Robust Possibility and Impossibility Results

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Abstract

In a general setting of private information, the possibility (impossibility) theorem is said to be valid, if a balanced and ex post efficient mechanism exists (does not exist) which agents voluntarily participate at. Possibility and impossibility results are called *robust* if they hold for all priors with independently distributed private information. The following papers, among others, implicitely contain such results: Myerson and Satterthwaite (1983), Güth and Hellwig (1986), Cramton, Gibbons and Klemperer (1987), McAfee (1991), Makowski and Mezzetti (1993), Williams (1994) and Schmitz (1998). The present paper introduces a criterion which does not depend on the prior distribution of information. This criterion allows, not only, to simplify drastically earlier proofs, but also to generalize the existing results in a substantial way.

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1 Introduction

Let us assume that a finite number of risk-neutral agents is facing a decision which affects individual profits (or utility). The shape of each profit function is private information of the corresponding agent. To solve their decision problem, the agents make use of a mechanism. Particular attention will be paid to mechanisms where participation remains voluntary. If a balanced mechanism, allowing for voluntary participation, exists which leads to the ex post efficient decision then the *possibility theorem* is said to be valid. The existence of such a mechanism may depend on the prior distribution of private information. If such a mechanism exists for all priors with independently distributed private information then the *robust possibility theorem* is said to hold. The *impossibility theorem* is valid if the possibility theorem is not. The *robust impossibility theorem*, finally, is valid if, for any priors with independently distributed private information, the impossibility theorem is valid.

The scope for possibility or impossibility results can be explored by looking at either one of the following two mechanisms. The *cheapest Groves mechanism* implements the ex post efficient decision in dominant strategies. Participation of all agents remains voluntary. The mechanism, of course, fails to be balanced such that losses or gains must be borne by the outside party which operates the mechanism. The cheapest Groves mechanism is, from the viewpoint of a risk-neutral outside operator, the cheapest to run among all such mechanisms. If the private information, at which participation constraints are binding, does not depend on the prior distribution then the robust possibility theorem (robust impossibility theorem) is valid if and only if the cheapest Groves mechanism never (always) runs a deficit. The possibility theorem (impossibility theorem) is valid if and only if the cheapest Groves mechanism leads, in expected terms, to a gain (loss) for the outside operator.

The *least dissent mechanism* is balanced, has telling the truth as a Bayesian Nash equilibrium and leads to the ex post efficient solution. Participation, however, need not be voluntary. At the private information where the interim expected net profit as compared to the outside option is lowest or, equivalently, where the dissent against participation attains its maximum, the least dissent mechanism leads, by definition, to the same maximum dissent for all agents. In other words, the maximum dissent from enforced participation is the same for all agents. If the private information, at which the dissent attains its maximum, does not depend on the prior distribution then the possibility theorem is valid if and only if, for a given prior distribution of private information, the least dissent mechanism does not violate any participation constraints, i.e. participation does never give rise to dissent. If this is the case for any prior with independently distributed private information then the robust possibility theorem is valid. The impossibility theorem is valid if, for a given prior distribution, the maximum dissent of each agent is positive, i.e. at least some of the participation constraints must be violated such that participation has to be enforced. Finally, the robust impossibility theorem is valid if some of the participation constraints are violated under the least dissent mechanism, no matter how the prior distribution of information looks like

To my knowledge, the explicit notion of a *robust* possibility theorem or impossibility theorem as introduced above is new. Nevertheless, implicitely, the existing literature contains many results which are robust. MYERSON AND SATTERTHWAITE (1983) have the robust impossibility theorem shown to be valid for the case of a seller owning an indivisible good and facing a potential buyer of his good; GÜTH AND HELLWIG (1986) have done the same for the case of an indivisible public good; MAKOWSKI AND MEZZETTI (1993) deal with the case of a seller of an indivisible private good facing an arbitrary number of potential buyers (private information is assumed to be identically distributed for all buyers); CRAMTON, GIBBONS AND KLEMPERER (1987) have the possibility theorem shown to be valid if initial endowments are equally distributed among agents (again, private information is assumed to be identically distributed for all agents); SCHMITZ (1998) has a similar result for the case of an indivisible public good; WILLIAMS (1994) has considered several buyers facing several potential buyers (private information of sellers and buyers, respectively, are assumed to be identically distributed); MCAFEE (1981), finally, explores the case of a seller of a divisible private good facing a single potential buyer.

All these papers – the list is not claimed to be complete – implicitely contain robust possibility or impossibility results in the sense of the present paper. They have in common to calculate, for a given prior distribution of information, the agents' interim expected net profits, taking the Bayesian incentive and participation constraints into account. The present paper, instead, introduces a criterion which does not depend on the underlying prior and which still allows to fully explore the scope for possibility and impossibility results, including robust ones. It turns out that the propagated shift of attention allows, not only, to simplify proofs drastically but also to extend the earlier results in a substantial way.

