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Extended Libor Market Models with Affine and Quadratic Volatility

by

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EXTENDED LIBOR MARKET MODELS WITH AFFINE AND QUADRATIC VOLATILITY

CHRISTIAN ZÜHLSDORFF

ABSTRACT. The market model of interest rates specifies simple forward or Libor rates as lognormally distributed, their stochastic dynamics has a linear volatility function. In this paper, the model is extended to quadratic volatility functions which are the product of a quadratic polynomial and a level-independent covariance matrix. The extended Libor market models allow for closed form cap pricing formulae, the implied volatilities of the new formulae are smiles and frowns. We give examples for the possible shapes of implied volatilities. Furthermore, we derive a new approximative swaption pricing formula and discuss its properties. The model is calibrated to market prices, it turns out that no extended model specification outperforms the others. The criteria for model choice should thus be theoretical properties and computational efficiency.

JEL Classification E43, G12, G13
Keywords forward Libor rates, Libor market model, affine volatility, quadratic volatility, derivatives pricing, closed form solutions, LMM

INTRODUCTION

In the following the term forward rate refers to a discretely compounded interest rate. Models for forward rates are called Libor market models because they describe the behaviour of rates which are directly observable in the market, e.g. 3- or 6-month Libor. This is a major advantage over classical short rate models or the Heath, Jarrow and Morton (1992) model for continuously-compounded rates. Continuously-compounded forward rates are a theoretical construct and are only available by some interpolation algorithm.

After the seminal article by Heath et al. (1992) who derive the no-arbitrage conditions for general whole-yield-curve models the next step in term structure modelling was the development of the Libor market models: Miltersen, Sandmann and Sondermann (1997) proved that by modelling (simple) forward rates as lognormal the no-arbitrage price of caps and floors are given by the Black (1976) formula used by practitioners and that this model can be specified in the Heath et al. (1992) framework. These results were extended by Brace, Gatarek and Musiela (1997) who derived an approximative formula for swaption prices and Jamshidian (1997) who introduced the swap market model.

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The pricing formulae derived in the cited articles rely on the assumption that the distribution of the forward rate (or the swap rate) under the pricing measure is lognormal. In an empirical analysis of the Libor market and the swap market model De Jong, Driessen and Pelsser (2001) find systematic pricing errors which can be explained by yield-spread and yield-curvature parameters.

Andersen and Andreasen (2000) were the first to introduce an extended Libor market model: the constant elasticity of variance (CEV) Libor market model with volatility being a power function. For this specification they derive a closed-form cap pricing formula and an approximative swaption pricing formula.

In the following we will propose other specifications of the volatility function which offer analytical pricing formulae and are easy to implement. In Zühlsdorff (1998) it was shown that the pricing partial differential equation (PDE) for a European contingent claim can be solved in a model for the (forward) price of an asset with a quadratic volatility function, the lognormal specification being a special case. This result is used in the first and second section of this paper to extend the Libor market model to quadratic volatility and derive closed-form pricing formulae for caps.

In the quadratic volatility setup the probability of attaining zero may be postive so we have to impose absorption as the only arbitrage-free boundary behaviour consistent with positive interest rates.

In section 4 we derive the dynamics of swap rates in the extended Libor market model. Given this, it is possible to show that the use of the same volatility function for forward as for swap rates is a good approximation of the correct swap volatility function for affine volatility models and approximately constant covariance factors. In this case the model gives closed-form solutions jointly for caps and for swaptions.

In section 5 different models are calibrated to market prices of caps and swaptions. All specifications perform better than the lognormal one, but no particular model outperforms the others.

Section 6 concludes. We will argue that the model with affine volatility is the most interesting one for further investigation from theoretical, computational and empirical points of view.

1. Extended LIBOR Market Models

We start the construction of the bond market with a *tenor structure*, an increasing list of maturity dates:

$$0 \leq t_1 < \cdots < t_{N+1} = T \qquad \delta_n = t_{n+1} - t_n > 0 \qquad n = 1, .., N$$

All the cashflows we will consider will take place at one of these dates. The daycount fractions δ_n are usually time intervals of length three or six months (up to daycount corrections). Our basic underlyings are the zero coupon bonds which mature at these dates:

$$P_n = P_n(t) = P(t, t_n) \qquad \text{for} \quad n = 1, .., N+1$$

If our model is arbitrage free, we can associate to each bond P_n a measure Q^n such that cash-flows discounted by P_n are local martingales under Q^n. The measure Q^n will be called the n-forward measure.

1.1. **Specification.** Given the tenor structure, the *simple compounding forward rates* or *Libor rates* L_n are defined by

$$L_n = \frac{P_n - P_{n+1}}{\delta_n P_{n+1}} \qquad \text{for} \quad n = 1, .., N.$$

It holds

$$P_n = P_{N+1} \prod_{j=n}^{N} (1 + \delta_j L_j) = P_1 \prod_{j=1}^{n-1} (1 + \delta_j L_j)^{-1} \qquad \text{for} \quad n = 1, .., N+1,$$

empty products are 1 by definition. As Libor rates are nominal rates, we impose the condition that they are non-negative. Or, we could say that we want the discount curve $x \mapsto P(t, t+x)$ to be non-increasing.

As the Libor rates are simple portfolios of discounted bonds, the usual argument applies: if the model allows the rate to attain zero, for no-arbitrage to hold it has to be absorbed in zero. Suppose that on the contrary the rate is reflected (immediately or after a stopping time):

$$L_n = 0$$

implies

$$P_{n+1} = P_n,$$

in that case any arbitrageur would buy P_n and sell P_{n+1} at zero cost and wait until

$$P_n > P_{n+1},$$

making a riskless profit by closing his position. For an elaboration of the same problem for continuously-compounded forward rates see Goldstein and Keirstead (1997).

Take now as given a stochastic base $(\Omega, P, \{\mathcal{F}_t\}_{0 \le t \le T})$ which satisfies the usual conditions and an adapted d-dimensional Wiener process W for an integer $d > 0$. Vectors will be column vectors and $^{\mathbf{t}}$ will denote transposition. We call a stochastic model for the dynamics of forward Libor rates of the form

$$dL_n = \cdots dt + p(L_n)\ \gamma_n^{\mathbf{t}} dW$$

an *extended Libor market model* if the following holds: p is a deterministic function and the γ_n are d-dimensional stochastic processes independent of the L_n. In the following the γs will be called the covariance factors of the model and p the volatility function. Given the extended dynamics, the covariation of the rates is

$$d\langle L_n, L_j \rangle = p(L_n)\ \gamma_n^{\mathbf{t}} \gamma_j\ p(L_j)\ dt.$$

1.2. **No-Arbitrage.**

Lemma 1.1. *The arbitrage-free dynamics of the Libor rates under the forward martingale measure Q^m with numeraire P_m is*

$$\begin{aligned} dL_n &= p(L_n)\,\gamma_n^{\mathsf{t}} dW^m + \sum_{j=1}^{n} \frac{\delta_j p(L_n) p(L_j)}{1+\delta_j L_j}\,\gamma_n^{\mathsf{t}}\gamma_j\,dt - \sum_{j=1}^{m-1} \frac{\delta_j p(L_n) p(L_j)}{1+\delta_j L_j}\,\gamma_n^{\mathsf{t}}\gamma_j\,dt \\ &= p(L_n)\,\gamma_n^{\mathsf{t}} \left(dW^m + \sum_{j=1}^{n} q(\delta_j, L_j)\ \gamma_j\,dt - \sum_{j=1}^{m-1} q(\delta_j, L_j)\ \gamma_j\,dt \right) \end{aligned}$$

with

$$q(\delta, L) = \frac{\delta p(L)}{1+\delta L}.$$

Empty sums are zero by definition.

Proof. We have to show that under the specified dynamics for L discounted bond prices are martingales, so the proof consists of verifying that the dynamics of the discounted bond prices

$$D_n = \frac{P_n}{P_m}$$

is

$$\frac{dD_n}{D_n} = \left(\sum_{j=n}^{N} q(\delta_j, L_j)\ \gamma_j - \sum_{j=m}^{N} q(\delta_j, L_j)\ \gamma_j \right)^{\mathsf{t}} dW^m.$$

The trivial but tedious calculation of the derivatives of the Ds with respect to the Ls and the application of Itô's formula is in appendix A.1. □

Now that we have know the arbitrage-free dynamics of the Libor rates, we have to check under which conditions the thus specified SDE is well-defined, ie, wether the arbitrage-free dynamics has a strong non-exploding solution.

Proposition 1.2. *Assume that the volatility function p is locally Lipschitz-continuous and that the covariation factors are deterministic functions such that*

$$-C \le \gamma_n(t)^{\mathsf{t}}\gamma_j(t) \le C \qquad \text{for all} \quad t \qquad \text{and} \quad 1 \le n, j \le N$$

for a constant $C \ge 0$. Then the model is well-defined under the terminal measure $Q^T = Q^{N+1}$.

The proof is in appendix A.2. Note that any continuously differentiable volatility function is locally Lipschitz-continuous, so the proposition covers a large class of possible specifications.

Under Q^{N+1} discounted bond prices D_n are local martingales and satisfy

$$D_{n+1} = \frac{P_{n+1}}{P_{N+1}} = \frac{1}{1+\delta_n L_n} \frac{P_n}{P_{N+1}} \le \frac{P_n}{P_{N+1}} = D_n$$

as by construction $L_n \ge 0$. Thus the specification satisfies the no-arbitrage condition of Musiela and Rutkowski (1997, 2.3). Using the backward induction technique from Musiela and Rutkowski

(1997, 4.1), the n-forward measure Q^n is specified by the Girsanov densities

$$\left.\frac{dQ^n}{dQ^{n+1}}\right|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^t q(\delta_n, L_n)\ \gamma_n^{\mathbf{t}} dW^{n+1}\right)$$

given the $n+1$-forward measure Q^{n+1}. We obtain the dynamics of lemma 1.1 for the associated Wiener processes

$$dW^n = dW^{n+1} - q(\delta_n, L_n)\ \gamma_n dt.$$

1.3. **Monte-Carlo Simulation.** The standard way to discretize the dynamics of L for Monte Carlo pricing are Euler steps:

$$\begin{aligned} L_n(t_{i+1}) &= L_n(t_i) + p(L_n(t_i))\gamma_n(t_i)^{\mathbf{t}} \\ &\left(\sqrt{\Delta_i}\eta_i + \sum_{j=1}^{n} q(\delta_j, L_j(t_i))\ \gamma_j(t_i)\ \Delta_i - \sum_{j=1}^{m-1} q(\delta_j, L_j(t_i))\ \gamma_j(t_i)\ \Delta_i\right) \end{aligned}$$

for independent d-dimensional normal random variates η_i. This Euler scheme is advocated as approximation by Hull and White (2000a) and Hull and White (2000b) in the lognormal case. They argue that one Euler step offers sufficient accuracy for practical purposes and is easy to implement. Hunter, Jäckel and Joshi (2001) propose a predictor-corrector Euler method which improves on this simple scheme.

