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Tymofiy Mylovanov, Thomas Tröger

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Bonn Graduate School of Economics
Department of Economics
University of Bonn
Adenauerallee 24 - 42
D-53113 Bonn

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A note on Zheng's conditions for implementing an optimal auction with resale

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Abstract

Zheng has recently proposed a seller-optimal auction for (asymmetric) independent-private-value environments where inter-bidder resale is possible. Zheng's construction requires novel conditions on the bidders' value distribution profile. We clarify the restrictions implied by these conditions. Given distributions for two bidders and the supports of the other bidders' distributions, Zheng's conditions uniquely determine the entire distribution profile. Moreover, if the bidders' distributions have the same support then Zheng's conditions imply that all distributions except one are identical, so that the final allocation is obtained after a single resale transaction, regardless of the number of bidders.

Keywords: independent private values, optimal auction, resale, inverse virtual valuation function

*Department of Economics, University of Bonn, Adenauerallee 24-42, 53113 Bonn. Email: mylovanov@uni-bonn.de and ttroeger@uni-bonn.de. We thank Rod Garratt, Georg Nöldeke, and Charles Zheng for helpful comments. Financial Support by the German Science Foundation (DFG) through SFB/TR 15 "Governance and the Efficiency of Economic Systems" is gratefully acknowledged.

1 Introduction

Zheng (2002) observes that "much of the auction design literature makes the unrealistic assumption that winning bidders cannot attempt to resell the good to losing bidders." As Zheng explains, the no-resale assumption is not innocuous. Unless the bidders are ex-ante symmetric, optimal auctions are typically inefficient in the sense that the winner is not always the bidder with the highest valuation (Myerson (1981)). This creates an incentive for the winning bidder to attempt to resell the good to one of the losers. The anticipation of resale can change bidding in the initial auction so that it ceases to be optimal.

Zheng proposes an alternative auction design that takes into account the inability of the initial seller to prohibit resale. He considers a sequential mechanism selection game where each current owner of the good chooses her sales mechanism knowing that the winner of today's mechanism will herself choose a sales mechanism that is optimal given that the next winner will choose an optimal sales mechanism, and so on.

Zheng establishes conditions on the profile of the distributions of the bidders' values such that the sequential mechanism selection game has an equilibrium where the initial seller obtains the same profit as when she can prohibit resale. Zheng's conditions have five parts: *Hazard Rate*, *Uniform Bias*, *Resale Monotonicity*, *Transitivity*, and *Invariance*. The first two are straightforward: Hazard Rate essentially requires smoothness of the bidders' distributions, and that every distribution has a weakly increasing hazard rate, while Uniform Bias requires that the bidders can be ranked according to their distributions' supports and hazard rates, where bidder 1 is the one who has the smallest support and the largest hazard rate.

Our purpose is to clarify the restrictions that Resale Monotonicity, Transitivity, and Invariance impose on the underlying distribution profile.[1] This is an open issue because the only examples of distribution profiles known to satisfy Zheng's conditions are uniform distributions (Zheng, 2002, Example 3).

[1]See Zheng (2002, p. 2213, p. 2215, and p. 2216) for explanations of why these conditions are crucial for his construction.

Technically, our crucial insight is that Resale Monotonicity and Invariance relate the bidders' distributions at points where the bidders tie with their virtual valuations. Hence, we analyze these conditions by using the bidders' inverse virtual valuation functions. We find that Resale Monotonicity is equivalent to a set of differential inequalities in terms of inverse virtual valuation functions, and Invariance implies a set of differential equations in terms of inverse virtual valuation functions (because of the stark implications of these equations, a separate analysis of Transitivity is unnecessary).

In 2-bidder environments, Zheng assumes Hazard Rate, Uniform Bias, and Resale Monotonicity, but not Transitivity or Invariance. We show that, given Hazard Rate and Uniform Bias, Resale Monotonicity is satisfied whenever the density of bidder 2's distribution is weakly decreasing (Proposition 1). If bidder 2's density is not weakly decreasing, then there exist bidder-1 distributions such that Resale Monotonicity holds as well as bidder-1 distributions such that Resale Monotonicity is violated (Proposition 2).

