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Optimal Choice From Known Rewards
With Uncertain Response

by

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Abstract

Consider a random permutation of a finite number n of known elements representing rewards. These rewards will be made or not made with certain known probabilities. At any stage a reward made can be accepted or rejected, there is no recall and only one reward can be accepted. The problem is to maximize the expected reward accepted.

We propose a computationally feasible approximation to the solution of the dynamic programming equation of the problem. Estimates of the error of the approximation are given recursively as well as in explicit form. Recursively the error of the approximation at any stage of the game is obtained in terms of the approximation and its error estimate at the following stage.

In numerical examples the goodness of the approximation and its error estimates are found by comparison with the optimal solution.

Key words: optimal stopping, optimal policy approximations, dynamic programming, uncertainty of selection.

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1 Introduction and Statement of the Problem

For a given finite set of positive, distinct, real numbers $S = \{x_1, ..., , x_n\}$ we consider a random permutation $X_1, ..., X_n$ and 0-1-valued variables $I(X_1), ..., I(X_n)$ satisfying

$$P\{I(X_i) = 1 \mid X_i = x^{(i)}, I(X_{i-1}), X_{i-1}, ..., I(X_1), X_1\} =$$

$$P\{I(X_i) = 1 \mid X_i = x^{(i)}\} = p(x^{(i)}) = p_i$$

for $x^{(i)} \in S$, where $p_1, ..., p_n$ are known. This model arises in problems of optimal choice from a finite, partially known population. As an example we consider an individual who sends out applications to each of n institutions for a priori known rewards. Responses come back in random order. If the ith response is from the jth institution, it contains reward $x^{(j)}$ with probability p_j or 0 otherwise. Upon receiving this response the applicant can accept it and stop or reject it and continue. Only one offer can be accepted, there is no recall and it is desired to obtain the maximal amount that can be achieved.

We are interested in approximating well known solutions to the optimal stopping problem of finding a stopping time τ_0 such that

$$\mathbf{E}X_{\tau_0} I(X_{\tau_0}) \geq \mathbf{E}X_\tau I(X_\tau)$$

for all stopping times τ, where $\mathbf{E}$ denotes the expected value.

This problem is only superficially similar to the secretary problem with uncertain employment as treated by Smith (1975) (see Freeman (1983) for a review of the secretary problem). The secretaries are presented in random order and their ranks are unknown to the observer. In the above problem the ranks, or more generally their utilities, are known but the choices can be made only with certain probabilities depending on the observations. Another related type of problem is the asset selling problem (see Bertsekas (1988)). This however is usually posed as an infinite problem: offers of any size will be made eventually (but losses are incurred while waiting for the occurrence of an offer).

For the secretary problem as presented here, i.e. with known rankings in unknown random order and uncertain employment acceptance, much more information on the future is available to the player at any stage of the game and needs to be processed. This makes the present problem more complex than the others mentioned above. An explicit solution is not known and the numerical solution of the corresponding dynamic programming equation, given next, becomes computationally impossible for moderately large n.

Define a sequence of functions $V^{(n-1)}(y_n), V^{(n-2)}(y_{n-1}, y_n), ..., V^{(0)}(y_1, ..., y_n)$ where $y_1, ..., y_n$ are any permutation of $x^{(1)}, ..., x^{(n)}$:

$$\begin{aligned}
V^{(n-1)}(y_n) &= y_n p(y_n) \\
V^{(n-2)}(y_{n-1}, y_n) &= \mathbf{E}(\max\{X_{n-1}I(X_{n-1}), V^{(n-1)}(X_n)\} \mid X_{n-2} = y_{n-2}, ..., X_1 = y_1) \\
&= \frac{1}{2} \sum_{y \in \{y_{n-1}, y_n\}} \{(y - V^{(n-1)}(y))^+ p(y) + V^{(n-1)}(y)(1 - p(y))\} \\
&\vdots
\end{aligned}$$

where z^+ denotes the positive part of z. Then it is well known that

$$\tau_0 = \inf\{i : X_i I(X_i) \geq V^{(i)}(y_{i+1}, ..., y_n)\}$$

is the optimal rule. From this it is clear that in order to compute $V^{(0)}$, the value of the game, we need to find

$$\binom{n}{n-1} + \binom{n}{n-2} + ... + \binom{n}{1} = 2^n - 2$$

values for $V^{(1)}, ..., V^{(n-1)}$. It is therefore important, from a practical point of view, to find approximate rules that are computationally feasible for large n and to estimate the error of the approximation involved.