Glasserman and Zhao (2000) argue that it is possible to avoid problems associated with the discretization of the drift. They propose to simulate other closely related martingales, e.g. discounted bond prices D_n. In a detailed analysis of the pricing error they find that under the terminal measure Q^{N+1} the martingales

$$X_n = \frac{D_n - D_{n+1}}{\delta_n} = L_n\ D_{n+1} = L_n \prod_{j=n+1}^{N} (1 + \delta_j L_j) = \frac{P_n - P_{n+1}}{\delta_n P_{N+1}}$$

give better results for derivatives pricing than other candidates. It holds

$$D_n = \frac{P_n}{P_{N+1}} = 1 + \sum_{j=n}^{N} \delta_j X_j.$$

Given the dynamics of the discounted bonds D, the dynamics of the X_n is

$$\begin{aligned} \frac{dX_n}{X_n} &= \frac{dD_n - dD_{n+1}}{\delta_n X_n} \\ &= \frac{1}{\delta_n X_n}\left(D_n \sum_{j=n}^{N} q(\delta_j, L_j)\ \gamma_j - D_{n+1} \sum_{j=n+1}^{N} q(\delta_j, L_j)\ \gamma_j\right)^{\mathbf{t}} dW^{N+1} \\ &= \left(\frac{D_n}{X_n}\frac{p(L_n)}{1+\delta_n L_n}\ \gamma_i + \sum_{j=n+1}^{N} q(\delta_j, L_j)\ \gamma_j\right)^{\mathbf{t}} dW^{N+1} \end{aligned}$$

so

$$\begin{aligned} dX_n &= X_n \left(\frac{p(L_i)}{L_i}\, \gamma_i + \sum_{j=i+1}^{n} q(\delta_j, L_j)\, \gamma_j \right)^{\mathbf{t}} dW^{N+1} \\ &= \left(D_{n+1} p(X_n/D_{n+1})\, \gamma_n + \sum_{j=n+1}^{N} \frac{\delta_j D_{j+1} p(X_j/D_{j+1})}{D_j}\, \gamma_j \right)^{\mathbf{t}} dW^{N+1}. \end{aligned}$$

For $n = N$ it holds

$$dX_N = dL_N = p(X_N)\, \gamma_N^{\mathbf{t}} dW^{N+1}.$$

For the lognormal specification $p(L) = L$ we obtain formula (19) of Glasserman and Zhao (2000):

$$\frac{dX_n}{X_n} = \left(\gamma_n + \sum_{j=n+1}^{N} \frac{\delta_j L_j}{1 + \delta_j L_j}\, \gamma_j \right)^{\mathbf{t}} dW^{N+1}$$

2. Caplet Pricing

A caplet is a call on a forward rate payed in arrears, ie, the contingent claim given by the payoff

$$\delta_n \left(L_n(t_n) - k \right)^+$$

at time t_{n+1}. It is used to hedge the buyer against upward moves of interest rates. We compute the no-arbitrage price of the caplet under the $n+1$-forward measure:

$$C_{aplet}(t) = \delta_n P_{n+1}(t) E^{n+1} \left[(L_n(t_n) - k)^+ \mid \mathcal{F}_t \right]$$

Using the results of Zühlsdorff (1998) we know that

$$E^{n+1} \left[(L_n(t_n) - k)^+ \mid \mathcal{F}_t \right] = V(t, L_n(t))$$

for a function V that solves the pricing PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2} \gamma(t)^2 p(L)^2 \frac{\partial^2 V}{\partial^2 L} = 0, \qquad V(T, L) = (L - k)^+.$$

Define the time change τ by

$$\tau^2(t) = \int_t^{t_n} \|\gamma_n(u)\|^2 du = \int_t^{t_n} \gamma_n(u)^{\mathbf{t}} \gamma_n(u)\, du.$$

The following specifications give tractable models with closed-form pricing rules for caplets.

2.1. **Log-normal.**

$$p(L) = L$$

This is the standard lognormal market model of Miltersen et al. (1997). The caplet is priced by the industry standard Black formula

$$\mathrm{BS}(k, L, \tau) = L\Phi(d + \tau/2) - k\Phi(d - \tau/2) \qquad d = \frac{\log L - \log k}{\tau}$$

which was derived in a seminal article by Fischer Black (1976). The Libor rate L_n is lognormally distributed under the $n+1$-forward measure, cf. the discussion for the quadratic specification with two real roots in section 2.4.

2.2. **Normal.** The class of Normal, Gaussian, or Bachelier models is specified by

$$p(L) = 1.$$

Libor rates L are normally, forwards (forward bond prices) are lognormally distributed. The price of a caplet is given by the Bachelier formula with absorption:

$$\mathrm{Bac}^{+}(k, L, \tau) = \mathrm{Bac}(k, L, \tau) - \mathrm{Bac}(k, -L, \tau)$$

with

$$\mathrm{Bac}(k, L, \tau) = \tau[d\Phi(d) + \phi(d)] \qquad d = \frac{L-k}{\tau}$$

2.3. **Affine.** The lognormal and the Gaussian model are special cases of affine volatility (AV) market models with

$$p(L) = L - l$$

where $l <= 0$ is a non-attainable boundary. The caplet's pricing formula is

$$\mathrm{AV}^{+}(k, L, \tau, l) = \mathrm{BS}(k-l, L-l, \tau) - \mathrm{BS}((k-l)(l-L)/l, -l, \tau).$$

2.4. **Quadratic Volatility with two real roots.**

$$p(L) = L(1 - L/u) \qquad 0 < u$$

Zero and u are non-attainable bounds for the Libor rates, the caplet price is

$$V = \mathrm{Q2}(k, L, \tau, u) = \mathrm{BS}(k(1-L/u), L(1-k/u), \tau).$$

This specification of the Libor dynamics is a very intuitiv one, it puts a natural upper bound on the forward interest rate and we know from the work of Ingersoll (1997) that the rate will never attain its lower bound zero or its upper bound u. Ingersoll (1997) used a dynamics like this for an exchange rate. Using the corollary from section 1 in Zühlsdorff (1998), we can easily recover formula (11) from Ingersoll (1997) for the transition density

$$P^{n+1}\left(L_n(t_n) \in dy \mid L_n(t) = x\right) = e^{-\tau^2/8}\sqrt{\frac{x(1-x)}{y^3(1-y)^3}}\,\varrho\left(\log\left(\frac{y}{1-y}\right); \tau, \log\left(\frac{x}{1-x}\right)\right)$$

where $\varrho(y; \tau, x)$ is the transition density of Brownian motion. Miltersen et al. (1997) showed that in the lognormal Libor market model the dynamics of the $n+1$–bond's forward price

$$D_{n+1} = \frac{P_{n+1}}{P_n}$$

under the n-forward measure Q^n is of the following form:

$$dD_{n+1} = -D_{n+1}\left(1 - D_{n+1}\right)\ \gamma_n^{\mathsf{t}} dW^n$$

Again, we find a transition density of the form given above, see equation (12) in Miltersen et al. (1997).

2.5. **Quadratic Volatility with two real roots and unbounded domain.** This specification has two real roots, the upper one in zero. The domain of the Libor rate is from zero to infinity.

$$p(L) = L(1 + L/d) \qquad d > 0$$

The value of the caplet is

$$\mathrm{Q2}_u(k, L, \tau, d) = \mathrm{BS}\ \big(\ k(L/d+1), L(k/d+1), \tau\ \big)\ .$$

2.6. **Quadratic Volatility with one real root.** The quadratic specification

$$p(L) = (L - l)^2$$

with root in l also allows for a closed-form caplet pricing formula

$$\mathrm{Q0}(k, L, \tau, l) = (L-l)(k-l)\left[\mathrm{Bac}^+\left(\frac{1}{L-l}, \frac{1}{k-l}, \tau\right) - \mathrm{Bac}^+\left(\frac{1}{L-l}, \frac{l-2L}{(L-l)(k-l)}, \tau\right)\right]$$

as the sum of two positive Bachelier formulae.

2.7. **Quadratic Volatility with no real root.**

$$p(L) = d\left[1 + \left(\frac{L-m}{d}\right)^2\right] \qquad d > 0$$

The parameter m defines the minimum of the parabola and d its slope.

$$V = \mathrm{Q0}(k, L, \tau, m, d) = \frac{e^{\tau^2/2}\sqrt{d}}{\pi \cos(z+a)} \sum_{n>0} c_n e_n(\tau^2) \sin_n(z)$$

$$a = \arctan(-m/d) \qquad z = \arctan\left(\frac{L-m}{d}\right) - a \qquad U = \pi/2 - a$$

$$e_n(\tau^2) = \exp(-n^2\pi^2\tau^2/2U^2) \qquad \sin_n(z) = \sin(n\pi z/U)$$

The constants c_n are given in appendix B, for the implementation see Zühlsdorff (1998).

2.8. **Constant Elasticity of Variance.** Andersen and Andreasen (2000) specify the constant elasticity of variance (CEV) market model where volatility is a power function:

$$p(L) = L^\alpha \qquad \text{for} \quad \alpha \geq 0$$

If $\alpha < 1$ it is possible that the forward rate attains zero and Andersen and Andreasen (2000) model this boundary arbitrage-free as absorbing. Like in the classical CEV models for assets of Cox and Ross (1976) they were able to derive closed-form cap prices. Define

$$a = \frac{k^{2(1-\alpha)}}{(1-\alpha)^2\tau^2} \qquad b = \frac{1}{1-\alpha} \qquad c = \frac{L^{2(1-\alpha)}}{(1-\alpha)^2\tau^2}$$

For $0 < \alpha < 1$ it holds

$$\mathrm{CEV}(k, L, \tau, \alpha) = L(1 - \chi^2(a, b+2, c)) - k\chi^2(c, b, a)$$

and for $1 < \alpha$

$$\mathrm{CEV}(k, L, \tau, \alpha) = L(1 - \chi^2(c, b, a)) - k\chi^2(a, 2-b, c)$$

where $\chi^2(\cdot, \nu, \xi)$ is the non-central chi-square distribution function with ν degrees of freedom and non-centrality parameter ξ.

