BONN ECON DISCUSSION PAPERS

Discussion Paper 9/2002

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by

Klaus Schürger

May 2002

Bonn Graduate School of Economics
Department of Economics
University of Bonn
Adenauerallee 24 - 42
D-53113 Bonn

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Klaus Schürger

Department of Statistics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany.
(e-mail: schuerg@finasto.uni-bonn.de)

Abstract. Let $S = (S_t),\ t = 0, 1, \dots, T$ (T being finite), be an adapted $\mathbb{R}^d$-valued process. Each component process of S might be interpreted as the price process of a certain security. A trading strategy $H = (H_t),\ t = 1, \dots, T$, is a predictable $\mathbb{R}^d$-valued process. A strategy H is called extreme if it represents a maximal arbitrage opportunity. By this we mean that H generates at time T a nonnegative portfolio value which is positive with maximal probability. Let F^e denote the set of all states of the world at which the portfolio value at time T, generated by an extreme strategy (which is shown to exist), is equal to zero. We characterize those subsets of F^e, on which no arbitrage opportunities exist.

Key words: Arbitrage, martingale measure

JEL Classification: G12,G13,D40

Mathematics Subject Classification (1991): 60G42

1 Introduction

A remarkable result due to Dalang, Morton and Willinger (cf.[1]) says that a finite-dimensional price process $S = (S_t)$ $(t = 0, 1, \ldots, T)$ defined on some probability space $(\Omega, \mathcal{F}, P)$ admits no arbitrage opportunities iff there exists a probability measure Q equivalent to P, under which (S_t) is a martingale. This remarkable result (the "first fundamental asset pricing theorem"), sometimes augmented by additional equivalent conditions, has been proved in many different ways (cf. [6], [3], [5], [2], [7], [4]).

In order to formulate one version of the first fundamental asset pricing theorem (cf.[4]), let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $(\mathcal{F}_t)$, $t = 0, 1, \ldots, T$ (T being finite) such that $\mathcal{F}_T = \mathcal{F}$. Let $S = (S_t)$ be an adapted d-dimensional process, i.e. $S_t = (S_t^1, \ldots, S_t^d)$ is $\mathcal{F}_t$-measurable. If all components are positive, then S_t^i might be interpreted as the price of a certain stock i at time t. Let $\mathcal{P}$ be the set of all predictable $\mathbb{R}^d$-valued processes $H = (H_t)(1 \le t \le T)$ ("trading strategies"), i.e. $H_t = (H_t^1, \ldots, H_t^d)$ is $\mathcal{F}_{t-1}$-measurable. For $H \in \mathcal{P}$ put

$$H \bullet S_t = \sum_{k=1}^{t} H_k \Delta S_k, \quad 0 \le t \le T \quad (\Delta S_k := S_k - S_{k-1}).$$

The scalar product $H_k \Delta S_k$ might be interpreted as the increment (at time k) of the value of a portfolio consisting during the time period $]k-1, k]$ of H_k^i shares of stock i. Let $\mathcal{P}_{\ge}$ denote the set of all $H \in \mathcal{P}$ such that $H \bullet S_T \ge 0$ a.s. Let L_+^0 denote the set of random variables which are a.s. nonnegative, and put

$$A_T = \{H \bullet S_T - \phi : \ H \in \mathcal{P}, \ \phi \in L_+^0\},$$
$$L_+^{(\delta)} = \{\xi : \ \xi \in L_+^0, \ \mathcal{P}(\xi > 0) \le \delta\} \quad (0 \le \delta \le 1).$$

(S_t) is said to satisfy the *no-arbitrage condition* if

$$H \bullet S_T = 0 \text{ a.s.} \qquad \text{for all } H \in \mathcal{P}_{\ge}.$$

This is equivalent to the condition

$$\text{(NA)} \qquad A_T \cap L_+^0 = L_+^{(0)}.$$

By $\bar{A}_T$ we denote the closure of A_T with respect to convergence in probability.

We will need the following version of the first fundamental asset pricing theorem (cf. [4]):

1.1 Theorem. *The following conditions are equivalent:*

(a) *(NA).*
(b) $A_T \cap L_+^0 = L_+^{(0)}$ *and* $A_T = \bar{A}_T$.
(c) $\bar{A}_T \cap L_+^0 = L_+^{(0)}$.
(d) *There exists a probability measure* $Q \sim P$ *with* $dQ/dP \in L^\infty$ *such that* (S_t) *is a* Q*-martingale.*

Here, $Q \sim P$ means that Q and P are *equivalent* (i.e. have the same null sets).

We put

$$\alpha(S) = \sup_{H \in \mathcal{P}_{\geq}} P(H \bullet S_t > 0). \tag{1.1.1}$$

Clearly (S_t) satisfies the no-arbitrage condition iff $\alpha(S) = 0$.

