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by

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Submodularity and the Evolution of Walrasian Behavior*

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Abstract

Vega-Redondo (1997) showed that imitation leads to the Walrasian outcome in Cournot Oligopoly. We generalize his result to aggregative quasi-submodular games. Examples are the Cournot Oligopoly, Bertrand games with differentiated complementary products, Common-Pool Resource games, Rent-Seeking games and generalized Nash-Demand games.

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1 Introduction

Vega-Redondo (1997) showed that imitators in a symmetric finite n-firm Cournot Oligopoly with strictly decreasing inverse demand for a homogeneous good converge to the Walrasian outcome. This result is rather striking since the Cournot Nash equilibrium appears to be very robust. Imitators mimic the action of a most successful player in the previous round. The adjustment process has inertia, that is not every period all players will adjust their actions. Players are allowed to make mistakes, i.e. with a small probability they randomize with full support. The imitation dynamics is a finite Markov chain that is perturbed by mistakes. Vega-Redondo (1997) showed that as the noise goes to zero, the unique invariant distribution converges to the Walrasian outcome. The key factor to understand this result is that a player adjusting towards the Walrasian outcome may decrease his payoff but decreases the opponents' payoffs even further.

We generalize Vega-Redondo's result to symmetric finite aggregative games that are quasi-submodular in a player's action and the aggregate of all players' actions. E.g., if a player prefers an action to a lower action for a given aggregate of all players' actions, then he must also prefer this action to the lower action for a lower aggregate. In short, we show that Vega-Redondo's result applies to a wider class of games than just Cournot games.

As in Vega-Redondo (1997), our analysis makes use of stochastic stability analysis. However, instead of using the basic graph theoretic arguments applied by Kandori, Rob, and Mailath (1993) and Young (1993), we employ as a short cut the concept of recurrent set (see Samuelson, 1997).

2 Submodularity

A lattice is a partially order set $\langle X, \succcurlyeq \rangle$ whose least upper bound and greatest lower bound are defined by $x' \vee x'' = \sup\{x', x''\}$ and $x' \wedge x'' = \inf\{x', x''\}$, for all $x', x'' \in X$ respectively. For example, if X is the product of several ordered sets, one may define $x' \vee x''$ (likewise $x' \wedge x''$) as the component-wise max (min) to define a lattice. Note that the direct product of a lattice is a lattice, i.e. if X is a lattice then so is $X^2 = X \times X$. A real valued function $f : X \longrightarrow \mathbb{R}$ on a lattice X is called submodular on X if $\forall x', x'' \in X$,

$$f(x' \wedge x'') + f(x' \vee x'') \leq f(x') + f(x''). \tag{1}$$

The function f is called strictly submodular if the inequality holds strictly for all unordered $x', x'' \in X$. The function f is called quasi-submodular on X if $\forall x', x'' \in X$,

$$f(x' \vee x'') \geq (>) f(x'') \implies f(x') \geq (>) f(x' \wedge x''), \tag{2}$$

$$f(x' \wedge x'') \geq (>) f(x'') \implies f(x') \geq (>) f(x' \vee x''). \tag{3}$$

Note that submodularity implies quasi-submodularity but not vice versa (see Topkis, 1998).

Definition 1 (Aggregative Quasi-Submodular game). *A symmetric (finite) strategic game $\Gamma = \langle N, S, a, \pi \rangle$ is called aggregative quasi-submodular if*

(i) $N = \{1, ..., n\}$ *is the finite set of players,*

(ii) the set of actions S_i, $\forall i \in N$, is a totally ordered (finite) lattice,

(iii) the aggregator $a_i : \times_{j \in N} S_j \longrightarrow T$, T being a totally ordered (finite) lattice, is strictly isotone and invariant to permutations of its arguments[1]*,*

[1]The function $a_i : \times_{j \in N} S_j \longrightarrow T$ is invariant to permutations of its arguments if

(iv) the payoff function $\pi_i : S \times T \longrightarrow \mathbb{R}$ is quasi-submodular in (s,t), $\forall i \in N$, i.e. $\forall (s',t'), (s'',t'') \in S \times T$,

$$\pi_i((s',t') \vee (s'',t'')) \geq (>) \pi_i(s'',t'') \Longrightarrow$$
$$\pi_i(s',t') \geq (>) \pi_i((s',t') \wedge (s'',t'')), \tag{4}$$
$$\pi_i((s',t') \wedge (s'',t'')) \geq (>) \pi_i(s'',t'') \Longrightarrow$$
$$\pi_i(s',t') \geq (>) \pi_i((s',t') \vee (s'',t'')). \tag{5}$$

(v) the action sets and payoff functions are symmetric, i.e. $S_i = S$ and $\pi_i = \pi$, $\forall i \in N$.

