Projektbereich A
Discussion Paper No. A–465

On the Stability of Intertemporal Equilibria with Rational Expectations"

by

Thorsten Hens *)

University of Bonn

November 1994

*) Financial support by Universidad Alicante and Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at the University of Bonn is gratefully acknowledged.

SONDERFORSCHUNGSBEREICH 303
Rheinische Friedrich-Wilhelms-Universität Bonn
Lennéstraße 37
D–53113 Bonn
Germany
Telefon (0228) 73 92 42
Telefax (0228) 73 79 40
E–Mail: with2 @ track.econ2.uni-bonn.de

Typeset in LaTeX by Dagmar Müller, Karsten Jädtke

Abstract

In this paper we propose a concept of stability for intertemporal equilibria with rational expectations: current period prices move proportionally to current period excess demand while future prices are formed according to the perfect foresight hypothesis. It is shown that this process is locally asymptotically stable if all goods are gross substitutes, or if the equilibrium has no trade. In general this process differs from a tâtonnement process in contingent contracts prices and from a tâtonnement in asset and spot market prices. It also differs from Hicks' and exceptional stability. In an intertemporal variant of Scarf's example on the instability of the Walrasian tâtonnement process it will be seen that the stability notion we propose is more stable than any other process investigated so far.

Keywords: Stability, rational expectations, general equilibrium

JEL: D52, D54, D84

1 Introduction

Studying the stability of intertemporal general equilibria can de done by staying within the Arrow-Debreu model. For this it is necessary to assume that there is a complete set of contingent contracts available at the outset of all times; spot markets do not reopen and agents do not need to form price expectations (cf. Debreu (1959) chapter 7). This version of an intertemporal general equilibrium model is formally equivalent to the static Arrow-Debreu model and one could therefore derive conditions for the stability of the intertemporal model from the extensive literature on the stability of the Arrow-Debreu model (cf. Hahn (1982) for a survey).

In a more realistic setting the system of contingent contracts is seriously incomplete; agents trade sequentially on reopening spot markets and they will have to form price expectations. Ever since Arrow (1953) (and more generally Magill and Shafer (1990)) it is known that if there are sufficiently many financial markets; then the equilibrium allocations of the incomplete markets model will coincide with those of the complete contingent contracts model, provided agents have correct price expectations. Thus under the perfect foresight hypothesis, from a static allocational point of view these two variants of an intertemporal general equilibrium model are equivalent. It is interesting to analyze whether these two models are dynamically equivalent as well, i.e. equivalent with respect to some adjustment process. This will, of course, depend on the concept of stability that is chosen in the comparison of the two.

We propose a new concept of stability that has some theoretical bearing on perfect foresight models: current period prices move proportionally to current period excess demand, while future price expectations are formed according to the perfect foresight hypothesis. This defines a tâtonnement process of short period equilibria with rational expectations. It can be seen as a consistent continuation of the study concerning the stability of short period equilibria with exogeneous expectations (cf. Enthoven and Arrow (1956), Arrow and Nerlove (1958), Arrow and Hahn (1971)). In the listed literature one assumes that "*The auctioneer operates in the present on the basis of the excess demands observable to him in the present*" (Arrow and Hahn (1971), page 310). Agents' present excess demand depends on price expectations that are modelled by some exogeneous continous functions of current period prices. Arrow and Hahn (1971) prove the stability of this short period tâtonnement assuming that there are no 'cross effects' on expectations, i.e. the price expectations of each commodity only depend on the present period price of that commodity. Stability then follows if all goods are gross substitutes and the Hicksian elasticity condition [1] is met, which it is e.g. with adaptive expectations (cf. Enthoven and Arrow (1956)). Following this line of research for analyzing the stability of intertemporal equilibria we suggest to take the same theoretical step as has been done in the transition from temporary equilibria towards rational expectations equilibria, i.e. to evoke the perfect foresight hypothesis in order to endogeneize the ad hoc expectation functions. We will then analyze the local asymptotic stability of this process.

Strictly speaking the perfect foresight hypothesis only requires that all agents know the future market clearing prices **in equilibrium** on current markets. Our concept thus extends the perfect foresight hypothesis locally to the surrounding neighbourhood of current period equilibrium prices. It is clear that evoking the perfect foresight hypothesis cannot possibly be justified by any reference to becoming more realistic than with exogeneous expectations. It should, however, be interesting to a theorist to examine what are the consequences for the stability of short period equilibria, once the arbitrariness of exogeneous expectations is resolved.

Given the Sonnenschein-Debreu-Mantel results we would, obviously, not expect the new process to be universally stable . However, as a first test it will be shown that the process is locally asymptotically stable if at equilibrium the normalized Jacobian of market excess demand (of the corresponding contingent

[1]The elasticity of price expectations with respect to current period prices is not greater than one.

contracts' model) is negative quasi-definite or negative diagonal dominant. The first case occurs e.g. at a no trade equilibrium or in a Hicksian economy, the second case occurs e.g. when all goods are gross substitutes. Thus for endogeneous expectations in the case of gross substitutes we achieve stability of the short period tâtonnement irrespectively of the Hicksian elasticity condition. Furthermore it follows that we achieve stability in the case of a Hicksian economy. This latter case is remarkable since Arrow and Hahn (1971) were not able to find conditions for the exogeneous expectation functions that would give stability in a Hicksian economy. Thus endogeneizing expectations can be seen as a way to solve this puzzle.

Having suggested a new notion of stability it is interesting to study how it compares with alternative stability concepts applicable in this setting. The alternative notions we consider are a tâtonnement process in asset-spot markets, a tâtonnement process in contingent contracts markets, Hicks' stability and expectational stability [2]. We will show that if at equilibrium the Jacobian of the contingent contracts' market model is symmetric, then, with respect to local asymptotic stability, all concepts are equivalent to our stability notion. However, as we will show, in general, each of the concepts will be different to any other. Furthermore in the comparison we will analyze for which processes the contingent-contracts, and the asset spot-markets model are dynamically equivalent. The analysis shows that this equivalence does not hold for tâtonnement stability, i.e. a tâtonnement in asset-spot market prices has different stability properties than a tâtonnement in contingent contracts' prices. The equivalence holds for Hicks' stability, expectational stability and for the adjustment process we propose.

Finally it is shown that in an intertemporal variant of Scarf's (1960) seminal example on the instability of the Walrasian tâtonnement process our new process is more stable than any of the other processes investigated because there are no parameter values of our example that make any of the other processes stable while our process is unstable. Yet on the other hand there are parameter values for which our process is stable whereas all other processes remain unstable.

2 The Model

There are two periods, $t = 0, 1$. Uncertainty arises because one of S possible states of the world occurs in the second period. In the first period there are L_1

[2]Expectational Stability goes back to Lucas(1978) and DeCanio(1979). For an application to general equilibria see Guesnerie (19??) and Balasko (1994).

commodities and in each state $s = 1, ..., S$ there are L_2 commodities. A consumption plan of agent $i = 1, ..., I$ is a non-negative vector describing the amount of the commodities available to the agent for consumption at the various dates and states of the world; the consumption set is then $\mathrm{IR}^{\mathrm{L}_1}_+ \times \mathrm{IR}^{\mathrm{SL}_2}_+$. Agents' endowments are denoted by ω^i, $i = 1, ..., I$. Each agent i evaluates consumption plans $x \in \mathrm{IR}^{\mathrm{L}_1}_+ \times \mathrm{IR}^{\mathrm{SL}_2}_+$ according to his utility function U^i . We make the standard assumptions on utility functions and endowments which guarantee differentiability of excess demand:

$A.1.$ $\omega^i \gg 0$ and U^i is smooth [3], $i = 1, ..., I.$

In order to transfer income across future states of the world, agents can buy and sell (without any short sale restrictions) $j = 1, ..., J$ nominal assets. Asset j pays off A^j_s units of account if state s occurs. Let $A \in \mathrm{IR}^{\mathrm{S}\times\mathrm{J}}$ denote the asset returns matrix with generic element A^j_s. Let $p_1 \in \mathrm{IR}^{\mathrm{L}_1}$ and $p_2 \in \mathrm{IR}^{\mathrm{SL}_2}$ denote spot prices and let $q \in \mathrm{IR}^{\mathrm{J}}$ denote asset prices. To abbreviate notation, for two vectors x and y that are partitioned across future states of the world, $x = (x_1, ..., x_S)$, $y = (y_1, ..., y_S)$ introduce $x2y$ to denote the S-vector of scalar products $(x_s \cdot y_s)$ $s = 1, ..., S$. Finally, let $\theta \in \mathrm{IR}^{\mathrm{J}}$ denote the portfolio of assets. The following maximization problem (M^i) summarizes agent $i's$ decision problem:

$$(M^i) \qquad \max_{\substack{x_1 \in \mathrm{IR}^{\mathrm{L}_1}_+ \\ \mathrm{x}_2 \in \mathrm{IR}^{\mathrm{SL}_2}_+ \\ \theta \in \mathrm{IR}^{\mathrm{J}}}} \quad U^i(x)$$

$$s.t. \quad p_1 \cdot x_1 + q \cdot \theta \leq p_1 \cdot \omega^i_1$$
$$p_2 2(x_2 - \omega^i_2) \leq A\theta$$

In order to ensure that asset demand is well defined we assume that there are no redundant assets, i.e.

A.2. rank A = J .

Note that the decision problem (M^i) has a solution if, and only if spot prices are strictly positive and asset prices are arbitrage free. Let Q denote the interior of the convex cone spanned by the rows of the pay-off matrix A, i.e. $Q := \{q \in \mathrm{IR}^{\mathrm{J}} | q = \mathrm{A}^{\mathrm{T}}\pi$ for some $\pi \in \mathrm{IR}^{\mathrm{S}}_{++}\}$. Then for $(q, p_1, p_2) \in Q \times \mathrm{IR}^{\mathrm{L}_1}_{++} \times \mathrm{IR}^{\mathrm{SL}_2}_{++}$ let $g^i(q, p_1, p_2)$, $z^i_1(q, p_1, p_2)$, $z^i_2(q, p_1, p_2)$ denote agent $i's$ excess demand of assets, the first period's and the second period's consumption goods, respectively. Given

[3] See e.g. Debreu (1972) for a precise definition of smooth preferences.

the assumptions A.1 and A.2 excess demand is differentiable. Let $G(q,p_1,p_2) := \sum_i g^i(q,p_1,p_2)$, $Z_1(q,p_1,p_2) := \sum_i z_1^i(q,p_1,p_2)$, $Z_2(q,p_1,p_2) := \sum_i z_2^i(q,p_1,p_2)$ denote market excess demand.

An asset-spot market equilibrium is an equilibrium of prices and price expectations (cf. Radner (1972)), i.e. a price vector $(\overset{*}{q},\overset{*}{p}_1,\overset{*}{p}_2)$, so that market excess demand is zero, i.e.

$$\begin{aligned} G(\overset{*}{q},\overset{*}{p}_1,\overset{*}{p}_2) &= 0 \\ Z_1(\overset{*}{q},\overset{*}{p}_1,\overset{*}{p}_2) &= 0 \qquad (EQ) \\ Z_2(\overset{*}{q},\overset{*}{p}_1,\overset{*}{p}_2) &= 0\ . \end{aligned}$$

The stability concept we propound requires regularity of $\partial_{p_2} Z_2(\overset{*}{q},\overset{*}{p}_1,\overset{*}{p}_2)$. Because assets are 'nominal', for this end we will assume that there are sufficiently many financial markets , i.e.

A.3. rank A = S .

Remark

Alternatively, for the purpose of regularity we could have chosen assets' payoffs to be denominated in consumption goods and could have kept the generality of insufficient asset markets. Since the comparison between the new stability concept and various other stability concepts including tâtonnement stability in the Arrow-Debreu contingent contracts' model constitutes an important point of our paper, we have to make assumption A.3 anyway in order to guarantee that the equilibrium points whose stability we compare coincide in the two models.

In a model with a complete set of contingent contracts an agent i faces the following decision problem [4]:

$$(M^i) \qquad \max_{\substack{\bar{x}_1 \in \mathbb{R}_+^{L_1} \\ \bar{x}_2 \in \mathbb{R}_+^{L_2}}} \quad U^i(\bar{x}_1,\bar{x}_2)$$

$$s.t. \quad \bar{p}_1 \cdot \bar{x}_1 + \bar{p}_2 \cdot \bar{x}_2 \quad \leq \quad \bar{p}_1 \cdot \omega_1^i + \bar{p}_2 \cdot \omega_2^i$$

In this decision problem $\bar{p}_t, \quad t = 1,2$ denotes the vector of contingent contracts prices and $\bar{x}_t,\ t = 1,2$ denotes the demand for contingent contracts. Let $\bar{z}_t^i(\bar{p}_1,\bar{p}_2),\ t = 1,2$ denote agent i's excess demand and let $\bar{Z}_t(\bar{p}_1,\bar{p}_2) := \sum_i z_t^i(\bar{p}_1,\bar{p}_2)$ denote market excess demand. A contingent contracts' equilibrium is a price vector $\overset{*}{\bar{p}}_1, \overset{*}{\bar{p}}_2$ so that

[4] Whenever we refer to the contingent contracts' model variables will carry an upper bar.

$$\begin{aligned} \bar{Z}_1(\overset{*}{\bar{p}}_1, \overset{*}{\bar{p}}_2) &= 0 \\ \bar{Z}_2(\overset{*}{\bar{p}}_1, \overset{*}{\bar{p}}_2) &= 0 \ . \end{aligned} \qquad (\overline{EQ})$$

To abbreviate notations, let $Z(q, p_1, p_2) = (Z_1(q, p_1, p_2), Z_2(q, p_1, p_2))$ and $\bar{Z}(\bar{p}_1, \bar{p}_2) = (\bar{Z}_1(\bar{p}_1, \bar{p}_2), \bar{Z}_2(\bar{p}_1, \bar{p}_2)$.

As Arrow (1953) has pointed out, for strictly positive spot prices and arbitrage-free asset prices, we have the identity [5]

(AE) $Z(A^T\pi, p_1, p_2) = \bar{Z}(p_1, \pi 2 p_2)$ for all $(\pi, p_1, p_2) \in \mathrm{IR}^{\mathrm{S}}_{++} \times \mathrm{IR}^{\mathrm{L_1 \times SL_2}}_{++}$.

Consequently, equilibrium allocations in the asset-spot market model will coincide with those of the contingent contracts' model.

Note that due to (AE) the Jacobians of market excess demand of the two models are intimately related to each other. It follows that for $q = A^T\pi$

$$\begin{aligned} \partial p_1 Z(q, p_1, p_2) &= \partial \bar{p}_1 \bar{Z}(p_1, \pi 2 p_2) \\ \partial p_2 Z(q, p_1, p_2) &= \partial \bar{p}_2 \bar{Z}(p_1, \pi 2 p_2)\Lambda(\pi) \end{aligned} \qquad (AE)$$

where Λ is a SL_2 diagonal matrix with π_s being on the diagonal of the s-th block. Thus at equilibrium the Jacobian of the asset-spot market model, $\partial_{p_2} Z_2(\overset{*}{q}, \overset{*}{p}_1, \overset{*}{p}_2)$ is regular if and only if $\partial \bar{p}_2 \bar{Z}_2(\overset{*}{p}_1, \overset{*}{\pi}\ 2\ \overset{*}{p}_2)$ is regular. In order to establish regularity one has to attempt to normalize prices. We choose the last commodity in the last state of period two as numeraire, i.e. we assume

(N.1) $p_{SL_2} \equiv 1$.