We will be interested in strategies which are extreme according to

1.2 Definition. We say that a strategy $H \in \mathcal{P}_{\geq}$ is *extreme* if

$$P(H \bullet S_T > 0) = \alpha(S). \tag{1.2.1}$$

If $\alpha(S) > 0$, then we might say that an extreme strategy represents a *maximal arbitrage opportunity.* It will be shown later (cf. Theorem 2.1) that an extreme strategy always exists, and that the set $\{H^e \bullet S_T > 0\}$ does (a.s.) *not* depend on the choice of the extreme strategy H^e. In the sequel H^e will always denote an extreme strategy.

This paper is devoted to characterizing all subsets F of the set $F^e = \{H^e \bullet S_T = 0\}$ such that (S_t) satisfies the *no-arbitrage condition on* F (this terminology will be explained below). In particular we shall determine the largest subset of F^e with this property. In general one cannot expect that (S_t) satisfies the no-arbitrage condition on F^e (assuming that $0 < P(F^e) < 1$). In fact, suppose e.g. that (S_t) is a P-martingale. Knowing in advance that the event F^e occurs may provide "too much information" on the evolution of (S_t). In that case the restriction of the process (S_t) to F^e is not a martingale. We will derive conditions which are necessary and sufficient for (S_t) to satisfy the no-arbitrage condition on F^e (cf. Theorem 2.14).

We shall close this section with explaining the terminology used in the last paragraph. If $\mathcal{G} \subset \mathcal{F}$ is any σ-algebra and $\emptyset \neq F \in \mathcal{F}$, then $F \cap \mathcal{G} := \{F \cap G : G \in \mathcal{G}\}$ denotes the *trace* of $\mathcal{G}$ on F ; $(F \cap \mathcal{F}_t)$ is the *trace* of the filtration $(\mathcal{F}_t)$ on F. Let $\mathcal{P}(F)$ denote the set of $\mathbb{R}^d$-valued processes $H = (H_t)$ $(1 \leq t \leq T)$ defined on F, which are predictable with respect to $(F \cap \mathcal{F}_t)$.

If $P(F) > 0$, then $\mathcal{P}_{\geq}(F)$ is the set of all $H \in \mathcal{P}(F)$ such that

$$H \bullet (S|F)_T \geq 0 \quad P(\cdot|F) - \text{a.s.}$$

($S|F = (S_t|F)$ denoting the process S restricted to F). Note that $S|F$ is adapted to $(F \cap \mathcal{F}_t)$. Correspondingly we say that S satisfies the *no-arbitrage condition on* F or (for short): NA *holds on* F if

$$H \bullet (S|F)_T = 0 \quad P(\cdot|F) - \text{a.s. for all } H \in \mathcal{P}_{\geq}(F).$$

For sets $A, B \in \mathcal{F}$ we write $A = B$ a.s. ($A \subset B$ a.s.) if, for their indicator functions I_A, I_B, we have $I_A = I_B$ a.s. ($I_A \leq I_B$ a.s.).

2 Characterization of sets on which NA holds

We show first that an extreme strategy always exists:

2.1 Theorem. (a) *An extreme strategy always exists.*
(b) *If H^* and H^{**} are extreme strategies, then*

(2.1.1) $$\{H^* \bullet S_T > 0\} = \{H^{**} \bullet S_T > 0\} \ a.s.$$

(c) *For any extreme strategy H^e we have*

(2.1.2) $$\{H \bullet S_t > 0\} \subset \{H^e \bullet S_t > 0\} \ a.s., \quad H \in \mathcal{P}_{\geq}.$$

Proof. (a) Let $H^{(n)} \in \mathcal{P}_{\geq}$ be such that

(2.1.3) $$P\Big(H^{(n)} \bullet S_t > 0\Big) \to \alpha(S) \quad (n \to \infty).$$

Put ($|\cdot|$ denoting the Euclidean norm)

$$\xi_n = \sum_{k=1}^{T} |H_k^{(n)}|$$

and choose numbers $c_n > 0$ such that $P(\xi_n > c_n) \leq 2^{-n}$. The Borel-Cantelli lemma implies that

$$\eta_t := \sum_{n=1}^{\infty} \frac{1}{c^n 2^n} \, |H_t^{(n)}| < \infty \text{ a.s.}, \quad 1 \leq t \leq T.$$

