Bonn Econ Discussion Papers

Discussion Paper 36/2002

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by

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November 2002

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February 2002
This version: November 2002

Abstract

Not only corporate but also sovereign debtors, in particular developing countries, may get into financial difficulties. Contrary to corporate issuers, they decide themselves if they continue to fulfill their debt obligations or convert their debt. I analyze the value of a default-risky sovereign bond in a setting in which foreign trade is reduced in case the country does not fulfill its obligations. Comparing the costs of debt service with the value of the punishment via foreign trade, the country voluntarily decides when to reorganize its debt. Knowing this threshold the value of a sovereign coupon-bond can be calculated.

Key words: Credit risk, sovereign debt, endogenous default

JEL Classification: G15, G33, F34, H63

*Financial help from Deutsche Forschungsgemeinschaft (DFG) is gratefully acknowledged.

1 Introduction

Bonds of sovereign issuers often are not default-riskless like those of western economies. This can be seen in Latin America and in Eastern Europe in the last years. It is not only corporate debt that is risky, sovereign borrowers may also suspend or even stop payments on their financial obligations. This may be sequel to revolutions in the borrowing country like in Russia, where after the October Revolution the new Soviet government decided to stop payments on old tsarist bonds in 1918. Normally sovereign debtors stop their payments on external debt because they have got into economical and financial difficulties. More than 80 % of sovereign defaults are connected to currency crises.[1] One example is the Russian Crisis in the summer of 1998, when the Russian Ruble was devaluated against the US Dollar and Russia defaulted on some Dollar-denominated bonds. Also some Mexican Peso-Crises in the 1980s and the early 1990s lead to defaults in Mexico as for example in 1982.

Sometimes currency crises are hidden behind a fixed exchange rate. In Argentina the Argentinean Peso has been linked to the US Dollar for the last ten years, and the Argentinean government promised to possess at least one US Dollar for each circulating Peso. However, in order to hold Pesos, banks offered higher interest rates for assets held in Pesos, making it difficult to sustain the fixed exchange rate. Yet, a devaluation increases the value of US-Dollar-denominated external debt. The system broke down in December 2001, when the Argentinean government stopped all payments on public external debt and then devaluated and floated the Argentinean Peso in 2002. This was somehow expected by financial markets, which have considered Argentina as almost insolvent for some months. So they traded Argentinean government bonds with a high risk premium. Also, the rating of these bonds became very low. Before Argentina suspended their payments, Argentinean bonds were only rated B by Standard and Poors, compared to AAA for German or US bonds. This is reflected in a very high yield-spread of Argentinean bonds with regard to default-riskless bonds. In November 2001 the spread reached more than 4000 basis points for a maturity of two years and continued growing in 2002 due to the ongoing crisis.

There are great differences between the yields of bonds of different sovereign borrowers in the world, even for the same rating-class.[2] These differences may reflect differences in the national interest rate policies of the central bank, but mostly they reflect the different risk premiums of different debtors due to different market opinions with respect to the default probabilities of the government bonds. All South and Latin American

[1] See Reinhart (2002).

[2] E. g. the credit spread between Swiss and French 5-year bonds, both rated AAA by Standard and Poors, is 150 basis points (27.02.2002).

sovereign debtors are rated BBB (Standard and Poors) or worse, whereas the credit spread varies between only 100 for Mexico and more than 2000 for Argentinean bonds. The spreads may vary a lot over time, e. g. Russian bonds that are rated B+ reduced the credit spread since the partial default in 1998 from more than 4000 to around 300 basis points now.

In contrast to corporate debtors a state that cannot pay its obligations or that is highly indebted cannot be liquidated; the country's assets cannot serve as collateral for the debt. What is more, there is no bankruptcy code for sovereign debtors. Sovereign debt is not a legally enforceable contract, although the IMF is planning to install a legal framework.[3] Sovereign debtors can decide themselves whether to stop or to continue their payments. Mostly they do not simply declare default but they try to convert their debt, e.g. to rearrange the time structure of payment obligations or to cut off the face value of the bonds in a way that the creditor has no real choice whether to accept the offer or not.

What forces sovereign debtors to pay their financial obligations? If a creditor cannot be sure that the debtor will repay his debt plus interest, he will not lend his money save with a risk premium. The premium rises in case a default occurs, and next time the debtor wants to borrow money, he has to pay higher interest rates. So a default increases the costs of future access to the credit market. This in turn will reduce the potential future wealth of the economy. The defaulting debtor also loses reputation, making it more difficult to get new money on the credit market. This may also reduce international trade opportunities, because trading partners are afraid that their exports cannot be paid. In an empirical analysis Rose (2002) shows that on average a debt-renegotiation with the Paris club[4] was followed by a reduction of bilateral trade of about eight percent a year in the following fifteen years.

In addition to that, own exports or foreign domiciled assets may be seized sequel to a default. Oil exporting countries like Venezuela can use exports as collateral, so in case of default parts of their exports are confiscated. This can link international delivery to lending to create an enforceable penalty reducing the shortfall-risk for the creditor. For example, in the second half of the 16^{th} century the Genoese lenders of King Philip II of Spain imposed great sanctions for non-payment. Conklin (1998) shows a sanction-

[3] As a response to the Argentinean crisis in December 2001 the IMF is trying to avoid great defaults in future by imposing an international insolvency procedure as for corporate debtors so that claims on sovereign debtors can also be prosecuted.