Without causing confusion, we will add an $\hat{}$ on top of a vector to denote that it has been truncated by its last component. Accordingly $\hat{}$ on a matrix means we have cancelled the last row, and column. Proposition 11 of Balasko (1994) establishes that for generic endowments the reduced Jacobian $\partial_{\hat{\bar{p}}_2} \hat{\bar{Z}}_2$ has maximal rank. Thus, because of (AE) $\partial \hat{p}_2 \hat{Z}_2$ inherits generic regularity from $\partial \hat{\bar{p}}_2 \hat{\bar{Z}}_2$. Furthermore we will normalize all asset prices to be one, i.e.

(N.2) $q_j = 1 \quad j = 1, ..., J$.

To abbreviate expressions let $\mathbb{1} \in \mathrm{IR}^{\mathrm{J}}$ denote the vector with every entry being one. Thus (N.2) can be rewritten as $q = \mathbb{1}$.

[5] $(\pi 2 p_2) \in \mathrm{IR}^{\mathrm{SL2}}_{++}$ is obtained from p_2 by multiplication of its components p_{sl} with π_s, $l = 1, ..., L_2$, $s = 1, ..., S$.

Remark

As a result of this normalization the reduced Jacobians of the two models are most similar compared to any other normalization. In Guesnerie and Hens (1994) it is shown that in a model without uncertainty (N.2) can equivalently be replaced by chosing any other first period consumption good as numeraire.

Now we are in a position to define the stability concept we would like to propose. Consider the equilibrium equations (EQ). Because of the regularity of $\partial_{\hat{p}_2}\hat{Z}_2(\mathbb{1}, \overset{*}{p}_1, (\hat{p}_2, 1))$ around $\overset{*}{p}_1$, a neighbourhood $N(\overset{*}{p}_1)$ and a continously differentiable function $\Psi_2 : N(\overset{*}{p}_1) \to \mathbb{R}^{\mathrm{SL}_1-1}_{++}$ exists so that

$\hat{Z}_2(\mathbb{1}, p_1, \Psi_2(p_1), 1) = 0$ for all $p_1 \in N(\overset{*}{p}_1)$.

The mapping Ψ_2 corresponds to Balasko's (1994) "expectations foreward correspondence" which he defines analogously for the contingent contracts model. So our stability concept is given by the dynamical system

(t) $\dot{p}_1\ (t) = Z_1(\mathbb{1}, p_1(\mathrm{t}), \Psi_2(p_1(\mathrm{t}), 1) \quad \mathrm{t} \in \mathbb{R}$.

In (t) current period prices are changed proportionally to current period excess demand while future prices are formed according to the perfect foresight hypothesis.

3 Gross Substitution and No-trade

As a first test we will show that the dynamical system (t) is locally asymptotically stable if in the contingent contracts model all goods are gross substitutes, or if there is no trade. To this end compute the Jacobian of the asset-spot market model at an equilibrium. Given the normalizations $(N.1)$ and $(N.2)$, we use Walras Law to cancel all asset markets as well as the market for the last commodity in the last state. This leads to a reduced Jacobian, $\hat{J}$, which will be partitioned across time periods. It is important to note how the Jacobian $\hat{J}$ is related to the Jacobian of the contingent contracts market model. This relationship is given in

Lemma 1

Let $J_1 = \partial_{p_1} Z_1, \hat{J}_2 = \partial_{\hat{p}_2} Z_1, \hat{J}_3 = \partial_{p_1}\hat{Z}_2, \hat{J}_4 = \partial_{\hat{p}_2}\hat{Z}_2$ and

let $\bar{J}_1 = \partial_{\bar{p}_1}\bar{Z}_1, \hat{\bar{J}}_2 = \partial_{\hat{\bar{p}}_2}\bar{Z}_1, \hat{\bar{J}}_3 = \partial_{\bar{p}_1}\hat{\bar{Z}}_2, \hat{\bar{J}}_4 = \partial_{\hat{\bar{p}}_2}\hat{\bar{Z}}_2$

then the normalized Jacobian of the asset-spot market model $\hat{J} = \begin{bmatrix} J_1 & \hat{J}_2 \\ \hat{J}_3 & \hat{J}_4 \end{bmatrix}$

and the normalized Jacobian of the contingent contracts model $\hat{\bar{J}} = \begin{bmatrix} \bar{J}_1 & \hat{\bar{J}}_2 \\ \hat{\bar{J}}_3 & \hat{\bar{J}}_4 \end{bmatrix}$ are related by the formula $\hat{J} = \hat{\bar{J}} \begin{bmatrix} I_{L_1} & \\ & \Lambda(\pi) \end{bmatrix}$

where I_{L_1} denotes the identity matrix of dimension L_1 and $\Lambda(\pi) \in \mathbb{R}^{L_2 S - 1}$ is a diagonal matrix with π_S being the diagonal entry in the s-th state, $s = 1, ..., S$.

The proof of Lemma 1 follows immediately from allocational equivalence, (AE).

The following convention is quite useful:

Definition

A matrix is said to be stable if every eigenvalue of the matrix has a negative real part.

Given this definition, from the theory of dynamical systems we notice that the adjustment process (t) is locally asymptotically stable if the following criterion is met

(s.c.t.)

> The dynamical system (t) is locally asymptotically stable if the matrix $H = J_1 - \hat{J}_2 \hat{J}_4^{-1} \hat{J}_3$ is stable. It is unstable if at least one eigenvalue of H has a positive real part.

Remark

Note that when using allocational equivalence the matrix H can be expressed via the contingent contracts Jacobian $\hat{\bar{J}}$. We then obtain $H = \bar{J}_1 - \hat{\bar{J}}_2 \hat{\bar{J}}_4^{-1} \hat{\bar{J}}_3$. Furthermore note that applying (t)-stability to the contingent contracts model, i.e. to the matrix $\hat{\bar{J}}$ would lead to exactly the same expression. Thus, with respect to (t)-stability the asset-spot and the contingent contracts model are equivalent.

The following proposition shows that the matrix H inherits negative diagonal dominance[6] and negative quasi-definiteness[7] from the matrix $\hat{\bar{J}}$.

[6] A matrix $A = (a_{ij})_{i,j=1,\dots,n}$ is negative diagonal dominant if $a_{ii} < 0$ and for some $\mathrm{w} \in \mathbb{R}^n_+$ $\mid a_{ii} \mid \mathrm{w}_i > \sum_{j \neq i} \mid a_{ij} \mid \mathrm{w}_j$, $i = 1, \dots, n$.

[7] A matrix A is negative quasi-definite if the symmetric matrix $A + A^T$ is negative definite.

Proposition 1

If at equilibrium the normalized Jacobian of the market excess demand in the contingent contracts model $\hat{J}$,is negative diagonal dominant or negative quasi-definite, then the matrix H is negative diagonal dominant respectively negative quasi-definite.

Proof

negative diagonal dominance:

Since a matrix A is negative diagonal dominant iff $-A$ is positive diagonal dominant, we will show that $-H$ is positive diagonal dominant if $-\hat{J}$ is positive diagonal dominant.

Let $-\hat{J}$ be diagonal dominant then its inverse $-\hat{J}^{-1}$ is diagonal dominant (cf. Horn and Johnson (1991) Theorem 2.5.12). The formula for the inverse of a partitioned matrix (cf. Horn and Johnson (1985) 0.7.3) specifies that $-H^{-1}$ is the Shur complement of $-\hat{J}$, i.e. the upper left block in the inverse of $-\hat{J}$. Since diagonal dominance is inherited by principal submatrices, $-H^{-1}$ is diagonal dominant and then so is $-H$ itself. Thus it remains to argue that $-H$ has a positive main diagonal. If $-\hat{J}$ is positive diagonal dominant, then $-\hat{J}$ is a P-matrix (cf. Murata (1977), Theorem 21 in chapter 1). From the formula of the inverse (cf. Horn and Johnson (1985), 0.8.2) we achieve that $-\hat{J}^{-1}$ has a positive diagonal, which is then inherited by $-H^{-1}$ and again by inversion this carries over to $-H$.

negative quasi-definiteness:

The reasoning for the case of negative quasi-definiteness is completely analogous, because the inverse of a positive quasi-definite matrix is positive quasi-definite (cf. Theorem 37, chapter 2 in Murata (1977)) and because positive quasi-definiteness is inherited by principal submatrices.