To tackle environments with $n \geq 3$ bidders, Zheng assumes Transitivity and Invariance in addition to Hazard Rate, Uniform Bias, and Resale Monotonicity. We show that for any given profile of supports for bidders 2 to $n-1$, and for any given distribution of bidder n, there exists at most one profile of distributions for bidders 2 to $n-1$ such that Zheng's conditions are satisfied (Proposition 3). This implies, together with Zheng (2002, Example 3), that if bidder n's distribution is uniform then the distributions of bidders 2 to $n-1$ must be uniform as well.

A particularly clear cut result holds if the distributions of bidders 2 to $n \geq 3$ have the same support and if Hazard Rate and Uniform Bias are taken as given. Then Resale Monotonicity, Transitivity, and Invariance are satisfied if and only if the distributions of bidders 2 to n are identical and have a weakly decreasing density (Proposition 4). Hence, if the distributions of bidders 2 to n have the same support and Zheng's assumptions are satisfied, then the final allocation is obtained after one resale transaction, just as in 2-bidder environments.

In Section 2 we introduce Zheng's Assumptions 1–5 and relate them to the bidders' inverse virtual valuation functions. Section 3 deals with 2-bidder environments. Environments with three or more bidders are treated in Section 4.

2 Zheng's assumptions and inverse virtual valuation functions

This note concerns Assumptions 1-5 of Zheng (2002). For the sake of brevity, we reiterate only those aspects of Zheng's model that are needed to state his assumptions. Consider an independent-private-value auction environment with $n \geq 2$ bidders. The distribution for the valuation of bidder $i = 1, \ldots, n$ is denoted F_i with support T_i.

Assumption 1 of Zheng consists of standard elements and needs no further discussion.

Assumption 1 (Hazard Rate) *For each player i, the support T_i of F_i is convex and bounded from below. If T_i is a non-degenerate interval, the density function f_i is positive and continuous on T_i and differentiable in its interior, and $(1 - F_i(t_i))/f_i(t_i)$ is a weakly decreasing function of t_i on T_i.*

We add the assumptions that for all i, the support T_i is non-degenerate and bounded, the derivative f_i' exists at the boundary of T_i, and f_i' is continuous on T_i. Let $\underline{t}_i = \min T_i$ and $\bar{t}_i = \max T_i$. Define the hazard rate $\lambda_i(t_i) = f_i(t_i)/(1 - F_i(t_i))$ for all t_i such that $F_i(t_i) < 1$.

The virtual valuation functions V_i $(i = 1, \ldots, n)$ are defined by $V_i(t_i) = t_i - (1 - F_i(t_i))/f_i(t_i)$ $(t_i \in T_i)$. Given the above assumptions, the derivative V_i' exists and is continuous and ≥ 1. Moreover,

$$V_i(T_i) = [V_i(\underline{t}_i), \bar{t}_i] \quad (i = 1, \ldots, n). \tag{1}$$

The inverse virtual valuation function V_i^{-1} is well-defined on $V_i(T_i)$. The derivative $(V_i^{-1})'$ is continuous and takes values in $(0, 1]$.

Assumption 2 of Zheng states that the bidders $i = 1, \ldots, n$ can be ranked in terms of the support T_i and of the virtual valuation function V_i. Observe that Assumption 2 is equivalent to hazard rate dominance if $T_1 = \ldots = T_n$.[2]

Assumption 2 (Uniform Bias) *For all $i, j = 1, \ldots, n$, if $i < j$ then $T_i \subseteq T_j$ and $V_i(x) \geq V_j(x)$ for all $x \in T_i$.*

By (1) and Assumption 2,

$$\forall i, j = 1, \ldots, n : \quad \text{if } i < j \text{ then } V_i(T_i) \subseteq V_j(T_j). \tag{2}$$

For $i < j$, let $\nu_{ij}(t_i) = V_j^{-1}(V_i(t_i))$. Zheng defines functions $\beta_{ij} : T_i \to T_j$ implicitly by

$$F_j(\beta_{ij}(t_i)) = F_j(\nu_{ij}(t_i)) + (\nu_{ij}(t_i) - t_i) f_j(\nu_{ij}(t_i)). \tag{3}$$

The β_{ij} functions play a central role in Zheng's equilibrium construction. He assumes the following conditions.