[4] The Paris Club is an informal group of creditors, amongst them are as well the greatest industrialized countries as developing countries. Renegotiations with the Paris Club are normally associated with IMF-programs.

based explanation for the successive debt renegotiations[5] during his reign.[6] However, 80% of sovereign debt of developing countries is not collateralized. Since World War II the importance of trade sanctions to enforce sovereign debt contracts grew as well as the importance of foreign trade of newly industrializing countries with industrialized countries.[7] International sanctions due to a default could be that the sovereign borrower will not get any new credit from international organizations like the IMF. Although sanctions are sometimes difficult to impose, a reduction of foreign trade and therefore a reduction of the economic activity of the defaulting country often is the consequence of a default.

In order to evaluate if a country has a debt burden that is too high and is therefore susceptible to defaults or debt rearrangements, there are several criterions. For developing countries the World Bank says that there is a debt overhang if the ratio of debt to GDP is more than 25%. Compared to the criteria of the European Monetary Union (EMU) allowing for a ratio of 60% just in order to ensure the stability of the monetary union this is a very low value. However, besides the level of output and the exchange rate the volatility of these variables plays a role, too. With a higher variation in the exchange rate it is more probable that the value of the future debt services increases so much that the state cannot afford interest payments and decides to declare default or to reorganize its debt.

The aim of this paper is to derive a valuation formula for defaultable sovereign bonds in a model taking not only the credit risk into account but also exchange-rate dynamics. Contrary to corporate issuers, sovereign countries decide themselves if they continue to fulfil their debt obligations or stop paying. Mostly they do not simply declare default but they try to convert their debt, i.e. to rearrange the time structure of payment-obligations. Note that this is realistic, because at the moment there are no legal enforcements, like a bankruptcy code to compel the borrower to fulfil its debt obligations.

In the corporate credit risk literature there are two main approaches to model the

[5] Altogether there were four renegotiations of debt during the reign of Philip II.

[6] King Philip II of Spain (1556-98) borrowed extensively during his reign to finance his military activities, therefore he needed to transfer specie from Spain to its troops fighting outside Spain. In the bankruptcy of 1575 when Spain reached its debt ceiling, the Genoese cut off new lending. Because Philip II broke off negotiations, the Genoese imposed an embargo on all currency transfers between Spain and its troops in the Low Countries, where he was engaged in a major campaign against Dutch rebels. Unpaid for more than a year the troops mutinied, leading to a great military defeat. Therefore Philip II capitulated to the lenders, leading to an agreement in 1578. For more details see Conklin (1998).

[7] Approximately 90% of Mexico's foreign trade is with industrialized countries (Bulow and Rogoff (1989)).

default event: so called intensity-based models and structural models. Intensity-based models model the default event via the first jump of a point process. They differ in the way the point process is modelled, e.g. Duffie and Singleton (1997) uses a general point process, whereas Lando (1998) uses a Cox-Process. In these models a given credit-spread-structure is reached via calibration of the model. These models can easily be applied to sovereign debt, but they do not use any relation between the default event and fundamentals like the exchange rate or the foreign reserves of the central bank. So they deliver no information about possible hedges. For rating models the small number of defaults in the last fifty years poses a problem, because this makes it difficult to estimate default probabilities and recovery rates for the different rating classes.

In structural models, going back to Merton (1974), a default occurs when the firm's assets are no longer sufficient to cover the debt obligations. So the default occurs if the firm value undershoots an exogenous threshold. In Merton (1974) this default only may take place at maturity. This was extended to cases in which the default is possible before maturity by Black and Cox (1976) and to stochastic interest rates by Longstaff and Schwartz (1995) as well as Schönbucher (2000) who derives a closed-form solution in a special case of stochastic interest rates.

We analyse the value of a default-risky sovereign bond issued in a foreign currency like Euro or US-Dollar in a structural model without interest rate risk in the style of Leland (1994), who derives an optimal capital structure and yield spreads for corporate bonds. In Leland (1994) the default triggering value is endogenously derived. The firm has a tax advantage of issuing debt. However, debts also increase the risk of default and therefore costs for a bankruptcy, diminishing the value of the firm. This trade-off leads to an optimal amount of debt, maximizing the firm value.

Applying this to a sovereign borrower, the problem is to find an equivalent for the value of the firm. We assume that the government wants to maximize a country's wealth, i. e. the capitalized value of all goods and services produced by the country minus the capitalized value of its debt obligations. This means that the government balances social costs of interest payment and economic consequences of declaring default. The sovereign country borrows money on the foreign capital market to finance domestic expenditures and international trade. The aim of borrowing is a higher wealth of the domestic economy, i.e. a higher domestic GDP. On the other hand the economically desired increase is associated with costs of debt and interest payments. So the government tries to reduce this burden and considers whether to stop paying interests or to continue. However, a default or another reduction in interest payments causes a lower domestic GDP in future periods. Thus, a default not only increases a country's wealth due to power payment obligations but also reduces its wealth because of its negative effects on the future output. Therefore the sovereign debtor should reorganize its debt if the reduction of the amount of future debt service payments exceeds the economic disadvantages the country has due to this unilaterally declared partial suspension of payment.

For modelling, a disadvantage of a macroeconomic variable like GDP is that it is not available on a daily basis and what is more, normally it is only available with a lag of at least two or three months, even in industrialized countries. Another difficulty is the problem of uncertainty about real GDP. Even if a developing country publishes data about its GDP, these values are not really reliable data. Besides output, the exchange rate also influences the relation between lost output and the costs of debt payment. The exchange rate is available on a daily basis, and furthermore there are more changes in the exchange rate than in GDP if there is no fixed-rate regime. So it is rather the exchange rate than a low output that pushes a sovereign country into a payment crisis. The dynamics of the exchange rate also allow hedging in a position linked with the default risk.

