge \mathbb{E}\psi_{T^0}$ for any monotone sequence of stopping times T^n taking values in $[0, T]$ and converging to T^0.

2 Optimal Consumption Plans

In this section, we are going to show how the utility gradient approach can be used in order to construct optimal consumption plans for the utility maximization problem (3). The starting point for this approach is the well–known principle that at optimum marginal utility from consumption does never exceed a certain fixed multiple of the costs of consumption, and that equality holds true between these quantities whenever consumption actually occurs. Formally, this means that for some Lagrange parameter $\lambda > 0$ an optimal plan (c^*, D^*) satisfies the first–order conditions

$$\nabla_c U(c^*, D^*)_t \le \lambda\phi_t \quad \text{with '=' whenever} \quad c_t > 0 \quad (0 \le t \le T) \tag{4}$$

and

$$\nabla_D U(c^*, D^*)_t \le \lambda\psi_t \quad \text{with '=' whenever} \quad dD^*_t > 0 \quad (0 \le t \le T)\,, \tag{5}$$

where '$dD^*_t > 0$' is short hand for 't is a time of increase for the non–decreasing process D^*'.

[1] It follows from concavity of $u(t, ., .)$ that $\partial_y u(t, i(t, \phi, y), y)$ is decreasing in y. As a consequence, the limits for $y \downarrow 0$ and $y \uparrow +\infty$ occurring in (2) do in fact exist.

Of course, in order to make use of these first order conditions, we have to compute the utility gradients $\nabla_c U$ and $\nabla_D U$ explicitly. This is particularly easy for the gradient with respect to consumption of the perishable good c. Indeed, since additional consumption of the perishable good affects marginal felicity only at those times where additional consumption actually occurs, marginal utility in the perishable good is given by the marginal period felicity, just like in the additive model with a single perishable good:

$$\nabla_c U(c, D)_t = \partial_c u(t, c_t, y_t^D)\,. \tag{6}$$

In contrast, additional purchases of durables at a certain point in time affect all future marginal felicities since such purchases increase the stock of durables at any time afterwards. Clearly, as the durable good depreciates at rate $\beta \geq 0$ this effect will decline accordingly and therefore we have to discount the future marginal felicities in the durable good. This leads us to the expression

$$\nabla_D U(c, D)_t = \mathbb{E}\left[\int_t^T \partial_y u(s, c_s, y_s^D) e^{-\beta(s-t)}\, ds \,\middle|\, \mathcal{F}_t\right] \tag{7}$$

for the marginal utility of additional purchases of the durable good at time t when otherwise following the consumption plan (c, D). The following lemma records this reasoning. It is the usual Kuhn–Tucker or saddle point theorem for the setup at hand.

Lemma 2.1 *A consumption plan* (c^*, D^*) *which exhausts the initial wealth,* $\Pi(c^*, D^*) = w$, *and satisfies the first–order conditions* (4) *and* (5) *solves the utility maximization problem* (3).

The next step in the utility gradient approach is to employ the explicit formulae for the utility gradients in order to determine the optimal consumption plan from the first order conditions (4) and (5). Again, this is particularly easy for the consumption plan for the perishable good: The Inada condition $\partial_c u(t, 0, y) = +\infty$ ensures that marginal utility is infinite whenever the agent does not consume the perishable good. By our formula for the utility gradient $\nabla_c U$ this is compatible with the first order condition (4) only if consumption of the perishable is made at every point time. Thus, an optimum (c^*, D^*) will satisfy

$$\partial_c u(t, c_t^*, y_t^{D^*}) = \lambda \phi_t \tag{8}$$

for every $t \in [0, T]$. This equation determines the consumption plan for perishable goods c^* as a function of time t, instantaneous state–price ϕ_t, and current stock level $y_t^{D^*}$:

$$c_t^* = i\left(t, \lambda\phi_t, y_t^{D^*}\right) , \tag{9}$$

where $i(t, ., y) = (\partial_c u(t, ., y))^{-1}$ is the inverse of marginal felicity with respect to the perishable good.

Finding an optimal plan for the durable good is more involved since the Inada condition $\partial_y u(t, c, 0+) = +\infty$ does *not* ensure that marginal utility in the durable good $\nabla_D U$ is infinite in periods where no durables are consumed. In fact, as we shall see, the optimal plan typically includes extended periods of time where no durables are purchased and where we have strict inequality '$<$' in the first order condition (5). The economic reason for this is that purchases of durable goods are irreversible and, thus, the agent cannot disinvest his stock of durables in order to benefit from 'high' prices for these goods in periods where the marginal utility from his stock of durables is comparably 'low'. As a technical consequence this entails that trying to proceed as for the perishable good, i.e., assuming equality in (5) and solving then for D^*, will not only be difficult but even impossible in general. Nevertheless it is possible to systematically deduce the optimal consumption plan D^* from an equality associated with the first order conditions (4) and (5). Instead of describing D^* directly, this equality characterizes an auxiliary process $L = (L_t)_{0 \le t \le T}$ which we call the *minimal stock level*. This adapted process describes a time–varying lower bound for the optimal stock of durables to be held by the agent at each point in time. More precisely, the agent's optimal plan for durables is to *track the minimal stock level* L, i.e., to refrain from any purchases of durables whenever the current stock $y_t^{D^*}$ is strictly above the time varying lower bound $L = (L_t)_{0 \le t \le T}$, and to purchase otherwise 'just enough' to ensure that $y_t^{D^*} = L_t$. Formally, this amounts to choose D^* as in the following definition.

Definition 2.2 *Let L be a progressively measurable process with upper–right continuous paths*[2]*. A plan D is said to track the process L if the associated stock level y^D takes the form*

$$y_t^{D^*} = e^{-\beta t} \max\{\eta, \sup_{0 \le v \le t} L_v e^{\beta v}\} \quad (t \in [0, T]) .$$

[2]Progressive measurability and upper–right continuity of L ensure that the running supremum $\sup_{0 \le v \le t}\{L_v e^{\beta v}\}$ is a right continuous, adapted process.

The process D which tracks a given level L is unique, as one can easily see from (1).

The next theorem shows that the optimal minimal stock level is characterized by a stochastic backward equation. This *minimal level equation* is specified in terms of the auxiliary function

$$v(t,\phi,l) \overset{\Delta}{=} \partial_y u(t, i(t,\phi,e^{-\beta t}l), e^{-\beta t}l)e^{-\beta t} \quad (t\in[0,T],\ \phi,l\in[0,+\infty))\,. \tag{10}$$

Theorem 2.3 *Suppose $L \geq 0$ is a progressively measurable process with upper–right continuous paths and $L(T)=0$ which solves the 'minimal level equation'*

$$\mathbb{E}\left[\int_s^T v(t,\lambda\phi_t, \sup_{s\leq v\leq t}\{L(v)e^{\beta v}\})\,dt \,\middle|\, \mathcal{F}_s\right] = \lambda\psi_s e^{-\beta s} \quad \textit{for all} \quad s\in[0,T) \tag{11}$$

where the function v is determined from agent's preferences via (10). *Denote by D^* the process which tracks the level L, and let c^* be given by* (9). *Then the consumption plan (c^*,D^*) solves the first–order conditions for optimality* (4) *and* (5).