Assumption 3 (Resale Monotonicity) *For all $i, j = 1, \ldots, n$, if $i < j$ then β_{ij} is weakly increasing.*

Assumption 4 (Transitivity) *If bidder i is ranked before bidder k ($i < j < k$), then for any t_j less than or equal to the supremum of the range of β_{ij}, $\beta_{ik}(\beta_{ij}^{-1}(t_j)) \geq V_k^{-1}(V_j(t_j))$.*

Assumption 5 (Invariance) *For all $w = 1, \ldots, n$, and $i, j > w$, if $t_i \leq \beta_{wi}(t_w)$ and $t_j \leq \beta_{wj}(t_w)$, then $V_i(t_i) \geq$ (resp. $=$)$V_j(t_j)$
implies $f_i(\nu_{wi}(t_w))/f_i(t_i) \geq$ (resp. $=$)$f_j(\nu_{wj}(t_w))/f_j(t_j)$.*[3]

We will evaluate the restrictions implied by these assumptions by utilizing the bidders' inverse virtual valuation functions. The first step towards analyzing Assumption 3 is to

[2]Hazard rate dominance is a stronger requirement than stochastic dominance and a weaker requirement than likelihood ratio dominance (see, e.g., Krishna (2002, Appendix B)).

[3]Zheng's paper contains a typo in Assumption 5 that is corrected here. He requires that ">" implies ">", but this is not needed and obviously is not meant because it would be violated by his own Example 3.

simplify the definition of the β_{ij} functions as follows.

Lemma 1 *For all $i, j = 1, \ldots, n$, if $i < j$ then for all $t_i \in T_i$,*

$$\beta_{ij}(t_i) = F_j^{-1}\left(1 - \frac{f_j(\nu_{ij}(t_i))}{\lambda_i(t_i)}\right) \quad \textit{if} \;\; t_i < \bar{t}_i,$$

and $\beta_{ij}(\bar{t}_i) = \bar{t}_j$.

Proof. Dividing (3) by $f_j(\nu_{ij}(t_i))$ and using the definition of V_j yields that (3) is equivalent to

$$\forall t_i \in [t_i^0, \bar{t}_i] : \; \frac{F_j(\beta_{ij}(t_i)) - 1}{f_j(\nu_{ij}(t_i))} \underbrace{-V_j(\nu_{ij}(t_i)) + t_i}_{=-V_i(t_i)+t_i} = 0. \tag{4}$$

The fact that $\beta_{ij}(\bar{t}_i) = \bar{t}_j$ follows because $-V_i(\bar{t}_i) + \bar{t}_i = 0$. The proof is completed by noting that $-V_i(t_i) + t_i = 1/\lambda_i(t_i)$ for all $t_i < \bar{t}_i$. *QED*

We can now provide a direct characterization of Resale Monotonicity in terms a set of differential inequalities involving inverse virtual valuation functions.

Lemma 2 *Let $i < j$. Consider distributions F_i and F_j satisfying Hazard Rate and Uniform Bias. Then β_{ij} is weakly increasing if and only if*

$$\forall v \in (V_i(\underline{t}_i), \bar{t}_i) : \; \frac{(V_i^{-1})'(v) - 1}{V_i^{-1}(v) - v} \leq \frac{2(V_j^{-1})'(v) - 1}{V_j^{-1}(v) - v}. \tag{5}$$

Proof. By Lemma 1, β_{ij} is weakly increasing if and only if the function

$$\frac{\lambda_i(t_i)}{f_j(\nu_{ij}(t_i))} \quad (t_i \in [\underline{t}_i, \bar{t}_i)) \tag{6}$$

is weakly increasing. Because V_i is strictly increasing,

$$\frac{\lambda_i(t_i)}{f_j(\nu_{ij}(t_i))} = \frac{\lambda_i(V_i^{-1}(V_i(t_i)))}{f_j(V_j^{-1}(V_i(t_i)))} \quad (t_i \in [\underline{t}_i, \bar{t}_i)),$$

and $\lambda_i(V_i^{-1}(v)) = 1/(V_i^{-1}(v) - v)$ for all $v \in [V_i(\underline{t}_i), \bar{t}_i)$. Hence, (6) shows that β_{ij} is weakly increasing if and only if

$$(V_i^{-1}(v) - v) f_j(V_j^{-1}(v)) \quad (v \in [V_i(\underline{t}_i), \bar{t}_i))$$