2

Note that negative quasi-definite and negative diagonal dominant matrices are stable[8]. Thus Proposition 1 gives two sufficient conditions for the stability of (t) . Furthermore note that, as we will show in section 4, stability of $\hat{J}$ itself does not guarantee stability of (t). Thus in this sense Proposition 1 gives the weakest conditions for the stability of (t).

We get the following two corollaries:

[8]Theorem 22, chapter 1 in Murata (1977); Theorem 39, chapter 2 in Murata (1977).

Corollary 1

At a no trade equilibrium the process (t) is stable.

Proof

At a no trade equilibrium the matrix $\hat{\bar{J}}$ is symmetric negative definite (cf. Balasko (1988), proposition 3.5.2). Thus, Propositon 1 applies.

2

Corollary 2

At a gross substitutes equilibrium[9] the process (t) is stable.

Proof

As Negishi (1958) has shown, gross substitutes combined with homogeneity, i.e. $\bar{J}\bar{p} = 0$ implies that $\hat{\bar{J}}$ is negative diagonal dominant. Thus Proposition 1 applies.

2

Part of the relevance of Proposition 1 can be seen from a comparison to Arrow and Hahn's (1971) analysis of short period equilibria with exogeneous expectations. Using our notation Arrow and Hahn (1971) consider the excess demand of the current period, $\bar{Z}_1$, as a function of current period prices $\bar{p}_1$ and future expected prices $\bar{p}_2$ which are resulting from expectations functions $\Psi_l : \mathbb{R}_{++} \to \mathbb{R}_{++}$, $\mathrm{l} = 1, \ldots, \mathrm{L} - 1$. These functions have no 'cross effects', i.e. $\bar{p}_{2l} = \Psi_{2l}(\bar{p}_{2l}), l = 1, \ldots, L$.[10] $\bar{Z}_1$ is assumed to be homogenous with degree zero in prices and satisfies Walras' Law, i.e. for all $\bar{p}_1, \bar{p}_2 \in \mathbb{R}^{\mathrm{L}}_{++}$ (H) $\bar{\mathrm{Z}}_1(\lambda\bar{\mathrm{p}}_1, \lambda\bar{\mathrm{p}}_2) = \bar{\mathrm{Z}}_1(\bar{\mathrm{p}}_1, \bar{\mathrm{p}}_2)$ for all $\lambda > 0$ and (W) $\bar{p}_1\bar{Z}_1(\bar{p}_1, \bar{p}_2) = 0$. Arrow and Hahn (1971) show that the dynamical system

$$(t') \qquad \dot{\bar{p}}(t) = \bar{Z}_1(\bar{p}_1(t), \Psi(\bar{p}_1(t)))$$

is globally stable if all goods are gross substitutions and for all prices $\bar{p}_1 \in \mathbb{R}^{\mathrm{L}}_{++}$ the Hicks elasticity condition

$$\varepsilon_l := \frac{d \log \bar{p}_{2l}}{d \log \bar{p}_{1l}} \leq 1, \quad l = 1, \ldots, L$$

is met for all commodities.

Note that with perfect foresight and $L = 1$ the elasticity of expectations is one because of the Homogenity property $\Psi(\lambda p_1) = \lambda\Psi(p_1)$ for all $\lambda > 0$

[9]A gross substitutes equilibrium is an equilibrium at which every off diagonal element of the Jacobian $\bar{J}$ is positive.

[10]In Arrow and Hahn (1971) there are the same number of commodities in both periods.

and all $p_1 > 0$. Thus for this case our result coincides with that of Arrow and Hahn (1971). With multiple commodities, however, the assumption of no cross derivatives on expectations is, in general, not compatible with the perfect foresight hypothesis and consequently there is no reason why Hicks' elasticity condition should be fulfilled with rational expectations.

Finally note that Enthoven and Arrow (1956) achieve global stability under the adaptive expectations' hypothesis

$$(a) \qquad \dot{\bar{p}}_{1l}(t) = \mu_j(\bar{p}_{1l}(t) - \bar{p}_{2l}(t)), \quad \mu_j > 0$$

if all goods are gross substitutes. But again this is not compatible with rational expectations.

Our result in the case of a symmetric negative definite Jacobian is of some interest since Arrow and Hahn (1971) were not able to extend their results based on exogeneous expectations to the case of a Hicksian economy. In a Hicksian economy consumers have identical and homothetic preferences. Recall that in this case the Jacobian at equilibria is symmetric negative definite. Arrow and Hahn (1971) state on page 312 "*Curiously enough, it does not seem possible to extend all these results to the Hicksian cases*" and in the same paragraph "*There may be a way of showing stability for the Hicksian case also, but we have not been able to find it.*" Thus endogeneizing expectations could be seen as an answer to this curious phenomenon.

4 Comparison of the stability concepts

In this subsection we will compare the new stability concept (t) with the stability of a tâtonnement in contingent contracts prices (c), Hicks' notion of perfect stability (h) and with expectational stability (e). We will show that if at equilibrium the Jacobian matrix of the contingent contracts model is symmetric, then the stability concepts (t), (c), (h) and (e) are all equivalent. By means of examples we will then demonstrate that in general however these concepts are all different.

Before doing so, consider a tâtonnement process in spot prices, p_1, and price expectations, p_2. Given the normalization rules $(N.1)$ and $(N.2)$ suppose first (second) period prices move proportionally to first (second) period excess demand, i.e. consider the dynamical system

$$(r) \qquad \dot{p}_1(t) = Z_1(\mathbb{1}, p_1(\mathrm{t}), \hat{p}_2(\mathrm{t}), 1)$$

$$\dot{\hat{p}}_2(t) = \hat{Z}_2(\mathbb{1}, p_1(\mathrm{t}), \hat{p}_2(\mathrm{t}), 1) \quad \text{for all} \quad \mathrm{t} \in \mathbb{R}.$$

The dynamical system (r) is locally asymptotically stable if and only if the corresponding Jacobian matrix is a stable matrix. Thus we have the stability criterion

(s.c.r.)

> the dynamical system (r) is locally asymptotically stable if the matrix $\hat{J}$ is a stable matrix. It is unstable if at least one of the eigenvalues of $\hat{J}$ has positive real part.

As the following argument shows, stability of (r) might depend on the choice of the asset structure A or alternatively (recall $q = A^T\pi$) on the choice of asset price normalization. This is a particular property of tâtonnement stability that does not arise for any of the other stability concepts (h), (c), (e), and (t).

To corroborate the claim, consider the case where there are no cross effects among the two time periods. Then applying Lemma 1 (s.c.r.) is satisfied if, and only if the matrix $\hat{\hat{J}}_0$ and the matrix $\hat{\hat{J}}_4$ are both stable. However, stability of the latter matrix might depend on the choice of π. Recall that for $L_2 = 3$, i.e. $\hat{\hat{J}}_4 \in \mathbb{R}^{2\times 2}$, the matrix $\hat{\hat{J}}_4$ is stable if and only if its trace is negative and its determinant is positive (these are the Routh-Hurwicz conditions specialized to a 2×2 matrix, cf. Murata (1977) p.93). The sign of the determinant is of course not affected by the choice of π. Thus if the diagonal entries of $\hat{\hat{J}}_4$ differ in sign, then the choice of π can affect the stability of $\hat{\hat{J}}_4$. The problem here is that a stable matrix might not be negative quasi-definite. Because if it were, we would know from Arrow and McManus (1958) (cf. Theorem 39' chapter 2 in Murata (1977)) that its stability is unaffected by postmultiplication by a positive diagonal matrix.