The paper is structured as follows. In section 2, we specify the evolution of the output of the sovereign country and its payment obligations. Furthermore the dynamics of the exchange rate are presented to link the debt denominated in foreign currency with the value of the output in domestic currency. Section 3 presents the decision of the government and the optimal decision about the default-threshold. Also the total value of the outstanding debt services in domestic currency is presented. In section 4 an explicit coupon-bond is valuated using the optimal threshold. Finally section 5 summarizes.

2 General dynamics

The aim of this paper is to value default-risky debt issued by sovereign borrowers. The debt is placed on international capital markets[8] that are assumed to be frictionless. Furthermore we assume perfect capital markets where continuous trading is possible. So there is risk-neutral valuation of financial contracts under the respective martingale measures on the domestic market of the borrower as well as on the foreign market. The link between the domestic market and the foreign market is given by the exchange rate dynamics.

[8]Normally this is done via banks. In addition, sovereign countries borrow directly from international institutions and other sovereign states, but this debt is not valued or traded on financial markets.

The government of the sovereign state issuing debt on international markets maximizes the wealth of the economy it represents, i.e. the capitalized value of all goods and services produced by the country minus the capitalized value of its debt obligations. This drives the decision of the government, whether it continues to fulfill its debt obligations or reorganizes its debt in order to reduce its interest burden. However, the latter incurs higher costs for the acquirement of future debt; sanctions are imposed and international trade is reduced. All these measures reduce the wealth of the economy. Comparing the costs of the debt service with the value of the punishment via a reduction of foreign trade, the government chooses an optimal critical value of output for which it reorganizes its debt. This means that the debtor voluntarily decides when to reorganize its debt, determining the default threshold on which the value of the debt depends endogenously.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a common filtered probability space with subjective probability-measure $\mathbb{P}$ and filtration $\mathcal{F}$. On both the domestic market and the foreign market there exists an equivalent martingale measure, $\mathbb{P}^D$ and $\mathbb{P}^*$ respectively. In the following, we define how the output of the borrowing country evolves over time, dependent on the decision of the state whether to pay its obligations or not. These obligations are quoted in foreign currency, so the exchange rate between domestic and foreign currency is introduced.

2.1 Dynamics of the output

Let Y_t^D denote the value of the output of the country of the sovereign debtor, denominated in national currency[9] at time t. Y_t^D is supposed to follow a standard diffusion equation:

$$dY_t^D = Y_t^D \mu_t^D dt + Y_t^D \sigma_t^D dW_t^D, \tag{1}$$

where μ_t^D is the growth rate of output, σ_Y^D a constant d-dimensional vector of volatilities and W_t^D a d-dimensional Brownian motion under the domestic equivalent martingale-measure $\mathbb{P}^D$. The growth rate of output is assumed to be constant and equal to the short rate r^D. This means that on average the economy grows with the rate of intertemporal substitution. The dot in $\sigma_Y^D \cdot dW_r^D$ stands for the Euclidian inner product in $\mathbb{R}^d$, therefore

$$\sigma_Y^D \cdot dW_t^D = \sum_{i=1}^{d} \sigma_Y^{D,j} dW_t^{D,i}. \tag{2}$$

[9]The superscript D stands for variables denominated in domestic currency, all variables in foreign currency are without a superscript.

So long as the sovereign borrower fulfills its payment obligations the output evolves according to (1). We assume that no real default is possible,[10] as the state only reorganizes its debt. For simplicity this reorganization[11] only may take place once. If the country does so, sanctions and a loss in reputation will take place. This loss raises the costs for the access to the capital market in the future, reducing future production possibilities. Also, it gets more difficult to acquire new credits from international organizations like the World Bank or the IMF. If the IMF cuts down all his credits this may result in severe problems for the government, which needs foreign currency for its national budget.

Furthermore, foreign trade is reduced in case of default or reorganization of debt. On the one hand potential exporters fear a seizing or a confiscation of their exports as payment in place of the payment the state did not make. On the other hand importers may not have enough foreign currency to pay for the imports and foreign exporters may shorten the time of payment making imports more expensive. Sanctions by international organizations or higher customs on foreign trade in order to punish the government are possible as well, but they are relatively rare, sanctions are mostly deployed for political or military reasons like the prevention of terrorism or human rights abuses.[12]

Most of these effects have a stronger impact in the near future. Over time the country gains back its reputation and therefore negative impacts of the default reduce. Furthermore sanctions only take place in the near future after a debt reorganization. Therefore we assume that output in periods after a reorganization at time τ reduces to

$$Y_t'^D \stackrel{\text{def}}{=} Y_t^D \left(1 - \gamma e^{-\delta(t-\tau)}\right), \tag{3}$$

where γ and δ are positive constants, $\gamma < 1$. The instantaneous reduction of output after renegotiation is reflected by the factor γ, higher values stand for higher reductions. The coefficient δ captures the persistence of this reduction, where lower values of δ mean longer persistence.[13] Therefore, the accumulated value of all goods and services

[10] So far in the majority of cases the borrowers have offered to the bondholders an exchange of the old bonds in new ones with new conditions like lower interest rate or face value or longer maturity, reducing the payment obligations of the debtor. It is also relatively rare that countries default on its international financial obligations, they typically renegotiate their debt through the Paris Club (see Rose (2002)).

[11] In the following, default always stands for a reorganization of debt.

[12] See Rose (2002).

[13] Rose (2002) estimates the reduction of bilateral trade following a debt renegotiation with the Paris Club. He finds a significant decline of eight percent in bilateral trade, persisting for around fifteen years. This reduction stays at a relative constant level but for longer lags the influence of a debt renegotiation on bilateral trade is not significant.

produced by the country subsequent to a default is reduced. Only in the long run the time path of output will reach the same value it would have reached in case no default had occurred.