Let $A_t := \{\eta_t < \infty\}$. Then $A_t \in \mathcal{F}_{t-1}$,

$$H_t^e := I_{A_t} \sum_{n=1}^{\infty} \frac{1}{c_n 2^n} \, H_t^{(n)}$$

is $\mathcal{F}_{t-1}$-measurable, and $H^e = (H_t^e) \in \mathcal{P}_{\geq}$. Since $P(H^e \bullet S_t > 0) \geq P(H^{(n)} \bullet S_T > 0)$ for all n, H^e is extreme by (2.1.3).
(b) If (2.1.1) does not hold, then

$$P\Big((H^* + H^{**}) \bullet S_T > 0\Big) > \alpha(S)$$

which is impossible since $H^* + H^{**} = (H_t^* + H_t^{**}) \in \mathcal{P}_{\geq}$. (c) is proved in the same way as (b). □

In the sequel H^e will always denote an extreme strategy. Note that, according to Theorem 2.1(b), the set $\{H^e \bullet S_t = 0\}$ does (a.s.) not depend on the choice of H^e.

The following example shows that, in general, an extreme strategy is not uniquely determined (up to multiplication by positive constants).

2.2 Example. ($T = d = 1$). Let $\Omega = \{\omega_1, \dots, \omega_4\}$, $\mathcal{F} = \{0,1\}^\Omega$ (the power set of Ω), and suppose that $P\{\omega\} > 0$, $\omega \in \Omega$. The filtration is given by $\mathcal{F}_0 = \sigma\Big\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\Big\}$, $\mathcal{F}_1 = \mathcal{F}$. Let $S_0 \equiv 0$, $S_1(\omega_1) = 1$, $S_1(\omega_2) = S_1(\omega_3) = 0$, $S_1(\omega_4) = -1$. Then the strategies H^*, H^{**} given by

$$\begin{array}{lcl} H_1^*(\omega_1) = H_1^*(\omega_2) = 1 & , & H_1^*(\omega_3) = H_1^*(\omega_4) = -1, \\ H_1^{**}(\omega_1) = H_1^{**}(\omega_2) = 2 & , & H_1^{**}(\omega_3) = H_1^{**}(\omega_4) = -1 \end{array}$$

are extreme, and $\{H^* \bullet S_1 > 0\} = \{H^{**} \bullet S_1 > 0\} = \{\omega_1, \omega_4\}$.

Our first characterization of sets on which NA holds is given by

2.3 Theorem. *Let $F \subset \{H^e \bullet S_T = 0\}$ be such that $P(F) > 0$. Then the following conditions are equivalent:*

(a) *NA holds on F.*

(b) *For every strategy $\widetilde{H} \in \mathcal{P}_\geq(F)$ there exists a strategy $H \in \mathcal{P}_\geq$ such that*

$$\widetilde{H} \bullet (S|F)_T = H \bullet S_T \text{ a.s. on } F. \tag{2.3.1}$$

Proof. (*b*) $\Rightarrow$ (*a*): Let $\widetilde{H} \in \mathcal{P}_\geq(F)$ be given, and let $H \in \mathcal{P}_\geq$ satisfy (2.3.1). Since H^e is extreme, we have $H \bullet S_T = 0$ a.s. on F, which, by (2.3.1), implies $\widetilde{H} \bullet (S|F)_T = 0$ a.s. on F.

(*a*) $\Rightarrow$ (*b*): Suppose that NA holds on F. This implies that, for any $\widetilde{H} \in \mathcal{P}_\geq(F)$, we have $\widetilde{H} \bullet (S|F)_T = 0$ a.s. on F. Hence (2.3.1) holds for $H = H^e$. □

2.4 Remark. Suppose that NA holds on F. Then, in general, it is not true that for any strategy $\widetilde{H} \in \mathcal{P}_\geq(F)$ there exists an extension $H \in \mathcal{P}_\geq$ of $\widetilde{H}$. In fact, in Example 2.2 we have that NA holds on $F^e = \{H^e \bullet S_T = 0\} = \{\omega_2, \omega_3\}$. Let $\widetilde{H} \in \mathcal{P}_\geq(F^e)$ given by $\widetilde{H}_1(\omega_2) = \widetilde{H}_1(\omega_3) = 1$. The only $\mathcal{F}_0$-measurable extension H of $\widetilde{H}$ is given by $H_1 \equiv 1$, and $H \notin \mathcal{P}_\geq$.

We shall now derive conditions for S to satisfy the no-arbitrage condition on a given set $F \subset \{H^e \bullet S_T = 0\}$, that involve martingale measures for S.