The paper is organized as follows. In section 2, the general setting is introduced. Moreover, the basic insights of incentive theory are briefly recalled. While this section may not present truly new results, the shift from traditional to robust possibility theorems or impossibility theorems will be emphasized. Section 3 deals with the case of an indivisible public decision. In section 4, a seller of an indivisible private good is facing an arbitrary number of potential buyers whereas, in section 5, the analysis is extended to the case where several sellers are present. In sections 3 to 5, the scope for robust possibility and impossibility results is fully specified. Since the prior distribution of information does not enter our criterion, these distribution functions need not be assumed to be identical. Generalizing CRAMTON ET AL. (1986) and SCHMITZ (1998), sections 3 to 5 also present possibility theorems which are shown to be valid if agents, ex ante, are able to specify a default decision properly. In the case of identical distribution functions it is the equal distribution of initial endowments which gives rise to the possibility theorem. Yet, in more general cases, the initial assignment of endowments required to ensure the validity of the possibility theorem may depend on the prior distribution of information such that the robust version of the possibility theorem may fail to hold. In any case, the result initially due to CRAMTON ET AL. (1987) can, not only, be established in a much simpler way but also, its generalized version is shown to hold far beyond the case of a single indivisible private good with identical prior distributions for which it has originally been established. Section 6 deals with the first order approach to possibility and impossibility results. Under Concluding Remarks, the results are summarized and their meaning for market type institutions, among them are double oral auctions and the explicit implementation of Walras and Lindahl equilibrium, will be explored.

2 The model

A set $I = \{1, ..., n\}$ of agents is facing a decision x from a set X of alternatives. X is either a finite set or a subset of an Euclidean space. The profit (or utility) $\pi_i(x, \beta_i)$ of agent i depends on the decision x as well as on his information parameter β_i which is drawn from an interval $[\beta_{iL}, \beta_{iH}]$ of the real line. Private information is independently distributed, the density function of β_i

being denoted by $f_i(\beta_i)$. If, in addition, agent i receives payment z_i his net profit (or quasi-linear utility) amounts to $\pi_i(x, \beta_i) + z_i$. For a given realization $\beta = (\beta_1, ..., \beta_n)$ of information parameters, the *ex post efficient decision* solves

$$x(\beta) \in \arg\max_x \sum_{i \in I} \pi_i(x, \beta_i) \quad (1)$$

such that the *ex post efficient surplus* is given by

$$W(\beta) = \sum_{i \in I} \pi_i(x(\beta), \beta_i). \quad (2)$$

The agents' decision problem is assumed to be solved by a mechanism which, in its general form, can be described by a game form

$$g : S_1 \times ... \times S_n \to X \times \Re^n$$

where agent i must choose his strategy s_i from strategy set S_i. At strategy profile $s = (s_1, ..., s_n)$, the allocation

$$g(s) = [X(s), Z_1(s), ..., Z_n(s)] \quad (3)$$

results with $X(s) \in X$ being the decision and $Z_i(s) \in \Re$ the payment to agent i.

Without such mechanism, by assumption, the outside option $x^0 \in X$ would be reached. The game form (3) allows for *voluntary participation* if each agent has one strategy $s_i^0 \in S_i$ at least for which

$$\pi_i(X(s_i^0, s_{-i}), \beta_i) \equiv \pi_i(x^0, \beta_i)$$

holds, no matter what strategies s_{-i} are chosen by the other agents. At the outside option, agent i receives zero payment, i.e. $Z_i(s_i^0, s_{-i}) \equiv 0$. It is the distinctive feature of market-type institutions that they allow for voluntary participation.

Since β_i is assumed to be private information of agent i, his strategy $s_i = s_i(\beta_i)$ can depend on β_i only. Suppose we are given a Bayesian Nash equilibrium $s(\beta) = (s_1(\beta_1), ..., s_n(\beta_n))$ which leads to the ex post efficient decision (1). Associated with such an equilibrium, there exists the direct mechanism

$$[x(\beta), z_1(\beta) = Z_1(s(\beta)), ..., z_n(\beta) = Z_n(s(\beta))] \quad (4)$$

which satisfies the *incentive constraints*

$$R_i(\beta_i) \equiv E_{-i}[\pi_i(x(\beta', \beta_i), \beta_i) + z_i(\beta)] \geq E_{-i}[\pi_i(x(\tilde{\beta}_i, \beta_{-i}), \beta_i) + z_i(\tilde{\beta}_i, \beta_{-i})] \tag{5}$$

for all $i \in I$ and all $\beta_i, \tilde{\beta}_i \in [\beta_{iL}, \beta_{iH}]$ as well as the *participation constraints*

$$R_i(\beta_i) \geq \pi_i(x^0, \beta_i) \tag{6}$$

for all $i \in I$ and all $\beta_i \in [\beta_{iL}, \beta_{iH}]$. In (5), E_{-i} denotes the expectation operator with respect to all information parameters but β_i. The direct mechanism (4) has telling the truth as a Bayesian Nash equilibrium and leads to the same allocation as the equilibrium $s(\beta)$ under the original game form (3). In this sense we can, without loss of generality, restrict attention to direct mechanisms which satisfy the incentive constraints (5) and the participation constraints (6). This, of course, is the well-known revelation principle.

