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Default Risk, Bankruptcy Procedures and the Market Value of Life Insurance Liabilities

by

An Chen, Michael Suchanecki

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Bonn Graduate School of Economics
Department of Economics
University of Bonn
Adenauerallee 24 - 42
D-53113 Bonn

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DEFAULT RISK, BANKRUPTCY PROCEDURES AND THE MARKET VALUE OF LIFE INSURANCE LIABILITIES

AN CHEN*
BONN GRADUATE SCHOOL OF ECONOMICS
UNIVERSITY OF BONN

MICHAEL SUCHANECKI‡
DEPARTMENT OF STATISTICS
UNIVERSITY OF BONN

Abstract. The topic of insolvency risk in connection with life insurance companies has recently attracted a great deal of attention. In this paper, the question is investigated of how the value of the equity and of the liability of a life insurance company are affected by the default risk and the choice of the relevant bankruptcy procedure. As an example, the U.S. Bankruptcy Code with Chapter 7 and Chapter 11 bankruptcy procedures is used. Grosen and Jørgensen's (2002) contingent claim model, implying only a Chapter 7 bankruptcy procedure, is extended to allow for more general bankruptcy procedures such as Chapter 11. Thus, more realistically, default and liquidation are modelled as distinguishable events. This is realized by using so–called standard and cumulative Parisian barrier option frameworks. It is shown that these options have appealing interpretations in terms of the bankruptcy mechanism. Furthermore, a number of representative numerical analyses and comparative statics are performed in order to investigate the effects of different parameter changes on the values of the insurance company's equity and liability, and hence on the value of the life insurance contract. To complete the analysis, the shortfall probabilities of the insurance company implied by the proposed models are computed and compared.

Keywords: Equity–Linked Life Insurance, Default Risk, Liquidation Risk, Contingent Claims Pricing, Parisian Options, Bankruptcy Procedures
JEL: G13, G22, G33
Subject and Insurance Branch Codes: IM10, IE10, IE50, IB10

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* (Corresponding author) Bonn Graduate School of Economics, University of Bonn, Adenauerallee 24–26, 53113 Bonn, Germany, Phone: +49–228–739229, Fax: +49–228–735048, E–mail: an.chen@uni–bonn.de.
‡ Department of Statistics, University of Bonn, Adenauerallee 24–26, 53113 Bonn, Germany, E–mail: michael.suchanecki@uni–bonn.de.

1. Introduction

The topic of insolvency risk in connection with life insurance companies has recently attracted a great deal of attention. Since the 1980s a long list of defaulted life insurance companies in Europe, Japan and the United States has been reported. A few examples from the United States are First Farwest Corp., Integrated Resource Life Insurance Co. and Pacific Standard Life Insurance Co. in 1989, Mutual Security Life Insurance Co. in 1990, First Executive Life Insurance Co. (this constituted the 12th largest bankruptcy in the United States in the period 1980–2005), First Stratford Life Insurance Co., Executive Life Insurance Company of New York, Fidelity Bankers Life Insurance Co., First Capital Life Insurance Co., Mutual Benefit Life Insurance Co. and Guarantee Security Life Insurance Co. in 1991, Fidelity Mutual Life Insurance Co. in 1992, Summit National Life Insurance Co., Monarch Life Insurance Co. and Confederation Life Insurance Co. in 1994, ARM Financial Group in 1999, Penn Corp. Financial Group in 2000, Conseco Inc. in 2002 (this constituted the 3rd largest bankruptcy in the United States in the period 1980–2005)[1] and Metropolitan Mortgage & Securities in 2004.[2] Table 1 provides more detailed information on the bankruptcy procedure and the number of days spent in default for some exemplary bankruptcies of life insurance companies in the United States.[3]

In Japan, the following life insurance carriers defaulted: Nissan Mutual Life in 1997, Chiyoda Mutual Life Insurance Co. and Kyoei Life Insurance Co. in 2000 and Tokyo Mutual Life Insurance in 2001. In Europe, there were the following most noticeable insolvency cases: Garantie Mutuelle des Fonctionnaires in France in 1993, the world's oldest life insurer Equitable Life (in the end only saved by a House of Lords' ruling) in the United Kingdom in 2000 and Mannheimer Leben (failed a fair value based solvency test, but recovered) in Germany in 2003. As the biggest corporate bankruptcy in Australia, HIH Insurance defaulted in 2001, mainly because of the inability to correctly estimate its liabilities (see Jørgensen (2004)).

Hence, it is worth having a close look at the bankruptcy procedures. We take the United States' Bankruptcy Code as an example. Similar bankruptcy laws are also applied in Japan and in France. In the U.S. Bankruptcy Code there are two possible procedures: Chapter 7 and Chapter 11 bankruptcy. It is generally assumed that a firm is in financial distress when the value of its assets is lower than the default threshold. With Chapter 7 bankruptcy, the firm is liquidated immediately after default, i.e., no renegotiations or reorganizations are

[1] Data taken from http://www.bankruptcydata.com.

[2] The life insurance insolvency cases up to 1994 are taken from the 1999 Special Comment "Life After Death" by Moody's Investors Service, Global Credit Research, available at http://www.moodys.com. The other cases are taken from the source mentioned in Footnote 1.

[3] These data are taken from Lynn M. LoPucki's Bankruptcy Research Database, http://lopucki.law.ucla.edu/index.htm.

American defaulted companies	Year	Bankruptcy code	Days spent in default
Executive Life Insurance Co.	1991	Ch. 11	462
First Capital Life Insurance Co.	1991	Ch. 11	1669
Monarch Life Insurance Co.	1994	Ch. 11	392
ARM Financial Group	1999	Ch. 11	245
Penn Corp. Financial Group	2000	Ch. 11	119
Conseco Inc.	2002	Ch. 11	266
Metropolitan Mortgage & Securities	2004	Ch. 11	n/a

TABLE 1. Some defaulted insurance companies in the United States.

possible. With Chapter 11 bankruptcy, first the reality of the financial distress is checked before the firm is definitively liquidated, i.e., the defaulted firm is granted some "grace" period during which a renegotiation process between equity and debt holders may take place and the firm is given the chance to reorganize. If, during this period, the firm is unable to recover then it is liquidated. Hence, the firm's asset value can cross the default threshold without causing an immediate liquidation. Thus, the default event is only signalled. For the above mentioned cases from the United States for which data were available, all of the life insurance companies filed for Chapter 11 bankruptcy and the "grace" period lasted from 119 days up to 1669 days. Such a bankruptcy procedure with a given "grace" period does not only exist in the United States, but also in Japan and in France. In France, a legal 3–month observation period before a possible liquidation is systematically granted to firms in financial distress by the courts. This period can be renewed once and can be exceptionally prolonged in the limit of six months. As these examples show, it is important to consider bankruptcy procedures that are explicitly based on the time spent in financial distress and to include such a "grace" period into the model if one wants to capture the effects of an insurance company's default risk on the value of its liabilities and on the value of the insurance contracts more realistically.

In the present article, we construct a contingent claim model along the lines of Briys and de Varenne (1994, 1997) and Grosen and Jørgensen (2002) for the valuation of the equity and the liability of a life insurance company where the liability consists only of the policy holder's payments. Their main contribution is to explicitly consider default risk in a contingent claim model to value the equity and the liability of a life insurance company. In Briys and de Varenne (1994, 1997), default can only occur at the maturity date, whereas in Grosen and Jørgensen (2002) default can occur at any time before the maturity date, i.e., they introduce the risk of a premature default to the valuation of a life insurance contract[4]. In order to model the default event, they build into the model a regulatory mechanism in

[4]Bernard et al. (2005a) recently extended this model by taking into account stochastic interest rates.

the form of an intervention rule, i.e., they add a simple knock–out barrier option feature to the different components of the insurance contract. The default event is defined so that the value of the total assets of the life insurance company must always be sufficient to cover the life insurance policy holder's initial deposit compounded with the guaranteed rate of return. Otherwise the firm defaults and is immediately liquidated. Absolute priority is assumed, i.e., the holder of the life insurance contract (= liability holder) has the first claim on the firm's assets. This corresponds to a Chapter 7 bankruptcy procedure, where default and liquidation times coincide.

However, as we have explained above, Grosen and Jørgensen's (2002) approach to modelling the insolvency risk does not reflect the reality well. Default and liquidation cannot be considered as equivalent events. We therefore extend their model in order to be able to capture the effects of the Chapter 11 (or of the other countries' codes corresponding to Chapter 11) bankruptcy procedure and to study the impact of a delayed liquidation on the valuation of the insurance company's liabilities and on the ex–ante pricing of the life insurance contracts. We do this by using so–called Parisian barrier option frameworks. Here we distinguish between two kinds of Parisian barrier options: standard Parisian barrier options and cumulative Parisian barrier options.

Assume, we are interested in the modelling of a Parisian down–and–out option. With standard Parisian barrier options, the option contract is knocked out if the underlying asset value stays consecutively below the barrier for a time longer than some predetermined time d before the maturity date. With cumulative Parisian barrier options, the option contract is terminated if the underlying asset value spends until maturity in total at least d units of time below the barrier. In a corporate bankruptcy framework these two Parisian barrier options have appealing interpretations. Think of the idea that a regulatory authority takes its bankruptcy filing actions according to a hypothetical default clock. In the case of standard Parisian barrier options, this default clock starts ticking when the asset price process breaches the default barrier and the clock is reset to zero if the firm recovers from the default. Thus, successive defaults are possible until one of these defaults lasts d units of time. One may say that in this case the default clock is memoryless, i.e., earlier defaults which may last a very long time but not longer than d do not have any consequences for eventual subsequent defaults. In the case of cumulative Parisian barrier options, the default clock is not reset to zero when a firm emerges from default, but it is only halted and restarted when the firm defaults again. Here d denotes the maximum authorized total time in default until the maturity of the debt. This corresponds to a full memory default clock, since every single moment spent in default is remembered and affects further defaults by shortening the maximum allowed length of time that the company can spend in default without being liquidated.[5] Thus, in the limiting case when d is set equal to zero (or is going to zero), we are back in the model of Grosen and Jørgensen (2002). Our model therefore encompasses that of Grosen and Jørgensen (2002) and also those of Briys and de Varenne (1994, 1997). Both kinds of Parisian options are of course not new in the literature on exotic options. They have been introduced by Chesney et al. (1997) and subsequently developed further in Hugonnier (1999), Moraux (2002), Anderluh and van der Weide (2004) and Bernard et al. (2005b).

There are two related papers in the credit risk literature analyzing the effects of bankruptcy procedures: Moraux (2003) extends the model of Black and Cox (1976) and models the value of debt and equity of a company in a structural model of credit risk when the default barrier is not an absorbing one. He is mainly concerned with valuing various forms of debt and analyzes the obtained credit spreads. François and Morellec (2004) perform a similar analysis in a time–independent framework extending Leland's (1994) model. However, these authors are more interested in credit spreads, debt subordination or agency conflicts. Bernard et al. (2005c) consider a model of bank deposit insurance with Parisian options.

The remainder of this article is structured as follows. In Section 2, we briefly review the model of Grosen and Jørgensen (2002) because we will place our model in almost the same basic setup. Moreover, we already introduce the standard Parisian barrier feature along the lines of Chesney et al. (1997). In the numerical analysis, in order to invert the Laplace transforms involved, we use the procedure introduced by Bernard et al. (2005b). Hence we are able to obtain approximate solutions for the components of the life insurance company's balance sheet and for the issued equity–linked life insurance contract. In the case of the cumulative Parisian barrier feature, we deduce quasi–closed–form solutions for the different components of the life insurance company's liabilities and the life insurance contract following and extending Hugonnier (1999) and Moraux (2002). In Section 3, we perform a number of representative numerical analyses and comparative statics for both cases in order to investigate the effects of different parameter changes on the value of the insurance company's equity and liability, and hence on the life insurance contract. In particular, we study the impact of the new regulation parameter d and compare it with the old regulation parameter η which determines the barrier level. In Section 4, we calculate the shortfall probabilities for both standard and cumulative Parisian options in order to analyze the incentives for the customers to engage in a life insurance contract in this model framework. Section 5 concludes.

[5] The real life bankruptcy procedures lie somewhere in between these two extreme cases.

2. Model

This section mainly consists of two parts. The first part reviews the basic model of Grosen and Jørgensen (2002) succinctly, and more importantly, the Parisian barrier option features are introduced to describe the different default and liquidation events. Accordingly, the rebate payment used by the above mentioned authors has to be altered because it does not make sense in our framework. The remaining part of this section focuses on the valuation of the life insurance company's equity and liability and of the issued life insurance contract.

