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Market Demand Functions in the CAPM

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Abstract

We demonstrate that in a CAPM economy Walras Law and the Tobin Separation Property characterize market demand on finite sets of prices. Consequently, for any number n there exist CAPM economies which have at least n equilibria and hence have n different beta pricing formulas. It is shown that the lower bound on the number of equilibria, n, is robust to pertubations of endowments.

Keywords: CAPM, Market demand, multiplicity of equilibria.
JEL class. No.: C 62, D 51, G 12.

1 Introduction

The Capital Asset Pricing Model (CAPM) is a rich source of intuition and the basis for many practical financial decisions. Being built around the means and covariances of the payoffs of securities, it finds its roots in Markowitz's (1952) description of the mean-variance portfolio selection problem. In its equilibrium form, the CAPM goes back to Sharpe (1964), Lintner (1965) and Mossin (1966). Given an assumption of "variance-aversion" on agents' preferences it can be shown that, for every announced price system, all agents will be satisfied with holding shares of the same two funds, the riskless asset and the price system (Tobin (1958)). At equilibrium the later fund can be replaced by the market portfolio. From this Mutual Fund property one then deduces a simple and extremely useful linear pricing relation expressed in terms of betas and rates of returns.

Recently new efforts have been devoted to the study of the CAPM as a general-equilibrium model in which the natural questions of existence (c.f. Nielsen (1990) and Allingham (1991)) and uniqueness (c.f. Nielsen (1988) and Dana (1994)) of an equilibrium are important. In this note we continue along this line of research and address the question of the structure of market demand in the CAPM. We will show that given any choice of a finite number of price systems and according demands satisfying Walras Law and the Tobin Separation Property there exist two variance-averse agents whose market demand coincides with the preassigned values. This result parallels results known in the general equilibrium literature (c.f. Shafer and Sonnenschein (1982) for a survey and Andreu (1982) for such a result on a finite number of prices).

Our result thus shows that the Tobin Separation Property is *the* additional structure which is gained in the CAPM over the classical general equilibrium model (c.f. Arrow and Debreu (1954)). As a corollary it is obtained that for any number n there exists a CAPM economy which has at least n equilibria. Across these equilibria relative prices of assets and asset allocations differ. Hence there are at least n different beta-pricing formulas. The lower bound on the number of equilibria, n, is shown to be robust to endowment pertubations.

2 The Model

Let $(M, \mathcal{M}, \mu)$ be a probability space. Consider $\mathcal{L}$, the space of real–valued measurable functions on $(M, \mathcal{M}, \mu)$. We endow $\mathcal{L}$ with the scalar product $x \cdot y = E(x, y)$ where $E(x,y) = \int_M x(m)y(m)d\mu$ and with the norm $\|x\| = \sqrt{E(x,x)}$.

For later reference we write the covariance of $x, y \in \mathcal{L}$ by $Cov(x, y) = E(x, y) - E(x)E(y)$, where $E(x) = \int_M x(m)d\mu$ is the expected value of x. The consumption set will be the subset of $\mathcal{L}$ with finite variance, $L^2(\mu) = \{x \in \mathcal{L} \mid \|x\|^2 < \infty\}$. The price space is the consumption space's dual space, also $L^2(\mu)$.

Let X denote the marketed subspace of the consumption space $L^2(\mu)$. We assume that X is a closed vector space. The space X can be interpreted as the set of linear combinations of an underlying set of securities in $L^2(\mu)$. Security markets are complete if $X = L^2(\mu)$; otherwise they are incomplete.

Let $\mathbb{1}$ be a riskless asset, in the sense that $\mathbb{1}(m) = 1$ for all m in M. We assume that $\mathbb{1} \in X$. It is convenient to decompose every $x \in L^2(\mu)$ as the sum of a riskless asset with the same mean as x and a vector that has mean 0, $x \equiv \bar{x}\mathbb{1} + \tilde{x}$, where $\bar{x} = E(x)$ and $E(\tilde{x}) = \tilde{x} \cdot \mathbb{1} = 0$.