The ex post efficient decision (1) and the incentive constraints (5) jointly determine, up to a constant of integration, the interim expected net profit $R_i(\beta_i)$ of agent i as defined in (5). Moreover, if the information parameter β_i^0 solves

$$\beta_i^0 \in \operatorname*{arg\,min}_{\beta_i \in [\beta_{iL}, \beta_{iH}]} \; E_{-i}[W(\beta)] - \pi_i(x_0, \beta_i) \tag{7}$$

then it is sufficient to check the participation constraint at β_i^0. For any given mechanism leading to interim expected net profit $R_i(\beta_i)$ of agent i, let

$$D_i(\beta_i, x^0) \equiv \pi_i(x^0, \beta_i) - R_i(\beta_i)$$

denote the *dissent* of agent i against using the mechanism as compared to the outside option. If the dissent is positive, i.e. if $D_i(\beta_i, x^0) > 0$, then the participation constraint is violated at β_i whereas, if $D_i \leq 0$, it is not. The dissent of agent i attains its maximum at information parameter β_i^0. If the participation constraint is binding at β_i^0, i.e. if the maximum dissent is zero, then the interim expected net profit of agent i is given by

$$R_i(\beta_i) = E_{-i}[W(\beta)] - E_{-i}[W(\beta_i^0, \beta_{-i})] \tag{8}$$

such that the expected gain, loss if negative, of an outside party operating the mechanism amounts to

$$E[-\sum z_i(\beta)] = E[\Delta(\beta, \beta^0)] - \Pi^0 \tag{9}$$

where, by definition,

$$\Delta(\beta, \beta^0) \equiv \sum_{i \in I} W(\beta_i^0, \beta_{-i}) - (n-1)W(\beta) \tag{10}$$

and

$$\Pi^0 \equiv \sum_{i \in I} \pi_i(x^0, \beta_i^0). \tag{11}$$

The expected gain (9) must be the same for any mechanism for which the participation constraints are binding at β_i^0 for all $i \in I$. The *cheapest Groves mechanism*

$$z_i^G(\beta) \equiv \sum_{j \neq i} \pi_j(x(\beta), \beta_j) - W(\beta_i^0, \beta_{-i}) + \pi_i(x^0, \beta_i^0) \tag{12}$$

is such a mechanism. It has telling the truth as dominant strategies and, if information parameters β are drawn, then the gain from running the cheapest Groves mechanism is

$$-\sum z_i^G(\beta) = \Delta(\beta, \beta^0) - \Pi^0$$

such that the expected gain amounts to (9). Notice, if the information parameter β_i^0 as defined by (7) does not depend on the prior distribution (as will be the case in many applications below), then this prior distribution need not be known to specify the mechanism (12). Moreover, from the view of a risk-neutral outside operator, implementing in dominant strategies comes, as compared to Bayesian implementation, at no extra cost.

For any direct mechanism (4) which is balanced, i.e. $\sum z_i(\beta) \equiv 0$, it follows that

$$\sum R_i(\beta_i^0) = E[\Delta(\beta, \beta^0)]. \tag{13}$$

If a mechanism is balanced, some of the participation constraints may be violated. The *least dissent mechanism* is, by definition, a balanced mechanism where the maximum dissent is equal for all agents. It can be constructed as follows. For any constant redistribution $\sum \gamma_i = 0$, the mechanism

$$z_i^{ld}(\beta) \equiv z_i^e(\beta_i) - \frac{1}{n-1} \sum_{j \neq i} z_j^e(\beta_j) + \gamma_i \tag{14}$$

where, by definition,

$$z_i^e(\beta_i) \equiv E_{-i}[\sum_{j \neq i} \pi_j(x(\beta), \beta_j)]$$

must be balanced and, hence, satisfies (13). Moreover, it has telling the truth as a Bayesian Nash equilibrium. Due to (13), finally, the constant redistribution can be specified such that, for

$$R_i^{ld}(\beta_i) \equiv E_{-i}[\pi_i(x(\beta)\beta_i) + z_i^{ld}(\beta)],$$

it holds that

$$R_i^{ld}(\beta_i^0) = \pi_i(x^0, \beta_i^0) + \frac{E[\Delta(\beta, \beta^0) - \Pi^0]}{n}.$$

Therefore, under the least dissent mechanism, the interim expected loss at information parameter β_i^0 as compared to the outside option, i.e. the maximum dissent, is the same for all agents. Since the mechanism is balanced, no outside operator is needed to cover losses from running the least dissent mechanism. Participation, however, may need to be enforced. To specify the mechanism (14), the prior distribution of information must be known.

The possibility theorem is valid in the sense of the introduction if and only if the expected gain (9) from operating the cheapest Groves mechanism is non-negative and if and only if the least dissent mechanism satisfies all participation constraints. If the information parameter, at which the dissents of agents attain their maximum, do not depend on the prior distribution of information then the robust possibility and impossibility theorem can equivalently be expressed in terms of both the cheapest Groves and the least dissent mechanism. In fact, the robust possibility theorem is valid if, for all β,

$$\Delta(\beta, \beta^0) - \Pi^0 \geq 0. \tag{15}$$

This is the case if and only if the cheapest Groves mechanism never, not just in expected terms, runs a deficit and if and only if, for all priors with independently distributed information, the least dissent mechanism satisfies all participation restraints. The impossibility theorem is valid if and only if the cheapest Groves mechanism runs an expected loss and if and only if the least dissent mechanism violates some participation constraints. The robust impossibility theorem, finally, is valid if, for all β,

$$\Delta(\beta, \beta^0) - \Pi^0 \leq 0 \text{ and } \Delta(\beta', \beta^0) - \Pi^0 < 0 \tag{16}$$

for some β'. This is the case if and only if the cheapest groves mechanism always, not just in expected terms, runs a deficit and if and only if, for all priors with independently distributed private information, the least dissent mechanism violates some of the participation constraints.