The present value of the expected[14] future output flow at time of default τ, given the current realization of Y_τ is given as

$$\begin{aligned}\mathbb{E}^D\left[\int_\tau^\infty e^{-\rho(s-\tau)}Y_s'^D ds\right] &= \mathbb{E}^D\left[\int_\tau^\infty e^{-\rho(s-\tau)}Y_s^D ds\right] - \mathbb{E}^D\left[\int_\tau^\infty \gamma e^{-(\rho+\delta)(s-\tau)}Y_s^D ds\right] \\ &= Y_\tau^D\left(\frac{1}{\rho - r^D} - \gamma\frac{1}{\rho+\delta-r^D}\right). \qquad (4)\end{aligned}$$

Thus, the present value of future output is smaller than in a case without international sanctions:

$$\begin{aligned}\mathbb{E}^D\left[\int_\tau^\infty e^{-\rho(s-\tau)}Y_s'^D ds\right] &= \frac{Y_\tau^D}{\rho - r^D}\left(1 - \gamma\frac{\rho - r^D}{\rho+\delta-r^D}\right) \\ &\stackrel{\text{def}}{=} \frac{Y_\tau^D}{\rho - r^D}\left(1-\eta^D\right), \qquad (5)\end{aligned}$$

where ρ stands for the discount factor of the government[15] and $\rho > r^D$, η^D is a positive constant.

2.2 Payment obligations

The sovereign borrower is assumed to have external debt with infinite maturity. This can be understood as a permanent rolling over of maturing debt so that no final payments exist. Except for this new debt due to the rolling over mechanism no new bonds are issued by the state. It is also assumed that no foreign debt is ever replaced by other, e.g. domestic bonds. The debt and all interest payments are denominated in foreign currency. For simplicity we assume that all debt is denoted in the same foreign currency. At each moment t in time the debtor has to pay interests in foreign currency C_t, as long as no reorganization has taken place. The payment obligation is assumed to be constant over time.

The government can decide to reorganize its debt once. That means that the face value and, as a consequence of that, total interest payments are reduced by a constant factor

[14] The expectations with respect to the domestic martingale-measure $\mathbb{P}^D$ are taken conditional on the information on the time τ-value of the output.

[15] If $\rho = r^D$, then the present value of the future output will be infinite, but the loss due to reorganization remains finite as long as $\rho + \delta > r^D$.

such that payment obligations at time t after the reorganization amount to ωC only, with ω such that $0 < \omega < 1$.
This reduction can also be understood as a permanent reduction of the future interest rates the borrower has to pay or a mixture of lower interest rates and lower face values.

2.3 Exchange rate

The foreign debt of the borrowing country is denominated in another currency than output. Thus, we need the exchange rate X_t between these two currencies that allows the conversion of foreign market cash flows into units of domestic currency. One unit of domestic currency is worth X_t units of foreign currency, where for reasons of arbitrage the dynamics of the exchange are

$$\frac{dX_t}{X_t} = \left(r - r^D\right) dt + \sigma_X \cdot dW_t^*. \tag{6}$$

W_t^* is a d-dimensional Brownian motion under the foreign equivalent martingale-measure $\mathbb{P}^*$, r the foreign short rate and σ_X a d-dimensional vector of volatilities. The connection between the two Brownian motions enables to rearrange the dynamics of the output. Therefore, it follows for the output-dynamics using the Brownian motion under the foreign equivalent martingale-measure and $dW^D = dW^* - \sigma_X dt$ that:

$$\frac{dY^D}{Y^D} = \left(r^D - \sigma_Y^D \cdot \sigma_X\right) dt + \sigma_Y^D \cdot dW^*. \tag{7}$$

To evaluate all dynamics in the same currency and under the same martingale measure let $Y_t := Y_t^D X_t$ be domestic output in foreign currency. With the Itô-formula the dynamics of Y_t are

$$\begin{aligned} dY_t &= d\left(Y_t^D X_t\right) \\ &= Y_t\left(r_t dt + \left(\sigma_Y^D + \sigma_X\right) dW_t^*\right) \\ &=: Y_t\left(r_t dt + \sigma_Y \cdot dW_t^*\right). \end{aligned} \tag{8}$$

3 The decision of the sovereign

The government decides itself when to reorganize its debt. Similar to Leland (1994) an optimal critical value for the domestic output is calculated, at which the organization should take place. To do that, we derive the general solution for the value of a financial contract dependent on the output process. With two explicit boundary conditions both

the value of the discounted future debt services and the present value of the future output is calculated, both dependent on the threshold triggering the reorganization. So finally the value maximizing the country's wealth can be determined.
For every financial contract $F(Y)$ conditional on the process Y following the Ito-formula the following must hold:

$$\begin{aligned} dF(Y) &= \frac{\partial F(Y)}{\partial t}dt + \frac{\partial F(Y)}{\partial Y}Yrdt \\ &+ \frac{\partial F(Y)}{\partial Y}Y\sigma_Y dW_t^* + \frac{1}{2}\frac{\partial^2 F(Y)}{\partial Y^2}\|\sigma_Y\|^2 Y^2 dt. \end{aligned} \tag{9}$$

Under risk-neutral valuation[16] the expected return on every financial contract must equal the risk free interest rate. So for a contract with constant payment outflow of f we get for the expected instantaneous changes

$$\begin{aligned} \mathbb{E}[dF(Y)] &= \frac{\partial F(Y)}{\partial t}dt + \frac{\partial F(Y)}{\partial Y}Yrdt + \frac{1}{2}\frac{\partial^2 F(Y)}{\partial Y^2}\|\sigma_Y\|^2 Y^2 dt + fdt \\ &\overset{!}{=} rF(Y)dt. \end{aligned} \tag{10}$$