A tâtonnement in contingent contracts prices, (c), given the choice of numeraire (N.2) is the dynamical system

$$(c) \qquad \dot{\bar{p}}_1(t) = \bar{Z}_1(\bar{p}_1(t), \hat{\bar{p}}_2(t), 1)$$

$$\dot{\hat{\bar{p}}}_2(t) = \hat{\bar{Z}}_2(\bar{p}_1(t), \hat{\bar{p}}_2(t), 1) \quad \text{for all} \quad t \in \mathbb{R}_+.$$

And we have the stability criterion

(s.c.c.)

> A tâtonnement in contingent contracts prices is locally asymptotically stable, if the matrix $\hat{\bar{J}}$ is a stable matrix. It is unstable if at least one eigenvalue of $\hat{\bar{J}}$ has a positive real part.

The contingent contracts process (c) is quite similar to the asset-spot market process (r). Note, however, that since the latter is affected by the choice of the asset structure A in general neither (s.c.r.) implies (s.c.c.) nor the reverse needs to hold. Again this can be demonstrated by the case where there are no cross effects between periods and let $L_2 = 3$. If the sign of the diagonal entries of $\hat{J}_4$ differs, we can have two cases:

Firstly, if the trace of $\hat{\bar{J}}_4$ is positive, (s.c.c.) is violated but (s.c.r.) might still hold for some particular choice of A (respectively π).

Secondly, if the trace of $\hat{\bar{J}}_4$ is negative, (s.c.c.) is satisfied but (s.c.r.) might not hold for a particular choice of A.

Thus although the two equilibrium concepts are equivalent from a static allocational point of view, they are different with respect to their tâtonnement dynamics.

According to Hicks (1939) an equilibrium in the markets for commodity j is said to be imperfectly stable if the markets for all the other commodities are held in equilibrium (with possible adjustment of the prices of these goods), and there is stability in the j-th market. A system of markets is perfectly stable if each market is imperfectly stable regardless of the number of other markets adjusted to equilibrium. Since we propose to consider the stability of current period markets given future markets remain in equilibrium, our stability concept definitely has some similarity with that of Hicks. As Samuelson (1947) has noted, Hicks stability (h) is void of any dynamical system. The well-known stability criterion for Hicks stability is

(s.c.h.)

> Given the normalization rules $(N.1)$ and $(N.2)$ the system of first and second period spot markets is Hicks stable (h) if the matrix $-\hat{J}$ is a P-matrix[11]. If at least one minor of $-\hat{J}$ is negative, then (h) is unstable.

The sign of the minors are unaffected by the postmultiplication by positive diagonal matrices. Thus by Lemma 1 (s.c.h.) occurs if, and only if $-\hat{\bar{J}}$ is a

[11] A P-matrix is a matrix with every principal minor being positive.

P-matrix. Consequently it does not matter for Hicks stability whether we consider the original asset-spot market system Z, or rather the contingent contracts market system $\bar{Z}$.

The last concept to be compared is that of expectational stability which was first formulated in aggregate macroeconomic models as a tâtonnement process in expectations (cf. Lucas (1978), DeCanio (1979)). As Guesnerie (1992) has noticed and Evans and Guesnerie (1993) have shown rigorously, in an appropriately formulated normal form game expectational stability is closely related to the notion of rationalizability introduced by Bernheim and Pearce. An equilibrium is said to be expectational stable, (e), if, for any non-trivial restrictions on expectations that are common knowledge, expectations converge to its equilibrium value. Every agent revises his expectations using the fact that all agents optimize and thus he is able to constrain the set of possible values of the endogeneous variables that rational agents can expect. Applied to our framework this would require each agent to engage in the following mental process: starting with a tentative set of price vectors for future period prices, the agent is able to deduce the set of current price vectors being compatible with the equilibrium in the first period given the tentative second period price vectors. For this tentative set of first period price vectors he can then deduce a new tentative set of second period price vectors. The iteration of this procedure defines a dynamical system and, as before, we are interested in its local asymptotic stability. Note that the iteration of sets converges to a single point if and only if each point of the initial set converges to that point. We will therefore only focus on iterating point expectations. Since the models Z and $\bar{Z}$ are srtictly equivalent, expectational stability can be defined for both of them.[12] Usually this leads to different dynamical systems, but, as is shown in Guesnerie and Hens (1994), for the price normalization used here they will coincide.

To define expectational stability formally, let resources w be chosen so that $\hat{J}_1$ and $\hat{J}_4$ are regular, (Balasko (1994) proves that this holds as a generic property in ω.). Then locally there is a function $\psi_2 : N(\overset{*}{p}_1) \subset \mathbb{R}^{L_1}_{++} \to \mathbb{R}^{L_2-1}_{++}$ so that

$$\hat{Z}_2(p_1, \psi_2(p_1), 1) = 0 \quad \text{for all} \quad p_1 \in N(\overset{*}{p}_1).$$

By analogy, there is a function $\psi_1 : N(\overset{*}{p}_2) \subset \mathbb{R}^{L_2-1}_{++} \to \mathbb{R}^{L_1}_{++}$ so that

$$Z_1(\psi_1(\hat{p}_2), \hat{p}_2, 1) = 0 \quad \text{for all} \quad \hat{p}_2 \in N(\overset{*}{p}_2).$$

The expectational dynamics are given by the recursion

$$(e) \qquad \hat{p}_2(t+1) = \psi_2(\psi_1(p_2(t))) \quad t \in \mathbb{N}$$

[12] Guesnerie (199) works with the asset-spot market model Z, whereas Balasko (1994) considers expectational stability in the contingent contracts model $\bar{Z}$.

Let $\rho(A)$ denote the max norm of the eigenvalues of A

The stability criterion for (e) is

(s.c.e.)

> expectational stability occurs if $\rho(\hat{J}_0^{-1}\hat{J}_1\hat{J}_4^{-1}\hat{J}_2) < 1$. The process (e) is unstable if at least one eigenvalue of $(\hat{J}_0^{-1}\hat{J}_1\hat{J}_4^{-1}\hat{J}_2)$ has norm greater than one.

Note that we could also have defined expectational stability in terms of contingent contracts excess demand. Due to Lemma 1 and the fact that $\Lambda(\pi)$ is cancelled out by $\Lambda(\pi)^{-1}$ this would lead to exactly the same stability criterion!

Now we are in a position to compare the stability notions (h), (c), (e), and (t). The first point to mention is that in the case of a symmetric Jacobian $\bar{J}$ all four concepts coincide.

Proposition 2

Suppose the Jacobian of the contingent contracts excess demand $\bar{J}$ is symmetric then the four stability concepts (h), (c), (e), and (t) are all equivalent.

Proof

First note that a symmetric matrix is a P-matrix if and only if it is positive definite (Theorem 35, chapter 2 in Murata (1977)). Furthermore a symmetric matrix is negative definite only if it is stable (cf. Horn and Johnson (1985)). To complete the equivalence recall from Horn and Johnson (1985) Theorem 7.7.6 that a symmetric matrix $M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$, where A and C are square, is positive definite if and only if $C - B^T A^{-1} B$ is positive definite which by Theorem 7.7.6 in Horn and Johnson (1985) is equivalent to $\rho(B^T A^{-1} B C^{-1}) < 1$.

2

In the general, asymmetric case all concepts are different. Yet before showing this, we observe that in the case where $\hat{J}$ is a 2×2 matrix (s.c.h.) implies (s.c.c.) and (s.c.t.).

Proposition 3

If the normalized Jacobian of the contingent contracts market excess demand $\hat{J}$ is of dimension 2 then Hicks stability (h) implies both, contingent contracts stability (c) and the stability of (t).