3 Indivisible Public Good

In this section, the set $I = \{1, ..., n\}$ of agents is assumed to face an indivisible public decision $x \in X = \{0, 1\}$. The valuation of agent i for decision $x = 1$ as compared to the status quo $x = 0$ is denoted by β_i such that profit functions can be expressed in the form $\pi_i(x, \beta_i) = \beta_i x$. The ex post efficient surplus (2) amounts to $W(\beta) = \max[\sum \beta_i, 0]$. The outside option coincides with the status quo $x^0 = 0$. Hence, for Π^0 as defined in (11), it holds that $\Pi^0 = 0$.Moreover, since $\partial W(\beta)/\partial\beta_i \geq 0$, it follows from (7) that the participation constraints are binding at the lowest valuations $\beta^0 = (\beta_1^0, ..., \beta_n^0) = \beta_L = (\beta_{1L}, ..., \beta_{nL})$. For $n = 2$, the setting corresponds to that of Myerson and Satterthwaite (1983) and, for $n \geq 2$, to that of Güth and Hellwig (1986).

The following theorem deals with the *sophisticated case* where the ex post efficient decision truly depends on the realization of the information parameters and where there is more than one-sided asymmetric information. It is characterized by the conditions

$$\sum_{i \in I} \beta_{iL} < 0 < \sum_{i \in I} \beta_{iH} \tag{17}$$

and

$$\#\{i \in I \mid \beta_{iL} < \beta_{iH}\} \geq 2. \tag{18}$$

For all other cases, the robust possibility theorem is valid as may be obvious and as will be shown later.

Theorem 1 *(sophisticated case)*

In the case of an indivisible public decision, the robust impossibility theorem is valid.

Proof:
The term (10) can be calculated as follows. Let $P(\beta) \equiv \{i \in I \mid W(\beta_{iL}, \beta_{-i}) > 0\}$ be the set of agents for which the ex post efficient decision would be $x = 1$ even at their lowest valuation β_{iL}. It follows that

$$\Delta(\beta, \beta_L) = \sum_{i \in P(\beta)} (\beta_{iL} - \beta_i + W(\beta)) - (n-1)W(\beta) =$$

$$\sum_{i \in P(\beta)} (\beta_{iL} - \beta_i) - (n - 1 - p(\beta))W(\beta) \tag{19}$$

where, by definition, $p(\beta) = \#P(\beta)$. It follows from (19) that $\Delta(\beta, \beta_L) \leq 0$ for all β such that $p(\beta) \leq n-1$. Moreover, for $p(\beta) = n$, it follows from (19) that $\Delta(\beta, \beta_L) = \sum_{i \in I} \beta_{iL}$ which, in the sophisticated case (17), is negative. Finally, for $\beta = \beta_H$, it follows from (17) and (19) that $\Delta(\beta_H, \beta_L) < 0$ if $p(\beta) \neq n-1$ whereas, if $p(\beta) = n-1$, then

$$\Delta(\beta, \beta_L) = \sum_{i \in P(\beta)} (\beta_{iL} - \beta_{iH}) < 0$$

as follows from (18). The theorem is established. ■

It also follows from (19) that the robust possibility theorem holds whenever (17) or (18) are violated. In fact, if $W(\beta_L) > 0$ then $p(\beta) \equiv n$ and, hence, $\Delta(\beta, \beta_L) \equiv \sum_{i \in I} \beta_{iL}$ whereas, if $W(\beta_H) = 0$ then $p(\beta) \equiv 0$ and $\Delta(\beta, \beta_L) \equiv 0$. If, finally, (18) is violated such that $W(\beta_{iL}, \beta_{-i}) = W(\beta)$ holds for all agents but one, say $i = 1$, then $\Delta(\beta, \beta_L) = W(\beta_{1L}, \beta_{-1}) \geq 0$. Therefore, unless we are facing the sophisticated case, the robust possibility theorem must hold.

These results are not new. But, by checking (16) directly and not, as in earlier proofs, in expected terms, skillful integration can be dispensed with. The approach simplifies the proof in a drastic way.

Let us consider next the case where the agents are able to specify ex ante a default decision $x^0 = prob\{x = 1\}$ which means, unless they reach agreement at the interim stage, the decision will be $x = 1$ with probability x^0 as specified ex ante. SCHMITZ (1998) has established, for $n = 2$ and for $n > 2$ but identical distribution functions, that, if specified properly, the default decision x^0 gives rise to a possibility theorem. The following theorem generalizes and simplifies his findings.

The interim expected net profit $R_i(\beta_i)$ is, up to a constant of integration, uniquely determined. Its derivative

$$R_i'(\beta_i) = E_{-i}[\partial W(\beta)/\partial \beta_i] = E_{-i}[x(\beta)]$$

is monotonically increasing, i.e. $R_i(\beta_i)$ is a convex function. The information parameter β_i^0 at which the participation constraint of agent i is binding (c.f. (7)) depends on the default decision x^0 and can be obtained as follows:

$$\beta_i^0 = \beta_i^0(x^0) = \begin{cases} \beta_{iL} & \text{if} \quad R_i'(\beta_{iL}) \geq x^0 \\ \beta_i^0 & \text{if} \quad R_i'(\beta_i^0) = x^0 \\ \beta_{iH} & \text{if} \quad R_i'(\beta_{iH}) \leq x^0 \end{cases}$$

Theorem 2 *There exists a default decision $x^0 \in [0,1]$ such that, for $\beta^0 = \beta^0(x^0)$,*

$$\Delta(\beta, \beta^0) \geq \sum_{i \in I} \pi_i(x^0, \beta_i) \tag{20}$$

holds for all β. At this default decision, the possibility theorem must be valid.