Proof : The process c^* satisfies the first order condition (4) by definition. To establish the other first order condition (5) at time $t\in[0,T)$, we note first that by definition of D^* we have for any $s\in[t,T]$

$$y_s^{D^*} \geq \bar{L}_{t,s} \overset{\Delta}{=} e^{-\beta s}\sup_{t\leq v\leq s}\{L_v e^{\beta v}\}\,. \tag{12}$$

Concavity of the felicity function u implies that the function $v(t,\phi,l)$ is decreasing in $l\in[0,+\infty)$ for fixed (t,ϕ). Therefore, the preceding estimate yields

$$\partial_y u(s,c_s^*,y_s^{D^*})e^{-\beta s} = v(s,\phi_s,y_s^{D^*}) \leq v(s,\phi_s,\bar{L}_{t,s})\,.$$

From this we obtain upon integration over $s\in[t,T]$ that

$$\nabla_D U(c^*,D^*)_t \leq \mathbb{E}\left[\int_t^T v(s,\phi_s,\bar{L}_{t,s})\,ds \,\middle|\, \mathcal{F}_t\right] e^{\beta t} = \lambda\psi_t \tag{13}$$

where the second equality follows from equation (11). This proves $\nabla_D U(c^*,D^*)\leq\lambda\psi$. Moreover, we know from the construction of D^* that whenever additional durables are purchased this is done so that $y_t^{D^*}=L_t$ holds true. Since this implies that equality holds true in (12) for any $s\geq t$, we infer that equality also holds true in (13) whenever $dD_t^*>0$. Hence, D^* satisfies the first order condition (5). $\square$

Theorem 2.3 establishes the usefulness of the concept of a minimal stock level of durables. The characterization of this process provided by the minimal level equation (11) can readily be used to check whether a given candidate actually is such a minimal level process (cf. Section 3 below). The theorem leaves open, however, how one can systematically construct such a minimal level L. In order to gain more insight into the structure of the minimal level process L, let us assume our agent enters the economy at some stopping time $S < T$. He then has to choose his consumption plan $(c^S, D^S) = (c_t^S, D_t^S)_{t\in[S,T]}$ for the remaining time period $[S,T]$. Suppose he has zero initial stock of durables at time S, $y_{S-}^{D^S} = 0$, and, finding this unsatisfactory, decides to immediately raise his stock to some state dependent, $\mathcal{F}_S$–measurable level $l > 0$. Suppose furthermore that after time S he refrains from any further purchases until at time $S' > S$ either the time horizon has elapsed, $S' = T$, or he again wishes to raise his stock of durables, $S' < T$. Between time S and S', his stock of durables will then evolve according to $y_t^{D^S} = le^{-\beta(t-S)}$, $t \in [S,S')$. We can therefore easily relate his marginal utility from purchasing durables at time S with the corresponding marginal utility at time S':

$$\begin{aligned}\nabla_D U(c^S,D^S)_S =&\mathbb{E}\left[\int_S^{S'} \partial_y u(t, c_t^S, le^{-\beta(t-S)})e^{-\beta(t-S)}\,dt\,\middle|\,\mathcal{F}_S\right]\\&+\mathbb{E}\left[e^{-\beta(S'-S)}\nabla_D U(c^S,D^S)_{S'}\,\middle|\,\mathcal{F}_S\right].\end{aligned}\tag{14}$$

Now, assume that the chosen consumption plan (c^S, D^S) is in fact optimal with Lagrange parameter λ, say. Then it follows from the first order condition (4) that

$$c_t^S = i(t, \lambda\phi_t, y_t^{D^S}) \quad (t \in [S,T])\,.\tag{15}$$

Moreover, the first order condition (5) yields that

$$\nabla_D U(c^S,D^S)_S = \lambda\psi_S \quad \text{and} \quad \nabla_D U(c^S,D^S)_{S'} = \lambda\psi_{S'}1_{\{S'<T\}}\,,\tag{16}$$

where the indicator function in the last term accounts for the possibility that in some states of the world S may already be the last time for purchases of durables (whence $S' = T$ and $\nabla_D U(c^S,D^S)_{S'} = \nabla_D U(c^S,D^S)_T = 0$). Plugging Eqs. (15) and (16) into (14) yields the relation

$$\begin{aligned}&\mathbb{E}\left[\int_S^{S'} v(t,\lambda\phi_t, le^{\beta S})\,dt\,\middle|\,\mathcal{F}_S\right]\\&\quad=\mathbb{E}\left[\lambda\psi_S e^{-\beta S} - \lambda\psi_{S'}e^{-\beta S'}1_{\{S'<T\}}\,\middle|\,\mathcal{F}_S\right]\end{aligned}\tag{17}$$

where v denotes the auxiliary function defined in (10).

Plainly, for any stopping time $S < T$ there may be many pairs (l, S') of $\mathcal{F}_S$–measurable random variables l and stopping times $S' > S$ which satisfy (17). In fact, it follows from the Inada conditions (2) that for any $S' > S$ in the class $\mathcal{S}'(S)$ of stopping times for which the right hand side in (17) is nonnegative, there exists an almost surely unique $\mathcal{F}_S$–measurable random variable $l = l_{S,S'}$ such that (17) holds true. Our preceding considerations suggest that all these random variables $l_{S,S'}$ are reasonable candidates for the optimal stock level our agent should choose when entering the economy at time S. Since we are interested in the minimal level of durable stock to hold at time S, it is natural to consider minimum or more precisely the essential infimum of all these random variables. In fact, in conjunction with Theorem 2.3 the following Theorem 2.4 shows that this infimum actually yields the optimal initial stock level to be chosen at time S:

Theorem 2.4 *There exists a progressively measurable process* $L = (L_t)_{0 \le t \le T} \ge 0$ *with upper–right continuous paths and* $L_T = 0$ *which solves the minimal level equation* (11). *At any stopping time* $S < T$ *this process is uniquely determined by*

$$L_S = \operatorname*{ess\,inf}_{S' \in \mathcal{S}'(S)} l_{S,S'} \,, \tag{18}$$

with $\mathcal{S}'(S) = \left\{ S' \textit{stopping time} : S' > S, \mathbb{E}\left[\lambda \psi_S e^{-\beta S} - \lambda \psi_{S'} e^{-\beta S'} 1_{\{S' < T\}} \,\middle|\, \mathcal{F}_S \right] > 0 \right\}$.

The proof is given in the appendix.

3 Explicit Solutions

In this section, we are going to show how explicit solutions to our minimal level equation can be obtained. In the first part, we are going to focus on computational aspects of this problem. We prove consistency of suitable discrete time approximations to the utility maximization problem, and we provide an efficient and easy to implement algorithm which computes the minimal level process in such a discrete time framework. In the second part, we are going to work in a 'homogeneous' continuous-time framework where the minimal level equation (11) can be verified directly for a suitable candidate. This allows us to determine the minimal level process in closed form and to describe explicitly the implied optimal consumption behavior.

3.1 Discrete Time and Numerical Solutions

In this section, we present an algorithm that calculates the optimal consumption plans for the corresponding utility maximization problem in discrete time. Moreover, we show that these discrete–time solutions converge to the continuous–time solution as the mesh of the grid vanishes. This is important in two regards. From a theoretical perspective, it is important to know that the discrete–time model converges to the continuous–time model; if that was not the case, the continuous–time model would be of limited value since our observations are clearly finite. From a computational perspective, it is very useful to have an algorithm that computes the optimal plans for *arbitrary* utility functions and underlying stochastics.

3.1.1 The Discrete–Time Model

Let us assume that the market for durable goods allows for purchases only at a finite number of times $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$. Then our agent's problem is to maximize his utility $U(c, D)$ not only subject to the budget constraint $\Pi(c, D) \leq w$, but also subject to the feasibility constraint that $dD_t > 0$ only for $t \in \tau \triangleq \{t_0, \ldots, t_{n+1}\}$. A nice feature of the present utility model is that allows for a simple proof of consistency with respect to such time discretizations if the state–price deflators ψ takes the form $\psi_t = \exp\left(-\int_0^t r_s\, ds\right) M_t$ for some integrable interest rate process $r \geq 0$ and a martingale M. This is the usual form for state–price deflators in financial economics.

Theorem 3.1 *As the mesh $\|\tau\| = \max_i |t_i - t_{i+1}|$ of the partition τ tends to zero, the utility which is obtained by the optimal plan for the discrete time problem tends to the utility obtained in the continuous time problem.*

The proof is given in the appendix.