Implementing the Constant Elasticity of Variance formula. Andersen and Andreasen (2000) propose to compute the χ^2 distribution function using the series expansion developed by Ding (1999):

$$\chi^2(x) = P\{\chi^2(v, \xi) \leq x\} = \sum_{j=0}^{\infty} v_j t_j$$

$$v_0 = u_0 = \exp(-\xi/2) \qquad v_j = v_{j-1} + u_j \qquad u_j = u_{j-1}\xi/2j \qquad t_j = \frac{t_{j-1}x}{\xi + 2j}$$

This expansion is theoretically very usefull, because the computation gives an explicit error bound

$$\varepsilon_n = \sum_{j=n}^{\infty} v_j t_j \leq \sum_{j=n}^{\infty} t_j \leq \frac{t_{n-1}x}{\xi + 2n - x},$$

but in our experience is too slow for practical uses. We prefer the improved first order Wiener germ approximation by Dinges (1989), which gives accurate values fast for small and large values of non-centrality. Define

$$\mu = \xi/\nu \qquad s = \frac{\sqrt{1 + 4x\mu/v} - 1}{2\mu}$$

and

$$S = v(s-1)^2 \left(\frac{1}{2s} + \mu - \frac{h(1-s)}{s} \right) + \frac{2(1+3\mu)^2}{9v(1+2\mu)^3} - \log\left(\frac{1}{s} - \frac{2}{s} \frac{h(1-s)}{1+2\mu s} \right)$$

for

$$h(y) = \frac{1}{y}\left[\left(\frac{1}{y} - 1 \right) \log(1-y) + 1 \right] - \frac{1}{2}$$

then

$$\chi^2(x) \cong \begin{cases} \Phi(\sqrt{S}) & s > 1 \\ 1/2 & s = 1 \\ \Phi(-\sqrt{S}) & s < 1 \end{cases}$$

For a tourough comparison of different approximation techniques for the χ^2 distribution function see Penev and Taykov (2000).

3. Examples of Implied Volatilities

As we had an extensive discussion of the different shapes of implied volatility for the quadratic specifications in Zühlsdorff (1998), we give here only a short comparison of the implied

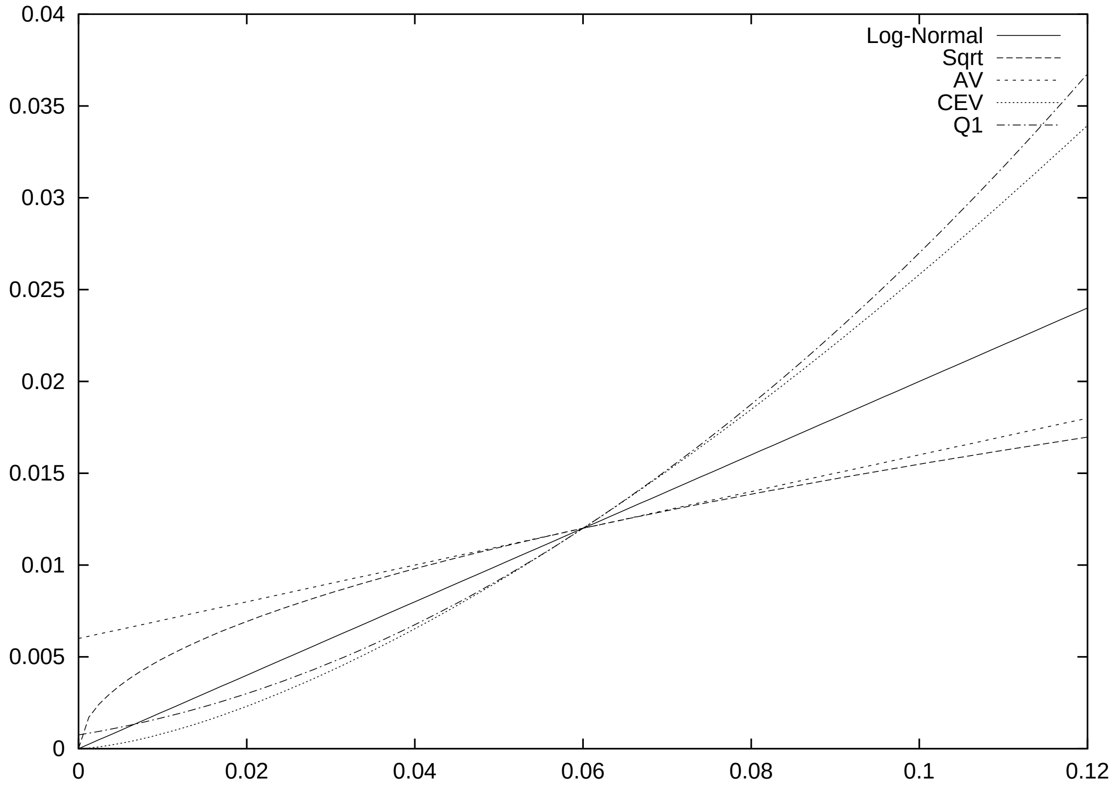

FIGURE 1. The CEV Volatility Functions and their Affine Approximation

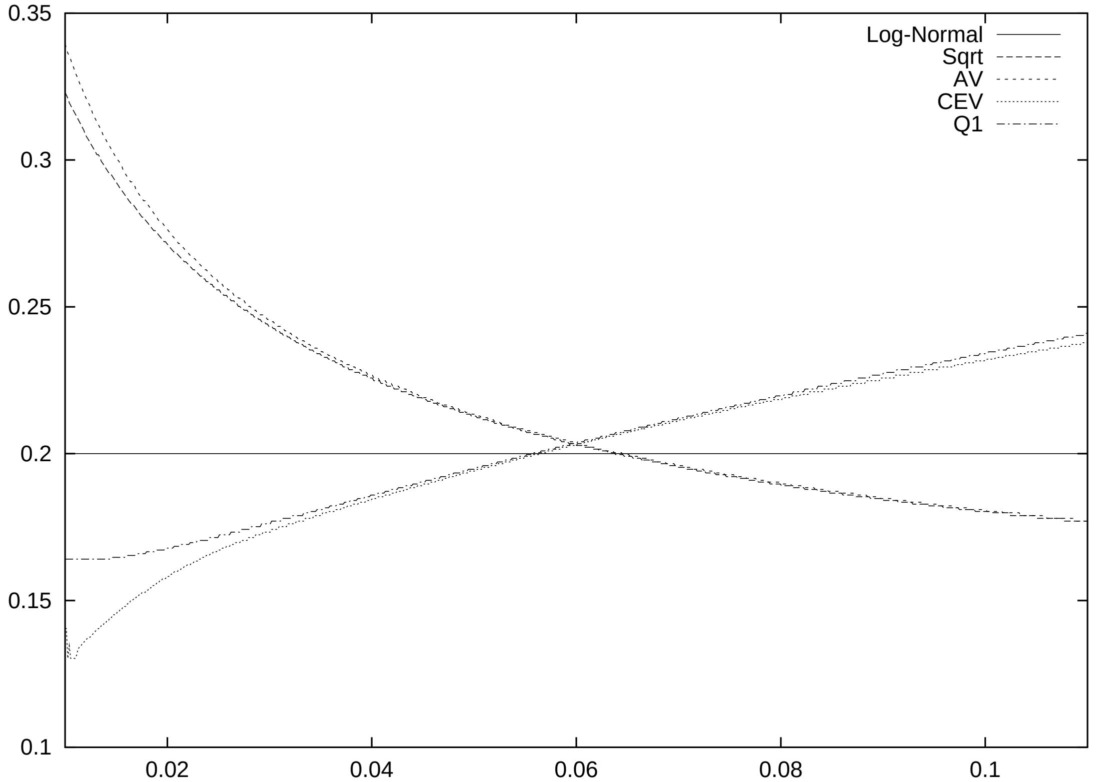

FIGURE 2. Implied Volatilities of CEV specifications and their Approximations

volatilities of quadratic and CEV models. Consider the following specifications of extended volatility:

Lognormal: with $\lambda = 0.2$
Sqrt: a CEV model with $\alpha = 1/2$ and $\lambda = .04899$
AV: affine with $l = 6\%$ and $\lambda = 0.1$
CEV: CEV with $\alpha = 1.5$ and $\lambda = 0.8165$
Q1: quadratic with one root, $l = -6\%$ and $\lambda = 1.875$

Suppose that the initial term structure of forward rates is constant 6%. The parameters for the examples are choosen such that the at-the-money volatility is 12%. The AV model approximates the Sqrt model and the $\mathrm{Q1}$ model the CEV specification. In figures 1 and 2 the volatility functions and their corresponding implied volatilities are plotted for a maturity of one year. The CEV prices and the prices of the $\mathrm{Q1}$ and AV specifications are distinguishable only for strikes far in- and out-of-the-money.

Using the results of an empirical comparison, we will argue in section 5.2 that for practical purposes the parametric form of the volatility function is irrelevant. The only important feature of any parametric form is the number of free parameters.

4. SWAPTION PRICING

A (payer) swap with fixed-leg k is given by the difference in cashflows between a floating investment in the rates L_n against one with the fixed rate k. This contract pays

$$\delta_n \left[L_n(t_n) - k\right] \quad \text{in} \quad t_{n+1} \quad \text{for} \quad n = 1, .., N.$$

The value of the swap is

$$\sum_{n=1}^{N} \delta_n \left(L_n - k\right) P_{n+1} = P_1 - P_{N+1} - kB \qquad \text{for} \quad B = \sum_{n=1}^{N} \delta_n P_{n+1}.$$

The bond B is an annuity which pays δ_n units of currency in t_{n+1} for each $n = 1, .., N$. The (equilibrium) swap rate or par rate κ is the value of k which makes the swap's present value zero:

$$\kappa = \frac{P_1 - P_{N+1}}{B} = \frac{\sum_{n=1}^{N} \delta_n P_{n+1}\, L_n}{B} = L^{\mathbf{t}}\omega$$

if we define weights $\omega_n = \delta_n P_{n+1}/B$. It holds $\mathbf{1}^{\mathbf{t}}\omega = 1$ where $\mathbf{1}$ is the N-dimensional unit vector. The formula shows two important properties of the swap rate:

- The swap rate is a weighted average of Libor rates.
- Under the pricing measure Q^B with numeraire B the swap rate is a local martingale. Jamshidian (1997) calls Q^B the forward swap measure.