Proof:
Without loss of generality we can focus on the sophisticated case such that (17) can be assumed to hold. If $x^0 = 0$ then $\beta^0 = \beta_L$ and, by (17), $\sum \beta_i^0 < 0$ whereas if $\beta^0 = \beta_H$ then $\sum \beta_i^0 > 0$. Therefore, by the intermediate value theorem, there exists a probability $x^0 \in (0,1)$ such that $\sum \beta_i^0 = 0$ and, hence, $\sum \pi_i(x^0, \beta_i^0) = 0$. For this realization β^0 of information parameters, if $W(\beta) = 0$ then $\Delta(\beta, \beta^0) = \sum W(\beta_i^0, \beta_{-i}) \geq 0$ whereas, if $W(\beta) > 0$, let be

$$P(\beta) \equiv \{i \in I \mid W(\beta_i^0, \beta_{-i}) > 0\}$$

and

$$N(\beta) \equiv \{i \in I \mid \beta_i^0 + \sum_{j \neq i} \beta_j \leq 0\} = I \backslash P(\beta). \tag{21}$$

It follows that

$$\Delta(\beta, \beta_0) = \sum_{i \in P(\beta)} (\beta_i^0 - \beta_i + W(\beta)) - (n-1)W(\beta) =$$

$$\sum_{i \in P(\beta)} (\beta_i^0 - \beta_i) - (n(\beta) - 1)W(\beta) \tag{22}$$

where, by definition, $n(\beta) = \# N(\beta)$. Similarly, (21) implies that

$$\sum_{i \in N(\beta)} (\beta_i^0 - \beta_i + W(\beta)) = \sum_{i \in N(\beta)} (\beta_i^0 - \beta_i) + n(\beta)W(\beta) \leq 0$$

and hence, by (22), that

$$\Delta(\beta, \beta^0) \geq \sum_{i \in P(\beta)} (\beta_i^0 - \beta_i) + W(\beta) + \sum_{i \in N(\beta)} (\beta_i^0 - \beta_i) = \sum_{i \in I} \beta_i^0 = 0$$

as was to be shown. ■

It seems fair again to claim that this proof is much simpler than that of SCHMITZ (1998). Moreover, since theorem 2 does not require distribution functions to be identical, the result is also more general.

Since (20) is shown to hold for all β, it also must hold in expected terms such that the possibility theorem is valid indeed. Notice, however, that the default decision x^0 leading to the possibility theorem depends on the prior distribution of information. As a consequence, it cannot be claimed that the robust possibility theorem would be valid.

4 Indivisible Private Good

Let us now assume that a seller ($i = 1$) of an indivisible private good is facing $n - 1$ potential buyers ($i = 2, ..., n$). The decision $x = (x_1, ..., x_n) \in \{0, 1\}^n$ requires to identify the agent i who receives the good ($x_i = 1$). Since only one unit of the good is available the constraint $\sum_{i \in I} x_i = 1$ must be met. The valuation of agent i of the good is denoted by β_i such that, presently, the profit functions are of the form $\pi_i(x, \beta_i) = \beta_i x_i$. The outside option $x^0 = (1, 0, ..., 0)$ corresponds to the decision where the seller keeps his good. The ex post efficient surplus amounts to $W(\beta) = \max[\beta_1, ..., \beta_n]$. Since $0 \leq \partial W(\beta)/\partial \beta_i \leq 1$ it follows that the seller's participation constraint is binding at the maximum valuation $\beta_1^0 = \beta_{1H}$ whereas the buyers' participation constraints are binding at their lowest valuation $\beta_i^0 = \beta_{iL}$ for $i = 2, ..., n$. Hence, for Π^0 as defined by (11), $\Pi^0 = \beta_{1H}$ must hold. The present setting corresponds to that of Makowski and Mezzetti (1993).

It proves useful to introduce the following notation. For any real valued vector $y = (y_1, ..., y_k)$ of arbitrary dimension, let $M(y) = \max[y_1, ..., y_k]$ denote the largest value and $m(y)$ the second largest value of all coordinates of y. More precisely, if $y_i = M(y)$ then $m(y) = M(y_{-i})$.

For the present setting, the *sophisticated case* is as follows. The valuation of the seller and of one buyer at least must be uncertain. Moreover, if the seller is at his lowest valuation while all buyers are at their highest valuation then it is efficient that one of the buyers receives the good. Finally, if the seller is at his highest valuation whereas all buyers are at their lowest valuations, then efficiency requires the seller to keep the good. More precisely, the sophisticated case can be summarized by the following condition:

$$\max[\beta_{1L}, M(\beta_{-1L})] < \min[\beta_{1H}, M(\beta_{-1H})] \tag{23}$$

In all other cases, the robust possibility theorem is valid as will be shown later in this section. The following theorem deals with the sophisticated case. The robust impossibility theorem is shown to be valid except for the case where there are two or more potential buyers and where the highest valuations of two buyers at least exceed the highest valuation of the seller.