It is easy to see that in the present discrete–time setting the first order condition for optimality (4) remains unchanged, and therefore, given a plan D^* for purchases of durables, the corresponding optimal consumption plan for perishable goods is described by (14). Also the first order condition (5) carries over to the present discrete–time setting with the only modification that $\nabla_D U(c^*, D^*) \leq \lambda\psi$ merely must hold true on $\tau \subset [0, T]$. As a consequence, the 'level principle' describing optimal consumption rules as in Theorem 2.3 still holds true: there is an adapted process $L = (L_t)_{t\in\tau}$ with $L_T = 0$ such that the optimal purchase plan for durables at some time $t \in \tau$ is to refrain from further purchases if the current stock of durables satisfies $y^{D^*}_{t-} \geq \psi_t$, and to buy otherwise just enough to ensure that $y^{D^*}_t = L_t$. The process L is characterized as the unique adapted solution to the *discrete–time minimal level equation*

$$\mathbb{E}\left[\int_s^T v(t, \lambda\phi_t, \max_{v \in \tau \cap [s,t]} \{L(v)e^{\beta v}\})\, dt \,\middle|\, \mathcal{F}_s\right] = \lambda \psi_s e^{-\beta s} \quad \text{for all} \quad s \in \tau \cap [0, T)\,. \tag{19}$$

3.1.2 Numerical Solution

There are several possibilities to compute the minimal level process L characterized by (19). The most obvious way is to solve successively for $L_{t_n}, L_{t_{n-1}}, \ldots, L_{t_0}$ in (19). This, however, would involve the calculation of increasingly complex conditional expectations and would thus result in a rather tedious, time–consuming task. On the other hand, one could use the discrete–time analogue of (18) and compute for each $S \in \tau$ the essential infimum of all the random variables $l_{S,S'}$ where S' varies over the stopping times with values in $\tau \cap (S, T]$ for which the right side in (17) is positive. These random variables are comparably easy to determine. However, the set of all stopping times S' to be considered can be huge so that the performance of such a naive approach would be pretty poor. It is therefore interesting to note that the number of stopping times S' to be taken into account to determine L_S from (17) can be reduced considerably. Indeed, as proved in Bank and Föllmer (2003), it suffices to consider for $S = t_i \in \tau$ the increasing sequence of stopping times $S'_0 \stackrel{\Delta}{=} t_{i+1} \leq S'_1 \leq \ldots$ where S'_{n+1} is obtained from S'_n by letting

$$S'_{n+1} \stackrel{\Delta}{=} \inf\{t \in \tau \cap (S'_n, T] : L_t e^{\beta t} \geq L_{S'_n} e^{\beta S'_n}\} \wedge T$$

on $\{\mathcal{F}_S\text{– ess inf}\, L_{S'_n} e^{\beta S'_n} = L_{S'_n} e^{\beta S'_n} < l_{S,S'_n} e^{\beta S'_n}\}$, provided this set has positive probability. [3] As soon as this is not the case, one can put $L_S = l_{S,S'_n}$ and continue with the calculation of L_S for $S = t_{i-1}$.

As an example, we study the importance of the *retirement effect* for both the stock level of the durable and the consumption of perishable goods. With a finite time horizon, the

[3] Here l_{S,S'_n} is defined as the unique $\mathcal{F}_S$–measurable solution l to (17) with $S' \stackrel{\Delta}{=} S'_n$ on the set where the right side in (17) is positive, and as $+\infty$ on the complement of this set; $\mathcal{F}_S\text{– ess inf}\, L_{S'_n} e^{\beta S'_n}$ denotes the conditional essential infimum of $L_{S'_n} e^{\beta S'_n}$, i.e., the smallest $\mathcal{F}_S$–measurable random variable dominating $L_{S'_n} e^{\beta S'_n}$.

minimal shadow level converges to zero as the horizon approaches. This implies that durables are not purchased after a certain age has been reached. In the example with deterministic prices for both goods considered in Figure 1, the horizon is 40 years, the expected lifetime of the durable is 20 years. The household stops buying after 24.5 years. The parameters in the model are chosen such that the consumption rate for perishable goods would be constant if there were no durables. However, as Figure 2 shows, the presence of the durable leads here to a time–varying consumption rate. It rises first, and drops later in life, as perishables and durables are complements in that model. The same effect can be observed in a model where prices fluctuate randomly, see Figure 3.

3.2 Cobb–Douglas Utility and log–Lévy Prices

We study now Cobb–Douglas preferences in a homogenous setting. Specifically, we assume that the time horizon is infinite, $T = \infty$. For simplicity, we set the initial stock level to $\eta = 0$[4]. The investor's felicity function is given by

$$\begin{aligned} u(t,c,y) &\triangleq e^{-\delta t}\frac{c^A y^B}{A+B} \\ &= e^{-\delta t}\frac{\left(c^\theta y^{1-\theta}\right)^{1-\alpha}}{1-\alpha}, \end{aligned}$$

for constants $\alpha, \delta > 0$, $0 < \theta < 1$, and $A = \theta(1-\alpha), B = (1-\theta)(1-\alpha)$. The parameters in the utility function may be interpreted in different ways. In one–period models, θ corresponds to the fraction of wealth a Cobb–Douglas agent spends for the perishable good. The parameter α describes in one–good models the inverse intertemporal elasticity of substitution as well as the degree of constant relative risk aversion. For our setting, α will describe the complementarity or substitutability between durable and perishable goods. The parameter δ measures the impatience of the investor.

We assume that price processes can be written as

$$\phi_t = e^{-X_t}, \psi_t = e^{-Y_t} \tag{20}$$

for a Lévy process (X, Y) with $X_0 = Y_0 = 0$. This covers a range of models. From finance models that date back at least to Merton (1971), we are familiar with the case of X and Y

[4]The results for general η are easily obtained and available from the authors upon request.

being Brownian motion with drift. The class of Lévy[5] processes includes also (compound) Poisson processes. The deterministic case, with $X_t = r^p t$ and $Y_t = r^d t$, belongs to it as well, and analyzed in detail below.

Before we present the general explicit solution in this case, one might want to recall that in the homothetic framework, it is enough to exhibit an optimal plan (c^*, D^*) for one specific wealth level w^* because the optimal plans for general wealth w are just constant multiples $\frac{w}{w^*}(c^*, D^*)$ of this one reference solution. To find this reference solution, let us introduce the minimal level

$$L_t^* = \left(\frac{e^{-\delta t}}{\phi_t^A \psi_t^{1-A}}\right)^{\frac{1}{1-A-B}} = \left(\frac{e^{-\delta t}}{\phi_t^{\theta(1-\alpha)} \psi_t^{1-\theta(1-\alpha)}}\right)^{\frac{1}{\alpha}} . \tag{21}$$

Set

$$m^* \triangleq \mathbb{E}\left[\int_0^\infty e^{-\frac{\delta}{1-A}t}\, \phi_t^{-\frac{A}{1-A}} \left\{\left(\sup_{u \le t} L_u^* e^{\beta u}\right) e^{-\beta t}\right\}^{-\frac{1-A-B}{1-A}} e^{-\beta t}\, dt\right] .$$

We will show below that L^* solves the minimal level equation (11) for the Lagrange multiplier

$$\lambda^* \triangleq \frac{A^A\, B^{1-A}}{A+B}\, (m^*)^{1-A} .$$

The corresponding optimal stock level is given by

$$y_t^* \triangleq \left(\sup_{u \le t} L_u^* e^{\beta u}\right) e^{-\beta t} ,$$

from which one obtains the corresponding plan D^* for durables via (1), that is

$$D_t^* \triangleq y_t^* + \int_0^t \beta y_s^* ds . \tag{22}$$

The corresponding optimal plan for perishables is, as we shall show,

$$c_t^* \triangleq \frac{A}{Bm^*}\, e^{-\frac{\delta}{1-A}t}\, \phi_t^{-\frac{1}{1-A}}\, (y_t^*)^{\frac{B}{1-A}} . \tag{23}$$

Finally, denote by $w^* \triangleq \Psi(c^*, D^*)$ the price of the candidate solution. We assume $w^* < \infty$.

Theorem 3.2 *The consumption plan* $\left(\frac{w}{w^*}\, c^*, \frac{w}{w^*}\, D^*\right)$ *is optimal for initial wealth* w*, where* D^* *is given by* (22) *and* c^* *by* (23).

[5]For a reference on Lévy processes, we suggest Bertoin (1996).

PROOF : Since the utility function is homothetic, it suffices to prove optimality for the specific wealth level w^*. In light of Theorem (2.3), we just have to show, therefore, that L^* solves the minimal level equation (11) for Lagrange multiplier λ^*.