Examples of the class of aggregative quasi-submodular games are as follows:

Example 1. (Cournot Oligopoly with a homogeneous good) The payoff function is $\pi(s,t) = p(t)s - c(s)$ with s being interpreted as a firm's quantity, p being a strictly decreasing inverse demand function and c a cost function (see also Schipper, 2001). The aggregator is simply the total quantity over all firms $a(s_1, ..., s_n) = \Sigma_{i \in N} s_i$.

Example 2. (Cournot Oligopoly with differentiated substitute products) The payoff function is $\pi_i(s_i,t) = p_i(s_i,t)s_i - c(s_i)$. Goods are substitutes if for example $p_i(s_i,t) = \beta\theta(\Sigma_{j=1}^n s_j^\beta)^{\theta-1} s_i^{\beta-1}$ with $0 < \beta\theta < 1$, $\theta < 1$ and $1 \geq \beta > 0$ (see Vives, 2000). The aggregator is $a(s_1, ..., s_n) = \Sigma_{j=1}^n s_j^\beta$.

Example 3. (Bertrand Oligopoly with differentiated complementary products) The payoff function is $\pi_i(s_i,t) = d_i(s_i,t)s_i - c(d_i(s_i,t))$ with

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$a_i(s_1, ..., s_n) = a_i(s_{b(1)}, ..., s_{b(n)})$ for all bijections $b : N \longrightarrow N$.

s_i being interpreted as the price for the good of firm i, d_i being the demand function of good i and c the cost function. Goods are complements if for example

$$d_i(s_i, t) = (\beta\theta)^{\frac{1}{1-\beta\theta}} \frac{s_i^{\frac{1}{\beta-1}}}{(\Sigma_{j=1}^n s_j^{\frac{\beta}{\beta-1}})^{\frac{1-\theta}{1-\beta\theta}}},$$

with $0 < \beta\theta < 1$, $\theta < 1$ and $\beta < 0$ (see Vives, 2000). The aggregator is $a(s_1, ..., s_n) = \Sigma_{j=1}^n s_j^{\frac{\beta}{\beta-1}}$.

Example 4. (Common-Pool Resource game) The payoff function is $\pi(s_i, t) = c(e - s_i) + \frac{s_i}{\Sigma_{j=1}^n s_j}[a\Sigma_{j=1}^n s_j - b(\Sigma_{j=1}^n s_j)^2]$ if $s_i > 0$ and $\pi(0, 0) = ce$ with $c, e, a, b \in \mathbb{R}_{++}$ (see Walker, Gardner, and Ostrom, 1990). Each appropriator $i \in N$ has an endowment e that can be invested in the Common-Pool Resource or in an outside activity with marginal payoff c. s_i denotes appropriator i's investment into the Common-Pool Resource, where $0 \leq s_i \leq e$. The return is $\frac{s_i}{\Sigma_{j=1}^n s_j}[a\Sigma_{j=1}^n s_j - b(\Sigma_{j=1}^n s_j)^2]$. The aggregator is $a(s_1, ..., s_n) = \Sigma_{j=1}^n s_j$.

Example 5. (Rent-Seeking game) The payoff function is $\pi(s, t) = \frac{s_i^r}{\Sigma_{j=1}^n s_j^r} v - s_i$ with $s_i \geq 0$ and $0 < r < 1$. Contestants compete for the rent v by bidding s_i. The probability of winning is $\frac{s_i^r}{\Sigma_{j=1}^n s_j^r}$ but the cost of bidding equals the bid (see Hehenkamp, Leininger, and Possajennikov, 2001). The aggregator is $a(s_1, ..., s_n) = \Sigma_{j=1}^n s_j^r$.