is weakly decreasing. Because (see, e.g., Krishna, 2002, p.255)

$$f_j(t_j) = \lambda_j(t_j) e^{-\int_{\underline{t}_j}^{t_j} \lambda_j(t)\mathrm{d}t} \quad (t_j \in [\underline{t}_j, \bar{t}_j))$$

and $\lambda_j(t_j) = 1/(t_j - V_j(t_j))$, the function β_{ij} is weakly increasing if and only if the function

$$Z(v) = \frac{V_i^{-1}(v) - v}{V_j^{-1}(v) - v} \; e^{-\int_{\underline{t}_j}^{V_j^{-1}(v)} \frac{1}{t - V_j(t)}\mathrm{d}t} \quad (v \in [V_i(\underline{t}_i), \bar{t}_i)) \tag{7}$$

is weakly decreasing. Because the derivatives f_j' and f_i' are continuous, the derivative Z' exists and is continuous. Hence, Z is weakly decreasing if and only if $Z' \leq 0$, which is equivalent to the condition (5). *QED*

The left-hand side of (5) equals the derivative of $\ln(V_i^{-1}(v) - v)$. Thus, Resale Monotonicity requires, for every $i < n$, that the logarithm of the difference between bidder i's actual valuation $V_i^{-1}(v)$ and her virtual valuation v is a sufficiently steep (downward-sloping) function of v.

It is not possible to simplify (5) by using additional properties of virtual valuation functions, because there are essentially no additional properties: any continuously differentiable function defined on an interval $[\underline{t}_i, \check{t}]$ ($\check{t} < \bar{t}_i$) with derivative not smaller than 1 and values below the identity function can be extended to the virtual valuation function of some distribution F_i satisfying Assumption 1 (see Krishna, 2002, p. 255).

The implications of Invariance derived in the following two lemmas are so strong that it is not necessary to deal with Transitivity separately.

Lemma 3 *Suppose that $n \geq 3$ and Hazard Rate, Uniform Bias, and Invariance hold. Then the densities $f_2, \ldots, f_{n-1}$ are weakly decreasing and the density f_n is weakly decreasing on $[V_n^{-1}(V_{n-1}(\underline{t}_{n-1})), V_n^{-1}(\bar{t}_{n-1})]$.*

Proof. Let $2 \leq i \leq n-1$. To show that f_i is decreasing, consider $t, t' \in [\underline{t}_i, \bar{t}_i]$ such that $t < t'$. Let, $j = n$, $w = 1$, $t_w = \bar{t}_1$, and $t_j = t_n = V_n^{-1}(V_i(t))$. By Assumption 2 and Lemma 1, $t, t' \leq \beta_{1i}(\bar{t}_1) = \bar{t}_i$ and $t_n \leq \beta_{1j}(\bar{t}_1) = \bar{t}_n$.

By Assumption 5, since $V_i(t) = V_n(V_n^{-1}(V_i(t))) = V_n(t_n)$,

$$\frac{f_i(V_i^{-1}(\bar{t}_1))}{f_i(t)} = \frac{f_n(V_n^{-1}(\bar{t}_1))}{f_n(t_n)}. \tag{8}$$

Similarly, by Assumption 5, since $V_i(t') > V_n(t_n)$,

$$\frac{f_i(V_i^{-1}(\bar{t}_1))}{f_i(t')} \geq \frac{f_n(V_n^{-1}(\bar{t}_1))}{f_n(t_n)}. \tag{9}$$

Combining (8) and (9), we obtain $f_i(t) \geq f_i(t')$.

To show that f_n is weakly decreasing on $[V_n^{-1}(V_{n-1}(\underline{t}_{n-1})), V_n^{-1}(\bar{t}_{n-1})]$, one repeats the above argument with $i = n$, $j = n-1$, and $t_n = V_{n-1}^{-1}(V_n(t))$. *QED*

The following lemma shows that Invariance implies certain differential equations for the inverse virtual valuation functions. In the proof of Proposition 3 below we will use these equations to compute the virtual valuation functions of the bidders 2 to $n-1$ from the virtual valuation function of bidder n.[4]