2.1. **Contract Specification.** As in the original work of Grosen and Jørgensen (2002), which is an extension of the early models merging default risk and life insurance contracts of Briys and de Varenne (1994, 1997), we assume that at time $t = 0$ the insurance company owns a capital structure as illustrated in the following balance sheet:

Assets	Liabilities
A_0	$E_0 \equiv (1-\alpha)A_0$ $L_0 \equiv \alpha A_0$
A_0	A_0

That is, for simplicity, we suppose that the representative policy holder (also liability holder) whose premium payment at the beginning of the contract constitutes the liability of the insurance company, denoted by $L_0 = \alpha A_0$, $\alpha \in [0, 1]$, and the representative equity holder, whose equity is accordingly denoted by $E_0 = (1-\alpha)A_0$, form a mutual company, the life insurance company. Through their initial investments in the company, both acquire a claim on the firm's assets for a payoff at maturity (or before maturity).

The following notations are used for the specification of the insurance contract:

T	:=	the maturity date
$L_T = L_0 e^{gT}$	:=	the guaranteed payment to the policy holder at maturity, where g is the minimum guaranteed interest rate
A_t	:=	the value of the firm's assets at time $t \in [0, T]$
δ	:=	the participation rate, i.e., to which extent the policy holder participates in the firm's surpluses at maturity.

Since an interest rate guarantee and the contribution principle which entitles the policy holder to a participation in the insurer's investment surpluses are common features of today's life insurance contracts, we consider the following simplified version of a participating life insurance contract incorporating all these features. The total payoff to the holder of such an insurance contract at maturity, $\psi_L(A_T)$, is given by:

$$\psi_L(A_T) = \delta[\alpha A_T - L_T]^+ + L_T - [L_T - A_T]^+.$$

This payment consists of three parts: a bonus (call option) paying to the policy holder a fraction δ of the positive difference of the actual performance of his share in the insurance company's assets and the guaranteed amount at maturity, a guaranteed fixed payment which is the initial premium payment compounded by the interest rate guarantee and a short put option resulting from the fact that the equity holder has a limited liability. In Grosen and Jørgensen (2002), a rebate payment,

$$\Theta_L(\tau) = \min\{L_0 e^{g\tau}, B_\tau\} = \min\{1, \eta\} L_0 e^{g\tau},$$

is offered to the liability holder in the case of a premature closure of the firm, where τ denotes the liquidation date. Analogously, the total payoff to the equity holder at maturity $\psi_E(A_T)$, is given by:

$$\psi_E(A_T) = [A_T - L_T]^+ - \delta[\alpha A_T - L_T]^+.$$

This payment consists of two call options: a long call option on the assets with strike equal to the promised payment at maturity, called the residual call, and a short call option offsetting exactly the bonus call option of the liability holder. For the equity holder a rebate is offered, too, in the case of a premature liquidation of the firm:

$$\Theta_E(\tau) = \max\{(\eta - 1), 0\} L_0 e^{g\tau}.$$

Grosen and Jørgensen (2002) model their regulatory intervention rule in the form of a boundary, i.e., an *exponential* barrier $B_t = \eta L_0 e^{gt}$ is imposed on the underlying asset value process, where η is a regulation parameter. When the asset price reaches this boundary, namely, $A_\tau = B_\tau$ with $\tau \in [0, T]$, the company defaults and is liquidated immediately, i.e., default and liquidation coincide. If the regulatory authority chooses $\eta \geq 1$, in the case of a liquidation, the liability holder obtains his initial deposit plus the accrued guaranteed interest up to the liquidation date. If an $\eta < 1$ is chosen, no such payment can be made to the full extent. Obviously, the specified contract contains standard down–and–out barrier options. Therefore the requirement $A_0 > B_0 = \eta L_0$ must be satisfied initially. It should be noted that in the case of a liquidation, any recovered funds will be distributed to the company's stake holders according to the usual procedure. The liability holder enjoys absolute priority, i.e., he has the first claim on the company's assets.

The bankruptcy procedure described above where default and liquidation occur at the same time corresponds to Chapter 7 of the U.S. Bankruptcy Code. As mentioned in the introduction, we generalize the model of Grosen and Jørgensen (2002) in order to allow for

Chapter 11 bankruptcy. This can be realized by adding a Parisian barrier option feature instead of the standard knock–out barrier option feature to the model. Before we come to this point, we have to make a small change on the rebate term of the issued contract. Both Parisian barrier option features could lead to the result that at the liquidation time the asset price falls far below the barrier value, which makes it impossible for the insurer to offer the above mentioned rebate. Hence, a new rebate for the liability holder is introduced to the model and it has the form of

$$\Theta_L(\tau) = \min\{L_\tau, A_\tau\},$$

where τ is the liquidation time. The rebate term implicitly depends on the regulation parameter η. Because of the following inequality

$$A_\tau \leq B_\tau = \eta L_\tau,$$

it is observed that for $\eta < 1$, the rebate corresponds to the asset value A_τ.

Correspondingly, the new rebate for the equity holder can be expressed as follows:

$$\Theta_E(\tau) = A_\tau - \min\{L_\tau, A_\tau\} = \max\{A_\tau - L_\tau, 0\},$$

i.e., the equity holder obtains the remaining asset value if there is any. Clearly, in the case of $\eta < 1$, all the asset value goes to the liability holder.

In this paper, we differentiate between two categories of Parisian barrier features:

- Standard Parisian barrier feature: This corresponds to a procedure where the liquidation of the firm is declared when the financial distress has lasted successively at least a period of length d.
- Cumulative Parisian barrier feature: This corresponds to a procedure where the liquidation is declared when the financial distress has lasted in total at least a period of length d during the life of the contract.

It is noted that the original model by Grosen and Jørgensen (2002) is a special case in both scenarios described above, namely when the time window d is set to 0. Observe that with $\eta \downarrow 0$, we are back in the model of Briys and de Varenne (1994), because in that situation premature default and liquidation are impossible.

2.2. **Valuation.** This subsection aims at valuing the liabilities of the life insurance company and of the issued life insurance contract. In the literature, different methods have been applied to value standard and cumulative Parisian options. The inverse Laplace transform method originally introduced by Chesney et al. (1997) is adopted to price the standard Parisian claims. The results of Hugonnier (1999) and Moraux (2002) and some newly derived extensions are used to value the cumulative Parisian claims.

In general, for the valuation framework, we assume a continuous–time frictionless economy with a perfect financial market, no tax effects, no transaction costs and no other imperfections. Hence we can rely on martingale techniques for the valuation of the contingent claim.

Under the equivalent martingale measure, the price process of the insurance company's assets $\{A_t\}_{t\in[0,T]}$ is assumed to follow a geometric Brownian motion

$$dA_t = A_t(rdt + \sigma dW_t),$$

where r denotes the deterministic interest rate, σ the deterministic volatility of the asset price process $\{A_t\}_{t\in[0,T]}$ and $\{W_t\}_{t\in[0,T]}$ the equivalent Q–martingale. Solving this differential equation, we obtain

$$A_t = A_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}.$$

2.2.1. *Standard Parisian Barrier Framework.* Before we come to the general valuation of standard Parisian barrier options, some special cases are considered:

- $A_t > B_t$ and $d \geq T - t$: In this case, it is impossible to have an excursion below B_t, between t and T, of length at least equal to d. Therefore, the value of a Parisian down–and–out call just corresponds to the Black–Scholes (Black and Scholes (1973)) price of a standard European call option.
- $d \geq T$: In this case the Parisian option actually becomes a standard call option.
- $A_t > B_t$ and $d = 0$: As already mentioned, this corresponds to the scenario which Grosen and Jørgensen (2002) introduced.

Apart from these special cases, the standard Parisian option is priced as follows. In the standard Parisian down–and–out option framework, the final payoff $\psi_L(A_T)$ is only paid if the following technical condition is satisfied:

$$T_B^- = \inf\{t > 0 | (t - g_{B,t}^A)\mathbf{1}_{\{A_t < B_t\}} > d\} > T \tag{1}$$

with

$$g_{B,t}^A = \sup\{s \leq t | A_s = B_s\},$$

where $g_{B,t}^A$ denotes the last time before t at which the value of the assets A hits the barrier B. T_B^- gives the first time at which an excursion below B lasts more than d units of time. In fact, T_B^- is the liquidation date of the company if $T_B^- < T$. It is noted that the condition in (1) is equivalent to

$$T_b^- := \inf\{t > 0 | (t - g_{b,t})\mathbf{1}_{\{Z_t < b\}} > d\} > T$$

where

$$g_{b,t} \;:=\; \sup\{s \leq t | Z_s = b\}; \qquad b = \frac{1}{\sigma}\ln\left(\frac{\eta L_0}{A_0}\right),$$

and $\{Z_t\}_{0\le t\le T}$ is a martingale under a new probability measure P which is defined by the Radon–Nikodym density

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_T} = \exp\left\{mZ_T - \frac{m^2}{2}T\right\}, \qquad m = \frac{1}{\sigma}(r - g - \frac{1}{2}\sigma^2),$$

i.e., $Z_t = W_t + m\,t$. The following derivation enlightens this equivalence argument:

$$\begin{aligned} g^A_{B,t} &= \sup\{s \le t : A_s = B_s\} \\ &= \sup\left\{s \le t : A_0 \exp\left\{(r - \frac{1}{2}\sigma^2)s + \sigma W_s\right\} = \eta L_0 e^{gs}\right\} \\ &= \sup\{s \le t : Z_s = b\} = g_{b,t}. \end{aligned}$$

Thereby we transform the event "the excursion of the value of the assets below the exponential barrier $B_t = \eta L_0 e^{gt}$" to the event "the excursion of the Brownian motion Z_t below a constant barrier $b = \dfrac{1}{\sigma}\ln\left(\dfrac{\eta L_0}{A_0}\right)$". This simplifies the entire valuation procedure. Under the new probability measure P the value of the assets A_t can be expressed as

$$A_t = A_0 \exp\left\{\sigma\, Z_t\right\} \exp\{g\,t\}.$$

It is well known that in a complete financial market, the price of a T–contingent claim with the payoff $\phi(A_T)$ corresponds to the expected discounted payoff under the equivalent martingale measure Q, i.e.,

$$E_Q\left[e^{-rT}\phi(A_T)\mathbf{1}_{\{T_B^- > T\}}\right].$$

This can be rephrased as follows:

$$e^{-(r+\frac{1}{2}m^2)T}E_P\left[\mathbf{1}_{\{T_b^- > T\}}\phi(A_0\exp\{\sigma Z_T\}\exp\{gT\})\exp\{mZ_T\}\right].$$

Therefore, the value of the liability of the life insurance company, i.e., the price of the issued life insurance contract is determined by:

$$\begin{aligned} V_L(A_0, 0) &= E_Q[e^{-rT}\left(\delta[\alpha A_T - L_T]^+ - [L_T - A_T]^+ + L_T\right)\mathbf{1}_{\{T_B^- > T\}}] \\ &\quad + E_Q[e^{-rT_B^-}\min\{L_{T_B^-}, A_{T_B^-}\}\mathbf{1}_{\{T_B^- \le T\}}] \\ &= \delta\,\alpha\, e^{-(r-g+\frac{1}{2}m^2)T}E_P\left[\left(A_0 e^{\sigma Z_T} - \frac{L_0}{\alpha}\right)^+ e^{mZ_T}\mathbf{1}_{\{T_b^- > T\}}\right] \\ &\quad - e^{-(r-g+\frac{1}{2}m^2)T}E_P\left[(L_0 - A_0 e^{\sigma Z_T})^+ e^{mZ_T}\mathbf{1}_{\{T_b^- > T\}}\right] + E_Q\left[e^{-rT}L_T\mathbf{1}_{\{T_b^- > T\}}\right] \\ &\quad + E_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\{mZ_{T_b^-}\}\min\{L_{T_b^-}, A_{T_b^-}\}1_{\{T_b^- \le T\}}\right] \\ &:= \delta\,\alpha\,\mathrm{PDOC}[A_0, B_0, \frac{L_0}{\alpha}, r, g] - \mathrm{PDOP}[A_0, B_0, L_0, r, g] + E_Q\left[e^{-rT}L_T\mathbf{1}_{\{T_b^- > T\}}\right] \\ &\quad + E_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\{mZ_{T_b^-}\}\min\{L_{T_b^-}, A_{T_b^-}\}1_{\{T_b^- \le T\}}\right]. \end{aligned}$$

It is observed that the price of this contingent claim consists of four parts: A Parisian down–and–out call option with strike $\frac{L_T}{\alpha}$ (multiplied by $\delta\,\alpha$), i.e., the bonus part, a Parisian down–and–out put option with strike L_T, a deterministic guaranteed part L_T which is paid at maturity when the value of the assets has not stayed below the barrier for a time longer than d and a rebate paid immediately when the liquidation occurs.