In the following section we construct an approximation to $V^{(0)}$ and to the optimal policy which at different degrees of precision can be implemented on computers of any size whereas the former cannot be implemented. We further obtain sharp upper bounds on the approximation. Moreover an estimate on the error of the approximation at any given stage of the game is obtained recursively in terms of the approximation and its error estimate at the following stage. In Section 3 it is numerically illustrated that the approximation can be surprisingly close, which is also reflected in the estimated error.

2 Approximation

The idea is to replace in the game $\{x_1, p(x_1), ..., x_n, p(x_n)\}$ certain groups of n_k states $x_{n_1}, ..., x_{n_k}$ by any one representative of them or by their mean and similary for their acceptance probabilities $p(x_{n_1}), ..., p(x_{n_k})$. These newly created states can recur n_k times. This way we obtain a computationally feasible approximation to the original problem, and to the error of the approximation.

2.1 Notation

More generally we introduce the alternative game $\{x_1^*, p(x_1^*), ..., x_n^*, p(x_n^*)\}$ where $x_k^* = x_k + \epsilon_k$ and $p(x_k^*) = p(x_k) + \delta_k$ for $k = 1, ..., n$ and the value of the optimal policy equal to $V_{\epsilon\delta}^{(0)}$. In here $\epsilon_k = \epsilon(x_k)$ and $\delta_k = \delta(x_k)$ are arbitrary such that $x_k^*, p(x_k^*) > 0$.

We also introduce the following notation: any state at stage $n-j$ of the game is characterized by a j-dimensional vector $\underline{x}$, respectively $\underline{x}^*$ with components $x_{n_1}, ..., x_{n_j}$ respectively $x_{n_1}^*, ..., x_{n_j}^*$ defining the future lying ahead of the stage. Consequently we denote the corresponding values by $V^{(n-j)}[\underline{x}]$ and $V_{\epsilon\delta}^{(n-j)}[\underline{x}^*]$. Also write $x \in \{\underline{x}\}$ instead of $x \in \{x_{n_1}, ..., x_{n_j}\}$ and with $\underline{x}_{-x}$ we denote the vector of dimension $j-1$ obtained by deleting the component x from $\underline{x}$ and similary for $\underline{x}^*_{-x^*}$. Moreover $\bar{p} = 1 - p$ and $x \vee y = \max\{x, y\}$.

With the above notation the optimal value $V^{(0)}$ can be written as the solution of the equation:

$$(j+1)V^{(n-j-1)}[\underline{x}] = \sum_{x > V^{(n-j)}[\underline{x}_{-x}]} (x \vee V^{(n-j)}[\underline{x}_{-x}])p(x) + \sum_{x \in \{\underline{x}\}} V^{(n-j)}[\underline{x}_{-x}]\bar{p}(x) \quad (2.1)$$

with initial conditions $V^{(n-1)}(x_k) = x_k p(x_k)$ for $k = 1, ..., n$ and similary for $V_{\epsilon\delta}^{(n-j+1)}[\underline{x}^*]$.

2.2 Global a Priori Error Estimates

Our first result gives a priori bounds on the approximation to the value $V^{(0)}$.

Theorem 2.1 For all $j = 1, ..., n$ and all $\underline{x}$ of dimension $n - j$

$$| V^{(n-j)}[\underline{x}] - V_{\epsilon\delta}^{(n-j)}[\underline{x}^*] | \leq \sum_{x \in \{\underline{x}\}} \{ | \delta(x) | x^* + | \epsilon(x) | p(x^*) \} .$$

For the proof of Theorem 2.1 we need the following lemma.

Lemma 2.1 If $c > 0$ then $a \vee (b + c) - a \vee b \leq c$.