By $\mathcal{M}_S$ we denote the set of all probability measures Q on $\mathcal{F}$ such that $Q \ll P$ (i.e. Q is absolutely continuous with respect to P), and Q is a martingale measure for S, i.e. each component process (S_t^i) is a Q-martingale. If $\mathcal{M}_S \neq \emptyset$, then we put

$$\mu(S) = \sup_{Q \in \mathcal{M}_S} P\Big(\frac{dQ}{dP} > 0\Big). \tag{2.4.1}$$

2.5 Definition. A probability measure $Q \in \mathcal{M}_S$ is called *extreme* if

$$P\Big(\frac{dQ}{dP} > 0\Big) = \mu(S). \tag{2.5.1}$$

2.6 Theorem. *Suppose that $\mathcal{M}_S \neq \emptyset$.*
(a) *An extreme probability measure always exists.*
(b) *If Q^* and Q^{**} are extreme probability measures, then*

$$\left\{\frac{dQ^*}{dP} > 0\right\} = \left\{\frac{dQ^{**}}{dP} > 0\right\} \quad a.s. \tag{2.6.1}$$

(c) *For any extreme probability measure Q^e we have*

$$\left\{\frac{dQ}{dP} > 0\right\} \subset \left\{\frac{dQ^e}{dP} > 0\right\} \quad a.s., \quad Q \in \mathcal{M}_S. \tag{2.6.2}$$

Proof. (a) Let $Q_1, Q_2, \ldots \in \mathcal{M}_S$ be such that

$$P\Big(\frac{dQ_n}{dP} > 0\Big) \to \mu(S) \quad (n \to \infty). \tag{2.6.3}$$

Let $c_1 > 0, c_2 > 0, \ldots$ be real numbers such that $c_1 + c_2 + \cdots = 1$. It is easy to check that $Q := c_1 Q_1 + c_2 Q_2 + \cdots$ is a probability measure such that $Q \ll P$ and

$$\frac{dQ}{dP} = \sum_{n=1}^{\infty} c_n \frac{dQ_n}{dP}$$

which, by (2.6.3), implies

$$P\Big(\frac{dQ}{dP} > 0\Big) \geq \mu(S). \tag{2.6.4}$$

Let

$$\xi := 1 + \sum_{t=0}^{T} \sum_{i=1}^{d} |S_t^i|.$$

Then $1 \leq \mathrm{E}_{Q_n}[\xi] < \infty$ ($\mathrm{E}_{Q_n}[\xi]$ denoting the expectation of ξ, taken with respect to Q_n). Hence, for

$$c := \sum_{n=1}^{\infty} \frac{1}{2^n \, \mathrm{E}_{Q_n}[\xi]}$$

we have $0 < c \leq 1$. Therefore, choosing

$$c_n := (c \, 2^n \, \mathrm{E}_{Q_n}[\xi])^{-1}$$

implies $Q \in \mathcal{M}_S$. By (2.6.4), Q is extreme. It is clear that (b) and (c) hold. □

In the sequel, Q^e will always denote an extreme probability measure. Note that the set $\{dQ^e/dP > 0\}$ does (a.s.) not depend on the choice of Q^e.

Finally, we consider a third kind of extreme objects. Let

$$\beta(S) = \sup_{\zeta \in \bar{A}_T \cap L^0_+} P(\zeta > 0). \tag{2.6.5}$$

Clearly,

$$0 \le \alpha(S) \le \beta(S) \le 1. \tag{2.6.6}$$

Note that, by Theorem 1.1,

$$\alpha(S) = 0 \quad \text{implies} \quad \beta(S) = 0. \tag{2.6.7}$$

2.7 Definition. A random variable $\zeta \in \bar{A}_T \cap L^0_+$ is called *extreme* if

$$P(\zeta > 0) = \beta(S). \tag{2.7.1}$$

2.8 Theorem. (a) *There always exists an extreme random variable of* $\bar{A}_T \cap L^0_+$.
(b) *If* ζ^* *and* ζ^{**} *are extreme, then*

$$\{\zeta^* > 0\} = \{\zeta^{**} > 0\} \quad a.s. \tag{2.8.1}$$

(c) *For any extreme random variable* ζ^e *we have*

$$\{\zeta > 0\} \subset \{\zeta^e > 0\} \quad a.s., \quad \zeta \in \bar{A}_T \cap L^0_+. \tag{2.8.2}$$

Proof. (a) Let $\zeta_n \in \bar{A}_T \cap L^0_+$ be such that

$$P(\zeta_n > 0) \to \beta(S) \quad (n \to \infty). \tag{2.8.3}$$

Choose numbers $c_n > 0$ such that $P(\zeta_n > c_n) \le 2^{-n}$. The Borel-Cantelli lemma implies that

$$\zeta^e := \sum_{n=1}^{\infty} \frac{1}{c_n 2^n} \zeta_n < \infty \quad \text{a.s.}$$

Clearly, by (2.8.3), $P(\zeta^e > 0) \ge \beta(S)$. Since $\bar{A}_T \cap L^0_+$ is a closed convex cone, $\zeta^e \in \bar{A}_T \cap L^0_+$, and ζ^e is extreme. (b) and (c) are obvious. □