Fortunately, there is no time dependency in the contracts here, so that the first addend cancels out. This makes it easy to find a solution. The explicit solution also depends on boundary conditions that remain to be specified later.
Therefore the following differential equation[17] must hold for the value of the future interest payments of the state $D(Y)$:

$$rYD_Y + \frac{1}{2}\|\sigma_Y\|^2 Y^2 D_{YY} + f - rD = 0. \tag{11}$$

The general solution of (11) is of the form

$$D(Y) = A_1 + A_2 Y^{\lambda_1} + A_3 Y^{\lambda_2}, \tag{12}$$

where the tqo exponents λ_1 and λ_2 are the zeros of the quadratic equation $r(\lambda - 1) + \frac{1}{2}\lambda(\lambda - 1)\|\sigma_Y\|^2$:

$$\lambda_{1,2} = \left(\frac{1}{2} - \frac{r}{\|\sigma_Y\|^2}\right) \pm \sqrt{\left(\frac{1}{2} - \frac{r}{\|\sigma_Y\|^2}\right)^2 + \frac{2r}{\|\sigma_Y\|^2}}. \tag{13}$$

[16]See Black and Cox (1976).
[17]Subscripts denote partial derivatives.

The positive zero λ_1 is equal to one and for the negative zero we get

$$\lambda_2 = -\frac{2r}{\|\sigma_Y^D + \sigma_X\|^2}. \tag{14}$$

Therefore, if we define $\lambda := \lambda_2$, the solution of the differential equation (11) simplifies to

$$D(Y) = A_1 + A_2 Y + A_3 Y^{\lambda}. \tag{15}$$

To calculate the explicit solution for the value of the outstanding debt services, two boundary conditions must be specified. The debtor chooses itself the critical value of the output triggering the default. If the output stays above this value, it will continue paying its payment obligations. When the threshold is reached from above, reorganization takes place. Thus, if the threshold is reached the discounted value of the future interest rate payments is $\omega C/\rho$.[18] Otherwise, if the output gets relatively high with respect to the critical value, reorganization gets more and more unlikely so the present value of the payment obligations tends to the value it would have if no reorganization was possible: C/ρ.
This gives the following two conditions:

(i) If Y reaches Y_B, then $D(Y_B) = \frac{\omega C}{\rho}$

(ii) If $Y \to \infty$, then $D(Y) \to \frac{C}{\rho}$

From (ii) follows $A_1 = \frac{C}{\rho}$ and $A_2 = 0$, because $\lambda < 0$, therefore condition (ii) means

$$D(Y_B) = \frac{\omega C}{\rho} = \frac{C}{\rho} + A_3 Y_B^{\lambda} \tag{16}$$

leading to $A_3 = -\frac{(1-\omega)C}{\rho}\frac{1}{Y_B^{\lambda}}$. Inserting into (15) gives the present value of the future obligation payments in foreign currency $D(Y) = \frac{C}{\rho} - \frac{(1-\omega)C}{\rho}\left(\frac{Y}{Y_B}\right)^{\lambda}$, and with the exchange rate X the present value of the debt services in domestic currency dependent on domestic output $Y^D = Y/X$ is

$$D^D\left(Y^D\right) = \frac{1}{X}\left[\frac{C}{\rho} - \frac{(1-\omega)C}{\rho}\left(\frac{Y^D X}{Y_B}\right)^{\lambda}\right]. \tag{17}$$

[18] The time τ value of a constant payment stream of ωC at all dates after τ with discount factor $\rho > 0$ is $\int_{\tau}^{\infty} e^{-\rho(\tau-t)}\omega C dt = \frac{\omega C}{\rho}$.

Analogously the discounted value of expected future output $G(Y)$ in foreign currency follows the differential equation (11) without a constant addend

$$rYG_Y + \frac{1}{2}\|\sigma_Y\|^2 Y^2 G_{YY} dt - rG = 0. \tag{18}$$

The general solution of (18) is $G(Y) = B_1 Y + B_2 Y^\lambda$.
If no reorganization of the debt takes place at all, the output level in foreign currency follows the dynamics according to (8) and at each point in time t the present value of the future goods and services of the economy knowing the actual output at time t is

$$\mathbb{E}\left[\int_t^\infty Y_s e^{-\rho(s-t)} ds\right] = \int_t^\infty Y_t e^{-(\rho-r)(s-t)} ds = \frac{Y_t}{\rho - r}, \tag{19}$$

where the discount factor the government ρ is assumed to be greater than the foreign short rate r, and expectations are taken under the foreign martingale-measure.
If the threshold Y_B is reached from above, reorganization takes place. Due to sanctions and a reduction in foreign trade as well as higher costs of refinancing at the capital market output falls in the following periods to a lower level[19] according to (3). Therefore, under the foreign martingale-measure the present value of the discounted future output in foreign currency at the moment Y_B is reached is $\frac{Y_B}{\rho-r}(1-\eta)$.[20] So the total discounted expected loss of the output is $\eta\frac{Y_B}{\rho-r}$, $0 < \eta < 1$. However, if the output gets very high, reorganization is extremely costly because the loss in output in case of default rises with the level of the output. So reorganization gets more and more unlikely if the output grows higher, such that the present value tends to the value in (19). So a default is more probable in poor developing countries than in rich industrial countries.
This leads to two boundary conditions for the present value of the expected future output in foreign currency:

[19] The reduced output also indices an increase in the price level, because of the macroeconomic relationship $VM = YP$, where V represents the velocity of money, M the amount of money in circulation and P the price level. Because V and M cannot adjust quickly, an output reduction induces an increase of the price level. This in term reduces the worth of the domestic currency, that means X will fall.