Proof of Lemma 2.2:

$$a \vee (b + c) - a \vee b = \begin{cases} 0 & \text{if} \quad a > b + c \\ c & \text{if} \quad b > a \\ b + c - a & \text{if} \quad b + c > a > b \end{cases}$$

$\square$

Proof of Theorem 2.1: We first consider the case $p(x_k) = p(x_k^*)$ for all $1 \leq k \leq n$. We proceed by induction. Obviously

$$V_\epsilon^{(n-1)}[x^*] = V^{(n-1)}[x] + | \epsilon | p$$

where we have omitted the argument in $p(x)$. Likewise we omit the argument in $\epsilon(x)$ and $\delta(x)$ below. Now suppose that

$$V_\epsilon^{(n-j)}[\underline{x}^*] \leq V^{(n-j)}[\underline{x}] + \sum_{x \in \{\underline{x}\}} | \epsilon | p$$

holds for any given j, where $j < n$. Then from (2.1) we have

$$(j+1) V_\epsilon^{(n-j-1)}[\underline{x}^*] = \sum_{x^* \in \{\underline{x}^*\}} \left((x^* \vee V_\epsilon^{(n-j)}[\underline{x}^*_{-x^*}]) p + V_\epsilon^{(n-j)}[\underline{x}^*_{-x^*}] \bar{p} \right)$$

$$\begin{aligned} &\leq \sum_{x^* \in \{\underline{x}^*\}} \left((x^* \vee \{ V^{(n-j)}[\underline{x}_{-x}] + \sum_{\{\underline{x}_{-x}\}} | \epsilon | p \}) \, p + \{ V^{(n-j)}[\underline{x}_{-x}] + \sum_{\{\underline{x}_{-x}\}} | \epsilon | p \} \bar{p} \right) \\ &\leq \sum_{x \in \{\underline{x}\}} \left((x \vee V^{(n-j)}[\underline{x}_{-x}]) p + V^{(n-j)}[\underline{x}_{-x}] \bar{p} \right) + \sum_{x \in \{\underline{x}\}} \left(\sum_{x \in \{\underline{x}\}} | \epsilon | p \right) \ , \end{aligned}$$

where the first inequality holds by induction hypothesis and the second follows from a twofold application of Lemma 2.1. Hence, dividing by $j+1$, we have shown that $V_\epsilon^{(n-j)}[\underline{x}^*] - V^{(n-j)}[\underline{x}] \leq \sum_{x\in\{\underline{x}\}} (\mid \delta \mid x^* + \mid \epsilon \mid p^*)$. By symmetry this implies $\mid V^{(n-j)}[\underline{x}] - V^{(n-j)}[\underline{x}^*] \mid \leq \sum_{x\in\{\underline{x}\}} (\mid \delta \mid x^* + \mid \epsilon \mid p^*)$.

Secondly we consider the case $x_k = x_k^*$ for all $1 \leq k \leq n$. We proceed by induction. Obviously

$$V_\delta^{(n-1)}[x] = V^{(n-1)}[x] + \mid \delta \mid x \ .$$

Now suppose that

$$V_\delta^{(n-j)}[\underline{x}] \leq V^{(n-j)}[\underline{x}] + \sum_{x\in\{\underline{x}\}} \mid \delta \mid x$$

holds for any given j, where $j < n$. Then

$$(j+1)V_\delta^{(n-j-1)}[\underline{x}] = \sum_{x\in\{\underline{x}\}} \Big((x \vee V_\delta^{(n-j)}[\underline{x}_{-x}])p^* + V_\delta^{(n-j)}[\underline{x}_{-x}]\bar{p}^* \Big)$$

$$\begin{aligned}
\leq \quad & \sum_{x\in\{\underline{x}\}} \left((x \vee \{V^{(n-j)}[\underline{x}_{-x}] + \sum_{y\in\{\underline{x}_{-x}\}} \mid \delta \mid y\}\,)\,p^* + \{V^{(n-j)}[\underline{x}_{-x}] + \sum_{y\in\{\underline{x}_{-x}\}} \mid \delta \mid y\}\bar{p}^* \right) \\
\leq \quad & \sum_{x\in\{\underline{x}\}} \Big((x \vee V^{(n-j)}[\underline{x}_{-x}]\,)\,p + V^{(n-j)}[\underline{x}_{-x}]\bar{p} \Big) \\
+ \quad & \sum_{x\in\{\underline{x}\}} \left(\sum_{y\in\{\underline{x}_{-x}\}} (\mid \delta \mid y) + (x \vee V^{(n-j)}[\underline{x}_{-x}]\,)\,\delta - V^{(n-j)}[\underline{x}_{-x}]\delta \right) \\
\leq \quad & (j+1)V^{(n-j)}[\underline{x}] + \sum_{x\in\{\underline{x}\}} (\sum_{x\in\{\underline{x}\}} \mid \delta \mid x),
\end{aligned}$$