In the sequel, ζ^e will always denote an extreme random variable of $\bar{A}_T \cap L^0_+$. Note that the set $\{\zeta^e = 0\}$ does (a.s.) not depend on the choice of ζ^e. Theorem 2.8(c) implies that

$$\{\zeta^e = 0\} \subset \{H^e \bullet S_T = 0\} \quad \text{a.s.} \tag{2.8.4}$$

2.9 Theorem. *Let* $F \in \mathcal{F}$ *be such that* $P(F) > 0$. *Then there is equivalence between:*
(a) NA *holds on* F.
(b) *There exists a probability measure* $Q \in \mathcal{M}_S$ *such that* $F = \{dQ/dP > 0\}$ *a.s.*

Proof. *(a)* $\Rightarrow$ *(b)*: Assume that NA holds on F. By Theorem 1.1 there exists a probability measure $\widetilde{Q}$ on $F \cap \mathcal{F}$ such that $\widetilde{Q} \sim P(\cdot|F)$, and $(S_t|F)$ is a $\widetilde{Q}$-martingale (the filtration being $(F \cap \mathcal{F}_t)$). Let Q denote the probability measure which is defined on $\mathcal{F}$ by $Q(A) = \widetilde{Q}(F \cap A)$, $A \in \mathcal{F}$. It is easy to see that $Q \in \mathcal{M}_S$, and $F = \{dQ/dP > 0\}$ a.s.

(b) $\Rightarrow$ *(a)*: Let $Q \in \mathcal{M}_S$ and put $F = \{dQ/dP > 0\}$. Let $\widetilde{Q}$ be the restriction of Q to $F \cap \mathcal{F}$. Then $\widetilde{Q} \sim P(\cdot|F)$, and $(S_t|F)$ is a $\widetilde{Q}$-martingale. Hence, by Theorem 1.1, NA holds on F. □

2.10 Corollary. *Let* $Q^e \in \mathcal{M}_S$ *be extreme. Then, for every* F *on which* NA *holds, we have*

$$F = \left\{ \frac{dQ^e(\cdot|F)}{dP(\cdot|F)} > 0 \right\} \quad a.s. \tag{2.10.1}$$

Proof. Let $F \in \mathcal{F}$ be such that $P(F) > 0$. Clearly

$$\frac{dQ^e(\cdot|F)}{dP(\cdot|F)} = \frac{P(F)}{Q^e(F)} \cdot \frac{dQ^e}{dP} I_F \quad \text{a.s.}$$

which implies that

$$\left\{ \frac{dQ^e(\cdot|F)}{dP(\cdot|F)} > 0 \right\} = F \cap \left\{ \frac{dQ^e}{dP} > 0 \right\} \quad \text{a.s.}$$

Hence if NA holds on F, then (2.10.1) follows from Theorem 2.9 and Theorem 2.6(c). □

2.11 Corollary. (a) *Suppose that* $\mathcal{M}_S \neq \emptyset$. *Then we have*

$$\left\{ \frac{dQ}{dP} > 0 \right\} \subset \{\zeta^e = 0\} \quad a.s, \quad Q \in \mathcal{M}_S. \tag{2.11.1}$$

(b) *If* NA *holds on* F, *then* $F \subset \{\zeta^e = 0\}$ *a.s.*

Proof. (a) Let $Q \in \mathcal{M}_S$ and put $F = \{dQ/dP > 0\}$. It follows from Theorem 2.9 that NA holds on F. Let $H^{(n)} \in \mathcal{P}$ and $\phi_n \in L^0_+$ be such that

$$\xi_n := H^{(n)} \bullet S_T - \phi_n \to \zeta^e \quad \text{in probability.}$$

Applying Theorem 1.1 to the restriction of (ξ_n) to F, we obtain that $\zeta^e = 0$ a.s. on F. This proves (2.11.1).
(b) This follows from Theorem 2.9 and (a). □

If the definition of $\bar{A}_T \cap L^0_+$ is based on a probability measure P^* (instead of P) we shall write $\bar{A}_T \cap L^0_+[P^*]$ instead of $\bar{A}_T \cap L^0_+$.
Our main result is