Various approaches are applied to valuing standard Parisian products, such as Monte–Carlo algorithms (Andersen and Brotherton–Ratcliffe (1996)), binomial or trinomial trees (Avellaneda and Wu (1999), Costabile (2002)), PDEs (Haber et. al (2002)), finite–element methods (Stokes and Zhu (1999)) or the implied barrier concept (Anderluh and van der Weide (2004)). In this article, we adopt the original Laplace transform approach initiated by Chesney et al. (1997). Later, in the numerical analysis, for inverting the Laplace transforms, we rely on the recently introduced and more easily implementable procedure by Bernard et al. (2005b). They approximate the Laplace transforms needed to value standard Parisian barrier contingent claims by a linear combination of a number of fractional power functions in the Laplace parameter. The inverse Laplace transforms of these functions are well–known analytical functions. Therefore, due to the linearity, the needed inverse Laplace transforms are obtained by summing up the inverse Laplace transforms of the approximate fractional power functions. In the following, we apply this technique to each component of the liabilities and of the issued contract.

It is well known that the price of a Parisian down–and–out call option can be described as the difference of the price of a plain–vanilla call option and the price of a Parisian down–and–in call option with the same strike and maturity date, i.e.,

$$\begin{aligned}
&PDOC[A_0, B_0, \frac{L_0}{\alpha}, r, g] \\
&= e^{-(r-g+\frac{1}{2}m^2)T} E_P\left[\left(A_0 e^{\sigma Z_T} - \frac{L_0}{\alpha}\right)^+ \exp\{mZ_T\}\mathbf{1}_{\{T_b^- > T\}}\right] \\
&= BSC[A_0, \frac{L_0}{\alpha}, r, g] - \underbrace{e^{-(r-g+\frac{1}{2}m^2)T} E_P\left[\left(A_0 e^{\sigma Z_T} - \frac{L_0}{\alpha}\right)^+ \exp\{mZ_T\}\mathbf{1}_{\{T_b^- \le T\}}\right]}_{:=PDIC[A_0,B_0,\frac{L_0}{\alpha},r,g]}.
\end{aligned}$$

The price of the plain–vanilla call option is obtained by the Black–Scholes formula as follows:

$$\begin{aligned}
\mathrm{BSC}[A_0, \frac{L_0}{\alpha}, r, g] &= E_Q\left[e^{-rT}\left(A_T - \frac{L_T}{\alpha}\right)^+\right] = A_0 N(d_1) - \frac{L_0}{\alpha} e^{-(r-g)T} N(d_2) \\
d_{1/2} &= \frac{\ln\left(\frac{\alpha A_0}{L_0}\right) + (r - g \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{(r - g \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.
\end{aligned}$$

Here $N(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ gives the cumulative distribution function of the standard normal distribution. Since it is valid that $\frac{L_0}{\alpha} \geq B_0$, $PDIC[A_0, B_0, \frac{L_0}{\alpha}, r, g]$ can be calculated as follows:

$$\mathrm{PDIC}[A_0, B_0, \frac{L_0}{\alpha}, r, g] = e^{-(r-g+\frac{1}{2}m^2)T} A_0 \int_k^{\infty} e^{my} \left(e^{\sigma y} - \frac{L_0}{\alpha} \right) h_1(T, y) dy$$

with $k = \frac{1}{\sigma} \ln \left(\frac{L_0}{\alpha \, A_0} \right) = 0$. The density $h_1(T, y)$ is uniquely determined by inverting the corresponding Laplace transform which is given by

$$\begin{aligned} \hat{h}_1(\lambda, y) &= \frac{e^{(2b-y)\sqrt{2\lambda}} \psi(-\sqrt{2\lambda} d)}{\sqrt{2\lambda} \psi(\sqrt{2\lambda} d)} \\ \text{with} \qquad \psi(z) &= \int_0^{\infty} x \exp\left(-\frac{x^2}{2} + zx \right) dx = 1 + z\sqrt{2\pi} e^{\frac{z^2}{2}} N(x), \end{aligned}$$

and λ the parameter of Laplace transform.

The Parisian down–and–out put option can be derived by the following in–out–parity:

$$\mathrm{PDOP}[A_0, B_0, L_0, r, g] := \mathrm{BSP}[A_0, L_0, r, g] - \mathrm{PDIP}[A_0, B_0, L_0, r, g].$$

Here $\mathrm{BSP}[A_0, L_0, r, g]$ gives the price of the plain–vanilla put option and $\mathrm{PDIP}[A_0, B_0, L_0, r, g]$ the price of the Parisian down–and–in put option. $\mathrm{BSP}[A_0, L_0, r, g]$ is derived by the Black–Scholes formula:

$$\begin{aligned} \mathrm{BSP}[A_0, L_0, r, g] &= E_Q \left[e^{-rT} \left(L_T - A_T \right)^+ \right] = L_0 e^{-(r-g)T} N(-d_2) - A_0 N(-d_1) \\ d_{1/2} &= \frac{\ln \left(\frac{A_0}{L_0} \right) + (r - g \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}. \end{aligned}$$

Due to the different possible choices of the η–value, different pricing formulas are obtained for the Parisian down–and–in put option. An $\eta < 1$, which leads to the fact that the strike is larger than the barrier, results in

$$\begin{aligned} \mathrm{PDIP}[A_0, B_0, L_0, r, g] &= e^{-(r-g+\frac{1}{2}m^2)T} \left(\int_{-\infty}^{b} e^{my} (L_0 - A_0 e^{\sigma y}) h_2(T, y) dy \right. \\ &\qquad \left. + \int_b^{k_1} e^{my} (L_0 - A_0 e^{\sigma y}) h_1(T, y) dy \right) \end{aligned}$$

with $k_1 = \frac{1}{\sigma} \ln \left(\frac{L_0}{A_0} \right)$. As before, $h_1(T, y)$ and $h_2(T, y)$ are calculated by inverting the corresponding Laplace transform. $\hat{h}_1(T, y)$ has the same value as before and the Laplace transform of $h_2(T,y)$ is given by

$$\begin{aligned}\hat{h}_2(\lambda,y) &= \frac{e^{y\sqrt{2\lambda}}}{\sqrt{2\lambda}\psi(\sqrt{2\lambda d})}+\frac{\sqrt{2\lambda}de^{\lambda d}}{\psi(\sqrt{2\lambda d})}\left(e^{y\sqrt{2\lambda}}\left(N\left(-\sqrt{2\lambda d}-\frac{y-b}{\sqrt{d}}\right)-N(-\sqrt{2\lambda d})\right)\right.\\ &\quad \left.-e^{(2b-y)\sqrt{2\lambda}}N\left(-\sqrt{2\lambda d}+\frac{y-b}{\sqrt{d}}\right)\right).\end{aligned}$$

Analogously, for the case of $\eta \geq 1$, the Parisian down–and–in put option has the form of

$$\mathrm{PDIP}[A_0,B_0,L_0,r,g] = e^{-(r-g+\frac{1}{2}m^2)T}\int_{-\infty}^{k_1} e^{my}(L_0-A_0e^{\sigma y})h_2(T,y)dy.$$

The third term in the payoff function can be calculated as follows:

$$\begin{aligned}E_Q[e^{-rT}L_T\mathbf{1}_{\{T_b^->T\}}] &= e^{-rT}L_T - E_Q[e^{-rT}L_T\mathbf{1}_{\{T_b^-\leq T\}}]\\ &= e^{-rT}L_T\left[1-e^{-\frac{m^2T}{2}}\left(\int_{-\infty}^{b} h_2(T,y)e^{my}dy+\int_b^{\infty} h_1(T,y)e^{my}dy\right)\right].\end{aligned}$$

As mentioned before, in the numerical analysis, we adopt the technique developed by Bernard et al. (2005b) to invert $\hat{h}_1$ and $\hat{h}_2$.

In the calculation of the expected rebate, distinction of cases becomes necessary again. For the case of $\eta < 1$, the liability holder will get $A_{T_b^-}$ if an early liquidation occurs. Therefore, the expected rebate can be calculated as follows:

$$\begin{aligned}& E_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\{mZ_{T_b^-}\}\min\{L_{T_b^-},A_{T_b^-}\}1_{\{T_b^-\leq T\}}\right]\\ =\ & A_0E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}\exp\{(m+\sigma)Z_{T_b^-}\}1_{\{T_b^-\leq T\}}\right]\\ =\ & A_0E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}1_{\{T_b^-\leq T\}}\right]E_P\left[\exp\{(m+\sigma)Z_{T_b^-}\}\right].\end{aligned}$$

The last equality follows from the fact that T_b^- and $Z_{T_b^-}$ are independent, which is shown in the appendix of Chesney et al. (1997). Furthermore, the corresponding laws for these two random variables are given in Chesney et al. (1997), too. As a consequence, we obtain

$$E_P\left[\exp\{(m+\sigma)Z_{T_b^-}\}\right] = \int_{-\infty}^{b} e^{(m+\sigma)x}\,\frac{b-x}{d}\,\exp\left\{-\frac{(x-b)^2}{2d}\right\}dx$$

and

$$E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}1_{\{T_b^-<T\}}\right] = \int_d^T e^{-\left(r-g+\frac{1}{2}m^2\right)t}\,h_3(t)\,dt,$$

where $h_3(t)$ denotes the density of the stopping time T_b^-. This can be calculated by inverting the following Laplace transform

$$\hat{h}_3(\lambda) = \frac{\exp\{\sqrt{2\lambda}b\}}{\psi(\sqrt{2\lambda d})}.$$

For the case of $\eta \geq 1$, we obtain

$$
\begin{aligned}
& E_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\{mZ_{T_b^-}\}\min\{L_{T_b^-}, A_{T_b^-}\}1_{\{T_b^-\leq T\}}\right] \\
= \; & A_0 E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}\exp\{(m+\sigma)Z_{T_b^-}\}1_{\{T_b^-\leq T\}}1_{\{Z_{T_b^-}\leq k_1\}}\right] \\
& + L_0 E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}\exp\{mZ_{T_b^-}\}1_{\{T_b^-\leq T\}}1_{\{k_1<Z_{T_b^-}<b\}}\right] \\
= \; & A_0 E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}1_{\{T_b^-\leq T\}}\right] E_P\left[\exp\{(m+\sigma)Z_{T_b^-}\}1_{\{Z_{T_b^-}<k_1\}}\right] \\
& + L_0 E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}1_{\{T_b^-\leq T\}}\right] E_P\left[\exp\{mZ_{T_b^-}\}1_{\{k_1<Z_{T_b^-}<b\}}\right].
\end{aligned}
$$

This can be calculated further similarly as in the case of $\eta < 1$. Inspired by Bernard et. al. (2005b), we invert $\hat{h}_3$ numerically in the same way.