We normalize prices so that $p = \mathbb{1} + \tilde{p}$, i.e. $\bar{p} = 1$ for all $p \in L^2(\mu)$ we consider. An agent $i = 1, \ldots, I$ is described by his endowments $w^i \in X$ and his utility function $u^i : X \to \mathbb{R}$. The agent's decision problem is

$$\max_{x \in X, p \cdot x \leq p \cdot w^i} u^i(x)$$

Under the assumptions made, this decision problem can be derived from an agent's portfolio choice problem by recognizing that asset prices must be arbitrage free; i.e. they can be expressed in terms of state prices $p \in L^2(\mu)$. Furthermore, note that without loss of generality we can restrict state prices to ly in X. Components of p which are in $X^\perp$ do not contribute to the value of comsumption bundles in X.

Given this set-up, the essential assumption that makes the model a CAPM is the

Assumption 1 (mean–variance preferences)

Every agent $i = 1, \ldots, I$ has variance averse mean–variance preferences, i.e. there exist functions $v^i : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}, (\mu, \sigma) \mapsto v^i(\mu, \sigma)$ increasing in μ and decreasing in σ such that for all $x \in X$ $u^i(x) = v^i(\mu(x), \sigma(x))$, where $\mu(x) = E(x)$ and $\sigma(x) = \|\tilde{x}\|$.

In addition, for demand to be a well defined function, we introduce the following assumption which goes back to Sharpe (1964).

Assumption 2 (strict quasi–concavity)

For every agent $i = 1, \ldots, I$ the utility function v^i is continuous and strictly quasi–concave.

Definition. *A consumer that satisfies Assumptions 1 and 2 is called a CAPM consumer.*

3 The Main Result

The following proposition on demand of a CAPM consumer is due to Tobin (1958).

Proposition 1 (Tobin Separation Property)
For all $p \in X$ the demand of a CAPM consumer i lies in the span of $\mathbb{1}$ and p. More precisely it has the form

$$x^i(p) = \left(p \cdot w^i + \|\tilde{p}\| \phi^i(\tilde{p})\right) \mathbb{1} - \phi^i(\tilde{p}) \frac{\tilde{p}}{\|\tilde{p}\|} \tag{1}$$

where $\phi^i(\tilde{p})$ is a real valued non-negative function and $\phi^i(0) = 0$.

Proof. Let us first note that if $\tilde{p} = 0$, then the price is collinear to the riskless asset. In this case the riskless asset is the most preferred consumption plan in the budget set since any other plan would have the same mean and a greater variance. This means that $\phi(0) = 0$ and the formula still makes sense when $\tilde{p} = 0$.

Let $x \in \arg\max_{x \in X} u(x)$ s. t. $p{\cdot}x \leq p{\cdot}w$. Then by Assumption 1, $p{\cdot}x = p{\cdot}w$.

We first show that x is in the span of $\mathbb{1}$ and p, or – equivalently – $\mathbb{1}$ and $\tilde{p}$. Decompose $x = y + z$, where z is perpendicular to $\mathbb{1}$ and $\tilde{p}$. Write y in the form $y = \alpha \mathbb{1} - \phi \frac{\tilde{p}}{\|\tilde{p}\|}$, for some scalars α and ϕ. From this follows that $p \cdot z = 0$ so that y is also budget feasible. But from z being perpendicular to $\mathbb{1}$ it follows that $\bar{x} = \bar{y}$. Furthermore

$$\|\tilde{x}\|^2 = \|\tilde{y} + \tilde{z}\|^2 = \|\tilde{y}\|^2 + \|\tilde{z}\|^2 > \|\tilde{y}\|$$

because $\tilde{z}$ is perpendicular to $\tilde{y}$. Hence $z = 0$.