Example 6. (Generalized Nash-Demand game) The payoff function is $\pi(s, t) = p(t)s$. The demand of a player is s. The probability of getting the demand is $p(t)$ which is strictly decreasing in the total of demands of all players $\Sigma_{j=1}^n s_j$. The aggregator is $a(s_1, ..., s_n) = \Sigma_{j=1}^n s_j$.

3 Imitation Dynamics

Time is discrete and indexed by $\tau = 0, 1, 2, \ldots$.

Definition 2 (Imitator). [2] *An imitator $i \in N$ chooses with full support from the set*

$$D_I(\tau - 1) := \{s \in S : \exists j \in N \ s.t. \ s = s_j(\tau - 1) \ and \\ \forall k \in N, \pi_j(\tau - 1) \geq \pi_k(\tau - 1)\}. \quad (6)$$

An imitator mimics the action of the player(s) with highest payoff in the previous period. At every time $\tau = 1, 2, \ldots$, each player $i \in N$ is assumed to revise his former action $s_i(\tau - 1)$ with a common i.i.d. probability $\rho \in (0, 1)$ according to the imitation rule. Initially in $\tau = 0$ players start with any arbitrary action within the action set S.

The process induced by the imitation dynamics is a discrete time finite Markov chain on the state-space $S^n = \times_{i \in N} S_i$. Each state $\omega(\tau) = (s_1(\tau), s_2(\tau), \ldots, s_n(\tau))$ induces a profit-profile $(\pi_1(\tau), \pi_2(\tau), \ldots, \pi_n(\tau))$. The Markov operator is defined in the standard way as transition probability matrix $P = (p_{\omega\omega'})_{\omega,\omega' \in S^n}$ with $p_{\omega\omega'} = prob\{\omega' | \omega\}$, $p_{\omega\omega'} \geq 0$, $\omega, \omega' \in S^n$ and $\sum_{\omega' \in S^n} p_{\omega\omega'} = 1$, $\forall \omega \in S^n$.

At every output revision opportunity τ, each player follows the imitation rule with probability $(1 - \varepsilon)$, $\varepsilon \in (0, \eta]$, where η is small, but with probability ε he randomizes ("mutates") with full support S. This noise makes the perturbed Markov chain $P(\varepsilon)$ irreducible and ergodic. This implies that there exists a unique invariant distribution $\varphi(\varepsilon)$ on S^n (see for example Masaaki, 1997). We focus on the unique limiting invariant distribution φ^* of P defined by $\varphi(\varepsilon)P(\varepsilon) = \varphi(\varepsilon)$, $\varphi^* := \lim_{\varepsilon \to 0} \varphi(\varepsilon)$ and $\varphi^* P = \varphi^*$. This long run

[2] See also Vega-Redondo (1997), p. 378.

distribution determines the average proportion of time spent in each state of the state-space in the long run (see Samuelson, 1997, for an introduction).

Consider $\varepsilon = 0$ and define an absorbing set $A \subseteq S^n$ by

(i) $\forall \omega \in A$, $\forall \omega' \notin A$, $p_{\omega\omega'} = 0$ and

(ii) $\forall \omega, \omega' \in A$, $\exists m \in \mathbb{N}$, m finite, s.t. $p^{(m)}_{\omega\omega'} > 0$, $p^{(m)}_{\omega\omega'}$ being the m-step transition probability from ω to ω'.

Let Z be the collection of all A in S^n.

We call states ω and ω' adjacent if exactly one mutation can change the state from ω to ω' (and vice versa). The set of all states adjacent to the state ω is the single mutation neighborhood of ω denoted by $M(\omega)$. The basin of attraction of an absorbing set A is the set $B(A) = \{\omega \in S^n | \exists m \in \mathbb{N}, \exists \omega' \in A \text{ s.t. } p^{(m)}_{\omega\omega'} > 0\}$. A recurrent set R is a minimal collection of absorbing sets with the property that there do not exist absorbing sets $A \in R$ and $A' \notin R$ such that $\forall \omega \in A$, $M(\omega) \cap B(A') \neq \emptyset$. We will make use of following lemma.