[20] Equivalent to (4) is $\frac{1-\eta}{\rho-r} = \left(\frac{1}{\rho-r} - \gamma\frac{1}{\rho+\delta-r}\right)$.

(i) If $Y = Y_B$, then $G(Y_B) = \frac{Y_B}{\rho - r}(1-\eta))$

(ii) If $Y \to \infty$, then $G(Y) \to \frac{Y}{\rho - r}$

From (ii) it follows immediately that $B_1 = \frac{1}{\rho - r}$, therefore the first condition is

$$G(Y_B) = \frac{Y_B}{\rho - r}(1-\eta) \stackrel{!}{=} \frac{1}{\rho - r} Y_B + B_2 Y_B^{\lambda}, \tag{20}$$

giving as value for the second coefficient $B_2 = -\frac{\eta}{\rho - r} Y_B^{1-\lambda}$. So together with the two boundary conditions the solution for the present value of the expected future output in foreign currency is

$$G(Y) = \frac{Y}{\rho - r} - \frac{\eta}{\rho - r} Y_B \left(\frac{Y}{Y_B}\right)^{\lambda} \tag{21}$$

or, in domestic values,

$$G^D(Y^D) = \frac{Y^D}{\rho - r} - \frac{\eta}{\rho - r} Y_B \left(\frac{Y^D X}{Y_B}\right)^{\lambda}. \tag{22}$$

The problem for the sovereign borrower is to decide at which level Y_B it should reorganize its debt. The government tries to maximize the wealth of the domestic economy, so it does not like the loss in output that appears after a reorganization of debt. However, it also seeks to reduce the burden on the economy due to the interest payments. Clearly, it should never declare default if the expected loss in output after reorganization compared to the case in which it would never reorganize its debt is higher than the total expected gain from reorganization. The expected loss depends on Y_B, so a critical value $\bar{Y}_B$ can be derived. For values of the output in foreign currency above $\bar{Y}_B$ reorganization would never be optimal. So a reorganization is too costly if

$$\frac{\eta}{\rho - r} Y_B > \frac{C(1-\omega)}{\rho}, \tag{23}$$

leading to the critical value

$$\bar{Y}_B = C \frac{(1-\omega)(\rho - r)}{\eta \rho}. \tag{24}$$

But if the output in foreign currency reaches this value from above the sovereign borrower has not only the choice between reorganization now or never, it can also wait a little bit more hoping that the output will raise again and default if it falls more. So the wealth of the economy may be increased if the government decides to continue paying its debt obligations. The value of $\bar{Y}_B$ maximizing the wealth of the economy is the value that maximizes the difference $G(Y) - D(Y)$:

$$\begin{aligned} & \max_{Y_B} G(Y) - D(Y) \\ = \quad & \max_{Y_B} \frac{Y}{\rho - r} - \frac{\eta}{\rho - r} Y_B \left(\frac{Y}{Y_B}\right)^{\lambda} - \frac{C}{\rho} + \frac{C\,(1-\omega)}{\rho} \left(\frac{Y}{Y_B}\right)^{\lambda}. \end{aligned} \tag{25}$$

The first order condition leads to

$$\begin{aligned} & \frac{\partial G(Y) - D(Y)}{\partial Y_B} \overset{!}{=} 0 \\ \Leftrightarrow \quad & -\frac{\eta}{\rho - r} \left(\frac{Y}{Y_B}\right)^{\lambda} + \frac{\lambda \eta}{\rho - r} Y_B \frac{Y^{\lambda}}{Y_B^{\lambda+1}} - \lambda \frac{C\,(1-\omega)}{\rho Y_B} \left(\frac{Y}{Y_B}\right)^{\lambda} = 0. \end{aligned} \tag{26}$$

and therefore to an optimal reorganization threshold of

$${Y_B}^* = \frac{\lambda}{\lambda - 1} \frac{C\,(1-\omega)}{\eta} \frac{\rho - r}{\rho}. \tag{27}$$

Because of $\lambda < 0$ this value is smaller than $\bar{Y}_B$, so it is optimal not to stop the interest payments immediately if $\bar{Y}_B$ is reached but to continue further until the output in foreign currency reaches the smaller value ${Y_B}^*$, i.e.

$${Y_B}^* = \frac{2r}{\left\|\sigma_Y^D + \sigma_X\right\|^2 + 2r} \frac{C\,(1-\omega)}{\eta} \frac{\rho - r}{\rho}. \tag{28}$$

For the sovereign borrower itself the critical value in domestic currency is more important:

$$\begin{aligned} Y_B^{D*} &= \frac{1}{X_t} \frac{2r}{\left\|\sigma_Y^D + \sigma_X\right\|^2 + 2r} \frac{C\,(1-\omega)}{\eta} \frac{\rho - r}{\rho} \\ &= \frac{C\,(1-\omega)}{X_t} \frac{2r}{\left\|\sigma_Y^D + \sigma_X\right\|^2 + 2r} \frac{\rho + \delta - r}{\gamma \rho}. \end{aligned} \tag{29}$$