A straightforward calculation shows that the minimal level equation reduces to

$$\mathbb{E}\left[\int_t^\infty e^{-\frac{\delta}{1-A}s-\frac{B\beta}{1-A}s}\phi_s^{-\frac{A}{1-A}}\left(\sup_{t\le u\le s} L_u^* e^{\beta u}\right)^{-\frac{1-A-B}{1-A}}\middle|\mathcal{F}_t\right] = \lambda^{\frac{1}{1-A}}\frac{(A+B)^{\frac{1}{1-A}}}{A^{\frac{A}{1-A}}B}\psi_t e^{-\beta t}\,.$$

Write LS for the left side of the equation, $l_t^* \overset{\Delta}{=} \log\left(L_t^* e^{\beta t}\right)$, and set $Z_t^1 \overset{\Delta}{=} \frac{1}{1-A}\delta t + \frac{B}{1-A}\beta t + \frac{A}{1-A}X_t$. With these definitions, we have

$$LS = \mathbb{E}\left[\int_t^\infty e^{-Z_s^1 - \sup_{t\le u\le s}\frac{1-A-B}{1-A}l_u^*}ds\,\middle|\,\mathcal{F}_t\right]\,, \tag{24}$$

As a linear combination of Lévy processes, (Z^1, l) is a Lévy process. We may therefore apply the strong Markov property and get

$$LS = \left(\mathbb{E}\int_0^\infty e^{-Z_s^1 - \sup_{0\le u\le s}\frac{1-A-B}{1-A}l_u^*}ds\right)e^{-Z_t^1 - \frac{1-A-B}{1-A}l_t^*} = m^*\psi_t e^{-\beta t}$$

Therefore, L^* solves indeed the minimal level equation for the appropriate choice of λ, which is

$$\lambda^* = \frac{A^A\,B^{1-A}}{A+B}\,(m^*)^{1-A}\quad.$$

□

The optimal shadow stock level depends on the ratio of personal discount factor $e^{-\delta t}$ and a geometric weighted average of prices. Higher impatience implies a lower optimal shadow level, as the investor transfers wealth to the present. Naturally, the shadow level is decreasing in the durable's price. Depending on the sign of A, durable and perishable goods are substitutes or complements. If $A < 0$ (or $\alpha > 1$), the shadow level for the durable good increases in the perishable's price, and the goods are substitutes. For $A > 0$ or $\alpha < 1$, they are complements. Thus, in addition to the familiar interpretation of α as relative risk aversion or inverse intertemporal elasticity of substitution, a third interpretation arises here: The value of α triggers complementarity or substitutability.

It is important to stress, however, that a local change in the shadow level need not influence the observed purchasing behavior. This occurs only when the change in the shadow level is positive, and this positive that the corresponding process $(L_t e^{\beta t})$ reaches a new running maximum. In contrast to time–separable models, the purchasing behavior is not myopic. Instead, current prices and the history of prices as summarized by the running maximum of the process $(L_t e^{\beta t})$ form a sufficient statistic for consumption decisions. The depreciation rate β determines the frequency of new durable purchases, as the following consideration shows. New durables are bought whenever the process $(L_t e^{\beta t})$ reaches a new maximum. With a higher β, this occurs more frequently. At the same time, the purchased amount of durables has to be smaller, of course; we have a higher frequency and a lower amplitude of purchases when the depreciation rate increases.

Graphic Illustrations As an illustration, we consider the following case. The price of the perishable good is deterministic, $\phi_t = e^{-r^p t}$ with $r^p = 3\%$. The price of the durable good is an exponential Brownian motion of the type $\psi_t = e^{-\zeta W_t - \zeta^2 t/2 - r^d t}$ for a Brownian motion W, volatility $\zeta = 8\%$ and interest rate $r^d = 5\%$. For the case of complements ($\alpha = 0.5$), Figure 5 shows the positive correlation of the consumption rate of the perishable good and the stock level of the durable. In particular, good news on the durable market, resulting in new purchases of the good, have a considerable impact on the consumption rate of the perishable good. On the other hand, the picture changes dramatically if perishables are a substitute for durables ($\alpha = 3.5$), as one can see from Figure 6.

The Figures 7 and 8 illustrate the impact of the expected lifetime of the durable good on optimal plans. Of course, a higher lifetime leads to less frequent purchases. In the case of substitutes, the corresponding impact on the consumption rate of the perishable good is dramatic.

The Figures 9 and 10 illustrate the role of the parameter θ. When θ is close to 1, the perishable's consumption rate is determined almost entirely by the perishable's price. For small θ, the stock of the durable has an impact.

The Deterministic Case A study of the deterministic case might improve one's understanding of the optimal consumption behavior. Assume that there is a constant own interest rate for the perishable good r^p as well as an own interest rate r^d for the durable, that is $X_t = r^p t, Y_t = r^d t$. Then the optimal shadow level satisfies $\log L^*_t = const + \frac{1}{1-A-B}\left(Ar^p + (1-A)r^d - \delta\right)t$. Thus, the convex combination $\bar{r} \stackrel{\Delta}{=} Ar^p + (1-A)r^d$ of interest rates and the time preference rate δ govern the optimal shadow stock level. Patient investors ($\delta \leq \bar{r}$) exhibit an increasing L, and thus the actual stock level coincides with the shadow level: The investor buys a bulk of a certain size at time 0 and increases the stock continuously at rate $\beta + \frac{\bar{r}-\delta}{1-A-B}$ afterwards in order to keep track of the increasing shadow level. This is, by the way, the solution one obtains in the frictionless world where the stock can be resold without any transaction costs. Impatient investors ($\delta > \bar{r}$) also buy a bulk at time 0; however, their optimal shadow level is decreasing. When the durable depreciates slowly ($\beta < \frac{\delta-\bar{r}}{1-A-B}$), the investor's stock level remains always above the optimal shadow level after the initial purchase, and he never buys the durable again. For relatively short–lived durables ($\beta > \frac{\delta-\bar{r}}{1-A-B}$), the investor has to compensate the large losses due to depreciation by purchasing continuously at rate $\beta - \frac{\delta-\bar{r}}{1-A-B}$ in order to keep the actual stock equal to the optimal shadow level. The reader is referred to Bank and Riedel (2000) for comparison with the case of one durable good.

Appendix

Proof of Theorem 2.4 : The proof consists essentially in an application of the results in Bank and ElKaroui (2002). To see this, put $X \triangleq \lambda\psi 1_{[0,T)}$ and define

$$f(\omega,t,l) \triangleq \begin{cases} v(t,\lambda\phi_t(\omega),-1/l) & \text{if } l<0\,, \\ -e^{-t}l & \text{if } l\geq 0\,. \end{cases}$$

By our assumptions on ψ, ϕ, and u, these choices of X and f satisfy the assumptions made in Bank and ElKaroui (2002). Indeed, ψ is lower–semi continuous in expectation and dominated by an integrable random variable, thus of 'class (D)'. Moreover, $f(.,.,l)$ is progressively measurable (even optional) and $\mathbb{P}\otimes dt$–integrable for fixed l; for fixed (ω,t), $f(\omega,t,.)$ is strictly decreasing from $+\infty$ to $-\infty$ over $\mathbb{R}$ since v is strictly decreasing by strict concavity of $u(t,.,.)$ with $v(t,\phi,0+)=+\infty$ and $v(t,\phi,+\infty)=0$ for any (t,ϕ) by assumption. Hence, we may apply Theorem 2 in Bank and ElKaroui (2002) to obtain existence of a progressively measurable process $\tilde{L}$ with upper–right continuous paths such that

$$X_S = \mathbb{E}\left[\int_S^T f(t,\sup_{S\leq v\leq t}\tilde{L}_v)\,dt \,\middle|\, \mathcal{F}_S\right] \tag{25}$$

for any stopping time $S<T$. Theorem 1 in the same paper yields a characterization of this process: $\tilde{L}_S = \operatorname{ess\,inf}_{S'} \tilde{l}_{S,S'}$ where S' varies over all stopping times taking values in $(S,T]$

and where $\tilde{l}_{S,S'}$ is the unique $\mathcal{F}_S$–measurable solution to

$$\mathbb{E}\left[\int_S^T f(t,\tilde{l}_{S,S'})\,dt \,\middle|\, \mathcal{F}_S\right] = \mathbb{E}\left[X_S - X_T \,|\, \mathcal{F}_S\right] . \tag{26}$$

Since $X > 0$ on $[0,T)$, it follows that $\tilde{L} < 0$ almost surely. Hence, putting $L_t \triangleq -e^{-\beta t}/\tilde{L}_t$ allows us to transform equation (25) into our minimal level equation (11). Similarly, the characterization (18) is obtained from (26) by noting that $l_{S,S'} = -e^{-\beta S}/\tilde{l}_{S,S'}$ satisfies (17) if the right side in (26) (or equivalently (17)) is non-negative. □