Lemma 4 *Suppose that $n \geq 3$ and Hazard Rate, Uniform Bias, and Invariance hold. Then, for $i = 2, \ldots, n-1$,*

$$\forall v \in [V_i(\underline{t}_i), \bar{t}_i) : \ \frac{2(V_i^{-1})'(v) - 1}{V_i^{-1}(v) - v} = \frac{2(V_n^{-1})'(v) - 1}{V_n^{-1}(v) - v}. \tag{10}$$

Proof. Let $i \in \{2, \ldots, n\}$ and $w = 1$. By (2) there exist for any $v \in [V_i(\underline{t}_i), \bar{t}_i]$ types $t_i \in T_i$ and $t_n \in T_n$ such that $v = V_i(t_i) = V_n(t_n)$. By Lemma 1, $t_i \leq \bar{t}_i = \beta_{1i}(\bar{t}_1)$ and $t_n \leq \bar{t}_n = \beta_{1n}(\bar{t}_1)$. Hence, Assumption 5 with $t_1 = \bar{t}_1$ yields that

$$\forall v \in [V_i(\underline{t}_i), \bar{t}_i] : \ \frac{f_n(V_n^{-1}(v))}{f_i(V_i^{-1}(v))} = \frac{f_n(\nu_{1n}(\bar{t}_1))}{f_i(\nu_{1i}(\bar{t}_1)))} =: k_i. \tag{11}$$

Because (see, e.g., Krishna (2002, p. 255))

$$f_n(t) = \lambda_n(t) e^{-\int_{\underline{t}_n}^{t} \lambda_n(t')\mathrm{d}t'} \quad (t \in [\underline{t}_n, \bar{t}_n))$$

[4] It is tempting to think that setting $V_i^{-1}(v) = V_n^{-1}(v)$ for all $v \in [V_i(\underline{t}_i), \bar{t}_i)$ solves the differential equation in Lemma 4. However, this is wrong unless $\bar{t}_i = \bar{t}_n$ because $V_n^{-1}(v) > v$ for all $v < \bar{t}_n$ and $V_i^{-1}(\bar{t}_i) = \bar{t}_i$.

and $\lambda_n(t) = 1/(t - V_n(t))$,

$$f_n(V_n^{-1}(v)) = \frac{1}{V_n^{-1}(v) - v} e^{-\int_{\underline{t}_n}^{V_n^{-1}(v)} \frac{1}{t' - V_n(t')} \mathrm{d}t'}.$$

An analogous formula holds for $f_i(V_i^{-1}(v))$. Hence, (11) implies

$$\forall v \in [V_i(\underline{t}_i), \bar{t}_i): \quad \frac{1}{V_n^{-1}(v) - v} e^{-\int_{\underline{t}_n}^{V_n^{-1}(v)} \frac{1}{t' - V_n(t')} \mathrm{d}t'} = k_i \frac{1}{V_i^{-1}(v) - v} e^{-\int_{\underline{t}_i}^{V_i^{-1}(v)} \frac{1}{t' - V_i(t')} \mathrm{d}t'}. \tag{12}$$

Taking the derivative on both sides of (12), we obtain

$$\forall v \in [V_i(\underline{t}_i), \bar{t}_i): \quad \frac{2(V_n^{-1})'(v) - 1}{(V_n^{-1}(v) - v)^2} e^{-\int_{\underline{t}_n}^{V_n^{-1}(v)} \frac{1}{t' - V_n(t')} \mathrm{d}t'} = k_i \frac{2(V_i^{-1})'(v) - 1}{\left(V_i^{-1}(v) - v\right)^2} e^{-\int_{\underline{t}_i}^{V_i^{-1}(v)} \frac{1}{t' - V_i(t')} \mathrm{d}t'}. \tag{13}$$

Dividing (13) by (12) yields (10). *QED*

3 Two-bidder environments

In environments with two bidders, the conditions assumed by Zheng are Hazard Rate, Uniform Bias, and Resale Monotonicity. Our first result shows that Resale Monotonicity is satisfied whenever the distribution of bidder 2 has a weakly decreasing density. One can prove this by observing that the left-hand side of (5) with $i = 1$ is ≤ 0 and the right-hand side of (5) with $j = 2$ is ≥ 0. Below we provide an alternative proof that uses Lemma 1.