Theorem 3 *(sophisticated case)*

In the case of a single indivisible private good, the following must be true:

(i) The robust possibility theorem never holds.

(ii) If $n = 2$ then the robust impossibility theorem always holds.

(iii) If $n \geq 3$ then the robust impossibility theorem is valid if and only if $m(\beta_{2H}, ..., \beta_{nH}) \leq \beta_{1H}$.

Proof:
In the present setting, the term $\Delta(\beta, \beta^0)$ can be calculated as follows:

$$\Delta(\beta, \beta^0) = M(\beta_{1H}, \beta_{-1}) + \sum_{i \geq 2} M(\beta_{iL}, \beta_{-i}) - (n-1)M(\beta)$$

To simplify, let $m \in \{2, ..., n\}$ be the buyer with the highest valuation, i.e. $\beta_m = M(\beta_{-1})$. It then follows that $M(\beta_{iL}, \beta_{-i}) = M(\beta)$ for all $i \geq 2$ but $i \neq m$. Hence

$$\Delta(\beta, \beta^0) = M(\beta_{1H}, \beta_{-1}) + M(\beta_{mL}, \beta_{-m}) - M(\beta). \tag{24}$$

Claim 1: There exists β such that $\Delta(\beta, \beta^0) < \Pi^0 = \beta_{1H}$.
Proof of claim 1: The sophisticated case (23) allows to find a feasible vector β of information parameters such that

$$\max[\beta_{1L}, M(\beta_{-1L})] < \beta_1 < m(\beta_{-1}) < M(\beta_{-1}) < \min[\beta_{1H}, M(\beta_{-1H})]$$

holds. In this region, (24) amounts to

$$\Delta(\beta, \beta^0) = \beta_{1H} + m(\beta_{-1}) - M(\beta_{-1}) < \beta_{1H}$$

as was to be shown. Claim 1 is established.
Claim 2: If there exists β such that $\Delta(\beta, \beta^0) > \Pi^0$ then $n \geq 3$ and $m(\beta_{2H}, ..., \beta_{nH}) > \beta_{1H}$.

Proof of Claim 2: If such a β does exist then, by monotonicity, $M(\beta_{mL}, \beta_{-m}) \leq M(\beta)$ and, by (24) $M(\beta_{1H}, \beta_{-1}) > \beta_{1H}$ such that $M(\beta_{-1}) > \beta_{1H}$ must hold. It then follows again from (24) that

$$\Delta(\beta, \beta^0) = M(\beta_{mL}, \beta_{-m}) > \beta_{1H}. \tag{25}$$

If it were true that $n = 2$ then (25) would imply that $M(\beta_{2L}, \beta_1) > \beta_{1H}$ and hence, since $\beta_1 \leq \beta_{1H}$, that $\beta_{2L} > \beta_{1H}$. This would be in contradiction with the sophisticated case (23). Therefore, if β exists such that $\Delta(\beta, \beta^0) > \Pi^0$ then $n \geq 3$.

Finally, again by (23) and since $\beta_1 \leq \beta_{1H} < M(\beta_{-1})$, it follows that $\beta_{mL} < \beta_{1H}$ and hence that $M(\beta_{-m}) = m(\beta) = m(\beta_{-1}) > \beta_{1H}$. Therefore, by monotonicity, $m(\beta_{2H}, ..., \beta_{nH}) > \beta_{1H}$ must hold. Claim 2 is established.

Claim 3: If $m(\beta_2, ..., \beta_n) > \beta_{1H}$ then $\Delta(\beta, \beta^0) = m(\beta_2, ..., \beta_n)$ and hence, by monotonicity, $m(\beta_{-1H}) = \Delta(\beta_H, \beta^0) > \beta_{1H}$.

Proof of Claim 3:

If $m(\beta_{-1}) > \beta_{1H}$ then, by (24), $\Delta(\beta, \beta^0) = \max[\beta_{mL}, m(\beta_{-1})]$. Since, by (23), $\beta_{mL} < \beta_{1H}$ and since, by assumption, $\beta_{1H} < m(\beta_{-1})$ it follows that $\Delta(\beta, \beta^0) = m(\beta_{-1})$. Claim 3 is established.

The above claims immediately lead to the proof of the theorem. In fact, (i) follows from claim 1, (ii) from claim 2 and (iii), finally, follows from claims 2 and 3 jointly. ■

Makowski and Mezzetti (1993) have considered the special case where all buyers' valuations are identically distributed on a common interval, say $[B_L, B_H]$. In this case, the sophisticated case (23) requires the seller's and the buyers' intervals of valuations to overlap. Then, according to theorem 3, if $n = 2$ or if $n \geq 3$ but $B_H \leq \beta_{1H}$ then the robust impossibility theorem is valid. In other words, the possibility theorem can only be valid if $n \geq 3$ and the maximum buyers' valuation B_H exceed the seller's maximum valuation β_{1H}. Theorem 3 contains this result due to Makowski and Mezzetti as a special case.