A (European payer) swaption with strike k and maturity t_1 is the right to enter a swap with fixed leg k starting in t_1. Its value at time $t \leq t_1$ is therefore

$$B(t)\ E^B\left[\frac{(1-P_{N+1}(t_1)-kB(t_1))^+}{B(t_1)} \mid \mathcal{F}_t\right] = B(t)\ E^B\left[(\kappa(t_1)-k)^+ \mid \mathcal{F}_t\right]$$

where E^B denotes expectation under the the forward swap measure Q^B. In practice, market participants use the log-normal model for the swap as for the forward rates, ie, they price swaptions as if

$$d\kappa = \kappa\, \gamma^{\mathbf{t}} d\widetilde{W}$$

where $\widetilde{W}$ is the Brownian motion associated with the forward swap measure Q^B. This yields the same closed-form solution, the Black formula, for the swaption price as for caplets, only with a different numeraire B which is the forward swap numeraire.

We will now derive the correct stochastic behaviour of the swap rate and then use this to find a well behaved approximation. Write the dynamics of the N-dimensional process $L = (L_1, .., L_N)^{\mathbf{t}}$ as

$$dL = \cdots dt + \Gamma d\widetilde{W}$$

for an $N \times d$–dimensional stochastic process Γ. If we specify an extended market model the components of Γ are given by

$$\Gamma_n^i = p(L_n)\gamma_n^i \qquad \text{for} \quad n = 1, .., N \qquad \text{and} \qquad i = 1, .., d.$$

To determine the volatility of the swap rate κ, we have to compute its derivative with respect to L. The rate κ is a linear function of ω and L, and the weight vector ω is again a function of L. In vector notation the differential of κ with respect to L is an N–dimensional vector

$$\frac{\partial \kappa}{\partial L} = \frac{\partial\left(L^{\mathbf{t}}\omega\right)}{\partial L} = \omega^{\mathbf{t}} + L^{\mathbf{t}}\omega_L \qquad \text{for} \quad \omega_L = \frac{\partial \omega}{\partial L} = \left[\frac{\partial \omega_j}{\partial L_n}\right]_{1\leq n\leq N}^{1\leq j\leq N}$$

or componentwise

$$\frac{\partial \kappa}{\partial L_n} = \omega_n + \sum_{j=1}^{N} L_j \frac{\partial \omega_j}{\partial L_n} \qquad \text{for} \quad n = 1, .., N.$$

Now we can compute the dynamics of the swap rate κ using the multi-dimensional Itô formula:

$$\begin{aligned} d\kappa &= \left(\omega^{\mathbf{t}} + L^{\mathbf{t}}\omega_L\right)\Gamma d\widetilde{W} \\ &= \sum_{i=1}^{d}\left(\omega^{\mathbf{t}} + L^{\mathbf{t}}\omega_L\right)\Gamma^i\, d\widetilde{W}^i \\ &= \sum_{i=1}^{d}\sum_{n=1}^{N}\left(\omega_n + \sum_{j=1}^{N} L_j \frac{\partial \omega_j}{\partial L_n}\right)\Gamma_n^i\, d\widetilde{W}^i \\ &= (G+H)^{\mathbf{t}} d\widetilde{W} \end{aligned}$$

for d-dimensional processes

$$G^{\mathbf{t}} = \omega^{\mathbf{t}}\Gamma \qquad \text{and} \qquad H^{\mathbf{t}} = L^{\mathbf{t}}\omega_L\Gamma.$$

This is the correct dynamics of κ under Q^B. To find a tractable approximation, we will first examine the second term H of its volatility with components

$$H^i = L^{\mathbf{t}}\omega_L\Gamma^i = \sum_{n=1}^{N}\sum_{j=1}^{N} L_j \frac{\partial\omega_j}{\partial L_n}\Gamma_n^i$$

As

$$\mathbf{1}^{\mathbf{t}}\omega = 1$$

the differential of this expression as a function of L is zero, therefore it holds

$$\mathbf{1}^{\mathbf{t}}\omega_L g = 0 \qquad \text{for all} \quad g \in \mathrm{I\!R}^N.$$

This implies that if we add (or subtract) a certain level c to the Libor curve

$$\overline{L} = L + c\mathbf{1} = (L_1 + c, \dots, L_N + c)^{\mathbf{t}}$$

we get the same value

$$\overline{L}^{\mathbf{t}}\omega_L g = (L + c\mathbf{1})^{\mathbf{t}}\omega_L g = L^{\mathbf{t}}\omega_L g + c\mathbf{1}^{\mathbf{t}}\omega_L g = L^{\mathbf{t}}\omega_L g$$

for all directions $g \in \mathrm{I\!R}^N$. This means that the second term H of the swap rate volatility does not depend on the forward rate curve but on the deviation of the forward curve from a constant level. In particular, for a flat forward Libor curve the term vanishes. This fact does not depend on the special model we have choosen, the reasoning is independent of the form of Γ. As a first approximation of the swap rate dynamics we will just omit the second term H.

Now assume a deterministic extended market model, ie

$$\Gamma_n^i = p(L_n)\gamma_n^i.$$

Then the first component of the swap rate volatility, the d–dimensional process G, is a weighted sum of the Libor rate factors γ_n:

$$G^{\mathbf{t}} = \omega^{\mathbf{t}}\Gamma = \sum_{n=1}^{N}\omega_n p(L_n)\gamma_n^{\mathbf{t}} = p(\kappa)\sum_{n=1}^{N}\frac{\omega_n p(L_n)}{p(\kappa)}\gamma_n^{\mathbf{t}}$$

For this first term G of the swap rate dynamics, we propose an approximation which fixes the weights at the beginning of the life of the swaption:

$$G^{\mathbf{t}} \approx p(\kappa)\theta^{\mathbf{t}}\gamma = p(\kappa)\sum_{n=1}^{N}\theta_n\gamma_n^{\mathbf{t}} \qquad \text{for} \quad \theta_n = \frac{\omega_n(t_0)p(L_n(t_0))}{p(\kappa(t_0))}$$

where γ is the $N \times d$ matrix

$$\left(\gamma_n^i\right)_{1\le n\le N}^{1\le i\le d}.$$

The approximation is by construction exact in t_0. In their empirical analysis of the lognormal Libor market model, De Jong et al. (2001) state that *although swaption prices do depend on* *the correlation between interest rates of different maturities, this turns out to be a second order effect; swaption prices are primarily determined by the volatilities of interest rates.* Our approximation of G uses this observation by fixing the weights, which define the time dependent correlation, to be constant. The approximative swap rate dynamics

$$d\kappa = p(\kappa)\ \theta^{\mathbf{t}}\gamma\ d\widetilde{W}$$

can be used to pricing swaptions with the formulae from section 2, numeraire B, and

$$\tau_\kappa^2(t) = \int_t^{t_1} \|\theta^{\mathbf{t}}\gamma(u)\|^2\ du.$$

It is a stylized fact of the empirics of the term structure that the average level of the curve accounts for most of its variance. A first simple market model is therefore

- a one-facor model with
- a flat covariance factor $\gamma_n^1(t) = \gamma(t)$, and
- an affine volatility function $p(L) = L - l$.

This one-factor model has two inputs, its left boundary l and the level γ which may be time dependend. Notice that for this specification the approximation of G is exact: as for an affine function the weights sum to one

$$\sum_{n=1}^{N} \frac{\omega_n(t)\ p(L_n(t))}{p(\kappa(t))} = \frac{\sum_{n=1}^{N} \omega_n(t) L_n(t) - l}{p(\kappa(t))} = 1$$

it holds

$$\sum_{n=1}^{N} \frac{\omega_n(t)\ p(L_n(t))}{p(\kappa(t))}\gamma(t) = \sum_{n=1}^{N} \frac{\omega_n(t_0)\ p(L_n(t_0))}{p(\kappa(t_0))}\gamma(t) = \sum_{n=1}^{N} \theta_n \gamma(t) = \gamma(t).$$

Combining these observations, we see that if

- the initial term structure is flat, and
- we use an affine one-factor model with a flat (possibly time-dependent) factor

the proposed approximative swap rate dynamics is exact.

5. Empirical Results

In this section, we use data provided by Westdeutsche Landesbank (WestLB) on their website

`www.westlb.de/swaps/`

to calibrate different specifications of the extended market models. We use data for 385 days starting January 1, 1998 to July 15, 1999. For the calibrations the complete data set was used, no cleaning procedure (check for outliers, etc.) was performed.