We next deduce $\alpha = p \cdot w + \phi\|\tilde{p}\|$ from Walras' Law, $p \cdot y = p \cdot w$.

To show that ϕ is non negative, note that $E(x) = p \cdot w + \phi\|\tilde{p}\|$ and $Var(x) = \phi^2$. Thus, since Variance is disliked, $(p \cdot w + \|\tilde{p}\||\phi(\tilde{p})|)\mathbb{1} - |\phi(\tilde{p})|\frac{\tilde{p}}{\|\tilde{p}\|}$, is preferred to $(p \cdot w + \|\tilde{p}\|(-|\phi(\tilde{p})|))\mathbb{1} - (-|\phi(\tilde{p})|)\frac{\tilde{p}}{\|\tilde{p}\|}$ unless $\phi = 0$, where $|\phi|$ denotes the absolute value of ϕ. This implies that ϕ is non-negative.

Strict quasi–concavity implies the uniqueness of the optimal portfolio. This ends the proof of Proposition 1. □

The Tobin Separation Property is inherited by market demand $x(p) = \sum_{i=1}^{I} x^i(p)$. Taking the market clearing condition $x(p) = w$ into account (where $w = \sum_{i=1}^{I} w^i$) establishes the Mutual Fund Property. This property says that at equilibrium every agent holds a certain fraction of just two funds: the market portfolio w and the riskless asset $\mathbb{1}$.

The main result shows that the Tobin Separation Property is the chief additional structure which a CAPM demand has compared with the classical Arrow-Debreu model does not possess.[1]

Theorem 1 *For any market portfolio $w \in X$ with positive mean and variance and any finite set of prices in X with pairwise different norms any function that satisfies Walras Law and the Tobin Separation Property is the aggregate demand of two CAPM consumers.*

Proof. We prove that two CAPM consumers are enough to generate such functions $x(p)$. Recall that the Tobin Separation Property implies that there are non-negative values $\phi(\tilde{p}_1), \ldots, \phi(\tilde{p}_n)$, so that for $p = p_1, \ldots, p_n$

$$x(p) = (p \cdot w + \|\tilde{p}\|\phi(\tilde{p}))\mathbb{1} - \phi(\tilde{p})\frac{\tilde{p}}{\|\tilde{p}\|}\ .$$

We know from the Tobin Separation Property that the demand for a given price lies in the span of the riskless asset and the price vector. The solution to the individual optimisation problem is thus the same as the solution of the problem

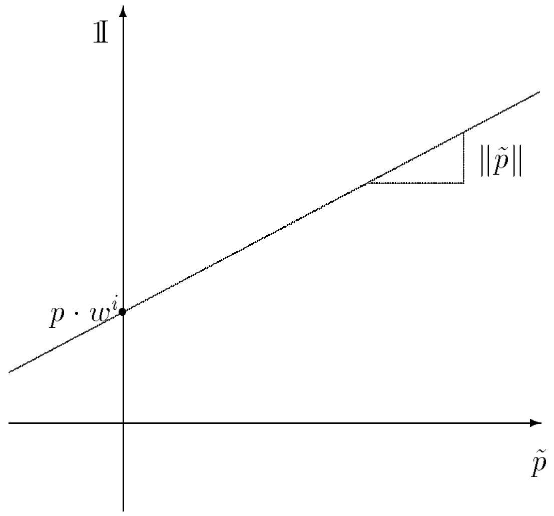
$$\max_{x \in \mathbb{R}\mathbb{1} + \mathbb{R}p,\ p\cdot x = p \cdot w^i} u^i(x)$$

The budget set is a line in this space, it is given by the parametrization $\phi \mapsto (p \cdot w^i + \|\tilde{p}\|\phi)\mathbb{1} - \phi\frac{\tilde{p}}{\|\tilde{p}\|}$. We even know that an optimal consumption plan corresponds to some non-negative ϕ. So that in the $\mathbb{R}\mathbb{1} + \mathbb{R}p$ plane which admits an orthonormal basis $\left(\mathbb{1}, -\frac{\tilde{p}}{\|\tilde{p}\|}\right)$, the solution to the agent's problem is the same as the solution of

$$\max_{\phi \geq 0} u^i\left((p \cdot w^i + \|\tilde{p}\|\phi)\mathbb{1} - \phi\frac{\tilde{p}}{\|\tilde{p}\|}\right).$$

Figure 1 shows the plane generated by $\mathbb{1}$ and $\tilde{p}$ for an typical consumer i.