Proof

Let $L_1 + (L_2 - 1) = 2$ then there are three possible cases:

1. $L_1 = 2$,

i.e. $\hat{J}$ reduces to the first period's Jacobian $J_1 = \begin{bmatrix} J_{11} & J_{12} \\ J_{13} & J_{14} \end{bmatrix}$

where $J_{1l}, l = 1, 2, 3, 4$ are scalars.

$(s.c.h.)$ is obtained iff $J_{11} < 0, J_{14} < 0$ and $\mid J_1 \mid > 0$.

$(s.c.c.) = (s.c.t.)$ is obtained iff $J_{11} + J_{14} < 0$ and $\mid J_1 \mid > 0$.

2. $L_1 = 1$,

   i.e. $\hat{J} = \begin{bmatrix} J_1 & \hat{J}_2 \\ \hat{J}_3 & \hat{J}_4 \end{bmatrix}$ where J_1 and $\hat{J}_k, k = 2, 3, 4$ are scalars.

   $(s.c.h.)$ is obtained iff $J_1 < 0, \hat{J}_4 < 0$ and $\mid \hat{J} \mid > 0$

   $(s.c.c.)$ is obtained iff $J_1 + \hat{J}_4 < 0$ and $\mid \hat{J} \mid > 0$

   $(s.c.t.)$ is obtained iff $J_1 - \frac{J_1 \hat{J}_2}{\hat{J}_4} < 0$

3. $L_1 = 0$,

   i.e. $\hat{J} = \begin{bmatrix} \hat{J}_{41} & \hat{J}_{42} \\ \hat{J}_{43} & \hat{J}_{44} \end{bmatrix}$ where $\hat{J}_{4l}, l = 1, 2, 3, 4$ are scalars.

   Now (t) is not defined and $(s.c.h.)$ implies $(s.c.c.)$ by the same argument as in case 1.

2

The following section deals with the fact that all stability concepts are generally different from one another. This point will be illustrated by a number of examples. Because of the Sonnenschein-Debreu-Mantel results it is not necessary to construct economies whose Jacobians possess the features needed for our examples, so that we can start working directly in terms of the Jacobian matrices. One can then construct the economies appropriately (Geanakoplos and Polemarchakis(1980)). Most of the examples can be given in the case $L_1 = L_2 - 1 = 1$, so that J_0, J_1, J_2andJ_4 are scalars. For $\hat{J} = \begin{bmatrix} J_0 & J_1 \\ J_2 & J_4 \end{bmatrix}$ the stability criteria then specialize to

(s.c.h.) $J_0 < 0, J_4 < 0, J_0 J_4 - J_1 J_2 > 0$

(s.c.c.) $J_0 + J_4 < 0, J_0 J_4 - J_1 J_2 > 0$

(s.c.e.) $\mid \frac{J_1 J_2}{J_0 J_4} \mid < 1$

(s.c.t.) $J_0 - \frac{J_1 J_2}{J_4} < 0$.

The following table clarifies most of the relations between (h), (c), (e) and (t). A '+' (-) sign in the table indicates that the corresponding stability criterium is satisfied (violated).

$\hat{J}\backslash^{\text{concept}}$	(h)	(c)	(e)	(t)	result
$\begin{bmatrix} -2 & 7 \\ -1 & 3 \end{bmatrix}$	-	-	-	+	$(s.c.t.)$ does not imply any of the others
$\begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix}$	+	+	-	+	$(s.c.h.)$ does not imply $(s.c.e.)$
$\begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$	-	+	-	-	$(s.c.c.)$ does not imply any of the others
$\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$	-	-	+	-	$(s.c.e.)$ does not imply any of the others

Thus it remains to show that $(s.c.h.)$ neither implies $(s.c.c.)$ [13] nor $(s.c.t.)$. For this let $L_1 + (L_2 - 1) = 3$, since the partition of $\hat{J}$ is irrelevant for (h) and (c) consider any matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then we get

(s.c.h.)

$$a_{11} < 0, a_{22} < 0, a_{33} < 0$$
$$a_{11}a_{22} > a_{12}a_{21}, a_{11}a_{33} > a_{22}a_{33}, a_{22}a_{33} > a_{23}a_{32}$$
$$\mid A \mid < 0$$

(s.c.c.)

$$a_{11} + a_{22} + a_{33} < 0$$
$$\mid 2A \cdot I \mid < 0,$$
$$\mid A \mid < 0$$

[13] Samuelson (1947) gives a 4×4 matrix which shows that (s.c.h.) does not imply (s.c.c.).

where

$$2A \cdot I = \begin{pmatrix} a_{11} + a_{21} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{pmatrix}$$

which is the bialternate product of A and I, cf. Murata(1977). Thus any matrix A with $a_{12} = a_{23} = a_{31} = 0, a_{11} < 0, a_{22} < 0, a_{33} < 0$ will work provided

$\mid A \mid = a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} < 0$

and

$\mid 2A \cdot I \mid = (a_{11} + a_{22})(a_{11} + a_{33})(a_{22} + a_{33}) - a_{13}a_{32}a_{21} > 0$

e.g. choose

$$A = \begin{pmatrix} -1 & 0 & -3 \\ -3 & -1 & 0 \\ 0 & -3 & -1 \end{pmatrix}.$$

To show that (h) does not imply (t) let $L_1 = 2$ and $L_2 - 1 = 1$. Then

$$\hat{J} = \begin{pmatrix} \hat{J}_{11} & 0 & \hat{J}_{21} \\ \hat{J}_{13} & \hat{J}_{14} & 0 \\ 0 & \hat{J}_{32} & \hat{J}_4 \end{pmatrix}$$

where again all entries are scalars and we get

(s.c.h.) $\hat{J}_{11} < 0, \hat{J}_{14} < 0, \hat{J}_4 < 0$ and $\hat{J}_{11}\hat{J}_{14}\hat{J}_4 + \hat{J}_{11}\hat{J}_{13}\hat{J}_{32} < 0$

(s.c.t.) $\hat{J}_{11} + \hat{J}_4 < 0$ and $\hat{J}_{11}\hat{J}_4 + \hat{J}_{11}\hat{J}_{32}\hat{J}_{13} > 0$.

Thus

$$\hat{J} = \begin{pmatrix} -1 & 0 & -2 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

satifies the requirements. Hence, by means of examples we have shown

Proposition 4

In general, the stability of the processes (t), (c), (h), (e) never implies the stability of any of the others.

5 Scarf's example

In the preceeding section we have shown that, without posing any restrictions on the set of economies, we are able to find examples which show that in general the four stability concepts are all different. We now take the other extreme point of view by comparing these concepts by means of a particular simple economy. This will deepen the understanding concerning the differences between these concepts.

The example we have chosen is an intertemporal variant of Scarf's (1960) famous example on the instability of the Walrasian tâtonnement process. Scarf's example was very important since it put an end to the search for general stability results in the complete markets model (cf. Fisher (1987)).

Moreover, the Scarf example is of topical importance as many of the proposed new stability concepts are usually first tested in Scarf's example (cf. Cartigny (1990), Flaschel (1991) and Herings (1994)) .

Finally, the cause of instability in Scarf's example is generally well understood. Since preferences are homothetic, the instability arises from wealth effects. In an exchange economy an individual's wealth is given by the evaluation of his endowments. The crucial parameters of the model will then be the distribution of agents' endowments (cf. Hirota (1981), Hildenbrand and Kirman (1988) and Hens and Hildenbrand (1994)).

We will consider an intertemporal variant of Scarf's example. To keep things simple there will be no uncertainty. Agents' preferences for second period consumption are adapted from Scarf (1960) and first period consumption is aggregated into a single commodity. Across periods agents' utility functions are additively separable with a common discount rate.

In the example there is a unique equilibrium. The Jacobian matrix at that equilibrium is of dimension 3 and not always symmetric, so that in principle all four stability concepts could be distinguished.