5.1. **Calibration.** In this subsection, we calibrate different models to the data. As our main interest is how the calibration is influenced by the choice of different volatility functions, we use

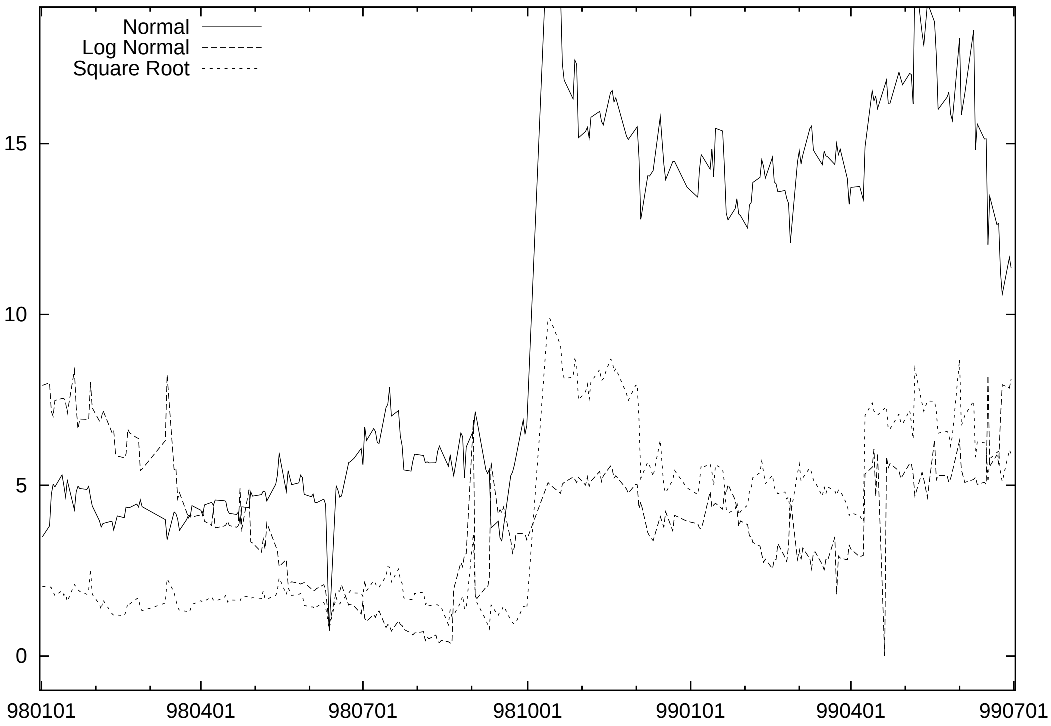

FIGURE 3. Fit to Caps 1998-01-01 / 1999-07-15

only a single flat covariance factor. The fit of all specifications could be improved considerably by making the factor time-dependent or by using a multi-factor model. The model is

$$dL = \cdots dt + p(L)\gamma dW.$$

The number of parameters of the model is one (for γ) plus the number of parameters for the specification of p. For example, the lognormal specification is a one-parameter model, the $\mathrm{Q2}$ specification is a two-parameter model.

First, we have to define how to measure the deviation of the models prices from the observed ones. Prices are quoted in basispoints. WestLB provides bid/ask prices and we take into account this additional information by considering only the deviation from the bis/ask bounds. Denote the observed cap ask price by a, the bid by b and the model cap price by c. For a given date, we minimize the distance d defined by

$$d = \sqrt{\frac{1}{\#\{b/a\}} \sum_{b/a} \max\{a - c, c - b, 0\}^2}$$

where the sum is over all bid/ask cap prices. The metric is such that any price which lies in the bid/ask spread counts as a perfect fit to the data. The sum is normalized by the number of bid/ask prices which is denoted by $\#\{b/a\}$. The quadratic metric was used as it is standard in the literature, see e.g. Amin and Morton (1994) or Christiansen and Struck Hansen (2001). We use the Powell algorithm for minimizing functions without computing derivatives, see section 10.6 in Press, Teukolsky, Vetterling and Flannery (1992).

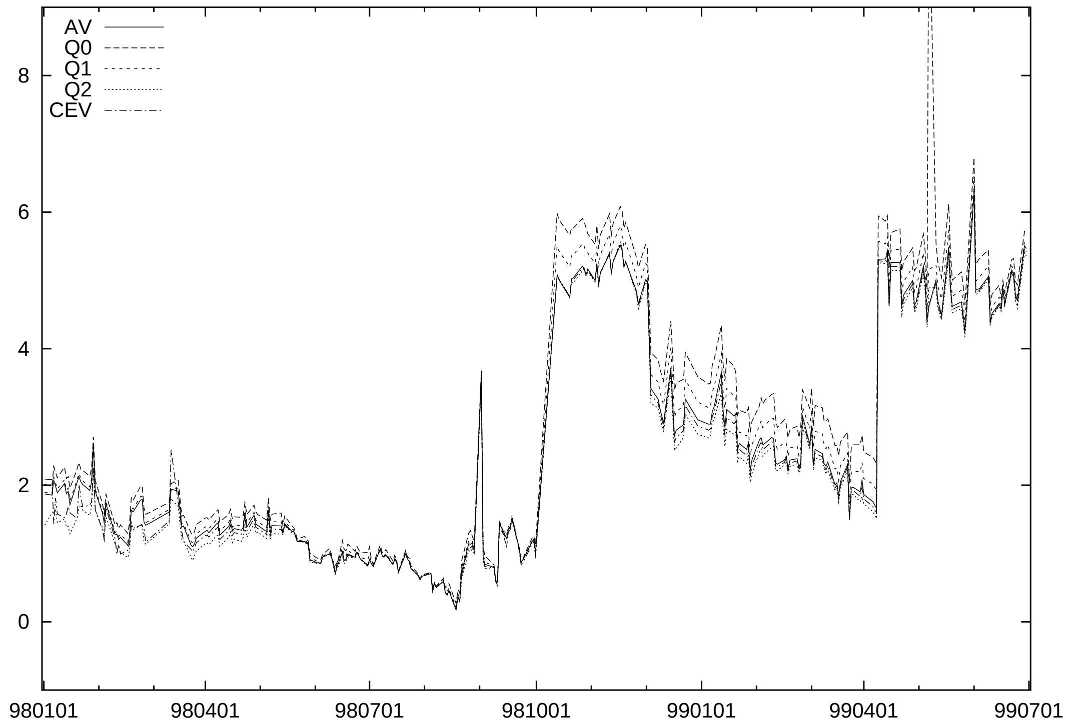

FIGURE 4. Fit to Caps 1998-01-01 / 1999-07-15

Figure 3 plots the fit of three simple models: Lognormal, Square-root (CEV with $\alpha = .5$ fixed) and Normal (Bachelier/constant volatility). We see that in the beginning of 1998 there is a big frown in the data as the Bachelier and the square-root specification fit the data better than the log-normal model. The situation reverses at the end of the year 1998 and during the beginning of 1999 where the Bachelier model performs very bad and the lognormal one is the best fit.

Figure 4 plots the fit of four extended models, AV (affine volatility), Q0 (quadratic with no root, the minimum of the parabola constraint to zero: $m = 0$), Q2 (quadratic with two roots), and CEV. Obviously, one degree of freedom is sufficient to fit the smile in the data. All four models perform equally well which leeds us to conclude that for a better fit one should use a more elaborate form of the covariance factors. Figure 5 plots the fit of the lognormal and the AV model to show the improvement offered by one additionel free parameter in the calibration. As the lognormal specification is a special case of the affine one, it does always fit the data better.

Now we investigate the approximative swap rate dynamics proposed in section 4. This approximation relies on the theoretical reasoning that the second term H of the swap rate dynamics should be small compared with the first term G. To verify this for the data used here, we took the AV model calibrated to the caps data on April 19, 1999. As we have only one flat factor, we are able to calculate the exact volatility of all swap rates using the result of section 4.

$$G = p(\kappa)\gamma \qquad H = \gamma \sum_{n=1}^{N} \sum_{j=1}^{N} L_j \, \frac{\partial \omega_j}{\partial L_n}$$