For the equity holder we have the following value for his contingent claim

$$
\begin{aligned}
V_E(A_0, 0) &= E_Q[e^{-rT}[A_T - L_T]^+\mathbf{1}_{\{T_B^- > T\}}] - E_Q[e^{-rT}\delta[\alpha A_T - L_T]^+\mathbf{1}_{\{T_B^- > T\}}] \\
&\quad + E_Q[e^{-rT_B^-}\max\{A_{T_B^-} - L_{T_B^-}, 0\}\mathbf{1}_{\{T_B^-\leq T\}}] \\
&= \mathrm{PDOC}[A_0, B_0, L_0, r, g] - \delta\,\alpha\,\mathrm{PDOC}[A_0, B_0, \frac{L_0}{\alpha}, r, g] \\
&\quad + E_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\{mZ_{T_b^-}\}\max\{A_{T_B^-} - L_{T_B^-}, 0\}1_{\{T_b^-\leq T\}}\right].
\end{aligned}
$$

It is composed of three parts: A Parisian down–and–out call option with strike L_T, called the residual claim, a short Parisian down–and–out call option with strike $\frac{L_T}{\alpha}$ (multiplied by $\delta\,\alpha$), i.e., the negative value of the liability holder's bonus option and a rebate paid immediately when the liquidation occurs. It is noted that the second component has already been calculated above. The first component is given by the price difference of the corresponding plain–vanilla and the Parisian down–and–in option. The price of the plain–vanilla option is described by

$$
\begin{aligned}
\mathrm{BSC}[A_0, L_0, r, g] &= E_Q\left[e^{-rT}\left(A_T - L_T\right)^+\right] = A_0 N(d_1) - L_0 e^{-(r-g)T}N(d_2) \\
d_{1/2} &= \frac{\ln\left(\frac{A_0}{L_0}\right) + (r - g \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.
\end{aligned}
$$

In order to calculate the relevant Parisian down–and–in option, again two cases are distinguished. For $\eta < 1$,

$$
\mathrm{PDIC}[A_0, B_0, L_0, r, g] = e^{-\left(r-g+\frac{1}{2}m^2\right)T}\int_{k_1}^{\infty} e^{my}(A_0 e^{\sigma y} - L_0)h_1(T, y)dy
$$

and for $\eta \geq 1$,

$$
\begin{aligned}
& \mathrm{PDIC}[A_0, B_0, L_0, r, g] \\
= \; & e^{-\left(r-g+\frac{1}{2}m^2\right)T}\left(\int_{k_1}^{b} e^{my}(A_0 e^{\sigma y} - L_0)h_2(T,y)\,d\,y + \int_{b}^{\infty} e^{my}(A_0 e^{\sigma y} - L_0)h_1(T,y)\,d\,y\right).
\end{aligned}
$$

Finally, we come to the value of the equity holder's rebate. Only in the case of $\eta \geq 1$, he would possibly obtain a rebate payment.

$$
\begin{aligned}
& E_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\{mZ_{T_b^-}\}\max\{A_{T_B^-} - L_{T_B^-}, 0\}1_{\{T_b^- \leq T\}}\right] \\
= \; & A_0 E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}\exp\{(m+\sigma)Z_{T_b^-}\}1_{\{T_b^- \leq T\}}1_{\{k_1 < Z_{T_b^-} < b\}}\right] \\
& - L_0 E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}\exp\{mZ_{T_b^-}\}1_{\{T_b^- \leq T\}}1_{\{k_1 < Z_{T_b^-} < b\}}\right].
\end{aligned}
$$

Further calculations can be done analogously to the derivation of the expected rebate for the liability holder.

2.2.2. *Cumulative Parisian Barrier Framework.* In this case the options are lost by their owners when the underlying asset has stayed below the barrier for at least d units of time during the entire duration of the contract. Therefore, the options do not lose their values when the following condition holds:

$$
\Gamma_T^{-,B} = \int_0^T \mathbf{1}_{\{A_t \leq B_t\}} dt < d,
$$

where $\Gamma_T^{-,B}$ denotes the occupation time of the process describing the value of the assets $\{A_t\}_{t\in[0,T]}$ below the barrier B during $[0,T]$. The condition is equivalent to

$$
\Gamma_T^{-,b} := \int_0^T \mathbf{1}_{\{Z_t \leq b\}} dt < d,
$$

where b and Z_t are the same value or process, respectively, as in the standard Parisian option case. Since τ is defined as the premature liquidation date, it implies:

$$
\Gamma_\tau^{-,b} := \int_0^\tau \mathbf{1}_{\{\tau \leq T\}} \mathbf{1}_{\{Z_t \leq b\}} dt = d.
$$

Consequently, we obtain the present value of the liability or of the contract issued to the policy holder in the cumulative Parisian framework:

$$
\begin{aligned}
V_L^C(A_0,0) &= E_Q[e^{-rT}\left(\delta[\alpha A_T - L_T]^+ + L_T - [L_T - A_T]^+\right)\mathbf{1}_{\{\Gamma_T^{-,b}<d\}}] + E_Q[e^{-r\tau}\Theta_L(\tau)] \\
&= E_Q[e^{-rT}\left(\delta[\alpha A_T - L_T]^+ - [L_T - A_T]^+\right)\mathbf{1}_{\{\Gamma_T^{+,b}\geq T-d\}}] + E_Q[e^{-rT}L_T\mathbf{1}_{\{\Gamma_T^{-,b}<d\}}] \\
&\quad + E_Q[e^{-r\tau}\Theta_L(\tau)] \\
&= e^{-(r-g+\frac{1}{2}m^2)T}\left(E_P\left[\delta\alpha\left(A_0e^{\sigma Z_T} - \frac{L_0}{\alpha}\right)^+ e^{mZ_T}\mathbf{1}_{\{\Gamma_T^{+,b}\geq T-d\}}\right]\right. \\
&\quad \left. - E_P\left[(L_0 - A_0e^{\sigma Z_T})^+ e^{mZ_T}\mathbf{1}_{\{\Gamma_T^{+,b}\geq T-d\}}\right]\right) + E_Q[e^{-(r-g)T}L_0\mathbf{1}_{\{\Gamma_T^{-,b}<d\}}] \\
&\quad + E_Q[e^{-r\tau}\min\{A_\tau, L_\tau\}] \\
&:= \delta\alpha\, C^+(0,A_0,\frac{L_0}{\alpha},B_0,T-d,r-g) - P^+(0,A_0,L_0,B_0,T-d,r-g) \\
&\quad + E_Q[e^{-(r-g)T}L_0\mathbf{1}_{\{\Gamma_T^{-,b}<d\}}] + E_Q[e^{-r\tau}\min\{A_\tau, L_\tau\}].
\end{aligned}
$$

Here, the first equality results from the equivalence of two events, i.e., the event that the occupation time of the asset process below the barrier is shorter than d during $[0,T]$ and the event that the occupation time of the asset price process above the barrier is longer than $T-d$, i.e.,

$$
\left\{\Gamma_T^{+,b} := \int_0^T \mathbf{1}_{\{Z_t>b\}}dt \geq T-d\right\} = \left\{\int_0^T \mathbf{1}_{\{Z_t<b\}}dt := \Gamma_T^{-,b} < d\right\}.
$$

First, let us consider the cumulative Parisian down–and–out call option. According to Hugonnier (1999) and the correction in Moraux (2002), the $(r-g,m)$ discounted price at time 0 of a cumulative Parisian call option with maturity T, strike $\frac{L_0}{\alpha}$, excursion level B_0, and window d is given by

$$
C^+\left(0,A_0,\frac{L_0}{\alpha},B_0,T-d,r-g\right) = e^{-(r-g+\frac{1}{2}m^2)T}\left(A_0\Psi^+_{m+\sigma}(T,k,b,T-d) - \frac{L_0}{\alpha}\Psi^+_m(T,k,b,T-d)\right)
$$

with $k = \frac{1}{\sigma}\ln\left(\frac{L_0/\alpha}{A_0}\right)$ and $b = \frac{1}{\sigma}\ln\left(\frac{\eta L_0}{A_0}\right)$. $\Psi^+_\mu(T,k,b,T-d)$ takes different values for different cases. The only interesting case for us is $b<0$, i.e. $B_0 < A_0$, and in this case $\Psi^+_\mu(T,k,b,T-d)$ assumes the following value:

$$
\begin{aligned}
&\Psi^+_\mu(T,k,b,T-d) \\
&= e^{\frac{\mu^2 T}{2}}\left[N\left(d^{\Xi(\mu)}\left(A_0, B_0\vee\frac{L_0}{\alpha},T\right)\right) - \left(\frac{B_0}{A_0}\right)^{2\mu/\sigma}N\left(d^{\Xi(\mu)}\left(\frac{B_0^2}{A_0}, B_0\vee\frac{L_0}{\alpha},T\right)\right)\right] \\
&\quad + \int_{T-d}^T ds\int_{k\wedge b}^b\left\{e^{\mu x}\gamma(b-x,-b,s,T-s)dx + \int_{k\vee b}^\infty e^{\mu x}\gamma(0,x-2b,s,T-s)dx\right\}, \qquad (2)
\end{aligned}
$$

where

$$\Xi(\mu) = \begin{cases} + & \text{if} \quad \mu = m + \sigma \\ - & \text{if} \quad \mu = m \end{cases}$$

$$\begin{aligned}
\gamma(a,b,u,v) &= \int_0^\infty \frac{(z+a)(z+b)}{\pi(uv)^{3/2}} \exp\left\{-\frac{(z+a)^2}{2v}\right\} \exp\left\{-\frac{(z+b)^2}{2u}\right\} dz \\
&= \frac{1}{\pi}\left\{\frac{av+bu}{(u+v)^2(uv)^{1/2}}\right\} \exp\left\{-\frac{a^2}{2v}-\frac{b^2}{2u}\right\} + \sqrt{\frac{2}{\pi}}\left(\frac{1}{u+v}\right)^{3/2} \\
&\quad \cdot \left(1-\frac{(b-a)^2}{u+v}\right) \exp\left\{-\frac{(b-a)^2}{2(u+v)}\right\} N\left(\frac{-au-bv}{(uv(u+v))^{1/2}}\right).
\end{aligned}$$

Second, let us consider the embedded cumulative Parisian down–and–out put option:

$$\begin{aligned}
&P^+(0, A_0, L_0, B_0, T-d, r-g) \\
&= A_0 L_0 \left[BSC\left(0, \frac{1}{A_0}, \frac{1}{L_0}, g-r\right) - C^+\left(0, \frac{1}{A_0}, \frac{1}{L_0}, \frac{1}{B_0}, d, g-r\right)\right],
\end{aligned}$$

where the put–call–symmetry is used. Furthermore, BSC is the Black–Scholes value of the corresponding call, i.e.,

$$\begin{aligned}
BSC\left(0, \frac{1}{A_0}, \frac{1}{L_0}, g-r\right) &= e^{(g-r)T}\frac{1}{A_0}N(d_1) - \frac{1}{L_0}N(d_2) \\
d_{1/2} &= \frac{\ln\left(\frac{L_0}{A_0}\right) + (g - r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.
\end{aligned}$$

And analogous to the call,

$$C^+\left(0, \frac{1}{A_0}, \frac{1}{L_0}, \frac{1}{B_0}, d, g-r\right) = e^{-\frac{1}{2}m_2^2 T}\left(\frac{1}{A_0}\Psi^+_{m_2+\sigma}(T, k_2, b_2, d) - \frac{1}{L_0}\Psi^+_{m_2}(T, k_2, b_2, d)\right)$$

with now $k_2 = -k_1 = \frac{1}{\sigma}\ln\left(\frac{A_0}{L_0}\right)$, $b_2 = -b = \frac{1}{\sigma}\ln\left(\frac{A_0}{\eta L_0}\right) > 0$ and $m_2 = \frac{1}{\sigma}(g - r - \frac{\sigma^2}{2})$. Hence, Ψ^+_μ owns a different value, namely,

$$\Psi^+_\mu(T, k_2, b_2, d) = \int_d^T ds \int_{k_2 \wedge b_2}^{b_2} \left\{e^{\mu x}\gamma(2b_2 - x, 0, s, T-s)dx + \int_{k_2 \vee b_2}^\infty e^{\mu x}\gamma(b_2, x - b_2, s, T-s)dx\right\}.$$

Third, we come to the valuation of the fixed payment. With a close look, the discounted expected fixed payment under martingale measure Q is nothing but the product of $e^{-(r-g)T}L_0$ and the price of a cumulative binary option paying 1 at maturity if the occupation time below the barrier is shorter than d. Hence, we can use the representation for the cumulative binary option derived in Hugonnier (1999) to obtain:

$$\begin{aligned}
&E_Q[e^{-(r-g)T}L_0\mathbf{1}_{\{\Gamma_T^{-,b}<d\}}] \\
&= e^{-(r-g+\frac{1}{2}m^2)T}L_0\Psi^+_m(T, -\infty, b, T-d),
\end{aligned}$$

where $\Psi^+_m(T, -\infty, b, T-d)$ takes its value according to (2).