Proof of Theorem 3.1 : Let $(c^\tau, D^\tau) \in \mathcal{B}(w)$ denote the optimal consumption plan in the constrained economy where the market for the durable good opens only at the dates $t \in \tau$. It is clear that $U(c^\tau, D^\tau) \leq U(c^*, D^*)$ where (c^*, D^*) denotes the optimal plan $(c^*, D^*) \in \mathcal{B}(w)$ in the unconstrained economy. To prove that $U(c^\tau, D^\tau) \to U(c^*, D^*)$ for $\|\tau\| \to 0$, define the constraint plan for durables $D^{*,\tau} \triangleq \sum_{i=0}^n D^*_{t_i} 1_{[t_i,t_{i+1})}$ and consider the rescaled plan $(k^\tau c^*, k^\tau D^{*,\tau})$ with $k^\tau \triangleq w/\Pi(c^*, D^{*,\tau})$. This plan is budget feasible in the constrained economy and its utility is therefore less than or equal to $U(c^\tau, D^\tau)$. Hence, it suffices to prove $\liminf_{\|\tau\|\to 0} U(k^\tau c^*, k^\tau D^{*,\tau}) \geq U(c^*, D^*)$. As $\|\tau\| \to 0$, the plan $D^{*,\tau}$ almost surely tends to D^* in every point t of continuity of D^*. This implies that almost surely

$$\int_0^T e^{-\int_0^t r_s\,ds}\,dD^{*,\tau}_t = e^{-\int_0^T r_s\,ds} D^{*,\tau}_T + \int_0^T D^{*,\tau}_t r_t e^{-\int_0^t r_s\,ds}\,dt$$

converges to $\int_0^T e^{-\int_0^t r_s\,ds}\,dD^*_t$ from below. Hence, the costs $\mathbb{E}\int_0^T \psi_t\,dD^{*,\tau}_t = \mathbb{E}\int_0^T e^{-\int_0^t r_s\,ds}\,dD^{*,\tau}_t$ for $D^{*,\tau}$ tend to the costs of D^* by monotone convergence. This entails $k^\tau \to 1$ for $\|\tau\| \to 0$. Hence, we also have $k^\tau D^{*,\tau} \to D^*$ almost surely in every point of continuity of D^*. This yields $u(t, k^\tau c^*_t, y_t^{k^\tau D^{*,\tau}}) \to u(t, c^*_t, y_t^{D^*})$ $\mathbb{P} \otimes dt$–almost everywhere, and so we have indeed $\liminf_{\|\tau\|\to 0} U(k^\tau c^*, k^\tau D^{*,\tau}) \geq U(c^*, D^*)$ by Fatou's lemma. □

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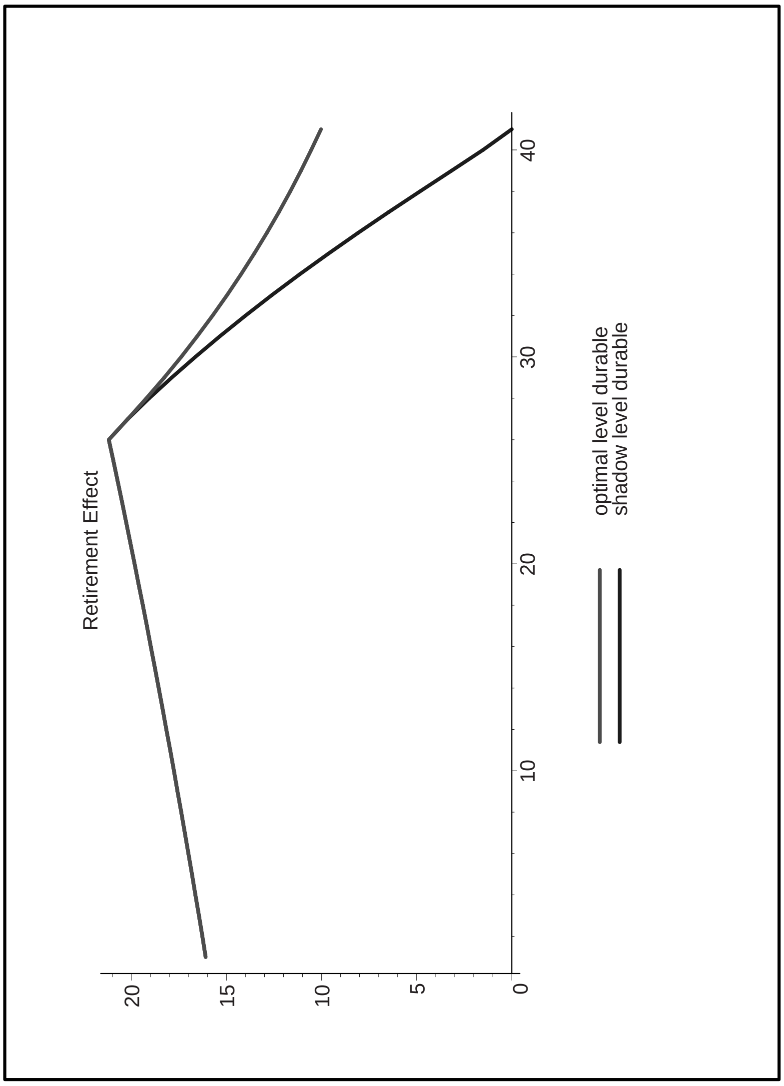

Figure 1: This picture illustrates the retirement effect for the optimal shadow and actual stock level of the durable in a deterministic setting. The horizon of the agent is 40 years. The consumer accumulates durables until a certain age (24), and afterwards, no further purchases of durables are made. The expected lifetime of the durable is $1/\beta = 20$. Parameters of the utility function are $\theta = .8, \alpha = .5, \delta = 0.02$. The price of the perishable follows $\phi_t = \exp(-r^p t)$ with $r^p = 0.02$. The price of the durable good is $\psi_t = \exp(-r^d t)$, with $r^d = 0.03$.

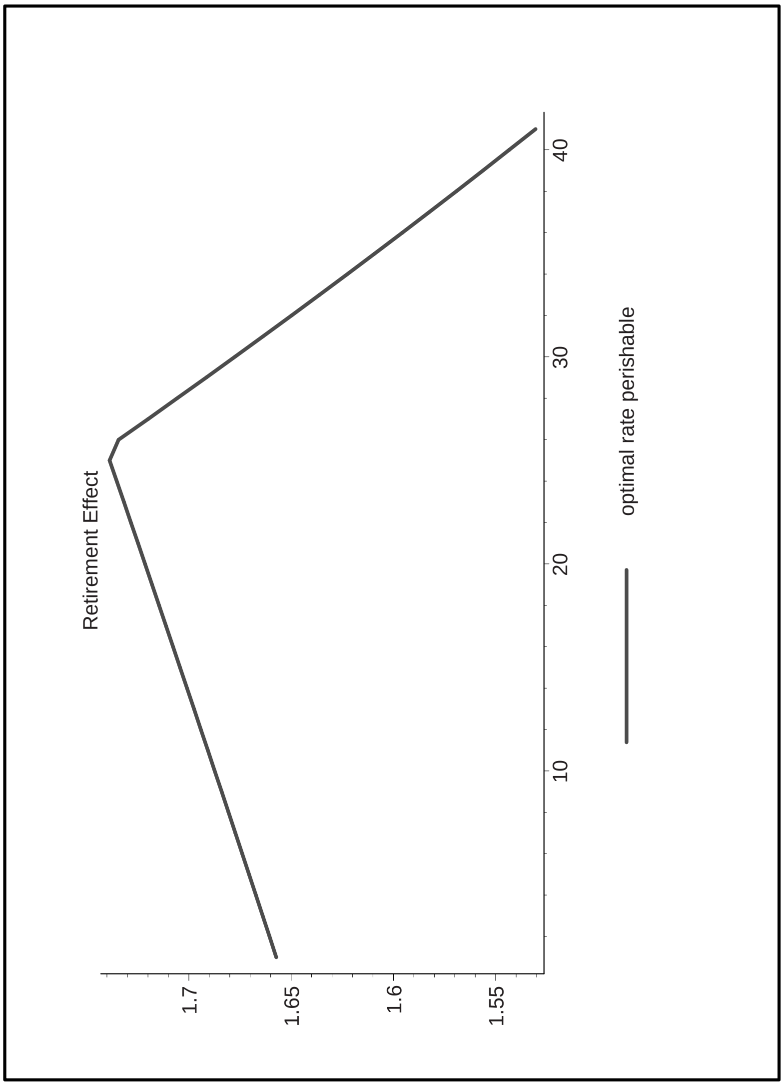

Figure 2: This picture illustrates the retirement effect for the optimal consumption rate of the perishable good for the same setting as Figure 1. Without durables, the consumption rate would be flat. The presence of durables induces a hump due to complementarity effects.