We get the following striking results:

Hicks stability is equivalent to contingent contracts stability, and these stability concepts are qualitatively equivalent to the stability in Scarf's original example. Up to the effect of the discounting for (c) and (h) we get instability for exactly the same distribution of endowments as in the atemporal Scarf example. The set of endowment distributions leading to instability is decreasing monotonically in the discount factor. Compared to (c) and (h) our new stability concept resembles more that of expectational stability (e). Both concepts are stable whenever (c) or (h) are stable; and there are parameter values for which (e) and (t) are stable but (c) and (h) are not. The stability of (e) and (t) is not monotone in the discount factor. We achieve stability for sufficiently high or low discounting. (t) differs from (e) only with respect to the value of the lower bound of the discount factor below which we achieve stability. This value is twice as large for (t) as it is for (e). Therefore we have managed to introduce a new process which proves to be the most stable one. Finally it is important to note that while (c) and (h) are unaffected by the choice of the numeraire in the second period, this normalization is crucial for (e) and (t) and it can be used to explain the instability of these two processes.

The details of the example are as follows:

There are three commodities tomorrow ($L_2 = 3$) and a single commodity today ($L_1 = 1$). There are three agents with preferences about date 2 consumption given by

$$U^i(x_2) := \min_{l \in \{1,2,3\} \setminus i} \{x_{2l}\} \quad i = 1, 2, 3.$$

Thus agent i wants to consume the commodities $\{1, 2, 3\} \setminus i$ in fixed proportion and he is not interested in having commodity i. These preferences are those chosen by Scarf (1960). With a denoting the discount factor common to all agents, an agent's decision problem in the contingent contracts model is given by

$$(\overline{M}^i) \qquad \operatorname*{Max}_{\bar{x}_1 \geq 0, \bar{x}_2 \geq 0} a \; log \bar{x}_1 + (1 - a) log U^i(\bar{x}_2)$$

$$s.t. \quad \bar{p}_1 \cdot \bar{x}_1 + \bar{p}_2 \cdot \bar{x}_2 \leq \bar{p}_1 \cdot \omega_1^i + \bar{p}_2 \cdot \omega_2^i.$$

From $(\bar{M})^i$ we derive the demand

$$\bar{x}_1^i = \frac{a(\bar{p}_1 \cdot \omega_1^i + \bar{p}_2 \omega_1^i)}{\bar{p}_1}$$

$$\bar{x}_{2l}^i = \begin{cases} \frac{(1-a)(\bar{p}_1 \cdot \omega_1^i + \bar{p}_2 \cdot \omega_2^i)}{\sum_{j \neq i} \bar{p}_{2j}} & l \neq i \\ 0 & l = i \end{cases}.$$

Thus for $\omega_1^i = 1, i = 1, 2, 3$ and $\sum_i \omega_l^i = 1 = \sum_l \omega_l^i$ we get the equilibrium prices $\overset{*}{p} = (a, 1 - a, 1 - a, 1 - a)$ and the equilibrium allocation is

$$\overset{*}{x}{}^i = 1, \overset{*}{x}{}^i{}_{2l} = \begin{cases} \frac{1}{2} & l \neq i \\ 0 & l = i \end{cases} \quad i = 1, 2, 3.$$

Since the market clearing conditions can be transformed into a linear system in prices which has maximal rank, this equilibrium is the unique equilibrium of the model.

Straightforward computations lead to the following expression for the Jacobian matrix at equilibrium

$$\bar{J} = \frac{1}{12}\left\{\begin{bmatrix} \frac{12(a-1)}{a} & 4 & 4 & 4 \\ 4 & \frac{2a}{a-1} & \frac{1-2a}{a-1} & \frac{1-2a}{a-1} \\ 4 & \frac{1-2a}{a-1} & \frac{2a}{a-1} & \frac{1-2a}{a-1} \\ 4 & \frac{1-2a}{a-1} & \frac{1-2a}{a-1} & \frac{2a}{a-1} \end{bmatrix} - 2\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \omega_1^1 & \omega_2^1 & \omega_3^1 \\ 0 & \omega_1^2 & \omega_2^2 & \omega_3^2 \\ 0 & \omega_1^3 & \omega_3^2 & \omega_3^3 \end{bmatrix}\right\}.$$

Note that the first matrix in this expression is symmetric negative definite. Thus stability is governed by the matrix of initial resources $W = (\omega_l^i)_{i,l=1,2,3}$. We will follow Hirota (1981) and restrict attention to double stochastic matrices W.

Note that in the space of these matrices there are the following six "corners":

n_1
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

n_2
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

n_3
$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

n_4
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

n_5
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

n_6
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

In Scarf's (1960) original example the chosen endowments were situated in the second corner i.e. n_2. For this choice each trajectory of the Walrasian tâtonnement process follows a closed orbit around the equilibrium. This is clearly not a robust choice of parameters, since in this case all eigenvalues have real part zero. The same is true for n_3. Thus with respect to the Walrasian tâtonnement process the equilibrium is neither stable nor unstable for n_2 and n_3. Whereas the equlibrium is stable for n_1 it is unstable for n_4, n_5, n_6 (cf. Hens and Hildenbrand (1994)). Our results on the stability of the various concepts are summarized in the following tables: table (c) table (h) table (e) table (t). In these tables the sign '+' (-) indicates that for the corresponding parameter values the process is stable (unstable). A '0' denotes that neither of the two is true.

(h)

a/n	1	2	3	4	5	6
0	+	+	+	-	-	-
.780415	+	+	+	-	-	-
.1	+	+	+	-	-	-
.13397415	+	+	+	-	-	-
.25	+	+	+	-	-	-
.5	+	+	+	0	0	0
.7	+	+	+	+	+	+
.75	+	+	+	+	+	+
.8	+	+	+	+	+	+
1	+	+	+	+	+	+

(c)

a/n	1	2	3	4	5	6
0	+	+	+	-	-	-
.780415	+	+	+	-	-	-
.1	+	+	+	-	-	-
.13397415	+	+	+	-	-	-
.25	+	+	+	-	-	-
.5	+	+	+	0	0	0
.7	+	+	+	+	+	+
.75	+	+	+	+	+	+
.8	+	+	+	+	+	+
1	+	+	+	+	+	+

(e)

a/n	1	2	3	4	5	6
0	+	+	+	+	+	+
.780415	+	+	+	0	+	0
.1	+	+	+	-	+	-
.13397415	+	+	+	-	+	-
.25	+	+	+	-	+	-
.5	+	+	+	0	0	0
.7	+	+	+	+	+	+
.75	+	+	+	+	+	+
.8	+	+	+	+	+	+
1	+	+	+	+	+	+

(t)

a/n	1	2	3	4	5	6
0	+	+	+	+	+	+
.780415	+	+	+	+	+	+
.1	+	+	+	+	+	+
.13397415	+	+	+	0	+	0
.25	+	+	+	-	+	-
.5	+	+	+	0	+	0
.7	+	+	+	+	+	+
.75	+	+	+	+	+	+
.8	+	+	+	+	+	+
1	+	+	+	+	+	+

Thus as long as $a > 0.5$ all concepts are stable. Smaller values of the discount factor a result in the instability of (c) and (h) for exactly the same endowment distributions as in Scarf's original example[14]. Restricting attention to the corners of the endowment space, (e) and (t) differ from these processess only with respect to the fifth corner for which they are stable as well. Furthermore, for $a < 0.134$ (t) becomes stable again and for $a < 0.078$ (e) becomes stable again. Note that tables (c)-(t) display the stability of the various concepts for endowment distributions being corners as described above. To display the behaviour for intermediate endowment distributions we produced some triangular plots (see the appendix). In the triangular plots each corner corresponds to one of the 6 corners in the parameter space and intermediate endowment distributions are convex combinations of the 3 corners chosen. A bright (dark) spot denotes a stable (unstable) configuration of parameters.