Earlier it was claimed that the robust possibility theorem is valid for all but sophisticated cases. Let us now establish this claim. If there is no uncertainty about the seller's valuation, i.e. if $\beta_{1L} = \beta_1 = \beta_{1H}$ then it follows from (24) that $\Delta(\beta, \beta^0) = M(\beta_{mL}, \beta_{-m}) \geq \beta_1 \geq \beta_{1H}$. If the seller's maximum valuation is not larger than all buyers' lowest valuations, i.e. if $\beta_{1H} \leq M(\beta_{-1L})$ then, by (24), $\Delta(\beta, \beta^0) = M(\beta_{mL}, \beta_{-m}) \geq M(\beta_L) = M(\beta_{-1L}) \geq \beta_{1H}$. For the remaining cases $M(\beta_{-1L}) = M(\beta_{-1H})$ and $M(\beta_{-1H}) \leq \beta_{1L}$ it can be shown that $\Delta(\beta, \beta^0) = M(\beta_{1H}, \beta_{-1}) \geq \beta_{1H}$. If this were not true then

$$M(\beta_{mL}, \beta_{-m}) = \max[\beta_1, \beta_{mL}, m(\beta_{-1})] < M(\beta) = \max[\beta_1, \beta_m, m(\beta_{-1})] \tag{26}$$

as follows from (24). But (26) can only hold if $\beta_m > \max[\beta_1, \beta_{mL}, m(\beta_{-1})]$ which, in case $M(\beta_{-1L}) = M(\beta_{-1H})$, would imply that $\beta_m = M(\beta_{-1L}) = M(\beta_{-1H}) = \max[\beta_{mL}, m(\beta_{-1})]$, a contradiction and which, in case $M(\beta_{-1H}) \leq \beta_{1L}$, would imply that $\beta_m = M(\beta_{-1H}) \leq \beta_{1L} \leq \beta_1$, a contradiction again. To summarize in all but sophisticated cases it holds that $\Delta(\beta, \beta^0) \geq \beta_{1H}$ which means that the robust possibility theorem is valid indeed.

To conclude this section, let us consider the case where the agents can ex ante specify a default decision $x^0 \in [0,1]^n$ which will be implemented unless they reach agreement at the interim stage. The i-th coordinate expresses the probability x_i^0 with which agent i would receive one unit of the good. Since one unit of the good is available, the constraint $\sum x_i^0 = 1$ must hold. Alternatively, x_i^0 could also be defined to be the fraction of the good which agent i initially owns. This interpretation is in line with Cramton et al. (1987) who have studied the present setting for the case where all agents' valuations are identically distributed. By making use of the method as propagated by the present paper, their result can be obtained in a much simpler way and, more important, it can be generalized to any collection of priors as long as private information remains independently distributed.

The interim expected net profit $R_i(\beta_i)$ is, as usual, uniquely determined up to a constant of integration by the incentive constraints and the priors. Its derivative

$$R_i'(\beta_i) = E_{-i}[\partial W(\beta)/\partial\beta_i] = E_{-i}[x_i(\beta_i)]$$

is monotonically increasing in β_i such that $R_i(\beta_i)$ is a convex function. The information parameters β_i^0 at which participation constraints are binding depends on the default decision in the following way:

$$\beta_i^0 = \beta_i^0(x_i^0) = \begin{cases} \beta_{iL} & \text{if } R_i'(\beta_{iL}) \geq x_i^0 \\ \beta_i^0 & \text{if } R_i'(\beta_i^0) = x_i^0 \\ \beta_{iH} & \text{if } R_i'(\beta_{iH}) \leq x_i^0 \end{cases}$$

Theorem 4 *There exists a default decision $x^0 \in [0,1]^n$, $\sum x_i^0 = 1$, such that*

$$\Delta(\beta, \beta^0) \geq \sum_{i \in I} \pi_i(x^0, \beta_i)$$

holds for all β and for $\beta^0 = \beta^0(x^0)$. At this default decision, the possibility theorem must be valid.

Proof:

We first deal with the case where all valuations are drawn from the some interval $[B_L, B_H]$. Notice, however, that density functions are not required to be identical.

Claim 1: There exists $B_L \leq B^0 \leq B_H$ and $x^0 \in [0,1]^n$, $\sum x_i^0 = 1$ such that all agents' participation constraints are binding at B^0, i.e. $\beta_i^0(x_i^0) = B^0$ for all $i \in I$.

Proof of Claim 1: If $\sum R_i'(B_L) \geq 1$ then choose x^0 such that $R_i'(B_L) \geq x_i^0$ for all $i \in I$ and $\sum x_i^0 = 1$. In this case, all participation constraints are binding at $B^0 = B_L$. Similarly, if $\sum R_i'(B_H) \leq 1$ then choose x^0 such that $R_i'(B_H) \leq x_i^0$ for all $i \in I$ and $\sum x_i^0 = 1$. In this case, all agents' participation constraints are binding at $B^0 = B_H$. In the remaining cases, by the intermediate value theorem, B^0 must exist such that $\sum R_i'(B^0) = 1$. If we choose $x_i^0 = R_i'(B^0)$ then all participation constraints are binding at $\beta_i = B^0$. In any case, claim 1 is established.