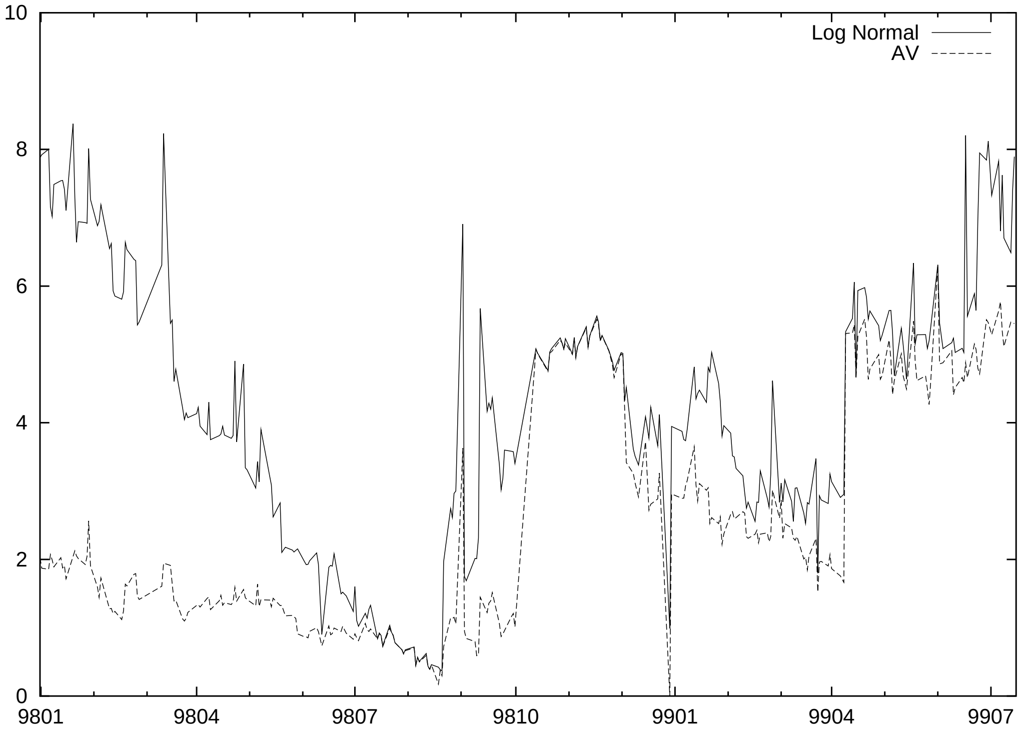

FIGURE 5. Fit to Caps 1998-01-01 / 1999-07-15

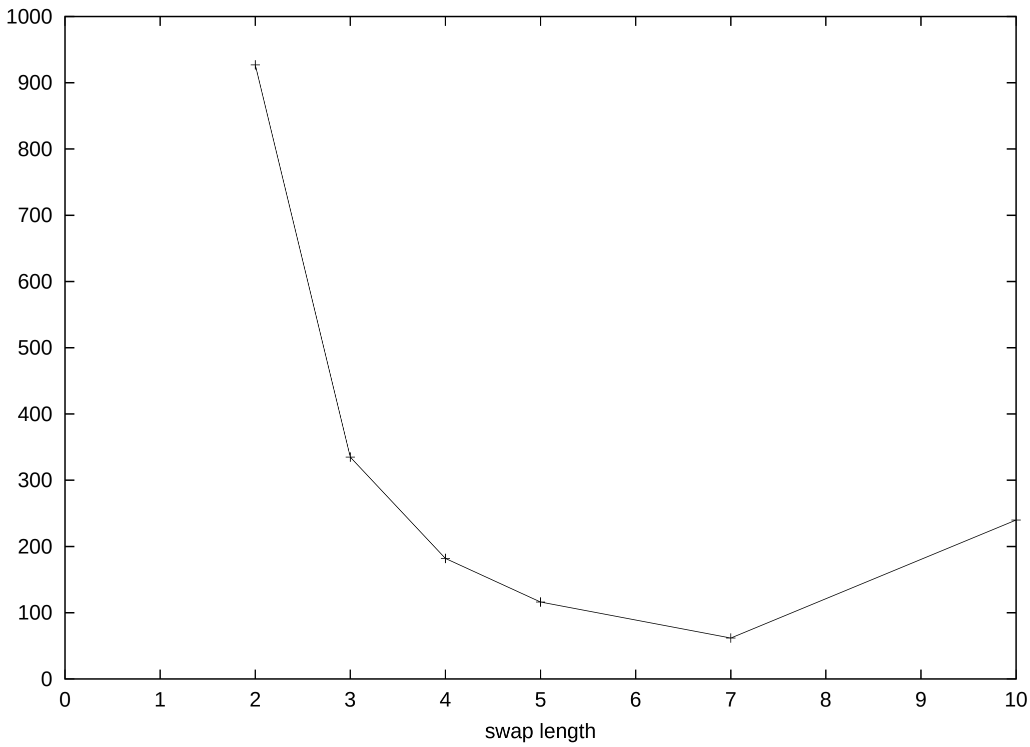

FIGURE 6. Quotient G/H on 1999-04-19

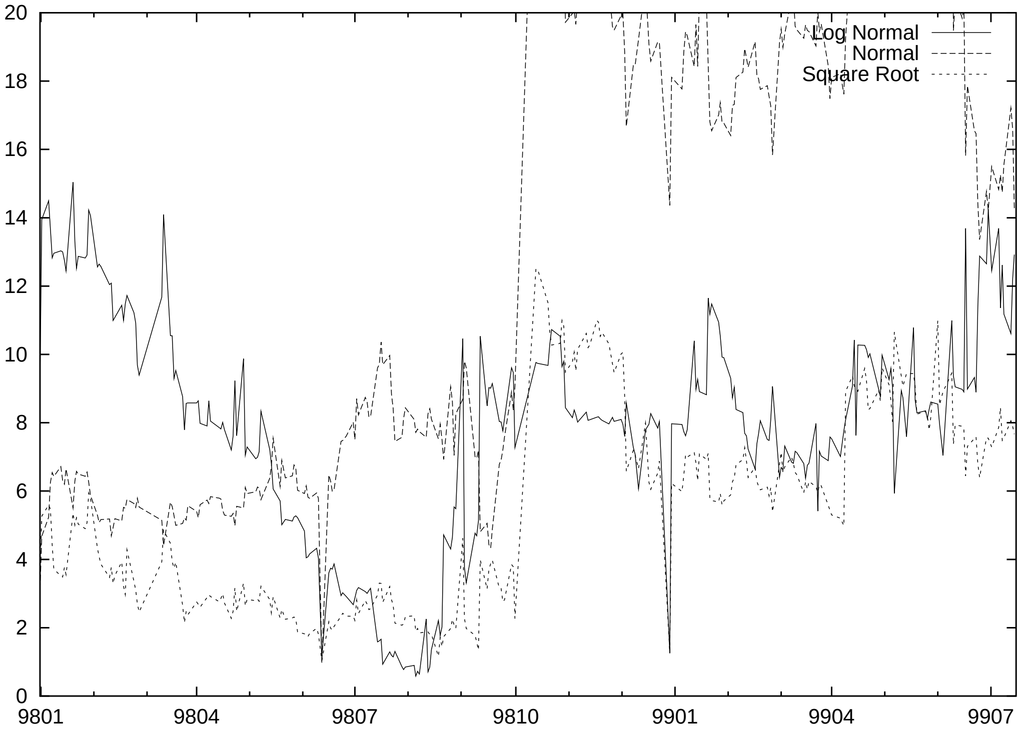

FIGURE 7. Fit to Swaptions 1998-01-01 / 1999-07-15

Given the calibrated parameters, figure 6 plots the factor

$$G/H = \frac{p(\kappa)}{\sum_{n=1}^{N}\sum_{j=1}^{N} L_j \frac{\partial \omega_j}{\partial L_n}}$$

by which the first term G is bigger than the second term H for swap of lengths up to ten years. As expected, we find that the omission of the second term is of no importance: even for a seven year swap the first term is still 62 times the value of the second one. The second term depends on the deviation of the forward curve from the constant level, so it is smaller for a shorter swap length. Notably, the swap rate with length one is equivalent to the spot Libor rate, so the approximation is exact, the second term H is zero, and the quotient is infinite.

After fitting the models to the caps data, we used the calibrated parameters to evaluate the fit to the swaption data using the approximative pricing formulae. That is, the parameters were not fitted to swaptions. We wanted to check wether a model calibrated to caps does also fit swaptions reasonably well, so we took the parameters of the caps fit and then evaluated the approximate price of the swaptions. Figures 7, 8, and 9 plot the fit of the models to swaption prices. All the statements for the fit to caps carry over to the swaption approximation, we observe the same smile and the fact that no extended model fits the data significantly better.

5.2. **Non-parametric Calibration.** In the their careful empirical study of the Libor market and the swap market model, De Jong et al. (2001) find systematic pricing errors which they explain by yield-spread and yield-curvature parameters. They argue that *it is more likely that*