Finally, we come to the derivation of the expected rebate payment:

$$E_Q[e^{-r\tau} \min\{A_\tau, L_\tau\}].$$

Above all, it is noted that τ can be described as the inverse of the occupation time d, namely,

$$\tau = \Gamma_-^{-1}(d) = \inf\{t \geq 0; \Gamma_t^{-,b} = d\}, \quad t \leq T.$$

Here two cases are distinguished: $\eta < 1$ and $\eta \geq 1$. First, let us look at the case of $\eta < 1$. In this case, the expected rebate is simplified to

$$E_Q[e^{-r\tau} A_\tau].$$

It can be further calculated as follows:

$$\begin{aligned}
E_Q[e^{-r\tau} A_\tau] &= A_0 E_P\left[e^{-(r-g+\frac{1}{2}m^2)\tau} e^{(\sigma+m)Z_\tau}\right] \\
&= A_0 e^{(\sigma+m)b} E_P\left[e^{-(r-g+\frac{1}{2}m^2)\tau} e^{(\sigma+m)(Z_\tau - b)}\right] \\
&= A_0 e^{(\sigma+m)b} E_P\left[e^{-(r-g+\frac{1}{2}m^2)\tau} e^{(\sigma+m)(Z_\tau^*)}\right] \\
&= A_0 e^{(\sigma+m)b} \int_{-\infty}^{0} \int_{d}^{T} e^{-(r-g+\frac{1}{2}m^2)s} e^{(\sigma+m)x} \\
&\quad \int_0^\infty \frac{|l-b||-x+l|}{\pi(s-d)^{3/2} d^{3/2}} \exp\left\{-\frac{(l-b)^2}{2(s-d)} - \frac{(-x+l)^2}{2d}\right\} dl\, ds\, dx
\end{aligned}$$

where $Z_{\tau^*} = Z_\tau - b$ is a $P-$Brownian motion with initial value $-b$. The first equality results from Girsanov's theorem, and the second and third step are done by using the argument that the law of a Brownian motion with initial value 0 staying below a negative barrier b should be equivalent to the law of a Brownian motion with initial value $-b$ staying below the barrier value of 0. The expression in the last integral gives the joint law of a Brownian motion with initial value $-b > 0$, the inverse of the occupation time of length d below 0 and the local time of this Brownian motion at the level 0, which is e.g. given as formula 1.1.5.8 in Borodin and Salminen (1996). In addition, we applied the results given in Chapter 6.3, Section C of Karatzas and Shreve (1991). By solving the integral with respect to the local time, we obtain the law of the Brownian motion and the inverse of the occupation time. Similarly, we can calculate the expected rebate payment for the case of

$\eta \geq 1$:

$$
\begin{aligned}
& E_Q[e^{-r\tau}\min\{A_\tau, L_\tau\}] \\
&= E_Q\left[e^{-r\tau}A_\tau \mathbf{1}_{\{Z_\tau<k_1\}}\right] + E_Q\left[e^{-r\tau}L_\tau \mathbf{1}_{\{k_1<Z_\tau<b\}}\right] \\
&= A_0 E_P\left[e^{-(r-g+\frac{1}{2}m^2)\tau}e^{(m+\sigma)Z_\tau}\mathbf{1}_{\{Z_\tau<k_1\}}\right] + L_0 E_P\left[e^{-(r-g+\frac{1}{2}m^2)\tau}e^{mZ_\tau}\mathbf{1}_{\{k_1<Z_\tau<b\}}\right] \\
&= A_0 e^{(\sigma+m)b}\int_{-\infty}^{\frac{1}{\sigma}\ln\frac{1}{\eta}}\int_d^T e^{-(r-g+\frac{1}{2}m^2)s}e^{(\sigma+m)x} \\
&\quad \cdot \int_0^\infty \frac{|l-b||-x+l|}{\pi(s-d)^{3/2}d^{3/2}}\exp\left\{-\frac{(l-b)^2}{2(s-d)} - \frac{(-x+l)^2}{2d}\right\} dl\, ds\, dx \\
&\quad + L_0 e^{mb}\int_{\frac{1}{\sigma}\ln\frac{1}{\eta}}^{0}\int_d^T e^{-(r-g+\frac{1}{2}m^2)s}e^{mx} \\
&\quad \cdot \int_0^\infty \frac{|l-b||-x+l|}{\pi(s-d)^{3/2}d^{3/2}}\exp\left\{-\frac{(l-b)^2}{2(s-d)} - \frac{(-x+l)^2}{2d}\right\} dl\, ds\, dx,
\end{aligned}
$$

where $k_1 = \dfrac{1}{\sigma}\ln\left(\dfrac{L_0}{A_0}\right)$ and $b = \dfrac{1}{\sigma}\ln\left(\dfrac{\eta L_0}{A_0}\right)$ as before. For the equity holder we have the following value for his contingent claim

$$
\begin{aligned}
V_E(A_0, 0)) &= E_Q[e^{-rT}[A_T - L_T]^+\mathbf{1}_{\{\Gamma_T^{-,b}<d\}}] - E_Q[e^{-rT}\delta[\alpha A_T - L_T]^+\mathbf{1}_{\{\Gamma_T^{-,b}<d\}}] \\
&\quad + E_Q[e^{-r\tau}\max\{A_\tau - L_\tau, 0\}].
\end{aligned}
$$

The value of the residual call is given by:

$$
\begin{aligned}
& E_Q[e^{-rT}[A_T - L_T]^+\mathbf{1}_{\{\Gamma_T^{-,b}<d\}}] \\
&= C^+(0, A_0, L_0, B_0, T-d, r-g) \\
&= e^{-(r-g+\frac{1}{2}m^2)T}\left(A_0\Psi^+_{m+\sigma}(T, k_1, b, T-d) - L_0\Psi^+_m(T, k_1, b, T-d)\right).
\end{aligned}
$$

Ψ^+_μ is given in (2). Again, the value of the short bonus option can be taken from the computations for the liability holder. Obviously, for the case of $\eta < 1$ the equity holder does not obtain any rebate payment. Consequently, we just look at the value of the equity holder's rebate when $\eta \geq 1$. Since the derivation is analogous to that for the policy holder, we jump to the result:

$$\begin{aligned}
&E_Q\left[e^{-r\tau}\max\{A_\tau - L_\tau, 0\}\right] \\
&= E_Q\left[e^{-r\tau}(A_\tau - L_\tau, 0)\mathbf{1}_{\{L_\tau < A_\tau < \eta L_\tau\}}\right] \\
&= A_0 e^{(\sigma+m)b}\int_{\frac{1}{\sigma}\ln\frac{1}{\eta}}^{0}\int_d^T e^{-(r-g+\frac{1}{2}m^2)s} e^{(\sigma+m)x} \\
&\quad \cdot \int_0^\infty \frac{|l-b||-x+l|}{\pi(s-d)^{3/2}d^{3/2}}\exp\left\{-\frac{(l-b)^2}{2(s-d)} - \frac{(-x+l)^2}{2d}\right\} dl\, ds\, dx \\
&\quad - L_0 e^{mb}\int_{\frac{1}{\sigma}\ln\frac{1}{\eta}}^{0}\int_d^T e^{-(r-g+\frac{1}{2}m^2)s} e^{mx} \\
&\quad \cdot \int_0^\infty \frac{|l-b||-x+l|}{\pi(s-d)^{3/2}d^{3/2}}\exp\left\{-\frac{(l-b)^2}{2(s-d)} - \frac{(-x+l)^2}{2d}\right\} dl\, ds\, dx.
\end{aligned}$$

In the next section, we calculate the contract for these two kinds of Parisian barrier frameworks numerically.

2.3. **Fair contract principle.** A contract is called fair if the accumulated expected discounted premium is equal to the accumulated expected discounted payments of the contract under consideration. This principle requires the equality between the initial investment of the policy holder and his expected benefit from the contract, namely the value of the contract equals the initial liability

$$V_L(A_0, 0) = \alpha A_0 = L_0.$$

Alternatively, we could also take the equity holder's point of view, since $A_0 = V_L(A_0, 0) + V_E(A_0, 0)$. Then,

$$V_E(A_0, 0) = (1-\alpha)A_0 = E_0.$$

Certainly, these equations hold for both standard and cumulative Parisian barrier claims.

3. Numerical Analysis

3.1. **Fair Combination Analysis.** According to the fair premium principle introduced in Section 2.3, we can determine the fair premium implicitly through a fair combination of the parameters. In this subsection, we mainly look at the fair combination of δ and g given various parameter constellations. As before, we consider two cases: standard and cumulative Parisian options.

3.1.1. *Standard Parisian Barrier Framework.* Again, two subcases are distinguished because different relations between the strike and the barrier require different valuation formulas.

(a) $\eta \in [0, 1] \quad \Longleftrightarrow \quad \dfrac{L_0}{\alpha} \geq L_0 \geq B_0$

Relation between the participation rate and the minimum guaranteed interest rate for a fair contract

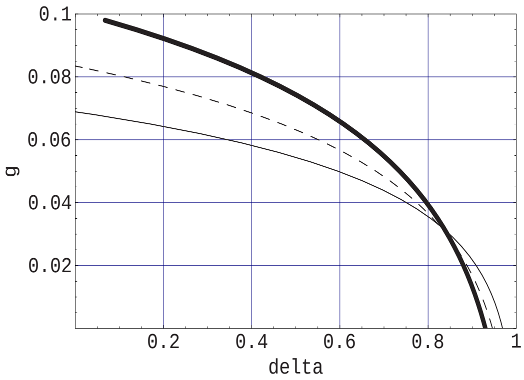

FIGURE 1. Relation between δ and g for different σ with parameters (case (a)): $A_0 = 100; L_0 = 80; \alpha = 0.8; r = 0.05; \eta = 0.8; T = 12; d = 1; \sigma = 0.15$ (solid); $\sigma = 0.20$ (dashing); $\sigma = 0.25$ (thick).

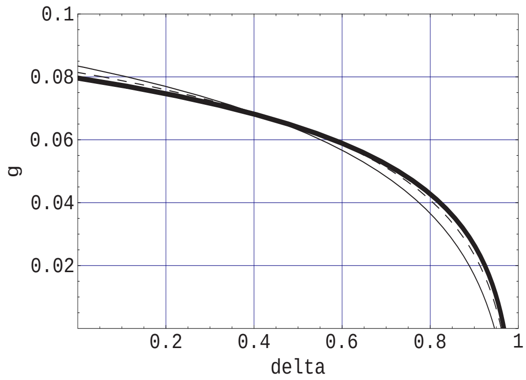

FIGURE 2. Relation between δ and g for different T with parameters (case (a)): $A_0 = 100; L_0 = 80; \alpha = 0.8; r = 0.05; \eta = 0.8; \sigma = 0.2; d = 1; T = 12$ (solid); $T = 18$ (dashing); $T = 24$ (thick).

Relation between the participation rate and the minimum guaranteed interest rate for a fair contract

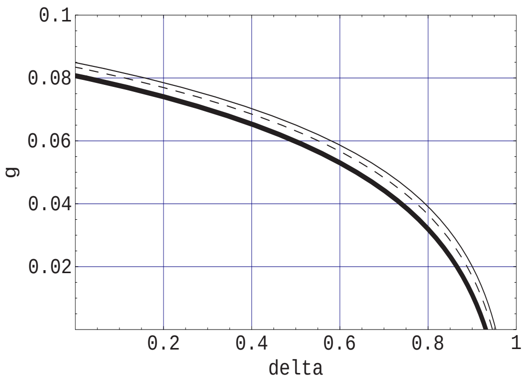

FIGURE 3. Relation between δ and g for different η with parameters (case (a)): $A_0 = 100; L_0 = 80; \alpha = 0.8; r = 0.05; T = 12; \sigma = 0.2; d = 1; \eta = 0.7$ (solid); $\eta = 0.8$ (dashing); $\eta = 0.9$ (thick).

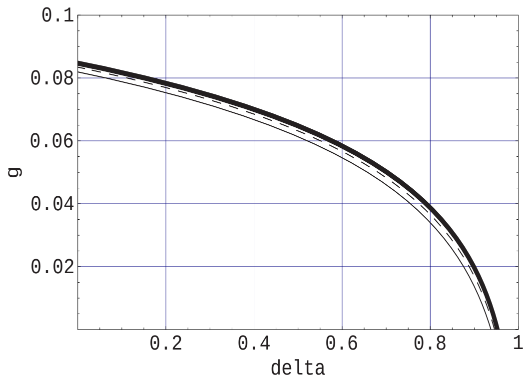

FIGURE 4. Relation between δ and g for different d with parameters (case (a)): $A_0 = 100; L_0 = 80; T = 12; \alpha = 0.8; \sigma = 0.2; \eta = 0.8; r = 0.05; d = 0.5$ (solid); $d = 1$ (dashing); $d = 2$ (thick).