[1] Note that Rolle's (1977) critique applies to the Mutual Fund Property and not to the Tobin Separation Property.



- Figure 1 -

The budget line in the $(1\!\!\mathrm{I}, \tilde{p})$ plane has intercept $p \cdot w^i$ with the $1\!\!\mathrm{I}$–axis and has slope $\|\tilde{p}\|$. It is important to note that because of Assumption 1 the indifference curves of the agent are the same in every $(1\!\!\mathrm{I}, \tilde{p})$ plane.

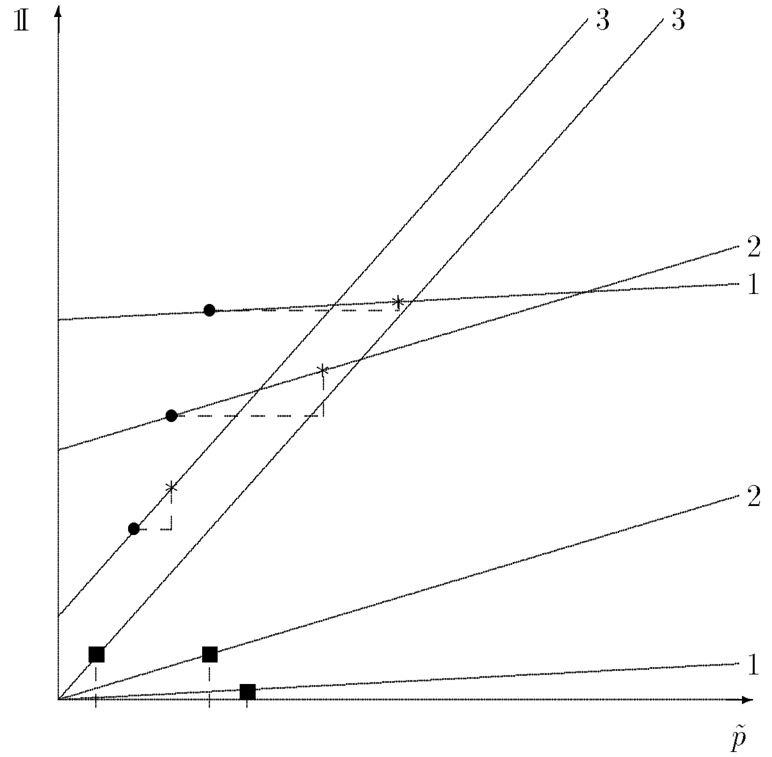
This remark justifies drawing all budget half–lines in the same plane. In this plane a vector x has two coordinates $\bar{x}$ and $\|\tilde{x}\|$. One may then define the budget set

$$B^i(p) = \{(\mu, \sigma) \in R^2 | \mu = (p \cdot w^i + \|\tilde{p}\|\sigma), \sigma \geq 0\}.$$

These half–lines may or may not intersect. The first agent will be given the endowment $w^1 = 0$. Then the half-lines $B^1(p_k)$ never intersect.

Take $w^2 = w$ as the second agent's endowment. For every $\tilde{p_k}$ define ψ_k as the minimal first coordinate of the intersection of $B^2(p_k)$ with $B^2(p_l), l \neq k$. Let furthermore $\phi^2(\tilde{p_k}) = \min(\psi_k/2, \phi(\tilde{p_k})/2)$. In the two dimensional space, the weak axiom of revealed preference is not violated for agent 2 if he is required to demand the point in $B^2(p_k)$ with coordinate $\phi^2(\tilde{p_k})$.