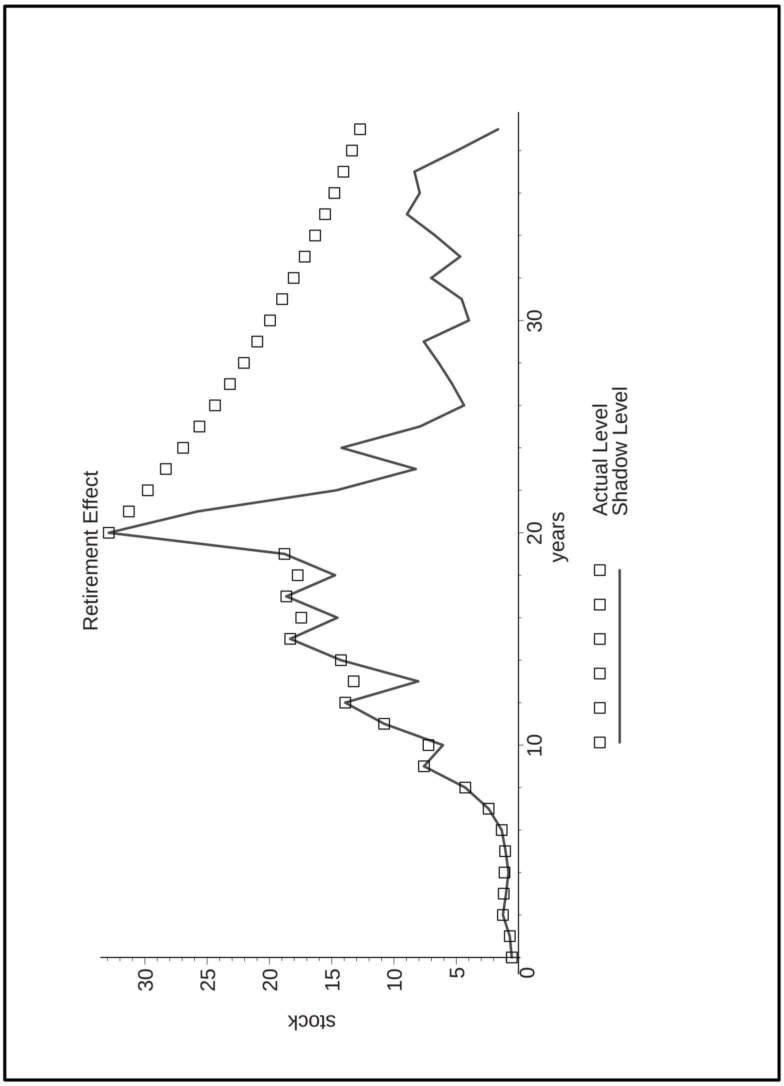

Figure 3: This picture illustrates the retirement effect for the durable good in a setting with stochastic prices. We use the same parameters as in Figure 1 and add volatiliy parameters $\xi = .05$ and $\zeta = .08$ to account for random price fluctuations from independent shocks. Please note that this is a discrete time example. Adjustments of the stock can be made at integer times only. Therefore, the shadow level should be plotted as a sequence of points, as we have done for the actual level. We think that the interpolated continuous polygon plot makes the picture more comprehensible. The same comment applies to the other illustrations, of course.

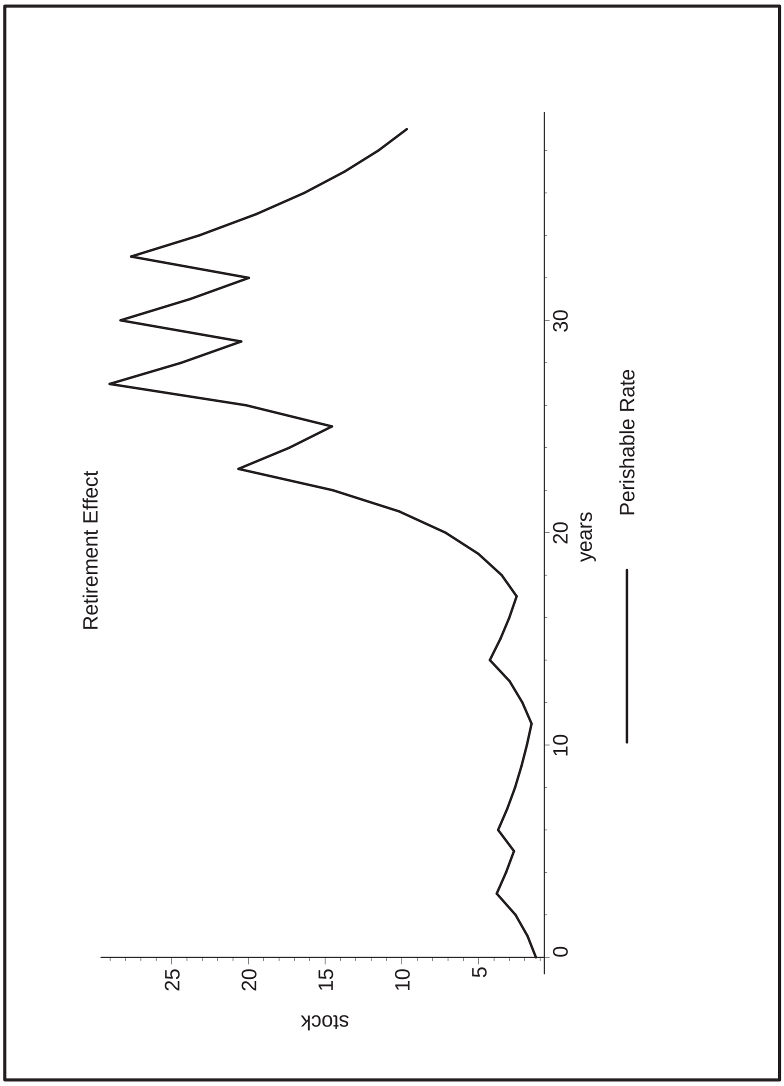

Figure 4: This picture illustrates the retirement effect for the perishable good in a setting with stochastic prices. We use the same parameters as in Figure 3. Due to favorable shocks, the perishable consumption rate does not decrease as early as in the deterministic example (Figure 1) where the retirement effect sets in in period 24. In this example, the retirement effect dominates after period 33.

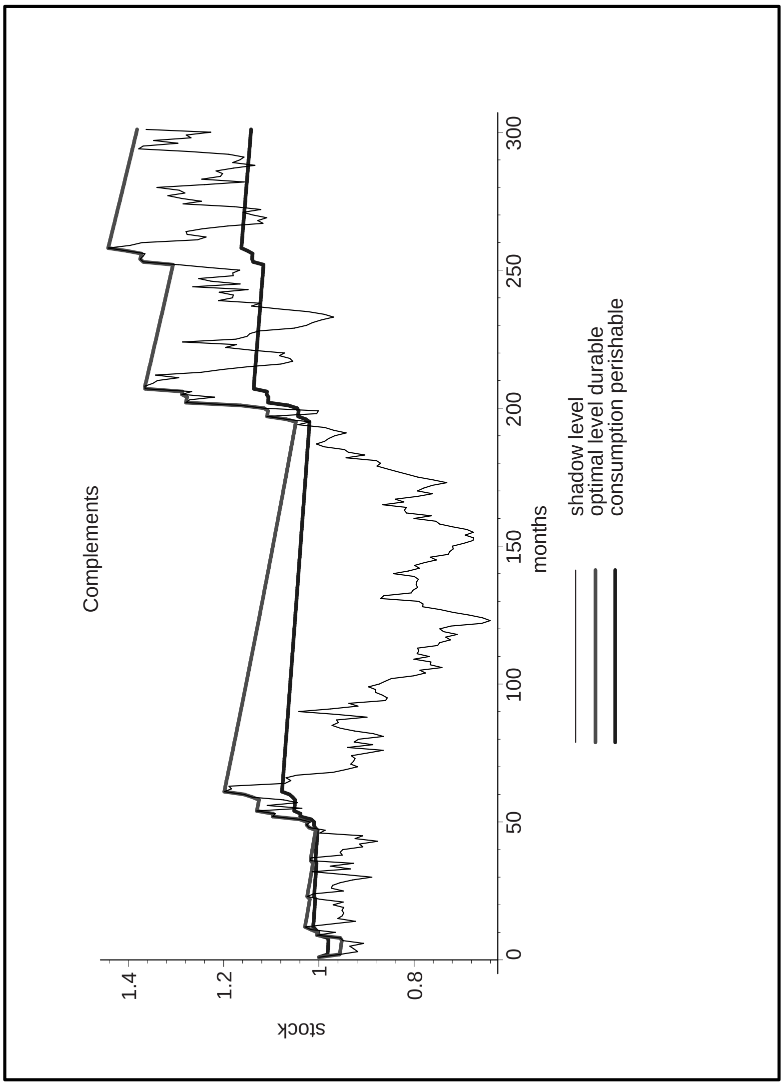

Figure 5: This picture illustrates the dynamics of durables and perishables when they are complements. Optimal shadow and actual stock level of the durable. Parameters are $\theta = .3, \alpha = .5, \beta = .01, \delta = 0.03$. The price of the perishable has $X_t = -r^p t$ with $r^p = 0.03$. The price of the durable good has $Y_t = \zeta W_t - \frac{\zeta^2}{2} t - r^d t$, with $\zeta = 0.08$ and $r^d = 0.05$. W is a Brownian motion.