where the first inequality holds by induction hypothesis and the second follows from an application of Lemma 2.1. This implies by symmetry that $\mid V^{(n-j)}[\underline{x}] - V_\delta^{(n-j)}[\underline{x}^*] \mid \leq \sum_{x\in\{\underline{x}\}} \mid \delta \mid x$. The statement of the theorem now follows since

$$\mid V_{\epsilon\delta} - V \mid \leq \mid V_{\epsilon\delta} - V_\epsilon \mid + \mid V_\epsilon - V \mid .$$

□

Remark: It is easily seen that the above bounds are sharp in certain trivial cases. Consider for example the space $\{x_i = i \text{ for } i = 1, \ldots, n\}$ with probabilities $p(x_i) = 0$ except for one i.

We obtain the following reduction in dimensionality by grouping. In the full space $V^{(0)} = V^{(0)}(x_1, \ldots, x_n)$ is an average over $\binom{n}{n-1}$ values of $V^{(1)}(x_2, \ldots, x_n)$, each of which in turn is an average of $\binom{n}{n-2}$ values of $V^{(2)}(x_3, \ldots, x_n)$ etc. Accordingly $d = \binom{n}{n-1} + \binom{n}{n-2} + \ldots + \binom{n}{1} = 2^n - 2$ possible sequences of $V^{(1)}, \ldots, V^{(n-1)}$ have to be computed. In the space reduced by forming j subgroups of size $n_1, \ldots, n_j$ the total number of sequences of length $\leq j$ is reduced to $\prod_{i=1}^{j}(n_i + 1) - 2$ (We have $n_1 + 1$ choices for the number of elements for the first subgroup, $n_2 + 1$ choices for the number of elements for the second subgroup etc.). In the case when every subgroup consists of only one element, this, of course, yields the number of computations for the full space.

2.3 Local Iterative Error Estimates

The following theorem leads to another estimate of the error of the approximation.

Theorem 2.2 For any $1 \leq j \leq n$ the following inequalities hold.

$$\sum_{x^* > V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}]} (V^{(n-j)}[\underline{x}_{-x}] - V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}])\bar{p}^* + \sum_{x^* \leq V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}]} (V^{(n-j)}[\underline{x}_{-x}] - V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}])$$

$$- \sum_{x^* > V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}]} \epsilon p^* - \sum_{x > V^{(n-j)}[\underline{x}_{-x}]} (x - V^{(n-j)}[\underline{x}_{-x}])\delta$$

$$\leq (j+1)(V^{(n-j-1)}[\underline{x}] - V_{\epsilon\delta}^{(n-j-1)}[\underline{x}^*]) \leq$$

$$\sum_{x > V^{(n-j)}[\underline{x}_{-x}]} (V^{(n-j)}[\underline{x}_{-x}] - V^{(n-j)}[\underline{x}^*_{-x^*}])\bar{p}^* + \sum_{x \leq V^{(n-j)}[\underline{x}_{-x}]} (V^{(n-j)}[\underline{x}_{-x}] - V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}])$$

$$- \sum_{x > V^{(n-j)}[\underline{x}_{-x}]} \epsilon p^* - \sum_{x > V^{(n-j)}[\underline{x}_{-x}]} (x - V^{(n-j)}[\underline{x}_{-x}])\delta.$$

Proof of Theorem 2.2: Using (2.1) we obtain

$$(j+1)(V^{(n-j-1)}[\underline{x}] - V_{\epsilon\delta}^{(n-j-1)}[\underline{x}^*]) = \sum_{x > V^{(n-j)}[\underline{x}_{-x}]} (x - V^{(n-j)}[\underline{x}_{-x}])p$$

$$- \sum_{x^*>V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}]} (x^* - V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}])p^* + \sum_{x\in\{\underline{x}\}} (V^{(n-j)}[\underline{x}_{-x}] - V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}])$$