Figure 2 illustrates this decomposition. In this figure the $*$ symbol denotes the market demand that is decomposed into the first and the second agent's demand, denoted by $\blacksquare$ and $\bullet$, respectively.



- Figure 2 -

We introduce now an auxiliary economy in which we shall only specify preferences of the agents on two goods (not endowments). The first good, the quantity of which is denoted by y_1 corresponds to $\bar{x}$ in $\mathcal{L}$, the second good, the quantity of which is denoted by y_2 is a linear decreasing function of $\|\tilde{x}\|$ of the form $y_2(x) = K - \|\tilde{x}\|$. To every utility function V^i in this auxiliary economy one associates one in the original economy by composition $(u^i(x) = v^i(y_1(x), y_2(x))$, $y_1(x) = \bar{x}$, $y_2(x) = K - \|\tilde{x}\| \geq 0$.) This will put us in the position to apply an extension of Afriat's theorem given by Chiappori and Rochet (1987).

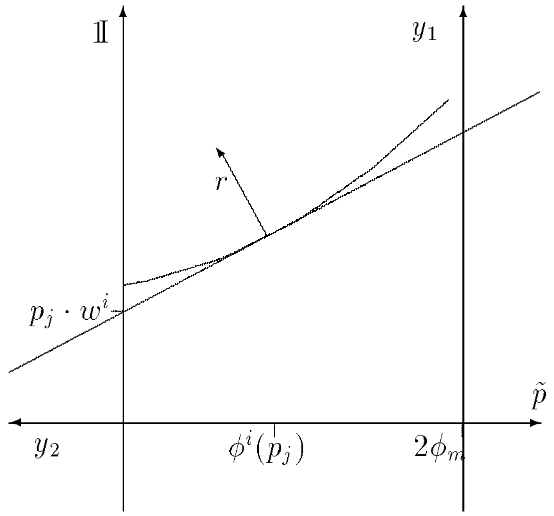
Let $\phi_m = \max\limits_{k=1,2,\, l=1,\ldots,n} \phi^k(p_l)$ and define the demand of the agent i in (y_1, y_2) coordinates as

$$y_l^i = \begin{pmatrix} p_k \cdot w^i + \|\tilde{p}_k\| \phi^i(\tilde{p}_k) \\ 2\phi_m - \phi^i(\tilde{p}_k) \end{pmatrix}.$$

The corresponding price vectors are

$$r_l = \begin{pmatrix} \|\tilde{p}_k\| \\ 1 \end{pmatrix}.$$

Figure 3 shows a typical consumer's decision in the former $(1\!\!1, \tilde{p})$ and the new (y_1, y_2) diagram.



- Figure 3 -

By construction, the demand of both agents satisfies the weak axiom of revealed preference which in two dimension is equivalent to the strong axiom (cf. Rose (1958)). Furthermore, we have chosen the demand of individuals so that it is different for all price vectors $p_1, \ldots, p_n$, so that the strong version of the strong axiom of revealed preference (cf. Chiappori and Rochet (1987)) is satisfied.

Hence, for each agent there is a continuously differentiable, strictly concave, monotonic utility function, v^i, rationalizing his demand behavior.