Proposition 1 *Suppose that $n = 2$. Consider any pair of distributions F_1 and F_2 satisfying Hazard Rate and Uniform Bias. If f_2 is weakly decreasing, then Resale Monotonicity is satisfied.*

Proof. Consider any pair of distributions F_1 and F_2 satisfying Assumption 1 and Assumption 2. Then, $\lambda_2(t_2)$ and $\nu_{12}(t_1)$ are increasing. If $f_2(t_2)$ is weakly decreasing on $[\underline{t}_2, \bar{t}_2]$, then $f_2(\nu_{12}(t_1))$ is weakly decreasing on $[\underline{t}_1, \bar{t}_1]$. Hence, β_{12} is weakly increasing by Lemma 1. *QED*

Things are less straightforward if bidder 2's density is not weakly increasing. The right-hand side of (5) with $j = 2$ is then < 0 for some $v = \check{v}$. Resale Monotonicity can still hold (for example, when both bidders have the same distribution $F_1 = F_2$). The result below shows, however, that one can always find bidder-1 distributions (with the same support as the bidder-2 distribution) such that Resale Monotonicity is violated. The proof works by constructing bidder 1's distribution such that the left-hand side of (5) with $i = 1$ equals 0 at $v = \check{v}$.

Proposition 2 *Suppose that $n = 2$. Consider any distribution F_2 that satisfies Hazard Rate. If f_2 is not weakly decreasing, then there exist distributions for bidder 1 such that Hazard Rate and Uniform Bias are satisfied, $T_1 = T_2$, and Resale Monotonicity is violated.*

Proof. Because f_2 is not weakly decreasing, there exists $\check{t}_2 \in (\underline{t}_2, \bar{t}_2)$ such that $f_2'(\check{t}_2) > 0$. Hence, $V_2'(\check{t}_2) > 2$. Thus, at $\check{v} = V_2(\check{t}_2)$,

$$(V_2^{-1})'(\check{v}) < 1/2. \tag{14}$$

Let $\bar{t}_1 = \bar{t}_2$ and $\underline{t}_1 = \underline{t}_2$. A continuous function $\lambda_1 : [\underline{t}_1, \bar{t}_1) \to [0, \infty)$ is the hazard rate of some distribution F_1 on $[\underline{t}_1, \bar{t}_1]$ if

$$\lim_{t \to \bar{t}_1} \int_{\underline{t}_1}^{t} \lambda_1(x) \mathrm{d}x = \infty.$$

(see, e.g., Krishna, 2002, p. 255). There exists a number $\check{\lambda} > \lambda_2(\check{t}_2)$ such that

$$\underline{t}_1 - \frac{1}{\check{\lambda}} < \underbrace{\check{t}_2 - \frac{1}{\lambda_2(\check{t}_2)}}_{=\check{v}} < \check{t}_2 - \frac{1}{\check{\lambda}}. \tag{15}$$

Let F_1 be a distribution on $[\underline{t}_1, \bar{t}_1]$ such that

$$\lambda_1(t) = \begin{cases} \check{\lambda} & \text{if } t \in [\underline{t}_1, \check{t}_2], \\ > \lambda_2(t) & \text{if } t \in (\check{t}_2, \bar{t}_1], \end{cases}$$

and such that λ_1 is weakly increasing and continuously differentiable. Given our construction Assumption 1 and Assumption 2 are satisfied. Also, $V_1(t_1) = t_1 - 1/\check{\lambda}$ for all $t_1 \in [\underline{t}_1, \check{t}_2]$.

Hence,

$$\forall v \in [\underline{t}_1 - \frac{1}{\check{\lambda}}, \check{t}_2 - \frac{1}{\check{\lambda}}] : \ V_1^{-1}(v) = v + \frac{1}{\check{\lambda}}. \tag{16}$$

By (15), $\check{v} \in [\underline{t}_1 - 1/\check{\lambda}, \check{t}_2 - 1/\check{\lambda}]$. Evaluating (16) yields that at $v = \check{v}$ the left-hand side of (5) with $i = 1$ equals 0. By (14), the right-hand side of (5) with $j = 2$ is < 0 at $v = \check{v}$. Hence, Assumption 3 is violated by Lemma 2. *QED*

4 Environments with three or more bidders

In environments with three or more bidders, the conditions assumed by Zheng are Hazard Rate, Uniform Bias, Resale Monotonicity, Transitivity, and Invariance.