Lemma 1 (Nöldeke and Samuelson). *Given a perturbed finite Markov chain, then at least one recurrent set exists. Recurrent sets are disjoint. Let the state ω be contained in the support of the unique limiting invariant distribution φ^*. Then $\omega \in R$, R being a recurrent set. Moreover, $\forall \omega' \in R$, $\varphi^*(\omega') > 0$.*

For a proof see for example Samuelson (1997), Lemma 7.1 and Proposition 7.7., proof pp. 236-238.

4 Result

Definition 3 (Walrasian outcome). *$\omega^* = (s_1^*, ..., s_n^*)$ is a Walrasian outcome if for $t^* = a(\omega^*)$,*

$$\pi(s^*, t^*) \geq \pi(s, t^*), \forall s \in S. \tag{7}$$

The Walrasian outcome describes a solution in which the player does not perceive the externality of his action. An example is price-taking behavior.

Theorem. *Given imitators with inertia and noise in an aggregative quasi-submodular game, let the Walrasian outcome $\omega^* \in S^n$ exist uniquely. Then $\varphi^*(\omega^*) = 1$.*

The proof follows from below lemmatas. Recall that Z is the collection of absorbing sets.

Lemma 2. *$Z = \{A_\omega = \{\omega\} : \omega = (s, ..., s) \in S^n$ for some $s \in S\}$.*

Proof. By symmetry of Γ, we have by D_I for every $\omega = (s, ..., s) \in S^n$ that $p_{\omega\omega} = 1$ and $p_{\omega\omega'} = 0$, $\forall \omega' \neq \omega$. Conversely, since at any τ and i.i.d. probability $\rho > 0$, there is positive probability that all firms adjust towards the same action in $D_I(\tau - 1)$ given any arbitrary $\omega(\tau - 1)$. □

Lemma 3. *$M(\omega) \cap B(\{\omega^*\}) \neq \emptyset$, $\forall \{\omega\} \in Z \backslash \{\omega^*\}$.*

Proof. By assumption ω^* is unique and by Lemma 2, $A_{\omega^*} = \{\omega^*\}$. Consider any absorbing set (state) $A \neq A_{\omega^*}$. We claim that starting in any $A \neq \{\omega^*\}$ a single (suitable) mutation can lead the dynamics to $B(\{\omega^*\})$. It is sufficient to show that $\forall s \in S$, $s \neq s^*$, $k \in \mathbb{N}$, $k \leq n$,

$$\pi(s^*, t) > \pi(s, t), \tag{8}$$

with $t = a(s_1^*, ..., s_k^*, s_{k+1}, ..., s_n)$. a is strictly isotone and invariant to permutations of it's arguments. π is quasi-submodular. Set $s^* \equiv s'$, $s \equiv s''$, $t \equiv t'$ and $t^* \equiv t''$. Note that the left-hand side of Formulas (4) and (5) is the definition of the Walrasian outcome in the Inequality (7) whereas the right-hand side is the above Inequality (8) (for $s \prec s^*$ the upper Formula (4) and for $s \succ s^*$ the lower Formula (5)). Setting $k = 1$ yields the desired claim and completes the proof of the lemma. □

Lemma 4. *$M(\omega^*) \cap B(A) = \emptyset$, $\forall A \in Z$, $A \neq \{\omega^*\}$.*

Proof. By setting $k = n - 1$ in Inequality (8), it follows that more than one mutation is needed to escape A_{ω^*} since players setting s^* are still better off after just one mutation. □

From previous lemmata follows that $R = \{\omega^*\}$. Thus by Lemma 1, $\varphi^*(\omega^*) = 1$. This completes the proof of the Theorem.

Note that just a single suitable mutation is required to trigger the convergence to the long run outcome. Hence, the convergence is rather fast compared to many results in the literature obtained by the same method.

5 Conclusions

We generalize Vega-Redondo's (1997) result to a class of aggregative quasi-submodular games. Examples of this class are many games with strategic substitutes. The result provides an evolutionary foundation for Walrasian behavior in an important class of non-cooperative games. Schipper (2001) also uses quasi-submodularity to prove that imitators are strictly better off than are best-response-players in Cournot oligopoly. This result too applies to aggregative quasi-submodular games.

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