The continuously differentiable and strictly convex mean–variance utility function is given by the function v^i restricted to $\mathbb{R}_+ 1\!\!1 + \mathbb{R}_+ \tilde{p}$. Hence, a continuous and convex utility function $v^i(\mu, \sigma)$ exists so that for every $i = 1, 2$ and l,

$$\left(\left(p_l \cdot w^i + \|\tilde{p}_l\| \phi^i(\tilde{p}_l)\right), \phi^i(\tilde{p}_l)\right) = \arg \max_{(\mu,\sigma) \in B^i(p_l)} v^i(\mu, \sigma) .$$

Now when considering the two agents in the original good space

$$\left(p_l \cdot w^i + \|\tilde{p}_l\| \phi^i(\tilde{p}_l)\right) 1\!\!1 - \left(\phi^i(\tilde{p}_l)\right) \frac{\tilde{p}_l}{\|\tilde{p}_l\|} = \arg \max_{x \in X, p \cdot x \leq p \cdot w} u^i(x) .$$

Summing the demand of the two agents ends the proof of the theorem.

□

Theorem 1 parallels the Sonnenschein-Mantel-Debreu results which show that on a compact set of prices any continuous function which satisfies Walras Law can be generated as the market excess demand of an Arrow-Debreu exchange economy with as many agents as the dimension of the commodity space. In our result we only need two consumers to generate an infinite dimensional market demand. This is possible because due to the mean-variance assumption the commodity space we work with becomes effectively two dimensional. A more substantial difference is that we do not impose any non-negativity constraints on individual demands. Introducing such constraints in the CAPM would rule out the important case of normal distributions and moreover would conflict with the Tobin-sepatation properly. With non-negativity constraints market demand has some additional properties (cf. Shafer and Sonnenschein (1982)) and in this case market excess demand has some additional properties when - as it is in our theorem - aggregate endowments are given at the outset (cf. Koch (1989)). The most important difference is that our theorem holds on a finite set of prices only. This assumption is however essential since not every function $\phi(p)$ can be decomposed as a (finite) sum of functions of $p \cdot w^i$ and $\|\tilde{p}\|$. Note that sums of the form $\sum_{i=1}^{n} \phi^i(p \cdot w^i, \|\tilde{p}\|)$ cannot generate arbitrary cross derivatives of all order. For our result it is not essential that in the finite set of prices we consider, no pair of prices has the same norm. In our construction we need this asssumption because it might otherwise happen that prices with the same norm give the same value to the market portfolio and yet market demand is different for these two prices. Following our construction, however individual demand has to be the same for both prices since in the $(\mathbb{1}, \tilde{p})$ space very agent's budget lines are identical in the two situations. We can dispense with this assumption at the expense of assuming that $\phi(p)$ is bounded away from zero. If this is the case, we choose a vector $\hat{w} \in X$ such that for any such prices the value of $\hat{w}$ differs. We then endow agent one with $\frac{\bar{w}}{2}\mathbb{1} + \varepsilon\hat{w}$. Concerning agent 2, now being endowed with $w - (\frac{\bar{w}}{2}\mathbb{1} + \varepsilon\hat{w})$, we follow the same construction as before. Note however that the budget lines of agent 1 might intersect now. Letting ε tend to zero we can ensure that these intersections tend to the $\mathbb{1}$ axis. Consequently, there is some $\varepsilon > 0$ for which the residual market demand which agent 1 has to consume is larger than the largest value of these intersections. The proof is then completed as previously.

4 Number of CAPM-equilibria

In this section we show that for any number n there exist CAPM economies with at least n equilibria. The equilibria will differ in relative asset prices as well as in asset allocations. Hence there are at least n different beta-pricing relations. Furthermore, the lower bound on the number of equilibria, n, is robust to endowment pertubations.

Let the market portfolio be

$$w = \bar{w}\mathbb{1} - \tilde{w} \;, \bar{w} > 0 \;, \|\tilde{w}\| > 0 \;.$$

In terms of Proposition 1, a CAPM-equilibrium is a price system, p, such that

$$\left(p \cdot w + \|\tilde{p}\| \sum_i \phi^i(\tilde{p})\right) \mathbb{1} - \frac{\sum_i \phi^i(\tilde{p})}{\|\tilde{p}\|}\tilde{p} = \bar{w}\mathbb{1} - \tilde{w} \;. \tag{2}$$