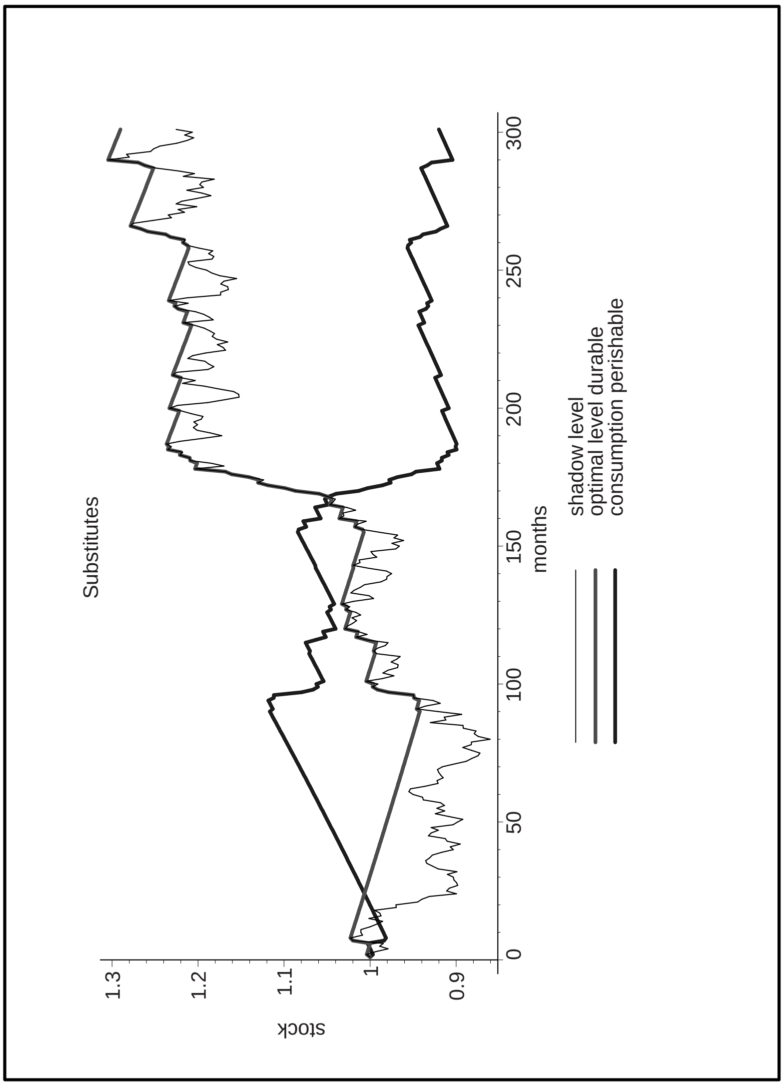

Figure 6: This picture illustrates the dynamics of durables and perishables when they are substitutes. Optimal shadow and actual stock level of the durable. Parameters are $\theta = .3, \alpha = 3.5, \beta = .01, \delta = 0.03$. The price of the perishable has $X_t = -r^p t$ with $r^p = 0.04$. The price of the durable good has $Y_t = \zeta W_t - \frac{\zeta^2}{2}t - r^d t$, with $\zeta = 0.08$ and $r^d = 0.05$. W is a Brownian motion.

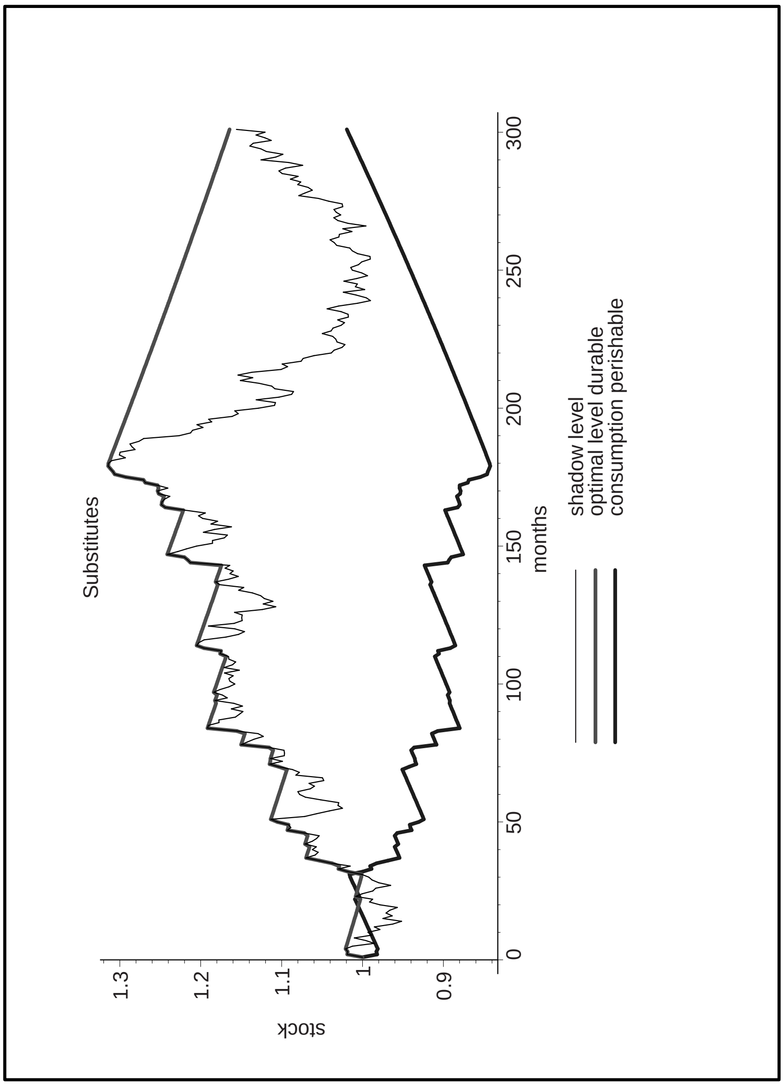

Figure 7: This picture illustrates the impact of expected lifetime of the durable good on consumption plans. Expected lifetime is 100 months. Optimal shadow and actual stock level of the durable. Parameters are $\theta = .3, \alpha = 3.5, \beta = .01, \delta = 0.03$. The price of the perishable has $X_t = -r^p t$ with $r^p = 0.04$. The price of the durable good has $Y_t = \zeta W_t - \frac{\zeta^2}{2}t - r^d t$, with $\zeta = 0.08$ and $r^d = 0.05$. W is a Brownian motion.