2.12 Theorem. *Suppose that* $P\{\zeta^e = 0\} > 0$. *Then:*
(a) *There exists a probability measure* $Q^* \in \mathcal{M}_S$ *such that*

$$\left\{ \frac{dQ^*}{dP} > 0 \right\} = \{\zeta^e = 0\} \quad a.s. \tag{2.12.1}$$

and

$$\frac{dQ^*}{dP} \in L^\infty. \tag{2.12.2}$$

(b) *If* $Q^e \in \mathcal{M}_S$ *is extreme, then*

$$\left\{ \frac{dQ^e}{dP} > 0 \right\} = \{\zeta^e = 0\} \quad a.s.$$

(c) $\{\zeta^e = 0\}$ *is (a.s.) the largest set on which* NA *holds.*

Proof. (a) Let $G^e := \{\zeta^e = 0\}$. In the sequel, we shall assume that $P(G^e) < 1$. (If $P(G^e) = 1$, then the desired result follows from Theorem 1.1.) First note that

$$\xi \in \bar{A}_T \cap L^0_+ \quad \text{implies that } \xi = 0 \text{ a.s. on } G^e. \tag{2.12.3}$$

Let us show that, for the probability measure $P^* = P(\cdot|G^e)$ (defined on $\mathcal{F}$), we have

$$\zeta \in \bar{A}_T \cap L^0_+[P^*] \quad \text{implies that } \zeta = 0 \quad P^*\text{-a.s.} \tag{2.12.4}$$

In order to show this, let $\zeta \in \bar{A}_T \cap L^0_+[P^*]$ be fixed. Then $\zeta \geq 0$ a.s. on G^e, and there exist $H^{(n)} \in \mathcal{P}$ and random variables ϕ_n such that $\phi_n \geq 0$ a.s. on G^e and, for any $\delta > 0$,

$$P\left(\left\{|H^{(n)} \bullet S_T - \phi_n - \zeta| > \delta\right\} \cap G^e\right) \to 0 \quad (n \to \infty). \tag{2.12.5}$$

Choose numbers $a_n > 0$ such that

$$P\left(H^{(n)} \bullet S_T + a_n\zeta^e < -1/n \,\middle|\, \Omega \setminus G^e\right) \leq 2^{-n}, \quad n \geq 1. \tag{2.12.6}$$

Note that, by (2.12.5), for any $\delta > 0$,

$$P\left(\left\{|H^{(n)} \bullet S_T + a_n\zeta^e - \phi_n - \zeta| > \delta\right\} \cap G^e\right) \to 0 \quad (n \to \infty). \tag{2.12.7}$$

Applying the Borel-Cantelli lemma with respect to the probability measure $\widetilde{P} = P(\cdot|\Omega \setminus G^e)$ (defined on $\mathcal{F}$), we obtain, by (2.12.6), that

$$\widetilde{P}\left(H^{(n)} \bullet S_T + a_n\zeta^e \geq -1/n \quad \text{for all sufficiently large } n\right) = 1. \tag{2.12.8}$$

Put

$$\widetilde{\phi}_n = \begin{cases} \phi_n & \text{on} \quad G^e \\ (H^{(n)} \bullet S_T + a_n\,\zeta^e)^+ & \text{on} \quad \Omega \setminus G^e \end{cases}$$

and

$$\widetilde{\zeta} = \begin{cases} \zeta & \text{on} \quad G^e \\ 0 & \text{on} \quad \Omega \setminus G^e. \end{cases}$$

Let us show that

$$\widetilde{\zeta} \in \bar{A}_T \cap L^0_+. \tag{2.12.9}$$

In fact, (2.12.8) implies

$$H^{(n)} \bullet S_T + a_n\zeta^e - \widetilde{\phi}_n \to \widetilde{\zeta} \quad \widetilde{P}\text{-a.s.}$$

which, in turn, gives for any $\delta > 0$,

$$P\left(\left\{|H^{(n)} \bullet S_T + a_n\zeta^e - \widetilde{\phi}_n - \widetilde{\zeta}| > \delta\right\} \cap (\Omega \setminus G^e)\right) \to 0.$$

Combining this with (2.12.7) and passing to a subsequence (if necessary), we therefore arrive at

$$H^{(n)} \bullet S_T + a_n \zeta^e - \widetilde{\phi}_n \to \widetilde{\zeta} \quad \text{a.s.} \tag{2.12.10}$$

There exist $\widehat{H}^{(n)} \in \mathcal{P}$ and $\widehat{\phi}_n \in L^0_+$ such that

$$P\left(|\widehat{H}^{(n)} \bullet S_T - \widehat{\phi}_n - a_n \zeta^e| > 1/n\right) \le 2^{-n}, \quad n \ge 1.$$

Using the Borel-Cantelli lemma once more, shows that (2.12.10) entails (2.12.9) which, combined with (2.12.3), implies $\zeta = 0$ a.s. on G^e. This proves (2.12.4). Hence, by Theorem 1.1, there exists a probability measure

$$Q^* \sim P^* \tag{2.12.11}$$

such that (S_t) is a Q^*-martingale, and

$$dQ^*/dP^* \in L^\infty. \tag{2.12.12}$$

Since $P^* \ll P$, it follows from (2.12.11) that $Q^* \in \mathcal{M}_S$. Finally, (2.12.1) and (2.12.2) are easily obtained from (2.12.11) and (2.12.12). (b) follows from (a) and Corollary 2.11. (c) is a consequence of (b) and Theorem 2.9. □

At the end of this section we shall outline an alternative proof of Theorem 2.12(a) which is based on a generalization of a certain version of Yan's [8] theorem.