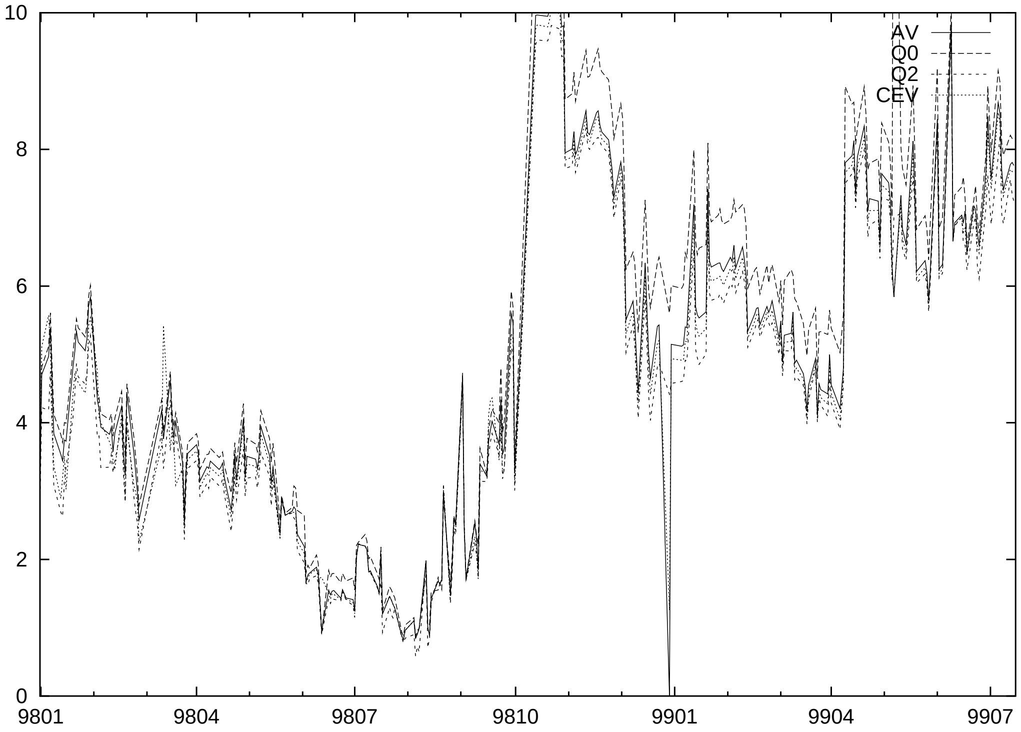

FIGURE 8. Fit to Swaptions 1998-01-01 / 1999-07-15

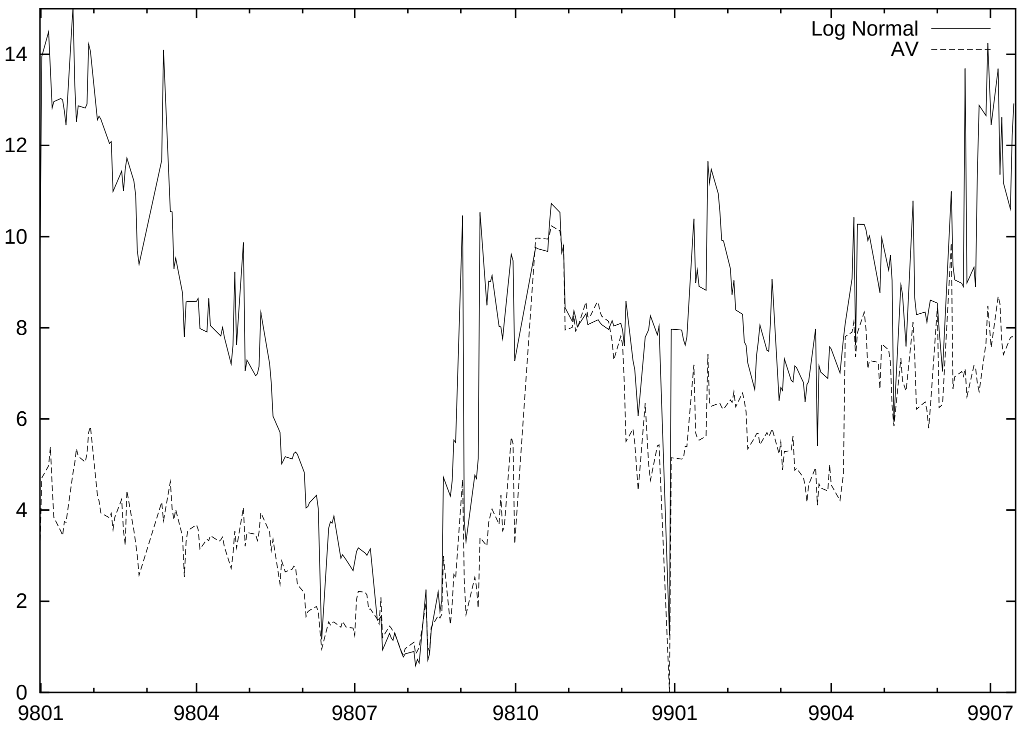

FIGURE 9. Fit to Swaptions 1998-01-01 / 1999-07-15

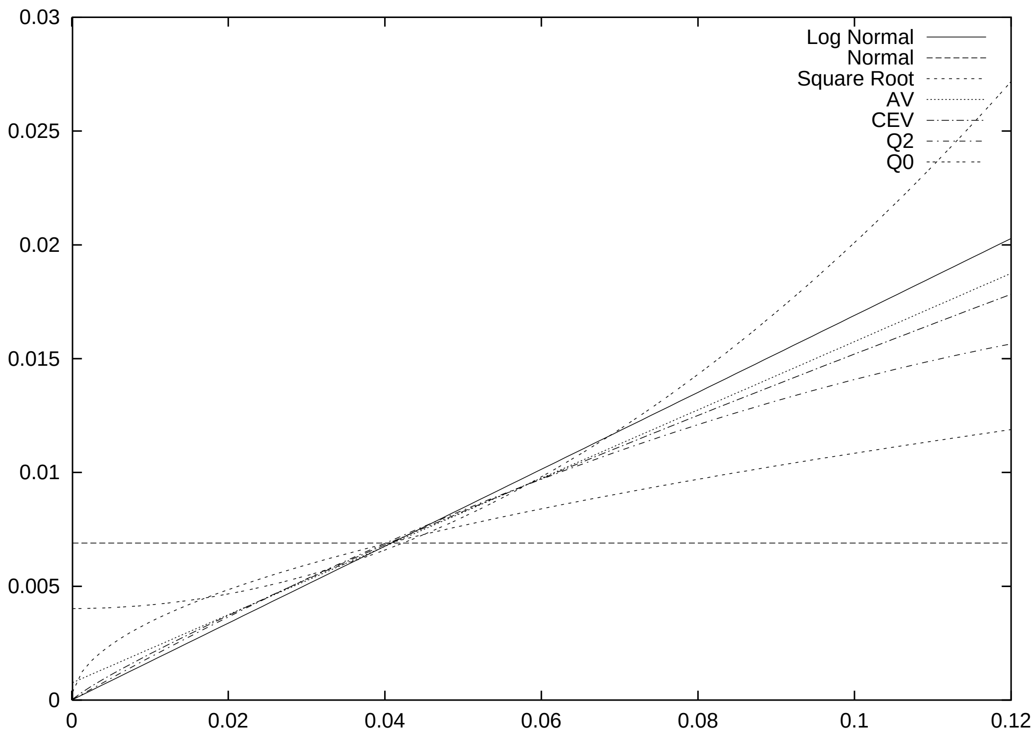

FIGURE 10. Fitted Volatility Functions 1999-04-19

the correlation between pricing errors and yield-curve variables is the result of misspecified volatility functions, and, in particular, the assumption of lognormally distributed interest rates.

In this subsection, we will argue that in fact the fit of a model does not depend on its parametric form. For every specification we see that independent of its parametric form the first free parameter of the model is fitted to a specific volatility value. If a second free parameter is available, it is fitted to a specific slope.

Figures 10 plots the volatility of the fitted models for April 19, 1999. Obviously, the calibration of any model finds the same value for instantaneous volatility (0.068) at a certain critical Libor level (4.1%). For specifications which have an second free parameter (AV, CEV, $\mathrm{Q2}$, $\mathrm{Q0}$) the calibration procedure does also find the same slope (0.14) at that level.

To show that this fact holds for the whole data set, we first define the critical Libor rate: it is computed as the intersection point of the calibrated volatility with the calibrated Bachelier volatility γ^{Bac}. Take for example the affine model.

$$\gamma^{\mathrm{AV}}(L^{\mathrm{AV}} - l^{\mathrm{AV}}) = \gamma^{\mathrm{Bac}} \quad \Longleftrightarrow \quad L^{\mathrm{AV}} = \frac{\gamma^{\mathrm{Bac}}}{\gamma^{\mathrm{AV}}} + l^{\mathrm{AV}}$$

where $\gamma^{\mathrm{AV}}, l^{\mathrm{AV}}$ and γ^{Bac} are the calibrated model parameters. The critical point L^{AV} is a kind of implied at-the-money point of the calibrated model. Figure 11 plots the critical Libor value for three two-parameter specifications for all observation dates and figure 12 plots the slope at that point. The plots confirm that the calibration procedure fits the same empirical volatility

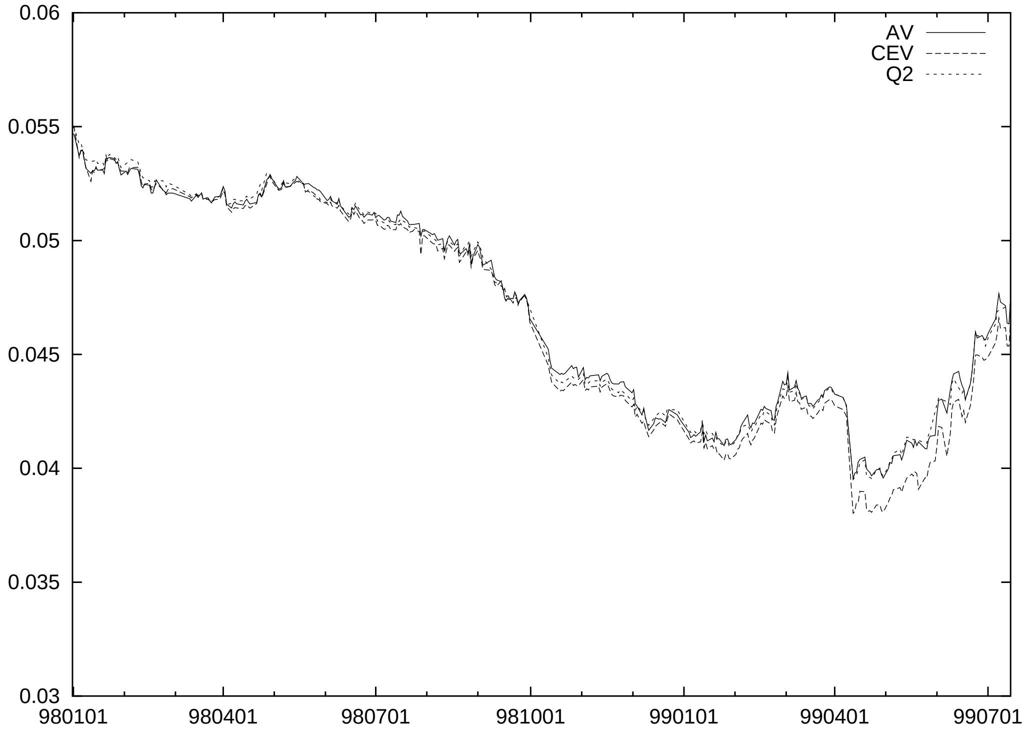

FIGURE 11. Intersection Point with Bachelier Specification

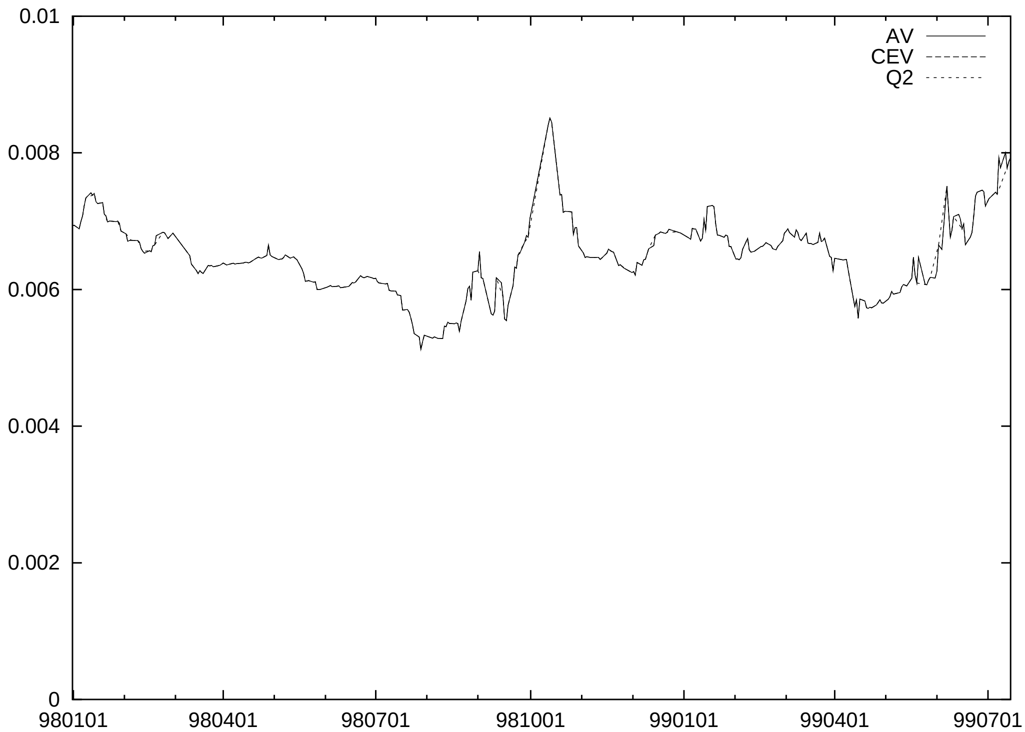

FIGURE 12. Slope at Intersection Point

function independently of its parametric form. This holds especially for the slope which is indistinguishable for the different calibrations.

Further, we see that the slope is less than one for the whole time interval from January 1, 1998, to July 15, 1999. This indicates systematic deviation from the log-normal market model to specifications with smaller slope for the volatility function. This corresponds to a frown in implied volatilities. De Jong et al. (2001) obtained analoguous results. They estimated values around 0.7 for the exponent parameter of the CEV market model.

6. Conclusion

The class of extended Libor market models offers a great variety of term structure models which allow for closed-form solutions for cap prices as well as theoretically and empirically well-founded approximations for swaption prices. They fit observed market prices better than the log-normal model without loosing tractability.