Hence equilibrium prices must be such that $\tilde{p}$ is colinear to $\tilde{w}$, say $\tilde{p} = \beta\tilde{w}$ for some $\beta \geq 0$. Theorem 1 has shown that on finite domains the function $\phi(\tilde{p}) := \sum_i \phi^i(\tilde{p})$ does only need to satisfy the non-negativity condition $\phi(\tilde{p}) \geq 0$. Thus leaving the decomposition of $\phi(\tilde{p})$ to Theorem 1, a CAPM-equilibrium is completely described by a pair of non-negative scalars (β, ϕ). Consequently it is obtained

Corollary 1

For any number n there exist CAPM-economies with at least n equilibria. Furthermore, the lower bound on the number of equilibria, n, is robust to pertubations of the endowments.

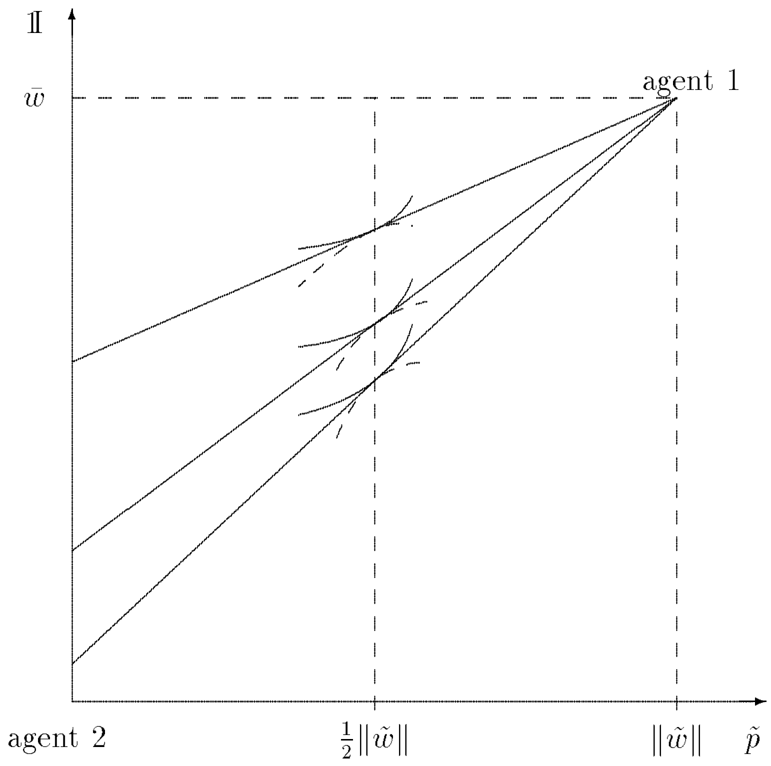
Proof. Consider the pairs $(\beta_k, \|\tilde{w}\|)_{k=1,\ldots,n}$ where $\beta_k > 0 \;,\quad k = 1,\ldots,n$ are n different scalars. Then $p_k = \mathbb{1} + \beta_k\tilde{w} \;,\quad k = 1,\ldots,n$ are n different equilibrium price vectors. Indeed, $p_k \cdot w = (\mathbb{1} + \beta_k\tilde{w})(\bar{w}\mathbb{1} - \tilde{w}) = \bar{w} - \beta_k\|\tilde{w}\|^2$ and $\|\tilde{p}_k\| = \beta_k\|\tilde{w}\|$. Thus $\sum_i \phi^i(\tilde{p}) = \|\tilde{w}\|$ solves equation (2).

$$(\bar{w} - \beta_k\|\tilde{w}\|^2 + \beta_k\|\tilde{w}\|\|\tilde{w}\|)\mathbb{1} - \frac{\|\tilde{w}\|\beta_k}{\beta_k\|\tilde{w}\|}\tilde{w} = \bar{w}\mathbb{1} - \tilde{w} \;.$$