2.13 Corollary. *The following conditions are equivalent:*
(a) $\mathcal{M}_S \neq \emptyset$.
(b) $P(\zeta^e = 0) > 0$.

Proof. (a) ⇒ (b): This follows from Corollary 2.11.
(b) ⇒ (a): This follows from Theorem 2.12. □

2.14 Theorem. *Suppose that $P(H^e \bullet S_T = 0) > 0$. Then there is equivalence between:*
(a) NA *holds on* $\{H^e \bullet S_T = 0\}$.
(b) $\{H^e \bullet S_t = 0\} = \{\zeta^e = 0\}$ *a.s.*
(c) $\bar{A}_T \cap L^0_+ \subset L^{(\alpha(S))}_+$ *($\alpha(S)$ given by (1.1.1)).*

Proof. (a) ⇒ (b): Assume that NA holds on $\{H^e \bullet S_T = 0\}$. It follows from Theorem 2.9 and Corollary 2.11 that

$$\{H^e \bullet S_T = 0\} = \{dQ^*/dP > 0\} \subset \{\zeta^e = 0\} \quad \text{a.s.}$$

for some $Q^* \in \mathcal{M}_S$. This implies (b).
(b) ⇒ (a): This follows from Theorem 2.12.

(c) ⇔ (b): Note that (c) is equivalent to

$$P(\zeta^e > 0) = \alpha(S) = P(H^e \bullet S_T > 0).$$

By Theorem 2.8 this is equivalent to (b). □

The subsequent examples show that, in the case $0 < \alpha(S) < 1$, each of the relations $\alpha(S) = \beta(S)$, $\alpha(S) < \beta(S) < 1$ and $\beta(S) = 1$ is possible.

2.15 Example. ($T = 1, d = 3$; $0 < \alpha(S) < \beta(S) < 1$). Let $\Omega = \Omega_1 \cup \cdots \cup \Omega_4$ (a disjoint union) where $\Omega_1 = \{\omega_{1m} : m \geq 1\}$ and $\Omega_i = \{\omega_i\}$ $(i = 2, 3, 4)$. The filtration is given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F} = \{0,1\}^\Omega$. Let $P\{\omega\} > 0$, $\omega \in \Omega$. (S_t) is given by $S_0 \equiv (0,0,0)$, $S_1(\omega_{1m}) = (1/m, -1, 0)$ $(m \geq 1)$, $S_1(\omega_2) = (0,1,0)$, $S_1(\omega_3) = (0,0,1)$, $S_1(\omega_4) = (0,0,-1)$. A strategy H belongs to $\mathcal{P}_{\geq}$ iff $H_1 \equiv (a,0,0)$ for some $a \geq 0$. Hence $\{H^e \bullet S_1 > 0\} = \Omega_1$, and $0 < \alpha(S) < 1$. On $\Omega_2 \cup \Omega_3 \cup \Omega_4$ NA does not hold (consider $H \in \mathcal{P}_{\geq}(\Omega_2 \cup \Omega_3 \cup \Omega_4)$, given by $H_1 \equiv (0,1,0)$). On the other hand, NA holds on $\Omega_3 \cup \Omega_4 = \{dQ/dP > 0\}$, $Q \in \mathcal{M}_S$ given by $Q\{\omega_3\} = Q\{\omega_4\} = 1/2$. Hence $0 < \alpha(S) < \beta(S) < 1$. Modifying this example in an obvious way, gives an example for which $0 < \alpha/S) = \beta(S) < 1$.

2.16 Example. ($T = 1, d = 2$; $0 < \alpha(S) < \beta(S) = 1$). Let $\Omega = \Omega_1 \cup \Omega_2$ (a disjoint union) where $\Omega_1 = \{\omega_{1m} : m \geq 1\}$ and $\Omega_2 = \{\omega_2\}$. The filtration is given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F} = \{0,1\}^\Omega$. Let $P\{\omega\} > 0$, $\omega \in \Omega$. (S_t) is given by $S_0 \equiv (0,0)$, $S_1(\omega_{1m}) = (1/m, -1)$ $(m \geq 1)$ and $S_1(\omega_2) = (0,1)$. It is easy to see that $H \in \mathcal{P}_{\geq}$ implies $H \bullet S_1 = (0,0)$ on Ω_2. Furthermore, $\{H^e \bullet S_1 > 0\} = \Omega_1$. On the other hand, let $H^{(n)} \in \mathcal{P}$ and $\phi_n \in L^0_+$ be given by $H^{(n)}_1 \equiv (n,1)$, $\phi_n(\omega_{1m}) = (n/m - 2)^+$ $(m \geq 1)$, and $\phi_n(\omega_2) = 0$. Since $H^{(n)} \bullet S_1(\omega) - \phi_n(\omega) \to 1$ $(\omega \in \Omega)$, we have $P(\zeta^e > 0) = \beta(S) = 1$. Hence, by Corollary 2.13, $\mathcal{M}_S = \emptyset$.