(b) $\eta \in \left[1, \frac{1}{\alpha}\right] \quad \Longleftrightarrow \quad \frac{L_0}{\alpha} \geq B_0 \geq L_0.$

We start our analysis with four graphics for the first subcase. The relation between the participation rate δ and the minimum guarantee g for different volatilities is demonstrated in Figure 1. First, it is quite obvious to observe a negative relation between the participation rate and the minimum guarantee (decreasing concave curves), which results from the fair contract principle. Similarly to Grosen and Jørgensen (2002), for smaller values of δ ($\delta < 0.83$), either higher values of g or of δ are required for a higher volatility in order to make the contract fair. For higher values of δ ($\delta > 0.83$), this effect is reversed. As the volatility goes up, the value of Parisian down–and–out call increases, while the value of the Parisian down–and–out put increases with the volatility at first and then decreases (hump–shaped). The value of the fixed payment goes down and the rebate term behaves similarly to the Parisian down–and–out put, i.e., goes up at first then goes down after a certain level of volatility is reached. For the low values of δ, the fixed payment dominates, therefore a positive relation between δ and σ (also g and σ) is generated. On the contrary, the reversed effect is observed for high values of δ. Therefore, a volatility–neutral fair combination of $(\delta^*, g^*) \approx (0.83, 0.033)$ is observed.

Figure 2 gives the relation between δ and g for different maturity dates T. The value of the Parisian down–and–out call rises with the time to maturity (positive effect), while the value of the Parisian down–and–out put increases with the time to maturity for a while then decreases (hump–shaped). For the chosen parameter values, the put value begins to go down when the maturity time is chosen larger than 3 years. Hence, this value decreases with T locally[6] (positive effect). The expected value of the fixed payment declines when the issued contracts have a longer duration (negative effect), while the expected rebate payment increases (positive effect). Before a certain δ is reached, namely, $\delta < 0.47$, the positive effect dominates the negative one. The reversed effect is observed for $\delta > 0.47$. Hence, a T–neutral fair combination is also observed here. It is worth mentioning that the magnitude of the effect of T is quite small, because the three curves almost overlap.

How δ (or g) changes with η is illustrated in Figure 3. First of all, it is noted that different η–values lead to different values of the barrier ($B_0 = \eta L_0$). In Grosen and Jørgensen (2002), the liability holder benefits much from a higher regulation parameter η because higher values of η provide the liability holder a better protection against losses. The same effect can also be found here. As the barrier is set higher, the values of Parisian down–and–out call and put decrease, so does the value of fixed payment. In contrast, the expected value of the rebate increases with the barrier. In all, the contract value rises when the barrier is set higher. This is why the solid curve ($\eta = 0.7$) lies above the thick one ($\eta = 0.9$). However, the effect is not as large as in the case of a standard knock–out barrier option (the distances among these three curves are not that big) because the introduction of the

[6] Because the three T values applied in Figure 2 are $T = 12$, 18 and 24, all of them are larger than $T = 3$.

Parisian barrier feature diminishes the knock–out probability (the factor d, i.e., the length of the excursion reduces the effect caused by the magnitude of the barrier). This positive effect of η (barrier) on the contract value becomes more obvious when the length of excursion d is smaller. Apparently, the adjustment of the parameter d has a considerable impact on the effect of η. Therefore, the regulator controls the strictness of the regulation by adjusting these two parameters. Later, Tables 2–4 will show a more intuitive effect of these two parameters.

The last figure for the first case exhibits how the contract value changes with the length of excursion d. Since it is the main concern of this paper to capture the effect of d, three tables are listed (Tables 2–4) for this purpose. Table 2 helps to understand the following argument. Obviously, a positive relation exists between the Parisian down–and–out call and the length of excursion (positive effect). The longer the allowed excursion, the larger the value of the option. In fact, the value of the call does not change much with the length of excursion when a certain level of d is reached, i.e., the value of the Parisian down–and–out call is a concave increasing function of d. The put option changes with the length of excursion in a similar way. It increases with d but the extent to which it increases becomes smaller after a certain level of d is reached. The fixed payment arises only when the asset price process does not stay below the barrier for a time longer than d. Hence, as the size of d goes up, the probability that the fixed payment will become due increases. Consequently the expected value of the fixed payment rises. Its magnitude is bounded from above by the payment $L_T e^{-rT}$. In contrast, the rebate payment appears only when the considered insurance company is liquidated, i.e., when the asset price process stays below the barrier for a time period which is longer than d. Therefore, the longer the length of excursion, the smaller the expected rebate payment. To sum up, the entire contract value diminishes with the length of excursion, i.e., the contract can only remain fair when a high d is combined with a high participation rate or a high minimum interest rate guarantee.

The same figures are provided for case (b) where the barrier value is larger than L_0. Since most of the graphics are similar to those of case (a), we do not want to repeat all the details. However, some further differences are discovered when the effect of d on the contract value is considered. In comparison with case (a), the length of excursion d shows a bigger effect here (the curves are more distant here). In case (b), the Parisian down–and–out call option exhibits considerably smaller values for very small values of d. This fact becomes especially evident for d near zero, since the barrier level is much higher in the present case (barrier $\geq L_0$) than in case (a) (barrier $< L_0$). It is well known that higher barriers lead to lower prices for down–and–out options (negative effect). If smaller values of d are used, this negative effect of the barrier cannot be reduced or even offset by the positive effect of d. Second, an extraordinarily small value of the expected fixed payment and on the contrary an extraordinary big value of the expected rebate are

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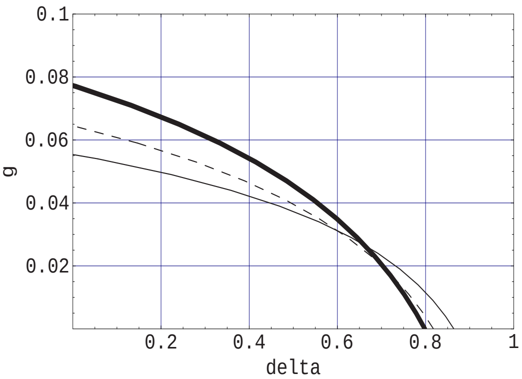

FIGURE 5. Relation between δ and g for different σ with parameters (case (b)): $A_0 = 100; L_0 = 80; \alpha = 0.8; r = 0.05; \eta = 1.2; T = 12; d = 1; \sigma = 0.15$ (solid); $\sigma = 0.20$ (dashing); $\sigma = 0.25$ (thick).

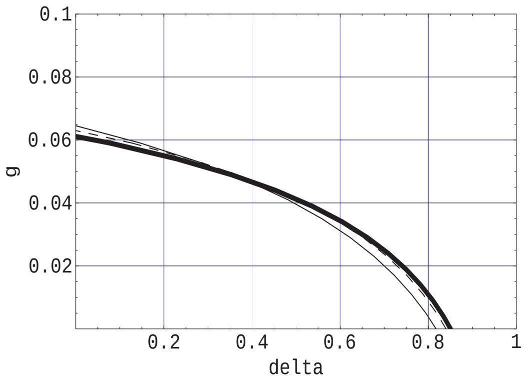

FIGURE 6. Relation between δ and g for different T with parameters (case (b)): $A_0 = 100; L_0 = 80; \alpha = 0.8; r = 0.05; \eta = 1.2; \sigma = 0.2; d = 1; T = 12$ (solid); $T = 18$ (dashing); $T = 24$ (thick).

observed for d close to zero. Altogether, very small values of d, say close to zero, combined with high barrier levels cause small contract values. This is the reason why a relatively more pronounced effect of d results for the case of $\eta \geq 1$ (c.f. Tables 3–4).

3.1.2. *Cumulative Parisian Barrier Framework.* As for the standard Parisian barrier options discussed above, in the cumulative Parisian option framework, a negative relation between the participation rate and the minimum interest rate guarantee is observed. Due to the fact that different η–values require the use of different valuation formulas, again two subcategories can be distinguished: (a) $\eta \in [0,1]$ ($\Leftrightarrow \frac{L_0}{\alpha} \geq L_0 \geq B_0$) and (b) $\eta \in [1, \frac{1}{\alpha}]$ ($\Leftrightarrow \frac{L_0}{\alpha} \geq B_0 \geq L_0$). For each of these subcategories, four figures are plotted. We illustrate how the participation rate and the minimum interest rate guarantee (δ and g) change with the volatility (σ), the maturity date (T), the regulation parameter (η) and the length of excursion (d). Since most of the results are similar to the standard Parisian option case, we only discuss the points where we observe differences. In the following we first consider category (a).

Overall, it is observed that in this case the resulting values for the fair participation rate are slightly smaller than those in the standard Parisian option case. Although this difference can hardly be seen in the graphics, it is observable in Tables 2–4. It is justified as follows. The cumulative Parisian down–and–out call, the down–and–out put and the fixed payment assume smaller values than the corresponding standard Parisian contingent

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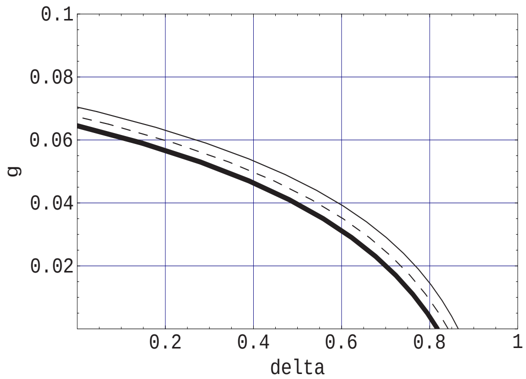

FIGURE 7. Relation between δ and g for different η with parameters (case (b)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $T = 12$; $\sigma = 0.2$; $d = 1$; $\eta = 1.1$ (solid); $\eta = 1.15$ (dashing); $\eta = 1.2$ (thick).

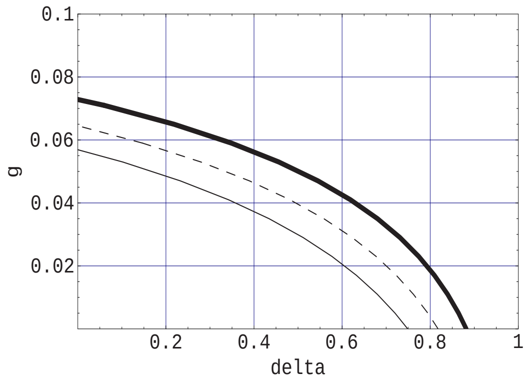

FIGURE 8. Relation between δ and g for different d with parameters (case (b)): $A_0 = 100$; $L_0 = 80$; $T = 12$; $\sigma = 0.25$; $\eta = 1.2$; $r = 0.05$; $\alpha = 0.8$; $d = 0.5$ (solid); $d = 1$ (dashing); $d = 2$ (thick).

claims. This is due to the fact that the knock–out probability becomes higher in the cumulative case, given the same parameters. This is quite obvious because the knock–out condition for standard Parisian barrier options is that the underlying asset stays *consecutively* below barrier for a time longer than d before the maturity date, while the knock–out condition for cumulative Parisian barrier options is that the underlying asset value spends until the maturity *in total* d units of time below the barrier. In contrast, the expected cumulative rebate part of the payment assumes larger values, because it is contingent on the reversed condition compared to the other three parts of the payment. Moreover, (usually) the total effect of these other parts together dominates that of the rebate.

Figure 9 depicts how the participation rate δ (or the minimum guarantee g) varies with the volatility. The figure is very similar to Figure 1. The fair combinations of g and δ for different maturity dates T are plotted in Figure 10, which resembles Figure 2.

How the regulation parameter η influences the fair combination of δ and g is demonstrated in Figure 11. In contrast to the standard Parisian case (Figure 3), η has a bigger impact on the fair parameter combination: the differences of the three curves are more pronounced. Intuitively, it is clear that the value of cumulative Parisian barrier options depends more on the magnitude of the barrier than the value of standard Parisian barrier options does.

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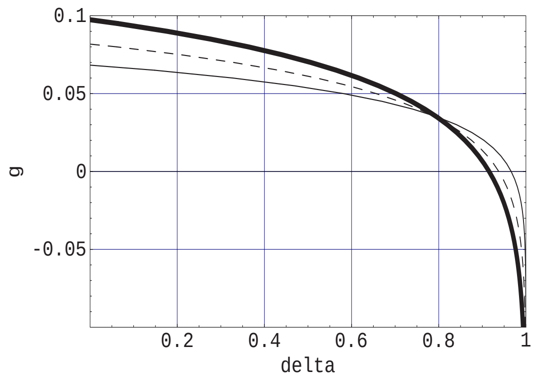

FIGURE 9. Relation between δ and g for different σ with parameters (case (a)): $A_0 = 100; L_0 = 80; \alpha = 0.8; r = 0.05; \eta = 0.8; T = 12; d = 1; \sigma = 0.15$ (solid); $\sigma = 0.20$ (dashing); $\sigma = 0.25$ (thick).