Dropping all superscripts from now on, the above sum can be shown to be equal to $\sum_{i=1}^{5} I_i$, where

$$\begin{aligned}
I_1 &= - \sum_{x(\epsilon)>V_{\epsilon\delta}[\underline{x}^*_{-x^*}]} (V[\underline{x}_{-x}] - V_{\epsilon\delta}[\underline{x}^*_{-x^*}])p^* \\
I_2 &= \sum_{x\in\{\underline{x}\}} V[\underline{x}_{-x}] - V_{\epsilon\delta}[\underline{x}^*_{-x^*}] \\
I_3 &= \left(\sum_{V[\underline{x}_{-x}]+\epsilon<x^*\leq V_{\epsilon\delta}[\underline{x}^*_{-x^*}]} - \sum_{V_{\epsilon\delta}[\underline{x}^*_{-x^*}]<x^*\leq V[\underline{x}_{-x}]+\epsilon} \right) (x - V[\underline{x}_{-x}])p^* \\
I_4 &= - \sum_{x>V[\underline{x}_{-x}]} (x - V[\underline{x}_{-x}])\delta \\
I_5 &= - \sum_{x^*>V_{\epsilon\delta}[\underline{x}^*_{-x^*}]} \epsilon p^*.
\end{aligned}$$

An upper bound to this expression is given as follows. First note that

$$I_3 \leq \left(\sum_{V[\underline{x}_{-x}]+\epsilon<x^*\leq V_{\epsilon\delta}[\underline{x}^*_{-x^*}]} - \sum_{V_{\epsilon\delta}[\underline{x}^*_{-x^*}]<x^*\leq V[\underline{x}_{-x}]+\epsilon} \right) (V_{\epsilon\delta}[\underline{x}^*_{-x^*}] - V[\underline{x}_{-x}] - \epsilon)p^*$$

This is absorbed into I_1 and I_5 and after combining I_1 with I_2 we obtain the upper bound described in the theorem. To obtain the lower bound we observe that $I_3 \geq 0$; hence it can be omitted. □

A particular case of this theorem arises when for all $j \geq j_0$ for some j_0 we have $x \leq V^{(n-j)}[\underline{x}_{-x}]$ and $x^* \leq V^{(n-j)}[\underline{x}^*_{-x^*}]$ at least for those x, x^* for which $x \neq x^*$ or $p(x) \neq p(x^*)$. This condition might well be satisfied at early stages of the game i.e. when future expectations exceed the returns from any of the states which are lumped into subgroups by the approximation. For this case the inequalities take a particularly simple form such that an alternative upper bound in $\mid V^{(0)} - V_{\epsilon\delta}^{(0)} \mid$ can be computed. This bound is sharper than that given by Theorem 2.1 and is obtained by simply solving the iterated relations explicitly from stage j_0 onwards. This is the content of the following Corollary.

Corollary 2.2 Suppose that $x > V^{(n-j_0)}[\underline{x}_{-x}]$ for some j_0 and for all $\underline{x}$, and similarly $x(\epsilon) > V_{\epsilon\delta}^{(n-j_0)}[\underline{x}^*_{-x^*}]$. Then

$$| V^{(0)} - V_{\epsilon\delta}^{(0)} | \leq \frac{1}{\binom{n}{j_0}} \sum | V^{(n-j_0)} - V_{\epsilon\delta}^{(n-j_0)} |$$

where the sum is taken over all $\binom{n}{j_0}$ combinations of j_0 terms out of $\{x_1, \ldots, x_n\}$.

Notes:

1. $| V^{(n-j_0)} - V_{\epsilon\delta}^{(n-j_0)} |$ can be estimated using Theorem 2.1. This way we obtain an improved, explicit estimate of the error of approximation.

2. If $x_k^* = x_k$ and $p(x_k^*) = p(x_k)$ for all k except when $k = k_1, \ldots, k_{n_0}$ for fixed n_0 and if the number of choices n is increased by successively adding elements of the game, such that

   $$\max_{1 \leq i \leq n_0} \{x_{k_i}\} < V^{(n-j)}(\underline{x}_{-x_{k_i}}) \text{ for } j \geq j_0$$

   for some j_0 and similarly for x^* and $V_{\epsilon\delta}^{(n-j_0)}$. It follows from the above corollary that in this case the error becomes negligible, i.e.

   $$| V^{(0)} - V_{\epsilon\delta}^{(0)} | \to 0 \quad \text{as} \quad n \to \infty\,.$$