From a practical point of view all extended models perform equally well. The one-factor AV specification fulfils the swaption pricing assumptions exactly, so it theoretically offers the best consistency of cap and swaption prices within one market model specification.

AV, $\mathrm{Q1}$ and $\mathrm{Q2}$ models are easier to implement as these specifications give analytical closed-form solutions to the pricing equations. For the CEV model we have to compute the non-central chi-square distribution function for noncentrality values up to $600-800$ which is difficult to implement. For the $\mathrm{Q0}$ model we have to implement a sine expansion which involves the calculation of several hundred sine values.

The empirical results indicate that a one-parameter volatility function offers enough degrees of freedom to capture the smiles in the data offered by WestLB. The calibration of a set of different parametric market models to the data revealed that the fit does not depend on a certain parametric form but on the volatility value and slope at a certain critical Libor rate. For a better fit we would have to consider more elaborate covariance structures. In the log-normal case De Jong et al. (2001) argue that a mean-reverting one-factor model

$$\gamma_i(t) = \gamma(t - t_i) = \gamma_0 e^{-\kappa(t_i - t)} \qquad \gamma_0, \kappa \geq 0$$

is a good choice for their data (US term structure July 1995 - September 1996). Christiansen and Struck Hansen (2001) investigate the log-normal market model using T-bill options and find no significant differences in the properties of three different covariance factor specifications. Further empirical research could include studies like Amin and Morton (1994) or Bühler, Uhrig-Homburg, Walter and Weber (1999) on the hedging properties of the different models.

Combining the observations and results, we conclude that the affine Libor market model is the most interesting extended model for further research: it is as fast to evaluate as the standard lognormal model and it offers a theoretically well-founded approximative swaption pricing formula.

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APPENDIX A. PROOFS FOR SECTION 1

Itô's formula. We use the following version of the multi-dimensional Itô formula, see e.g. Karatzas and Shreve (1991, 3.3.6): if

$$dX_n = \mu_n \, dt + \gamma_n^{\mathbf{t}} dW \qquad 1 \le n \le N$$

then for any function $F \in \mathcal{C}^2(\mathbb{R}^N)$ it holds

$$\begin{aligned} dF(X) &= \sum_{n=1}^{N} \left(F_n(X)\, dX_n + \frac{1}{2} \sum_{j=1}^{N} F_{nj}(X)\, d\langle X_n, X_j \rangle \right) \\ &= \sum_{n=1}^{N} \left(F_n(X)\, dX_n + \frac{1}{2} F_{nn}(X)\, d\langle X_n, X_n \rangle + \sum_{j=n+1}^{N} F_{nj}(X)\, d\langle X_n, X_j \rangle \right) \end{aligned}$$

with

$$F_n = \frac{\partial F}{\partial x_n} \qquad \text{and} \qquad F_{nj} = \frac{\partial^2 F}{\partial x_n \partial x_j} \, .$$

The equality holds because $\langle X_n, X_j \rangle = \langle X_j, X_n \rangle$ and $F_{nj} = F_{jn}$.

A.1. **Proof of Lemma 1.1.** The derivatives of the Ds with respect to the Libor rates are given by:

$$D_n = \frac{P_n}{P_m} = \begin{cases} \prod_{j=n}^{m-1} (1 + \delta_j L_j) & n < m \\ \prod_{j=m}^{n-1} (1 + \delta_j L_j)^{-1} & m < n \\ 1 & m = n \end{cases}$$

$$\frac{\partial D_n}{\partial L_k} = \frac{\delta_k D_n}{1 + \delta_k L_k} \begin{cases} 1 & n \le k < m \\ -1 & m \le k < n \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial^2 D_n}{\partial^2 L_k} = 2 \frac{\delta_k^2 D_n}{(1 + \delta_k L_k)^2} \begin{cases} 1 & m \le k < n \\ 0 & \text{otherwise} \end{cases}$$

and for $l \neq k$

$$\frac{\partial^2 D_n}{\partial L_k \partial L_l} = \frac{\delta_k \delta_l D_n}{(1 + \delta_k L_k)(1 + \delta_l L_l)} \begin{cases} 1 & n \le k, l < m \quad \text{or} \quad m \le k, l < n \\ 0 & \text{otherwise} \end{cases}$$

For $n = m$ obviously

$$dD_n = 0.$$

The multi-dimensional Itô formula gives for $n < m$

$$\begin{aligned}\frac{dD_n}{D_n} &= \sum_{k=n}^{m-1}\left(\frac{\delta_k}{1+\delta_k L_k}dL_k + \sum_{l=k+1}^{m-1}\frac{\delta_k\delta_l}{(1+\delta_k L_k)(1+\delta_l L_l)}\langle L_k, L_l\rangle\right)\\ &= \sum_{k=n}^{m-1} q(\delta_k, L_k)\ \gamma_k^{\mathsf{t}}\left[dW^m - \sum_{l=k+1}^{m-1} q(\delta_l, L_l)\ \ \gamma_l + \sum_{l=k+1}^{m-1} q(\delta_l, L_l)\ \gamma_l\right]\\ &= \sum_{k=n}^{m-1} q(\delta_k, L_k)\ \ \gamma_k^{\mathsf{t}} dW^m.\end{aligned}$$

For $m < n$ follows analogously

$$\frac{dD_n}{D_n} = -\sum_{k=m}^{n-1} q(\delta_k, L_k)\ \gamma_k^{\mathsf{t}} dW^m.$$

So the discounted asset prices are local martingales for the specified dynamics of the Libor rates which shows that the model is arbitrage-free. □

A.2. **Proof of Proposition 1.2.** The condition

$$L_n \geq 0 \qquad \text{for all} \quad n$$

is equivalent to

$$D_1 \geq \cdots \geq D_N \geq 1 \qquad \text{as} \qquad D_n = \frac{P_n}{P_{N+1}}.$$

We prove the strong existence of the D_n recursively and write W for W^{N+1}.

Start $n = N$**:** It has to hold

$$dD_N = D_N q(\delta_N, L_N)\gamma_N^{\mathsf{t}} dW = \delta_N p\left(\frac{D_N - 1}{\delta_N}\right)\gamma_N^{\mathsf{t}} dW.$$

By the general existence proposition of Zühlsdorff (1998) we know that under our assumptions this SDE has a strong non-exploding solution. If at some point in time D_N attains one, we take it to be absorbed there. This is eqivalent to absorption of L_N in zero.

Step $n < N$**:** Given the strong solutions L_j for $j > n$, define for any bound $b > 0$ the stopping time

$$\theta_b = \inf\{t \mid \max\left(L_{n+1}, .., L_N\right) \geq b\}$$

and the measurable set

$$\Omega_b = \{\theta_b \leq T\}.$$

The Libor rates $L_{n+1}, .., L_N$ are bounded by b for $t \leq \theta_b$. By strong existence of the L_j for all $j > n$ we know that

$$Q^{N+1}\left(\Omega_b\right) \xrightarrow[b\to\infty]{} 1.$$

We will show that the solution for D_n exists on each of the sets Ω_b and thus globally. The dynamics of D_n is

$$dD_n = \left(\delta_n D_{n+1} p\left(\frac{D_n - D_{n+1}}{\delta_n D_{n+1}}\right)\gamma_n + D_n \sum_{j=n+1}^{N} q(\delta_j, L_j)\,\gamma_j\right)^{\mathbf{t}} dW$$

The second term is Lipschitz-continuous on the set Ω_b by construction:

$$\left\| x \sum_{j=n+1}^{N} q(\delta_j, L_j)\,\gamma_j - y \sum_{j=n+1}^{N} q(\delta_j, L_j)\,\gamma_j \right\| \le C|x-y| \max_{0\le L\le b} p(L) \sum_{j=n+1}^{N} \delta_j \qquad \text{a.s.}$$

As we have $D_{n+1} \le D_n$, for

$$1 \le D_{n+1} \le x, y \le m$$

the first term is locally Lipschitz

$$\left\| \delta_n D_{n+1} p\left(\frac{x - D_{n+1}}{\delta_n D_{n+1}}\right)\gamma_n - \delta_n D_{n+1} p\left(\frac{y - D_{n+1}}{\delta_n D_{n+1}}\right)\gamma_n \right\| \le \delta_n m \left\|\gamma_n\right\| K_m |x-y|$$

where K_m denotes the Lipschitz-constant for p on $0 \le x, y \le m$. Thus we can define D_n on each Ω_b as the solution to the SDE which fulfils the conditions of the existence proposition from Zühlsdorff (1998). If at some point in time $D_n = D_{n+1}$ we take it to be equal to D_{n+1} afterwards which is equivalent to absorption in zero of L_n. □

Appendix B. Coefficients of the Q0 pricing formula

We are in the case of of the general quadratic model with no real roots and boundary $(0, \infty)$, see Zühlsdorff (1998, App. B). The coefficients are defined by:

$$A = \frac{\pi}{2} \qquad a = \arctan(-m/d) \qquad Z(x) = \arctan\left(\frac{x-m}{d}\right) - a$$

$$K = Z(k) \qquad U = Z(u) = \frac{\pi}{2} - a$$

$$S(n,z) = \int \sin(z+a)\sin_n(z)\,dz = -\frac{\nu_n^+ \sin(a - \nu_n^- z) + \nu_n^- \sin(a + \nu_n^+ z)}{2q_n}$$

$$C(n,z) = \int \cos(z+a)\sin_n(z)\,dz = -\frac{\nu_n^+ \cos(a - \nu_n^- z) + \nu_n^- \cos(a + \nu_n^+ z)}{2q_n}$$

$$\nu_n^\pm = \frac{n\pi}{U} \pm 1 \qquad q_n = \frac{n^2\pi^2}{U^2} - 1 = \nu_n^+ \nu_n^-$$

$$\begin{aligned}
a_n &= \left((m-k)S_n - dC_n\right)\Big|_{Z(k)}^{d\pi} \\
b_n &= n(-1)^{n+1}\int_0^t e^{(n^2-1)\lambda/2d^2}\,d\lambda = \begin{cases} t & n = 1 \\ n(-1)^{n+1} d^2 \frac{e^{(n^2-1)t/2d^2} - 1}{n^2-1} & n > 1 \end{cases} \\
c_n &= a_n + b_n
\end{aligned}$$