The following result confirms Zheng's own assessment that Invariance is "very restrictive" (2002, p. 2217): if supports are given, then the distribution for bidder n pins down a unique candidate distribution for every bidder 2 to $n-1$ (but imposes no restriction on bidder 1's distribution).

Proposition 3 *Let $n \geq 3$. Then, for any given profile of supports $T_2 \subseteq \ldots \subseteq T_{n-1}$, and any distribution F_n, there exists at most one profile $F_2, \ldots, F_{n-1}$, such that Hazard Rate, Uniform Bias, and Invariance hold.*

Proof. Let $i \in \{2, \ldots, n-1\}$. Given the virtual valuation function V_n, we show that there exists at most one distribution F_i such that Hazard Rate, Uniform Bias, and Invariance hold. Because we want to apply Lemma 4, we consider the linear (inhomogeneous) ordinary differential equation

$$2g'(v) - 1 = (g(v) - v)h(v), \tag{17}$$

where

$$h(v) := \frac{2(V_n^{-1})'(v) - 1}{V_n^{-1}(v) - v} \quad (v \in [V_n(\underline{t}_n), \bar{t}_n)).$$

We distinguish cases where $\bar{t}_i < \bar{t}_n$ and where $\bar{t}_i = \bar{t}_n$.

Let $\bar{t}_i < \bar{t}_n$. Then the function h is continuous at $v = \bar{t}_i$. Hence, standard results on differential equations imply that the equation (17) together with the boundary condition $g(\bar{t}_i) = \bar{t}_i$ has a unique solution g_i on $[V_n(\underline{t}_n), \bar{t}_i]$.

Now consider two distributions F_i and $\check{F}_i$ for bidder i with support T_i such that Hazard Rate, Uniform Bias, and Invariance hold; denote by V_i and $\check{V}_i$ the corresponding virtual valuation functions. Without loss of generality, $V_i(\underline{t}_i) \leq \check{V}_i(\underline{t}_i)$. Using Lemma 4,

$$\forall v \in [\check{V}_i(\underline{t}_i), \bar{t}_i) : \; V_i^{-1}(v) = g_i(v) = \check{V}_i^{-1}(v). \tag{18}$$

Setting $v = \check{V}_i(\underline{t}_i)$, (18) implies $V_i^{-1}(\check{V}_i(\underline{t}_i)) = \check{V}_i^{-1}(\check{V}_i(\underline{t}_i)) = \underline{t}_i$, hence $\check{V}_i(\underline{t}_i) = V_i(\underline{t}_i)$. Together with (18) we obtain $V_i = \check{V}_i$ and thus $F_i = \check{F}_i$.

Let $\bar{t}_i = \bar{t}_n$. Because $h(v) \to \infty$ as $v \to \bar{t}_i$, standard uniqueness results for differential equations do not apply. However, $g = V_n^{-1}$ obviously solves (17) on $[V_n(\underline{t}_n), \bar{t}_n)$ and satisfies the boundary condition $g(\bar{t}_n) = \bar{t}_n$. Next, we show

(*) if a function k solves (17) on $[\underline{v}, \bar{t}_n)$ for some $\underline{v} < \bar{t}_n$, satisfies the boundary condition $k(\bar{t}_n) = \bar{t}_n$, and has the following additional property,

$$\exists \hat{v} < \bar{t}_n \forall v \in [\hat{v}, \bar{t}_n) : \; k(v) \leq V_n^{-1}(v), \tag{19}$$

then $k = V_n^{-1}$ on $[\underline{v}, \bar{t}_n)$.

Because $V_n^{-1} - k$ solves the homogeneous differential equation $2g'(v) = g(v)h(v)$,

$$k(v) = \Delta \; e^{\int_{V_n(\underline{t}_n)}^{v} \frac{h(w)}{2} \mathrm{d}w} + V_n^{-1}(v) \quad (v \in [\underline{v}, \bar{t}_n)) \tag{20}$$

for some $\Delta \in \mathbb{R}$. By (19), $\Delta \leq 0$. Moreover,

$$k'(v) = \Delta \frac{h(v)}{2} \; e^{\int_{V_n(\underline{t}_n)}^{v} \frac{h(w)}{2} \mathrm{d}w} + (V_n^{-1})'(v). \tag{21}$$