Proof of Corollary 2.2: It follows from Theorem 2.2 that under the assumptions of the corollary for all $j \geq j_0$ we have for the negative parts $(\ldots)^-$

$$-(j+1)(V^{(n-j-1)}[\underline{x}] - V_{\epsilon\delta}^{(n-j-1)}[\underline{x}^*])^-$$

$$\geq - \sum_{x > V^{(n-j)}[\underline{x}_{-x}]} (V^{(n-j)}[\underline{x}_{-x}] - V_{\epsilon\delta}[\underline{x}^*_{-x^*}])^- \bar{p}^* - \sum_{x \leq V^{(n-j)}[\underline{x}_{-x}]} (V^{(n-j)}[\underline{x}_{-x}] - V_{\epsilon\delta}[\underline{x}^*_{-x^*}])^-$$

and a similar relation holds for the positive parts. This implies

$$(j+1) \mid V^{(n-j-1)}[\underline{x}] - V_{\epsilon\delta}^{(n-j-1)}[\underline{x}^*] \mid \leq \sum_{x \in \{\underline{x}\}} \mid V^{(n-j)}[\underline{x}_{-x}] - V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}] \mid .$$

It can be shown by induction that this implies the statement of the corollary. □

Motivated by Theorem 2.2 we propose the following recursive relation to estimate the error between $V^{(n-j)}[\underline{x}]$ and $V_{\epsilon\delta}^{(n-j)}[\underline{x}^*]$ by

$$(j+1) \mid V^{(n-j-1)}[\underline{x}] - V_{\epsilon\delta}^{(n-j-1)}[\underline{x}^*] \mid \leq \tag{2.2}$$

$$\begin{aligned} & \sum_{x^* > V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}]} \mid V^{(n-j)}[\underline{x}_{-x}] - V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}] \mid \bar{p}^* \\ + & \sum_{x^* \leq V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}]} \mid V^{(n-j)}[\underline{x}_{-x}] - V_{\epsilon\delta}[\underline{x}^*_{-x^*}] \mid \\ + \mid & \sum_{x^* > V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}]} \epsilon p^* \mid \\ + \mid & \sum_{x^* > V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}]} (x^* - V_{\epsilon\delta}^{(n-j)}[\underline{x}^*_{-x^*}])\delta \mid . \end{aligned}$$

This estimate has been tested numerically in Section 3 and found to perform satisfactorily.

3 Numerical Examples

We consider a game in which the "acceptance probabilities" are the reciprocals of the rewards. This game is approximated by lumping the third and fourth elements, x_3, x_4 into one state which can occur twice. This state is chosen in the first approximation as x_3 and in the second approximation as $(1/2)(x_3+x_4)$. Correspondingly, the acceptance probabilities $p(x_3)$ and $p(x_4)$ are replaced by $p(x_3)$ and by $(1/2)(p(x_3)+p(x_4))$, (see Table I).

In Table II the values of the exact game and both the approximations are shown at each possible state of the games and their error estimates (2.2) are compared with the true approximation error. To save space this is only done for a game of size 5, i.e. $x_1, \ldots, x_5$. As expected, the true error increases with decreasing the stage and increasing the number of possible future outcomes, and at every stage the true error depends on what future is open to the player. Though the approximate error lies well below that specified in Theorem 2.1 (=.67), the true error can still be considerably smaller at certain states.

In Table III we compare the goodness of different approximations to the game which is now increasing successively from the initial five elements up to size ten with inverse acceptance probabilitiesi as shown in Table I. It can be seen that the goodness of the approximation might depend on the way the elements are lumped into repeated states and their acceptance probabilities. It is recommended that a group of several states be replaced by their mean rather than by any of them and similarly for the acceptance probabilites. Also it can be seen from Table III that the approxmation gains in precision if the size of the game grows in the sense of Corollary 2.2.

	Exact game									
x	1	2	3	4	5	6	7	8	9	10
$p(x)$	1.000	0.500	0.333	0.250	0.200	0.167	0.143	0.125	0.111	0.100
	1. Approximation									
x^*	1	2	3	3	5	6	7	8	9	10
$p(x^*)$	1.000	0.500	0.333	0.333	0.200	0.167	0.143	0.125	0.111	0.100
	2. Approximation									
x^*	1	2	3.5	3.5	5	6	7	8	9	10
$p(x^*)$	1.000	0.500	0.292	0.292	0.200	0.167	0.143	0.125	0.111	0.100

Table I: The full game and different approximations.