This value can be reached due to a fall of domestic output, but also because a devaluation of the home currency increases the critical value. This is due to the payment obligations in foreign currency. In case of a devaluation the domestic output loses in

value relatively to the interest obligations. Thus, in case of a currency crisis a default or reorganization becomes more likely to occur. Of course this value is proportional to the payment obligation in domestic currency, so only the relation between output and interest obligations measured in the same currency matters.
Obviously the critical value depends negatively on the exchange rate. Higher uncertainty as well of the future output as of the exchange rate dynamics also lowers the threshold ${Y_B}^*$. If the volatility of output is higher, the chance of a strong economic rebound is higher, so the loss in future output in consequence of a reorganization of the debt are higher as well. Analogous, if the sovereign borrower already is in financial distress, a higher volatility of the exchange rate increases the probability that the interest payments in foreign currency will get much cheaper in future times.
Furthermore, the parameters determining the reduction of the output after a reorganization influence the threshold: A higher value of γ, meaning a higher reduction of output due to a default, increases the costs of default, therefore reduces the critical value ${Y_B}^*$. Similarly, higher values of δ reduce the time, the reorganization has a negative impact on the output and therefore reduces the costs of reorganization. That means ${Y_B}^*$ is increasing in δ.
Because of $\delta > r$, ${Y_B}^*$ depends negatively on the government's discount-rate ρ. The greater the discount-rate is, the less the government is interested in the future output development and payment obligations. The reduction of the payment obligation remains the same over time, but the output reductions subsequent to a default are higher in the nearer future. Therefore, with a higher discount-rate the discounted benefit due to the renegotiation carries less weight relatively to the discounted costs in terms of lower output. Hence if the sovereign borrower has a high discount-rate, he should reorganize his debt at higher output-levels than if he has a higher appreciation of the future.
The influence of the interest rate on the threshold value is not clear. For low interest rates the critical value of the output is increasing in r and for higher interest rates the threshold falls again. The highest value of the reorganization threshold is reached for an interest rate of

$$r^* = \sqrt{\frac{\left\|\sigma_Y^D + \sigma_X\right\|^2}{2}\left(\rho + \delta + \frac{\left\|\sigma_Y^D + \sigma_X\right\|^2}{2}\right)} - \frac{\left\|\sigma_Y^D + \sigma_X\right\|^2}{2}. \tag{30}$$

To calculate the total value of the country's payment obligation in domestic currency (29) must be inserted in (17). This gives

$$D^D\left(Y^D\right) = \frac{1}{X}\frac{C}{\rho} - \left(\frac{1}{X}\frac{C\left(1-\omega\right)}{\rho}\right)^{1-\lambda}\left(\frac{\lambda-1}{\lambda}\frac{\eta}{\rho-r}Y^D\right)^{\lambda}. \tag{31}$$

As long as $Y^D > Y_B^{D^*}$ the value

$$p := \left(\frac{Y^D X}{Y_B}\right)^{\lambda} = \left(\frac{Y^D}{Y_B^{D^*}}\right)^{-\frac{2r}{\|\sigma_Y\|^2}}, \tag{32}$$

$p \in [0,1]$, can be interpreted as the default probability. It is increasing in the output-volatility and decreasing in the actual output and the interest-rate. Rearranging the terms in (31) leads to an alternative formulation, where $\frac{1}{X}\frac{C}{\rho}$ is the payment obligation in case no reorganization takes place:

$$D^D\left(Y^D\right) = \frac{1}{X}\frac{C}{\rho}(1-p) + \frac{1}{X}\frac{C\omega}{\rho}p, \tag{33}$$

this means that for the default probability it is only the value of the output relative to the payment obligation that matters.

4 Valuation of a coupon bond

Total foreign debt of the state was assumed to live infinite or always be rolled over at maturity, with a constant payment obligation at each moment in time. We can regard total debt as the sum of a great number of different coupon bonds that are always rolled over at maturity, so there is no real maturity of the debt. If the number of these bonds and therefore the amount of interest payment dates is high enough, the assumption of a continuous payment stream can be used as approximation for this more realistic case. With this assumption the above model can be applied and an optimal default or reorganization threshold of the output, dependent on the exchange rate, can be derived.

With this threshold default probabilities can be calculated and therefore the value of a defaultable coupon-bond. To do that the coupon-bond with payments at intermediate dates is separated in different zero-coupon-bonds. Each of them has maturity at the respective coupon payment date and a face value equal to the coupon or the final payment at maturity.

Let the face value of the bond be K and the coupon be c. The different coupon payments are due at $t_1, \ldots, t_n = T$, where T is the maturity of the total bond. With this the value of this risky bond at time t is:

$$\bar{B}(t) = \sum_{i=1}^{n} K\bar{B}(t,t_i)\,1_{\{t \le t_i\}}c + K\bar{B}(t,T), \tag{34}$$

where the values with the bar denote risky values and those without default-riskless values, and $\bar{B}(t, t_i)$ is the value of a defaultable zero-coupon-bond at time t with maturity t_i and face value 1. The indicator-function only respects the case that the valuation date t is after the maturity of respective zero-coupon-bond.