By Lemma 3,

$$\forall t \in [V_n^{-1}(V_{n-1}(\underline{t}_{n-1})), V_n^{-1}(\bar{t}_{n-1})] : \; V_n'(t) \leq 2. \tag{22}$$

Because $\bar{t}_n = \bar{t}_i \leq \bar{t}_{n-1} \leq \bar{t}_n$, we have $V_n^{-1}(\bar{t}_{n-1}) = \bar{t}_n$ and thus

$$V_n^{-1}(V_{n-1}(\underline{t}_{n-1})) < V_n^{-1}(\bar{t}_{n-1}) = \bar{t}_n.$$

Hence, (22) implies $(V_n^{-1})'(v) \geq 1/2$ for all $v \in N := [V_{n-1}(\underline{t}_{n-1}), \bar{t}_n) \neq \emptyset$. Hence, $h(v) \geq 0$ for all $v \in N$. Hence, (21) implies $k'(v) \leq (V_n^{-1})'(v)$ for all $v \in N \cap [\underline{v}, \bar{t}_n)$. Together with

$k(\bar{t}_n) = \bar{t}_n = V_n^{-1}(\bar{t}_n)$ this implies $k(v) \geq V_n^{-1}(v)$ for all $v \in N \cap [\underline{v}, \bar{t}_n)$. Hence, $\Delta \geq 0$ by (20). Thus, $\Delta = 0$ and therefore $k = V_n^{-1}$, showing (*).

Now, as in the case $\bar{t}_i < \bar{t}_n$, consider two distributions F_i and $\check{F}_i$ for bidder i with support T_i such that Hazard Rate, Uniform Bias, and Invariance hold; denote by V_i and $\check{V}_i$ the corresponding virtual valuation functions. Without loss of generality, $V_i(\underline{t}_i) \leq \check{V}_i(\underline{t}_i)$. By Uniform Bias, $V_i^{-1}(v) \leq V_n^{-1}(v)$ and $\check{V}_i^{-1}(v) \leq V_n^{-1}(v)$ for all $v \in [\check{V}_i(\underline{t}_i), \bar{t}_n)$. By Lemma 4, both V_i^{-1} and $\check{V}_i^{-1}$ solve (17) on $[\check{V}_i(\underline{t}_i), \bar{t}_n)$. Hence, by (*),

$$\forall v \in [\check{V}_i(\underline{t}_i), \bar{t}_i) : \; V_i^{-1}(v) = V_n^{-1}(v) = \check{V}_i^{-1}(v),$$

which implies $F_i = \check{F}_i$ by the same arguments as in the case $\bar{t}_i < \bar{t}_n$. *QED*

It is interesting to contrast Proposition 3 with Zheng (2002, Example 3), where it is shown that Zheng's conditions are satisfied if every bidder's distribution is uniform (on a possibly different interval for each bidder). Proposition 3 reveals that if the distribution for bidder n is uniform, then Zheng's conditions are satisfied only if the distributions for bidders 2 to $n-1$ are uniform as well.

The last result concerns environments where the distributions of bidders 2 to n have the same support.

Proposition 4 *Let $n \geq 3$. Suppose that Hazard Rate and Uniform Bias hold, and $T_2 = \ldots = T_n$. Then Resale Monotonicity, Transitivity, and Invariance, are satisfied if and only if $F_2 = \ldots = F_n$ and the density f_i ($i \geq 2$) is weakly decreasing.*

Proof. "if": The verification of Assumption 3 is analogous to the proof of Proposition 1. Assumptions 4 and 5 are straightforward.

"only if": Suppose that Assumptions 1–5 hold. Consider any F_n. By Proposition 3 and "if", $F_2 = \ldots = F_n$. By Lemma 3, the density f_i ($i \geq 2$) is weakly decreasing. *QED*

On the one hand, Proposition 4 considerably extends Zheng (2002, Example 3), allowing for a large class of non-uniform distributions for bidders 2 to n, and any bidder-1 distribution that satisfies Hazard Rate. On the other hand, the condition that the distributions for bidders 2 to n are identical renders Zheng's n-bidder equilibrium construction essentially equivalent to the 2-bidder case: the resale mechanism used by bidder 1 is symmetric across bidders 2 to n so that the final allocation is obtained after one resale transaction.

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