Figure 4 illustrates these equilibria in an Edgeworth-Box diagram. Note that for the second agent's budget lines higher intercepts correspond to lower slopes so that budget lines actually intersect. It is easily veryfied that all budget lines

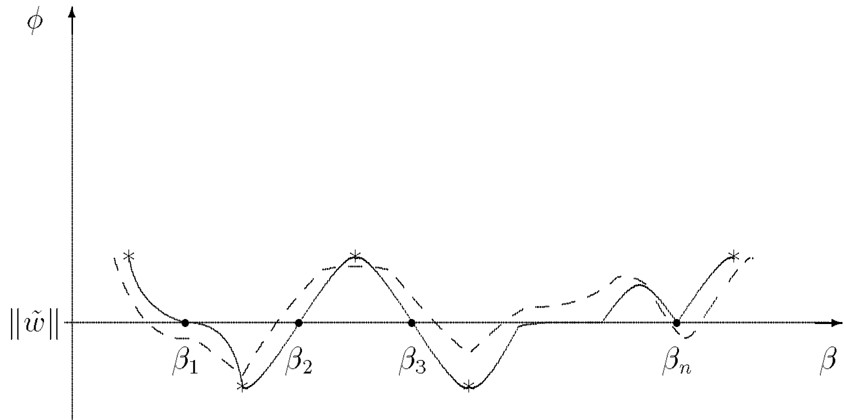
$$B^2(p) := \{(\mu, \sigma) \in \mathbb{R}^2 | \mu = (p \cdot \omega^2 + \|\tilde{p}\|\sigma \;,\quad \sigma \geq 0\}$$

intersect in the point $(\bar{w}, \|\tilde{w}\|)$.



- Figure 4 -

For the robustness of the lower bound on the number of equilibria consider Figure 5. There we have plotted ϕ against β. For $\beta_1, \ldots, \beta_n$ we have $\phi = \|\tilde{w}\|$, which is the equilibrium value.



- Figure 5 -

To make the lower bound on the number of equilibria robust, for intermediate values of β choose ϕ to be alternating between some positive and some negative value. Application of Theorem 1 to all these values (including the original values $\beta_1, \ldots, \beta_n$) gives a CAPM-economy with at least n equilibria. The market demand function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is given by the sum of the two agent's demand ϕ^1, ϕ^2 which by the Corollary in Chiappori and Rochet (1987) are continuous functions of income and (relative) prices, i.e.

$$\phi(\beta) = \phi^1(0, \beta\|\tilde{w}\|) + \phi^2(\bar{w} - \beta\|\tilde{w}\|^2, \beta\|\tilde{w}\|) .$$

Market excess demand is a function $Z(\beta, w) = \phi(\beta, w) - \|\tilde{w}\|$. Now suppose we did perturb w slightly to $\hat{w}$. As Z is continuous in w, for the intermediate values of β which we have chosen $Z(\beta, \hat{w})$ remains strictly positive respectively strictly negative. And as Z is continuous in β by the mean value theorem we still obtain at least n equilibria.

□

The equilibria which are constructed in Corollary 1 differ with respect to the size of the risk premium which is deducted from the expected value of an assets payoff. In the CAPM risk is measured by the covariance of an asset and the market portfolio. Recall that $w = \bar{w}\mathbb{1} - \tilde{w}$ and $p_k = \mathbb{1} + \beta_k\tilde{w}$.

Thus the value of an asset $x \in X$ is given by

$$p_k \cdot x = (\mathbb{1} + \beta_k\tilde{w}) \cdot x = \bar{x} + \beta_k\tilde{w} \cdot x$$

Note that $\tilde{w} \cdot x = cov(\tilde{w}, x) = -cov(w, x)$ thus

$$p_k \cdot x = \bar{x} - \beta_k cov(w, x) \tag{3}$$

Hence in every CAPM-equilibrium risk is measured in the same way. The risk premium however is indeterminate.

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