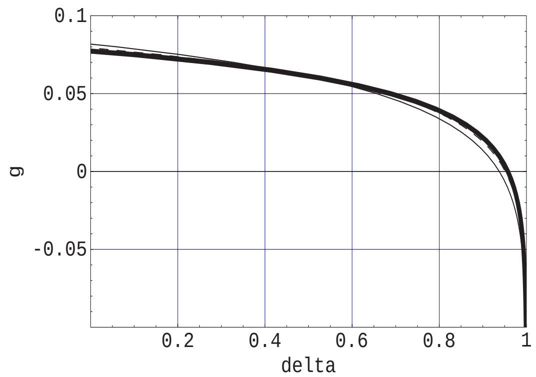

FIGURE 10. Relation between δ and g for different T with parameters (case (a)): $A_0 = 100; L_0 = 80; \alpha = 0.8; r = 0.05; \eta = 0.8; \sigma = 0.2; d = 1; T = 12$ (solid); $T = 18$ (dashing); $T = 24$ (thick).

Relation between the participation rate and the minimum guaranteed interest rate for a fair contract

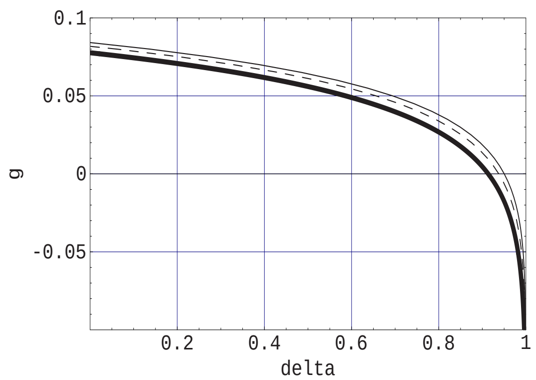

FIGURE 11. Relation between δ and g for different η with parameters (case (a)): $A_0 = 100; L_0 = 80; \alpha = 0.8; r = 0.05; T = 12; \sigma = 0.2; d = 1; \eta = 0.7$ (solid); $\eta = 0.8$ (dashing); $\eta = 0.9$ (thick).

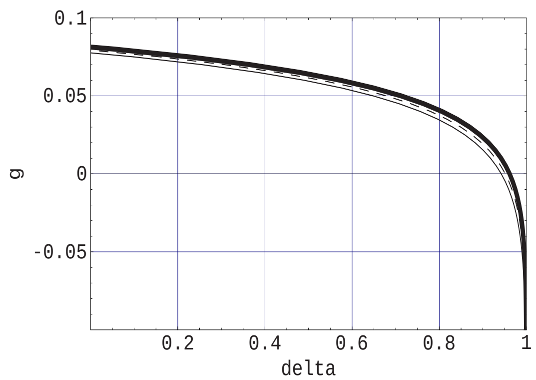

FIGURE 12. Relation between δ and g for different d with parameters (case (a)): $A_0 = 100; L_0 = 80; T = 12; \alpha = 0.8; \sigma = 0.2; \eta = 0.8; r = 0.05; d = 0.5$ (solid); $d = 1$ (dashing); $d = 2$ (thick).

Figure 12 illustrates the effect of the length of excursion d on the fair combination of δ and g. As in the standard Parisian case (c.f. Figure 4), the parameter d does not show

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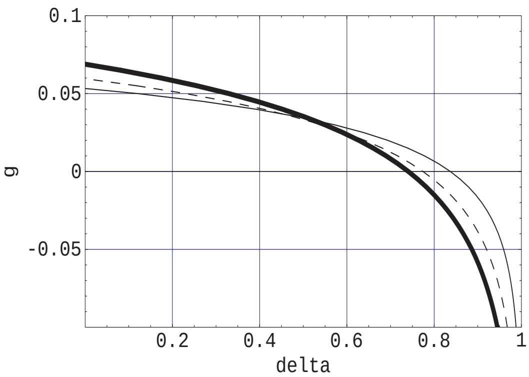

FIGURE 13. Relation between δ and g for different σ with parameters (case (b)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\eta = 1.2$; $T = 12$; $d = 1$; $\sigma = 0.15$ (solid); $\sigma = 0.20$ (dashing); $\sigma = 0.25$ (thick).

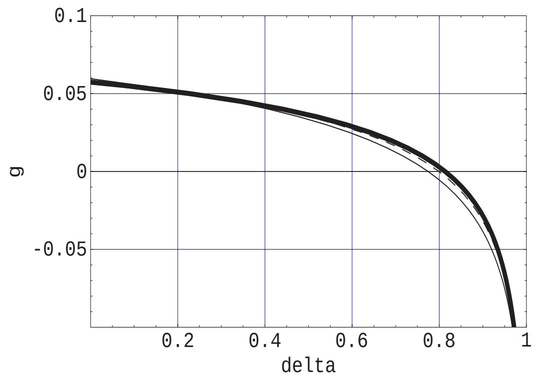

FIGURE 14. Relation between δ and g for different T with parameters (case (b)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\eta = 1.2$; $\sigma = 0.2$; $d = 1$; $T = 12$ (solid); $T = 18$ (dashing); $T = 24$ (thick).

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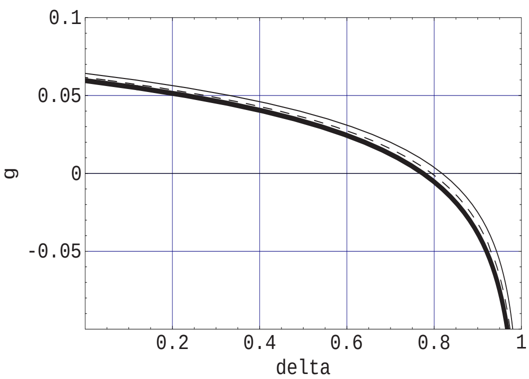

FIGURE 15. Relation between δ and g for different η with parameters (case (b)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $T = 12$; $\sigma = 0.2$; $d = 1$; $\eta = 1.1$ (solid); $\eta = 1.15$ (dashing); $\eta = 1.2$ (thick).

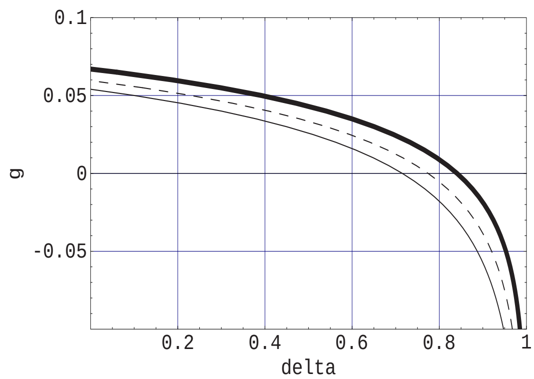

FIGURE 16. Relation between δ and g for different d with parameters (case (b)): $A_0 = 100$; $L_0 = 80$; $T = 12$; $\alpha = 0.8$; $\sigma = 0.2$; $\eta = 1.2$; $r = 0.05$; $d = 0.5$ (solid); $d = 1$ (dashing); $d = 2$ (thick).

a big influence (but bigger than in the standard Parisian case) on the fair combination of δ and g. All four parts of the payment change with d similarly to the standard Parisian case, namely the cumulative Parisian down–and–out call, the cumulative Parisian down–and–out put and the expected fixed payment go up when d is increased (positive effect). The opposite is true for the rebate part (negative effect). However, the magnitude of the changes in the values is bigger.

Figures 13–16 are plotted for the case where $\eta \in [1, \frac{1}{\alpha}]$. This parameter choice leads to a considerably higher barrier level, which reduces the values of the cumulative Parisian down–and–out call, the cumulative Parisian down–and–out put and of the expected payment to a big extent and increases the expected rebate part (c.f. Tables 2–4). Since Figures 13–16 are quite similar to Figures 5–8, we do not discuss them in detail.

3.2. **Value Decomposition For Fair Contracts.** In the above numerical analysis, it could be noticed that the choice of the η–parameter influences the effect of d. In the following, the separate effect of d and η is analyzed through some tables. In Tables 2–4 it is investigated how the fair participation rate and the different components of the liability holder's and the equity holder's payoff change with the length of excursion d for different η–values. Since we do not want to repeat the results of the last subsection, we just mention several important aspects and concentrate on the liability holder's claims. First, assume that the regulation parameter is set to be zero, which results in a barrier level of zero. It then follows that the length of excursion d has no effect on the components of the liability holder's payoff, because the asset price can never hit the barrier in this situation due to the log–normal assumption of the asset dynamics. That means, the insurance company never defaults and hence is never liquidated. Then we are back in the standard call and put case. Therefore, we obtain the same values for the standard and cumulative Parisian option, and also for Grosen and Jørgensen's (2002) case. Second, except in this extreme case, smaller participation rates result from the cumulative Parisian option framework than from the standard Parisian modelling given the same parameters. Obviously, for the same parameters, the cumulative down–and–out contingent claims exhibit smaller values than the standard Parisian ones. Third, we emphasize here that the effect of η is twofold. On the one hand, an increase in η leads to a rise of barrier level, which accelerates the default of the company, especially when d is set to a small value. On the other hand, a larger expected rebate results from a higher η. Finally, we summarize how different combinations of d and η affect the different components of the liability holder's payoff. If small η's ($\eta = 0.8 \,\mathrm{or}\, 0.9$) are combined with long d's (e.g. $d = 5$), the probability that the firm defaults before the maturity date is small. Hence, very high bonus values, very high expected fixed payments and very small rebate values are observed. As the barrier level rises gradually, the default probability climbs up, and so does the expected rebate. However, in the other extreme case, where high barrier levels (e.g. for the cases $\eta = 1.1$ and $\eta = 1.2$) are combined with a very short length of excursion (say $d = 0.25$ in a 20–year contract), relatively small bonus values, small fixed payments and relatively large expected rebate payments result.

4. Shortfall Probability

Until now we have not raised the question of how attractive the issued contract is to the liability holder. The liability holder might be interested in getting to know with exactly what probability he will get the rebate payment at the liquidation time instead of the contract value at the maturity date. Therefore, in this section, we would like to have a look at the shortfall probability, i.e., the probability of an early liquidation (liquidation occurs before the maturity date).

Obviously, it only makes sense to consider the shortfall probability under the subjective probability measure, under which the assets are assumed to evolve as follows:

$$dA_t = A_t(\mu\, dt + \sigma d\widetilde{W}_t)$$

where $\mu > 0$ is the instantaneous expected return of the asset and $\widetilde{W}_t$ is a martingale under the subjective measure. In the case of the standard Parisian framework, the shortfall probability is given by

$$\begin{aligned}\widetilde{P}^{SF} &= \widetilde{P}(T_B^- = \inf\{t > 0 | t - g_{B,t}^A 1_{\{A_t < B_t\}} > d\} \leq T) \\ &= e^{-\frac{\tilde{m}^2 T}{2}} \left(\int_{-\infty}^{b} h_2(T, y) e^{\tilde{m} y}\, d\, y + \int_b^{\infty} h_1(T, y) e^{\tilde{m} y}\, d\, y \right)\end{aligned}$$

with $\tilde{m} = \frac{1}{\sigma}(\mu - g - \frac{1}{2}\sigma^2)$.