We conclude this section with outlining an alternative proof of Theorem 2.12(a). We shall use the following result which generalizes a certain version of Yan's [8] theorem (compare Theorem 3.1 in [6]). Using similar arguments as in the proof of Theorem 3.1 in [6], we obtain

2.17 Theorem. *Let $0 \leq \epsilon < 1$ be fixed. Let $K \subset L^1$ be a convex cone which is closed with respect to the norm topology on L^1. Suppose that*

(2.17.1) $$K \supset -L^1_+ := \{\xi : -\xi \in L^1 \cap L^0_+\}$$

and

(2.17.2) $$K \cap L^1_+ \subset L^{(\epsilon)}_+.$$

Then there exists a random variable $\widehat{Z}$ such that

(2.17.3) $$0 \leq \widehat{Z} \leq 1 \quad a.s.,$$

(2.17.4) $$P(\widehat{Z} = 0) \leq \epsilon,$$

and

(2.17.5) $$\mathrm{E}[\xi \widehat{Z}] \leq 0, \quad \xi \in K.$$

If we have additionally that

$$K \cap L^1_+ \not\subset L^{(\delta)}_+ \quad \textit{for all} \quad 0 \le \delta < \epsilon, \tag{2.17.6}$$

then the above $\widehat{Z}$ *can be chosen in such a way that*

$$P(\widehat{Z} = 0) = \epsilon. \tag{2.17.7}$$

Proof of Theorem 2.12(a): We proceed similarly as in the proof of Theorem 1.1 in [4]. Fix any extreme random variable ζ^e. Choose any probability measure $\widehat{P} \sim P$ such that ζ^e and the random variables S^i_t are $\widehat{P}$-integrable, and

$$d\widehat{P}/dP \in L^\infty. \tag{2.17.8}$$

Note that $\bar{A}_T \cap L^0_+[\widehat{P}]$ equals $\bar{A}_T \cap L^0_+$, and ζ^e is also extreme with respect to $\widehat{P}$. Let $\widehat{\beta}(S) := \widehat{P}(\zeta^e > 0)$ and $K := \bar{A}_T \cap L^1(\widehat{P})$. Then K is a closed convex cone in $L^1(\widehat{P})$, and satisfies (2.17.1) as well as (2.17.2) and (2.17.6) for $\epsilon = \widehat{\beta}(S)$. According to Theorem 2.17 there exists a random variable $\widehat{Z}$ satisfying (2.17.3) and (2.17.5) with respect to $\widehat{P}$, and we have

$$\widehat{P}(\widehat{Z} = 0) = \widehat{\beta}(S). \tag{2.17.9}$$

Let $Q^* \ll \widehat{P}$ be given by $dQ^*/d\widehat{P} = \widehat{Z}$. Then, by (2.17.8), $dQ^*/dP \in L^\infty$. By (2.17.5), (S_t) is a Q^*-martingale, and $Q^* \in \mathcal{M}_S$. It follows from (2.17.9) and Corollary 2.11 that

$$\left\{ \frac{dQ^*}{dP} > 0 \right\} = \left\{ \frac{dQ^*}{d\widehat{P}} > 0 \right\} = \{\zeta^e = 0\} \quad \widehat{P} - \text{a.s.}$$

Since $\widehat{P} \sim P$, this shows that Q^* satisfies (2.12.1). □

3 Conclusion

The main objective of this paper has been to characterize those subsets of $F^e = \{H^e \bullet S_T = 0\}$, on which NA holds. In particular we showed that $\{\zeta^e = 0\}$ is the largest set with this property. Theorem 2.14 gives conditions which are necessary and sufficient for (S_t) to satisfy the no-arbitrage condition on F^e.

The intuitive reason for the fact that, in general, NA does not hold on F^e is the following: Knowing in advance that F^e occurs may provide "too much information" on the evolution of (S_t). Hence it would be interesting to find conditions equivalent to those in Theorem 2.14, which are formulated in terms of certain notions of information theory (e.g. entropy).

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