As starting values for (2.2) we used $\mid V^{(n-2)}[\underline{x}] - V_{\epsilon\delta}^{(n-2)}[\underline{x}^*] \mid$. This led to a considerable increase of precision over starting values choosen according to Theorem 2.1. Using $\mid V^{(n-j)}[\underline{x}] - V_{\epsilon\delta}^{(n-j)}[\underline{x}^*] \mid$ for $j > 2$ would lead to a further improvement on the error estimate of $\mid V^{(0)} - V_{\epsilon\delta}^{(0)} \mid$.

		1. Approximation			2. Approximation		
future	V(exc)	V(aprx)	Er(exc)	Er(aprx)	V(aprx)	Er(exc)	Er(aprx)
12	1.25	1.25	0	0	1.25	0	0
13	1.333	1.333	0	0	1.376	0.043	0.043
14	1.375	1.333	0.042	0.042	1.376	0.001	0.001
15	1.4	1.4	0	0	1.4	0	0
23	1.583	1.583	0	0	1.62	0.037	0.038
24	1.625	1.583	0.042	0.042	1.62	0.005	0.004
25	1.65	1.65	0	0	1.65	0	0
34	1.708	1.665	0.043	0.042	1.746	0.038	0.038
35	1.733	1.733	0	0	1.774	0.041	0.041
45	1.775	1.733	0.042	0.042	1.774	0.001	0.001
123	1.694	1.694	0	0	1.738	0.044	0.092
124	1.75	1.694	0.056	0.208	1.738	0.012	0.088
125	1.783	1.783	0	0	1.783	0	0
134	1.874	1.814	0.06	0.208	1.913	0.039	0.037
135	1.911	1.911	0	0	1.963	0.052	0.096
145	1.975	1.911	0.064	0.209	1.963	0.012	0.085
234	2.041	1.981	0.06	0.195	2.07	0.029	0.03
235	2.077	2.077	0	0	2.124	0.047	0.084
245	2.142	2.077	0.065	0.195	2.124	0.018	0.082
345	2.283	2.214	0.069	0.195	2.317	0.034	0.034
1234	2.104	2.036	0.068	0.241	2.133	0.029	0.054
1235	2.144	2.144	0	0	2.195	0.051	0.1
1245	2.217	2.144	0.073	0.246	2.195	0.022	0.108
1345	2.383	2.303	0.08	0.252	2.418	0.035	0.058
2345	2.475	2.392	0.083	0.243	2.507	0.032	0.054
12345	2.525	2.436	0.089	0.268	2.557	0.032	0.069

Table II: Exact Error (Er(exc)) and Estimated Error (Er(aprx)) of First and Second Approximation to the Game of Size 5 over all Stages.

		1. Approximation			2. Approximation		
space	V(exc)	V(aprx)	Er(exc)	Er(aprx)	V(aprx)	Er(exc)	Er(aprx)
5	2.5254	2.4364	0.0890	0.2679	2.5566	0.0313	0.0688
6	2.9472	2.8427	0.1046	0.2627	2.9778	0.0305	0.0755
7	3.3644	3.2434	0.1210	0.2629	3.3824	0.0180	0.1100
8	3.7868	3.6813	0.1055	0.2521	3.7787	0.0082	0.1155
9	4.2026	4.1268	0.0758	0.1990	4.1979	0.0047	0.0948
10	4.6249	4.5686	0.0563	0.1488	4.6225	0.0023	0.0748

Table III: Exact Error (Er(exc)) and Approximate Error (Er(aprx)) of the Approximation to the Value of the Game for Sizes 5 to 10, Using the First and Second Approximations.

Concluding Remarks

The program used for the above computations written in the C language is available from the first author. We also note that the probabilities p_i need not be assumed to be known but can rather be estimated during the process of receiving responses successively. For the case when $p_i < p_j$ for $x_i > x_j$ we may assume that the model

$$p(x_i) = 1 - \Phi\left(\frac{x_i - \mu}{\sigma}\right)$$

or

$$p(x_i) = \left[1 + \exp\left\{\frac{x_i - \mu}{\sigma}\right\}\right]^{-1}$$

applies, where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2\right\} du$$

and μ, σ are unknown. These parameters can be estimated by the method of maximum likelihood from at least two known responses and be updated successively as more observations become available. In this case the results obtained above have to be interpreted conditionally given μ, σ.

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