Combining stable with unstable corners will result in areas in these triangles of stability and instability respectively. It is interesting to point out that even combining stable corners can lead to instability for some convex combination of parameters and vice versa for instable corners. An instance of the latter occurs with the corners n_4, n_5, n_6. Here a convex combination of the corners means to distribute endowments more equally among consumers and as it was first pointed out by Hildenbrand and Kirman (1988) this leads to stability in the original Scarf example. The same is true for (c) and (h) in our extended version of Scarf's example. An instance for the opposite phenomenon occured again with combining corners n_4, n_5, n_6 in the case of (e) and (t). Although for a sufficiently small discount rate each corner is a stable endowment distribution, every convex combination does not necessarily lead to stability. In the middle of the triangle we obviously achieve stability again; this follows since the Jacobian is symmetrically negative definite orthogonal to aggregate endowments in this case (cf. Hildenbrand and Kirman (1988)). But in contrast to (c) and (h) there is no monotone increase of stability when endowments become more equally distributed.

By way of explanation of these complex results we would like to suggest the following:

(c) and (h) are similar in the sense that in contrast to (e) and (t) they do not use the intertemporal structure of the model explicitly. In this sense they are closer to the tâtonnement process in Scarf's original example. Instability therefore occurs if, and only if, the agents' marginal propensity to consume and the agents' endowments are too closely positively associated with each other

[14]Note that introducing the first period consumption stabilizes the equilibrium for the endowment corners n_2 and n_3.

(cf. Hens and Hildenbrand (1994)). In this sense instability is necessary for the instability of the processes (e) and (t). To get some explanation for the remaining difference to (c) and (h) consider the role of the numeraire in our example:

So far the third commodity was chosen as numeraire. If we were to choose the first commodity instead, then we would have to drop the second row and column of $\bar{J}$. Since the first matrix is not affected by this, we can focus on W. We observe that the new results can be obtained from the previous results by a shift of index of the endowment corners:

$n_1 \to n_1, n_2 \to n_2, n_3 \to n_3, n_4 \to n_5, n_5 \to n_6, n_6 \to n_4.$

Similarily, if we were to choose the second commodity as numeraire, the shift of indices would have to be

$n_1 \to n_1, n_2 \to n_3, n_3 \to n_2, n_4 \to n_4, n_5 \to n_6, n_6 \to n_6.$

Note that this does not change the stability of either (c) nor (h), because stable corners have been exchanged for stable corners and unstable corners have been exchanged for unstable corners. However the stability of (e) and (t) is affected by the choice of the numeraire. Note that going back to the asset-spot market model, changing the numeraire means changing the commodity in which the asset pay-offs are denominated. Thus a change in the stability properties of the equilibrium can be interpreted by changing the assets' pay-off pattern.[15] If commodity one were chosen, then the endowment corner n_4 would become unstable whereas n_5 would become stable. Similarily, for the second good as numeraire, n_5 would become stable whereas n_6 would become unstable. From this we can deduce the following 'rules for stability' concerning this intertemporal variant of Scarf's example:

(1) If a consumer who does not want to consume the numeraire owns all of it, the equilibrium is stable.

(2) If exactly one consumer owns all of what he does not want to consume and this is not the numeraire, the equilibrium is unstable.

[15]The importance of the choice of second period's numeraire for (e) has as well be noticed by Guesnerie and Hens (1994).

6 Appendix

7 References

Arrow. K.J. (1953) : Le role des valeurs bour sieres pour la repartition la meilleuve des risques; *Econometric, Colloques Internaux du Centre National de la Recherche Scientifique 11, 41-47,[English translation, Review of Economic Studies 31, (1964), 91-96.]*

Arrow,K.J. and F. Hahn (1971) : General Competitive Analysis; *Holden Day: San Francisco*

Arrow, K.J. and M. Nerlove (1958) : A note on expectations and stability; *Econometrica, 26, 297-305.*

Arrow, K.J. and M. Mc Manns (1958) : A note on dynamic stability; *Econometrica, 26, 448-454.*

Balasko, Y. (1988) : Foundations of the Theory of General Equilibrium; *Academic Press; London*

Balasko, Y. (1994) : The Expectational Stability of Walrasian Equilibria; *Journal of Mathematicl Economics, 23, 179-203.*

Cartigny, P. (1991) : Variations on a Theme from Scarf; *C.R.Q.E. Document de Travail, 90 AIS, Marseille.*

DeCanio, S. (1979) : Rational Expectations and Learning from Experiences; *Quarterly Journal of Economics, 93, 47-58.*

Debreu, G. (1959) : Theory of Value; *Wiley: New York.*

Debreu, G. (1972) : Smooth Preferences; *Econometrica 40(4): 603-615.*

Evans, G. and R. Guesnerie (1993) : Rationalizability, Strong Rationilzability, and Expectational Stability; *Games and Economic Behaviour 5, 632-646.*

Enthoven,A.C. and K.J. Arrow (1956) : A theorem on expectations and the stability of equilibrium; *Econometrica, 24, 288-293.*

Fisher,F.M. (1987) : Adjustment Processes and Stability; *in the New Palgrave;J. Eatwell, M. Milgate and P.Newman (eds.) pp. 26-29, London: Macmillian.*

Flaschel, P. (1991) : Zur Stabilitt von Tâtonnement-Prozessen mit derivativem Feedback; *mimeo University of Bielefeld.*

Geanakoplos, J.D. and H.M.Polemarchakis (1980) : On the disaggregation of excess functions; *Econometrica 48, 315-331.*

Guesnerie, R. (1994) : Expectational Stability...;

Guesnerie, R. (1992) : An Exploration of the Eductive justification of the Rational-Expectations Hypthesis; *American Economic Review Vo. 82, No. 5, 1254-1278.*

Guesnerie, R. and T. Hens (1994) : The Choice of the Numeraire for Expectational Stability; *mimeo DELTA, Paris.*

Hahn, F. (1982) : Stability; *Chapter 16 in Handbook of Mathematical Economics Vol. II, edited by K.J. Arrow and M.D. Intriligator, pp. 745-793, New York: Noth Holland.*

Herings, P. J-J. (1994) : A Globally and Universally Stable Price Adjustment Process; *mimeo, CentER, Tilburg University.*

Hens,T. and W. Hildenbrand (1994) : Stability of General Equilibrium: Simultaneous versus Sequential Adjustment Processe; *mimeo, University of Bonn.*

Hildenbrand, W. and A.P. Kirman (1988) : Equilibrium Analysis; *North Holland: Amsterdam.*

Hirota, M. (1981) : On the Stability of Competitive Equilibrium and the Patterns of Initial Holdings: An Example; *International Economic Review 22, 461-470.*

Hicks, J.R. (1939) : Value and Capital; *Oxford: Claredon Press.*

Horn, A.R. and C.R. Johnson (1985) : Matrix Analysis; *Cambridge University Press; Cambridge, U.K..*

Horn, A.R. and C.R. Johnson (1991) : Topics in Matrix Analysis; *Cambridge University Press; Cambridge, U.K..*

Lucas, R.E. (1978) : Asset Prices in an Exchange Economy; *Econometrica 46, 1429-1445*

Magill, M. and W. Shafer (1990) : Characterisation of generically complete real asset structures; *Journal of Mathematicl Economics, Vol. 19, Nos 1/2, 167-194.*

Murata, Y. (1977) : Mathematics for Stability and Optimization of Economic Systems; *Academic Press, New York.*

Negishi, T. (1958) : A Note on the Stability of an Economy Where All Goods Are Gross Substitutes; *Econometrica 26, 445-447.*

Radner, R. (1972) : Existence of Equilibrium of plans Pices and price Expectations; *Econometrica 40, No 2, 289-303.*

Samuelson, P.A. (1947) : Foundations of Economic Analysis; *Cambridge, Mars.:*

Harvard University Press

Scarf, H.E. (1960) : Some Examples of Global Instability of the Competitive Equilibrium; *International Economic Review 1, 157-172.*