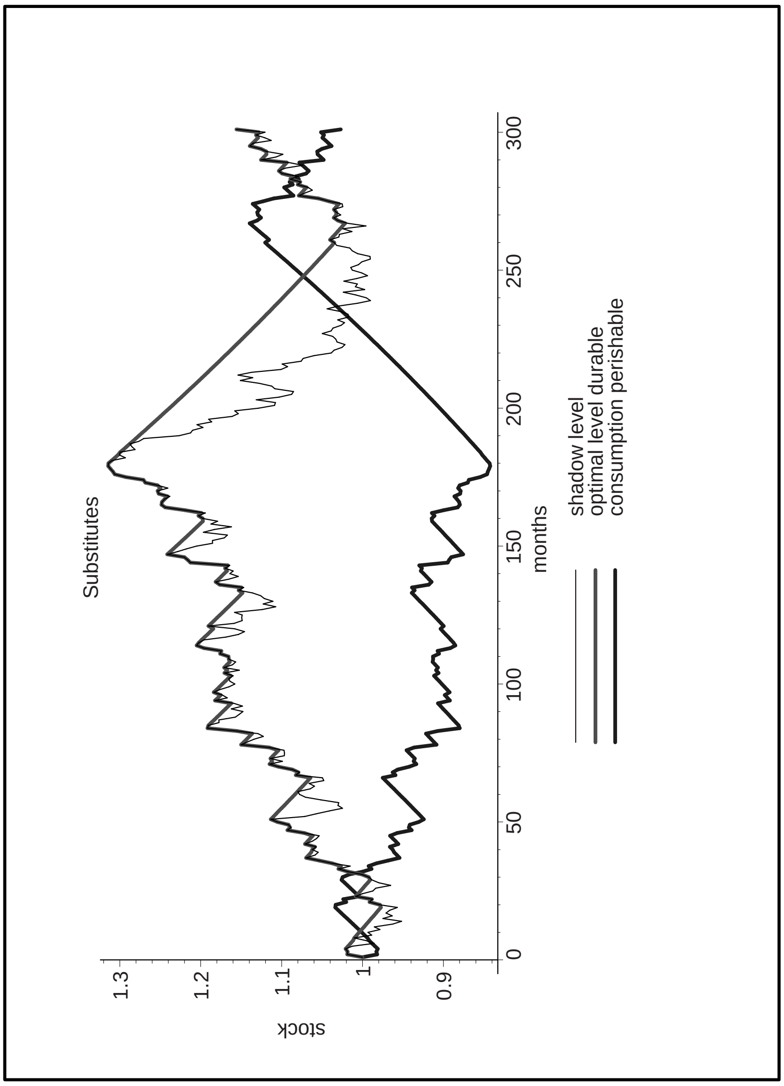

Figure 8: This picture illustrates the impact of expected lifetime of the durable good on consumption plans. Here, expected lifetime is 33 months. Optimal shadow and actual stock level of the durable. Parameters are $\theta = .3, \alpha = 3.5, \beta = .03, \delta = 0.03$. The price of the perishable has $X_t = -r^p t$ with $r^p = 0.04$. The price of the durable good has $Y_t = \zeta W_t - \frac{\zeta^2}{2} t - r^d t$, with $\zeta = 0.08$ and $r^d = 0.05$. W is a Brownian motion.

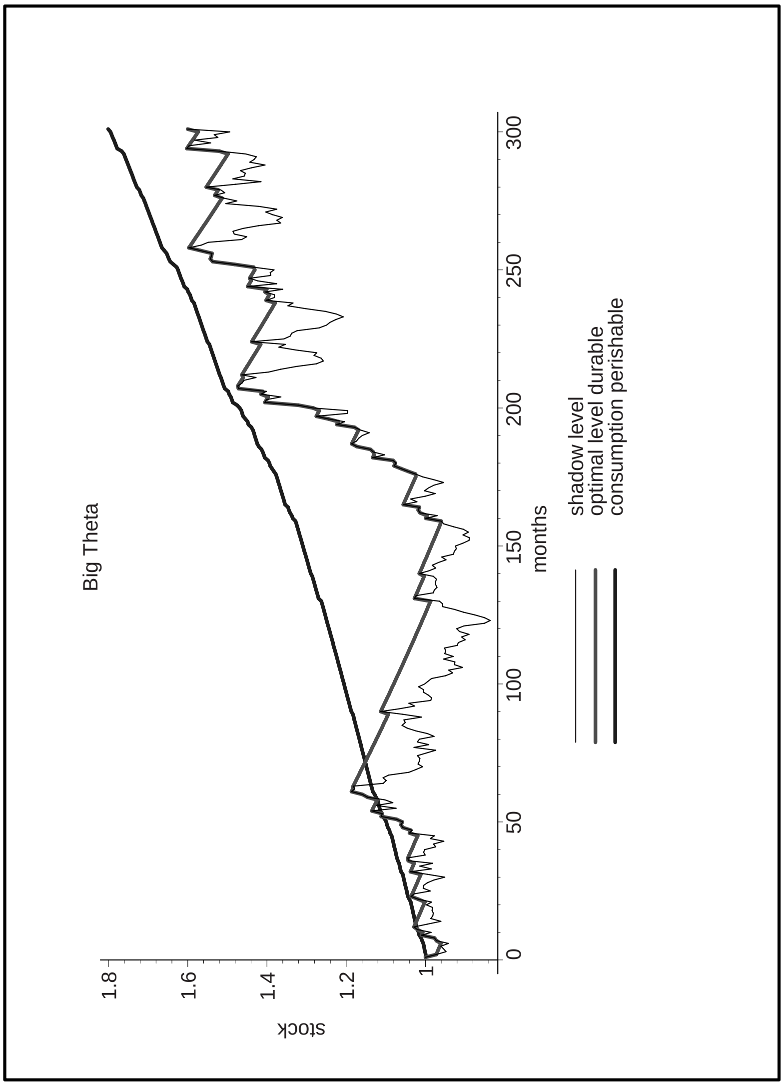

Figure 9: This picture illustrates the impact of the parameter θ. Parameters are $\theta = .9, \alpha = 0.5, \beta = .03, \delta = 0.03$. The price of the perishable has $X_t = -r^p t$ with $r^p = 0.04$. The price of the durable good has $Y_t = \zeta W_t - \frac{\zeta^2}{2} t - r^d t$, with $\zeta = 0.08$ and $r^d = 0.05$. W is a Brownian motion.

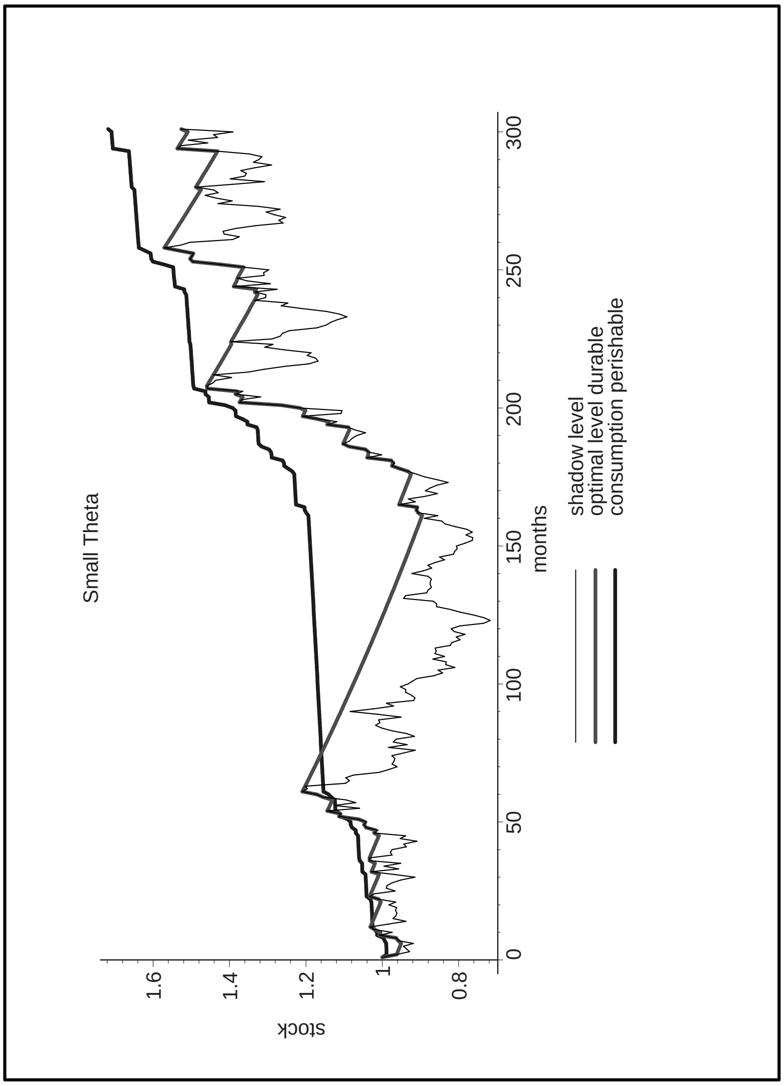

Figure 10: This picture illustrates the impact of the parameter θ. Parameters are $\theta = .5, \alpha = 0.5, \beta = .03, \delta = 0.03$. The price of the perishable has $X_t = -r^p t$ with $r^p = 0.04$. The price of the durable good has $Y_t = \zeta W_t - \frac{\zeta^2}{2} t - r^d t$, with $\zeta = 0.08$ and $r^d = 0.05$. W is a Brownian motion.