Each of these zero-coupon-bonds of the total bond can be valuated separately. The default or reorganization occurs, if $Y_t^D \leq \frac{Y_B}{X_t}$. In this case the write-down is ωK. Taking this into account, the value of one zero-coupon-bond with maturity t_i at time t is:

$$\begin{aligned} \bar{B}(0, t_i) &= B(0, t_i) - B(0, t_i)\,\omega P\left(\min_{s \leq t_i} Y_s^D \leq \frac{Y_B}{X_s}\right) \\ &= B(0, t_i)\left(1 - \omega P\left(\min_{s \leq t_i} Y_s^D \leq \frac{Y_B}{X_s}\right)\right) \qquad (35) \end{aligned}$$

under the foreign martingale-measure. So only the default probabilities need to be calculated, depending on the current values of the output and the exchange rate. This default probability is

$$P\left(\min_{t \leq t_i} Y_t \leq Y_B\right) = P\left(\min_{t \leq t_i} Z_t \leq \ln\left(\frac{Y_B}{Y_0^D X_0}\right)\right) \qquad (36)$$

with $Z_t := \ln\left(\frac{Y_t^D}{Y_0^D}\frac{X_t}{X_0}\right)$. So Z_t follows a Brownian motion with drift:

$$Z_t = \left(r - \frac{1}{2}\|\sigma_Y\|^2\right)t + \sigma_Y W_t \qquad (37)$$

If it is furthermore assumed that $d = 1$, then $Z_t = \left(r - \frac{1}{2}\sigma_Y^2\right)t + \sigma_Y W_t$. For a Brownian motion B_t with drift: $B_t = \mu t + \sigma W_t$ the distribution of the running minimum is known. Following Musiela and Rutkowski (1997) we get for every $y \leq 0$

$$P\left(\min_{s \leq t} B_s \leq y\right) = N\left(\frac{y - \mu t}{\sigma\sqrt{t}}\right) + e^{2\mu y \sigma^{-2}} N\left(\frac{y + \mu t}{\sigma\sqrt{t}}\right), \qquad (38)$$

where $N(\cdot)$ stands for the standard normal distribution function. With this the probability that the output reaches the reorganization threshold is

$$\begin{aligned} P\left(\min_{t \leq t_i} Z_t \leq \ln\left(\frac{Y_B}{Y_0^D X_0}\right)\right) &= N\left(\frac{\ln\left(\frac{Y_B}{Y_0^D X_0}\right) - \left(r - \frac{1}{2}\sigma_Y^2\right)t_i}{\sigma\sqrt{t_i}}\right) \\ &\quad + e^{\left(\frac{2r}{\sigma_Y^2} - 1\right)\ln\left(\frac{Y_B}{Y_0^D X_0}\right)} N\left(\frac{\ln\left(\frac{Y_B}{Y_0^D X_0}\right) + \left(r - \frac{1}{2}\sigma_Y^2\right)t_i}{\sigma\sqrt{t_i}}\right). \qquad (39) \end{aligned}$$

Putting together the relation between the value of one risky and one default riskless zero-coupon-bond can be calculated:

$$\begin{aligned} \frac{\bar{B}(0,t_i)}{B(0,t_i)} &= \left(1-\omega N\left(d_1^i\right)-\omega \exp\left(\left(\frac{2r}{\sigma_Y^2}-1\right)\ln\left(\frac{Y_B}{Y_0^D X_0}\right)\right)N\left(d_2^i\right)\right) \qquad (40)\\ \text{with } d_{1,2}^i &= -\frac{\left(\frac{Y_0^D X_0}{Y_B}\right)\pm\left(r-\frac{1}{2}\sigma_Y^2\right)t_i}{\sigma\sqrt{t_i}}, \end{aligned}$$

and therefore by summing up the different zero-coupon-bonds we get the total value of a default-risky sovereign coupon bond at initial time $t=0$, dependent on the current value of the output and the exchange rate:

$$\begin{aligned} \bar{B}(0) &= \sum_{i=1}^{n} K\bar{B}(0,t_i)\,c+K\bar{B}(0,T)\\ &= \sum_{i=1}^{n} KB(0,t_i)\,c\left(1-\omega N\left(d_1^i\right)-\omega\exp\left(\frac{2r}{\sigma_Y^2}-1\right)\ln\left(\frac{Y_B}{Y_0^D X_0}\right)N\left(d_2^i\right)\right)\\ &\quad +K\left(1-\omega N\left(d_1^n\right)-\omega\exp\left(\left(\frac{2r}{\sigma_Y^2}-1\right)\ln\left(\frac{Y_B}{Y_0^D X_0}\right)\right)N\left(d_2^n\right)\right)\\ \text{with } d_{1,2}^i &= -\frac{\left(\frac{Y_0^D X_0}{Y_B}\right)\pm\left(r-\frac{1}{2}\sigma_Y^2\right)t_i}{\sigma\sqrt{t_i}}. \end{aligned}$$

5 Conclusion

We present a structural model in the style of Leland (1994) to value credit risky bonds of a sovereign issuer. The sovereign state issues debt on the foreign capital market and has a constant payment obligation in foreign currency. The borrower himself decides when to declare default, i.e. when to reorganize its debt to lower the interest burden. The aim of the government is to maximize the wealth of the economy. There are no legal enforcements to compel the borrower to pay, but if he does not pay, this will have a negative impact on the country's foreign trade in the future. It will be more difficult and therefore more expensive to get new money on the international market, which, in turn, reduces the amount of future imports and the production possibilities for the economy. There might also be a direct effect on exports that could be confiscated in a

foreign country as substitution for outstanding debt payments.
Comparing the costs of the debt service with the value of the punishment via output and foreign trade reduction, the country voluntarily decides when to reorganize its debt, i.e. to declare default. The optimal threshold for domestic output is calculated, dependent on the exchange rate, the payment obligation, the write-down and some other parameters like the discount factor of the government. In a currency crisis the threshold can raise drastically when the exchange rate falls. This brings the threshold value closer to actual output and therefore increases the default risk.
In contrast to output the exchange rate is immediately observable on a daily basis. Therefore, a position in the country's debt on the foreign capital market can be hedged with the exchange rate. If in addition output is replaced by the domestic stock index that could serve as a proxy for expected future gains of the economy, a second hedging instrument is available for the debt of the sovereign issuer.

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