In case of the cumulative Parisian framework, the shortfall probability is determined by

$$\begin{aligned}\widetilde{P}^{SF} &= \widetilde{P}(\tau \leq T) = \widetilde{P}\left(\frac{1}{T} \int_0^T \mathbf{1}_{\{\widetilde{W}_u + \tilde{m}\, u \leq b\}} du \geq \frac{d}{T} \right) \\ &= \widetilde{P}\left(\frac{1}{T} \int_0^T \mathbf{1}_{\{\widetilde{W}_u - \tilde{m}\, u \leq -b\}} du \leq 1 - \frac{d}{T} \right) \\ &= 2 \int_0^{1 - \frac{d}{T}} \left\{ \left[\frac{n(-\tilde{m}\sqrt{T}\sqrt{1-u})}{\sqrt{1-u}} + (-\tilde{m}\sqrt{T}) N(-\tilde{m}\sqrt{T}\sqrt{1-u}) \right] \right. \\ &\quad \left. \cdot \left[\frac{1}{\sqrt{u}} n\left(\frac{(-b)/\sqrt{T} + \tilde{m}\sqrt{T} u}{\sqrt{u}} \right) + \tilde{m}\sqrt{T} e^{2\tilde{m} b} N\left(\frac{b/\sqrt{T} + \tilde{m}\sqrt{T} u}{\sqrt{u}} \right) \right] \right\} du.\end{aligned}$$

In the above derivation, Equation (12) of Takács (1996) is applied. In Table 5, several shortfall probabilities are calculated for both standard and cumulative Parisian frameworks. First, apparently, shortfall occurs with a higher probability in the case of cumulative than in that of standard Parisian options. This is due to the fact that the knock–out

	d	δ	BO	SP	CFP	RL	V_L	RC	SBO	RE	V_E
$\eta = 0 \Rightarrow$ Barrier $= 0$											
GJ		0.951	41.49	-5.39	43.90	0.00	80.00	61.49	-41.49	0.00	20.00
PA	0-T	0.951	41.49	-5.39	43.90	0.00	80.00	61.49	-41.49	0.00	20.00
CP	0-T	0.951	41.49	-5.39	43.90	0.00	80.00	61.49	-41.49	0.00	20.00
$\eta = 0.8 \Rightarrow$ Barrier $= 64$											
GJ		0.836	30.91	-0.03	19.84	29.28	80.00	50.91	-30.91	0.00	20.00
PA	0	0.836	30.91	-0.03	19.84	29.28	80.00	50.91	-30.91	0.00	20.00
	0.25	0.888	35.60	-0.15	24.13	20.42	80.00	55.60	-35.60	0.00	20.00
	0.50	0.902	36.94	-0.23	25.83	17.46	80.00	56.94	-36.94	0.00	20.00
	1.00	0.917	38.38	-0.40	28.29	13.73	80.00	58.38	-38.38	0.00	20.00
	1.50	0.926	39.20	-0.57	30.23	11.14	80.00	59.20	-39.20	0.00	20.00
	2.00	0.932	39.75	-0.73	31.41	9.57	80.00	59.75	-39.75	0.00	20.00
	5.00	0.945	41.02	-1.94	36.46	4.46	80.00	61.02	-41.02	0.00	20.00
CP	0	0.836	30.91	-0.03	19.84	29.28	80.00	50.91	-30.91	0.00	20.00
	0.25	0.874	34.26	-0.09	22.75	23.08	80.00	54.26	-34.26	0.00	20.00
	0.50	0.886	35.41	-0.13	23.93	20.79	80.00	55.41	-35.41	0.00	20.00
	1.00	0.901	36.81	-0.22	25.59	17.82	80.00	56.81	-36.81	0.00	20.00
	1.50	0.910	37.72	-0.30	26.85	15.73	80.00	57.72	-37.72	0.00	20.00
	2.00	0.917	38.39	-0.40	27.90	14.10	80.00	58.39	-38.39	0.00	20.00
	5.00	0.938	40.39	-1.03	32.43	8.21	80.00	60.39	-40.39	0.00	20.00
$\eta = 0.9 \Rightarrow$ Barrier $= 72$											
GJ		0.743	23.87	0.00	15.23	40.90	80.00	43.87	-23.87	0.00	20.00
PA	0	0.743	23.87	0.00	15.23	40.90	80.00	43.87	-23.87	0.00	20.00
	0.25	0.840	31.27	-0.04	20.14	28.63	80.00	51.27	-31.27	0.00	20.00
	0.50	0.865	33.44	-0.09	21.91	24.74	80.00	53.44	-33.44	0.00	20.00
	1.00	0.891	35.83	-0.20	24.23	20.24	80.00	55.83	-35.83	0.00	20.00
	1.50	0.905	37.23	-0.32	26.79	16.30	80.00	57.23	-37.23	0.00	20.00
	2.00	0.915	38.17	-0.44	29.10	13.17	80.00	58.17	-38.17	0.00	20.00
	5.00	0.940	40.52	-1.42	33.63	7.27	80.00	60.52	-40.52	0.00	20.00
CP	0	0.743	23.87	0.00	15.23	10.90	80.00	43.87	-23.87	0.00	20.00
	0.25	0.814	29.11	-0.02	18.47	32.44	80.00	49.11	-29.11	0.00	20.00
	0.50	0.836	30.93	-0.04	19.78	29.32	80.00	50.93	-30.93	0.00	20.00
	1.00	0.862	33.19	-0.08	21.63	25.26	80.00	53.19	-33.19	0.00	20.00
	1.50	0.878	34.68	-0.13	23.03	22.42	80.00	54.68	-34.68	0.00	20.00
	2.00	0.890	35.80	-0.18	24.21	20.18	80.00	55.80	-35.80	0.00	20.00
	5.00	0.925	39.24	-0.63	29.30	12.09	80.00	59.24	-39.24	0.00	20.00

Table 2. **Decomposition of fair contracts with** $A_0 = 100$, $r = 0.05$, $g = 0.02$, $\alpha = 0.8$, $\sigma = 0.2$, $T = 20$.

$\eta = 1 \Rightarrow$ Barrier = 80											
	d	δ	BO	SP	CFP	RL	V_L	RC	SBO	RE	V_E
GJ		0.569	14.50	0.00	10.71	54.79	80.00	34.50	-14.50	0.00	20.00
PA	0	0.569	14.50	0.00	10.71	54.79	80.00	34.50	-14.50	0.00	20.00
	0.25	0.764	25.29	-0.01	16.19	38.53	80.00	45.29	-25.29	0.00	20.00
	0.50	0.807	28.54	-0.02	18.40	33.08	80.00	48.54	-28.54	0.00	20.00
	1.00	0.851	32.19	-0.09	20.26	27.64	80.00	52.19	-32.19	0.00	20.00
	1.50	0.875	34.36	-0.18	21.95	23.87	80.00	54.36	-34.36	0.00	20.00
	2.00	0.891	35.83	-0.28	25.99	18.46	80.00	55.83	-35.83	0.00	20.00
	5.00	0.931	39.70	-0.93	29.68	11.55	80.00	59.70	-39.70	0.00	20.00
CP	0	0.569	14.50	0.00	10.71	54.79	80.00	34.50	-14.50	0.00	20.00
	0.25	0.715	22.10	0.00	14.22	43.69	80.00	42.10	-22.10	0.00	20.00
	0.50	0.756	24.77	-0.01	15.65	39.59	80.00	44.77	-24.77	0.00	20.00
	1.00	0.801	28.12	-0.03	17.65	34.26	80.00	48.12	-28.12	0.00	20.00
	1.50	0.829	30.36	-0.05	19.18	30.51	80.00	50.36	-30.36	0.00	20.00
	2.00	0.848	32.05	-0.08	20.46	27.57	80.00	52.05	-32.05	0.00	20.00
	5.00	0.906	37.44	-0.36	26.02	16.91	80.00	57.44	-37.44	0.00	20.00
$\eta = 1.1 \Rightarrow$ Barrier = 88											
GJ		0.540	9.10	0.00	6.31	64.58	80.00	22.64	-9.10	6.46	20.00
PA	0	0.540	9.10	0.00	6.31	64.58	80.00	22.64	-9.10	6.46	20.00
	0.25	0.659	18.23	0.00	12.05	49.72	80.00	37.53	-18.23	0.70	20.00
	0.50	0.726	22.43	0.00	14.28	43.29	80.00	42.09	-22.43	0.34	20.00
	1.00	0.795	27.41	-0.02	17.32	35.27	80.00	47.26	-27.41	0.15	20.00
	1.50	0.833	30.52	-0.08	19.70	29.28	80.00	50.42	-30.52	0.10	20.00
	2.00	0.858	32.70	-0.18	21.73	25.57	80.00	52.64	-32.70	0.06	20.00
	5.00	0.918	38.47	-0.48	28.13	13.40	80.00	58.45	-38.47	0.02	20.00
CP	0	0.540	9.10	0.00	6.31	64.58	80.00	22.64	-9.10	6.46	20.00
	0.25	0.612	14.97	0.00	10.06	54.97	80.00	33.06	-14.97	1.91	20.00
	0.50	0.666	18.06	0.00	11.58	50.36	80.00	36.77	-18.06	1.29	20.00
	1.00	0.731	22.26	-0.01	13.72	44.03	80.00	41.44	-22.26	0.82	20.00
	1.50	0.771	25.21	-0.02	15.34	39.47	80.00	44.60	-25.21	0.61	20.00
	2.00	0.799	27.48	-0.03	16.71	35.84	80.00	46.99	-27.48	0.49	20.00
	5.00	0.882	35.04	-0.20	22.66	22.50	80.00	54.84	-35.04	0.20	20.00

TABLE 3. **Decomposition of fair contracts with** $A_0 = 100$, $r = 0.05$, $g = 0.02$, $\alpha = 0.8$, $\sigma = 0.2$, $T = 20$.

$\eta = 1.2 \Rightarrow$ Barrier = 96											
	d	δ	BO	SP	CFP	RL	V_L	RC	SBO	RE	V_E
GJ		0.514	3.16	0.00	2.07	74.77	80.00	8.21	-3.16	14.95	20.00
PA	0	0.514	3.16	0.00	2.07	74.77	80.00	8.21	-3.16	14.95	20.00
	0.25	0.580	12.00	0.00	8.14	59.86	80.00	27.87	-12.00	4.13	20.00
	0.50	0.645	16.20	0.00	10.53	53.27	80.00	33.94	-16.20	2.26	20.00
	1.00	0.737	22.18	0.00	13.65	44.17	80.00	41.09	-22.18	1.09	20.00
	1.50	0.782	25.91	-0.02	16.08	38.01	80.00	45.20	-25.91	0.71	20.00
	2.00	0.818	28.83	-0.09	18.42	32.75	80.00	48.32	-28.83	0.51	20.00
	5.00	0.901	36.76	-0.10	24.71	18.53	80.00	56.61	-36.76	0.15	20.00
CP	0	0.514	3.16	0.00	2.07	74.77	80.00	8.21	-3.16	14.95	20.00
	0.25	0.561	9.16	0.00	6.00	64.84	80.00	21.91	-9.16	7.25	20.00
	0.50	0.607	12.09	0.00	7.60	60.30	80.00	26.82	-12.09	5.28	20.00
	1.00	0.677	16.57	0.00	9.85	53.58	80.00	33.04	-16.57	3.53	20.00
	1.50	0.725	19.96	0.00	11.56	48.48	80.00	37.28	-19.96	2.68	20.00
	2.00	0.759	22.67	-0.01	13.00	44.33	80.00	40.52	-22.68	2.16	20.00
	5.00	0.862	32.24	-0.10	19.27	28.59	80.00	51.33	-32.24	0.91	20.00

TABLE 4. **Decomposition of fair contracts with** $A_0 = 100, r = 0.05$, $g = 0.02$, $\alpha = 0.8$, $\sigma = 0.2$, $T = 20$.

	Shortfall Probability PA						Shortfall Probability CP					
σ	μ		η		d		μ		η		d	
	0.06	0.08	0.9	1.1	0.5	2	0.06	0.08	0.9	1.1	0.5	2
0.10	0.013	0.000	0.000	0.000	0.000	0.000	0.018	0.003	0.010	0.093	0.004	0.002
0.15	0.125	0.052	0.092	0.240	0.068	0.035	0.159	0.070	0.124	0.318	0.085	0.053
0.20	0.289	0.180	0.251	0.403	0.222	0.132	0.347	0.227	0.308	0.507	0.259	0.184

TABLE 5. Shortfall probabilities for standard and cumulative Parisian frameworks with parameters: $A_0 = 100, L_0 = 80$, $g = 0.02, T = 20$, $\mu = 0.08$, $\sigma = 0.2$, $\eta = 0.8$, $d = 1$.

condition is less demanding for the cumulative Parisian option. All the other effects, e.g. that the shortfall probability increases in σ and η and decreases in μ and d, are quite straightforward. Therefore, the insurance company can offer customers with different risk aversions (willingness to accept a certain shortfall probability) different insurance contracts according to varying parameter choices.

5. Conclusion

In the present article, we extend the model of Grosen and Jørgensen (2002) and investigate the question of how to value an equity–linked life insurance contract when considering the default risk (and the liquidation risk) under different bankruptcy procedures. In order to take into account the realistic bankruptcy procedure Chapter 11, these risks are modelled in both standard and cumulative Parisian frameworks. In the numerical analysis part, we perform several sensitivity analyses to see how the fair combinations of the participation rate and the minimum interest rate guarantee depend on the volatility of the company's assets, the maturity dates of the contract, the regulation parameter and the length of excursion. In addition, due to their importance, a number of tables are given which help to catch and to compare the effects of the two regulation parameters d and η. Furthermore, we consider how likely it is that the liability holder will obtain the rebate payment whose size is uncertain at the point in time when the contract is signed. Based on the analysis in Section 4, the insurance company can offer different contracts to customers with different willingness to accept certain shortfall probabilities.

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