Claim 2: If all participation constraints are binding at the same value $\beta_i^0 = B^0$ then, for $\beta^0 = (B^0, ..., B^0)$,

$$\Delta(\beta, \beta^0) \geq B^0 = \sum \pi_i(x^0, B^0).$$

Proof of Claim 2: Since all participation constraints are binding at the same value B^0, it follows that

$$\Delta(\beta, \beta^0) = \sum_{i \in I} M(B^0, \beta_{-i}) - (n-1)M(\beta). \tag{27}$$

If $B^0 < M(\beta) = \beta_m$ it follows from (27) that $\Delta(\beta, \beta^0) = M(B^0, \beta_{-m}) \geq B^0$. If $B^0 \geq M(\beta)$ then it follows from (27) that

$$\Delta(\beta, \beta^0) = \sum_{i \in I} B^0 - (n-1)M(\beta) \geq B^0 \tag{28}$$

such that claim 2 is established. Therefore, theorem 4 is shown to hold if all valuations are drawn from the same interval.

To finish the proof of the theorem for non-identical intervals of valuation, let $B_L = \min[\beta_{1L}, ..., \beta_{nL}]$ and $B_H = \max[\beta_{1H}, ..., \beta_{nH}]$ denote the lowest and highest valuation, respectively, of all agents. Take any sequence $f_i^k(\beta_i)$ of density functions on $[B_L, B_H]$ which converge to the given density functions $f_i(\beta_i)$ on $[\beta_{iL}, \beta_{iH}]$ as $k \to \infty$. According to the established part of the theorem, there exist default decisions $x^k \in [0,1]^n$, $\sum x_i^k = 1$, and a common value $B_L \leq B^k \leq B_H$ at which all participation constraints are binding, i.e. for all $i \in I$ and all k,

$$B^k \in \arg \min_{\beta_i \in [B_L, B_H]} R_i^k(\beta_i) - \beta_i x_i^k. \tag{29}$$

Moreover, for all β and k,

$$\Delta(\beta, \beta^k) \geq \sum \pi_i(x^k, B^k) \tag{30}$$

must hold. Without loss of generality, $x^0 = \lim x^k$ and $B^0 = \lim B^k$ can be assumed to exist. It follows from (29) that

$$B^0 \in \arg \min_{\beta_i \in [B_L, B_H]} R_i(\beta_i) - \beta_i x_i^0$$

and from (30) that the possibility theorem is valid in the sense that more incentive and participation constraints are met than what would be needed for the given priors on the subset $\times_{i=1}^{n} [\beta_{iL}, \beta_{iH}]$ of $[B_L, B_H]^n$. In this way, the theorem is shown to hold also for cases where the agents' valuations are drawn from different intervals. ■

Cramton et al. (1987) have assumed all agents' valuations to be distributed according to the same distribution function. In this case, the interim expected net profit as a function of valuation does not depend on the agent, i.e. $R_i(\beta_i) = R(\beta_i)$ for all $i \in I$, such that, at identical shares $x_i^0 = 1/n$, all agents' participation constraints are binding at the same value B^0. While the information parameter B^0 still depends on the prior distribution of information, the default decision $x_i^0 = 1/n$ does not. It then follows from (28) that the possibility theorem must hold if all agents initially own identical shares. Theorem 4 shows how this result due to Cramton et al. can be generalized to non-identical distributions of information. Asymmetric priors, of course, may require unequal redistribution of initial endowments in order to ensure the possibility theorem. Moreover, in general, the default decision depends on the priors such that the possibility theorem but not the robust possibility theorem can be shown to hold.

5 Several Buyers and Sellers

The setting of section 4 is now extended to the case where several sellers are present, each of which owns one unit of an indivisible private good. No agent desires to keep or to buy more than one unit of the good. The sets of sellers and of buyers are denoted by S and B, repsectively, their number being $n_S = \#S$ and $n_B = \#B$. The decision $x = (x_1, ..., x_n) \in \{0,1\}^{n=n_B+n_S}$ concerns the subset of agents who obtain the goods, $x_i = 1$ meaning that agent i is among them. Since there are n_S units of the good available the constraint $\sum x_i = n_S$ most hold. The valuation of agent i is denoted by β_i and, hence, $\pi_i(x, \beta_i) = \beta_i x_i$. It proves useful to introduce the following notation. Let

$$\text{rank } (\beta_i \mid \beta) \equiv \#\{j \in I \mid \beta_j > \beta_i \text{ or } \beta_j = \beta_i \text{ and } j \leq i\}$$

denote the rank of valuation β_i among $\beta_1, ..., \beta_n$ where, by assumption, the rank follows the rank of the index in case of ambiguities. Ex post efficiency requires that the agents

$$P(\beta) \equiv \{i \in I \mid \text{ rank } (\beta_i \mid \beta) \geq n_S\}$$

whose valuations are among the n_S highest receive one unit of the good. The ex post efficient surplus (see (2)) amounts to $W(\beta) = \sum_{i \in P(\beta)} \beta_i$.

The outside option consists of the initial owners keeping the goods, i.e. $x_i^0 = 1$ if $i \in S$ and $x_i^0 = 0$ if $i \in B$. Since $0 \leq \partial W / \partial \beta_i \leq 1$, it follows from (8) that, for sellers $i \in S$, the participation constraint is binding at the highest valuation $\beta_i^0 = \beta_{iH}$ whereas, for buyers $i \in B$, it is binding at the lowest valuation $\beta_i^0 = \beta_{iL}$. For Π^0 as defined by (11),

$$\Pi^0 = \sum_{i \in I} \pi_i(x^0, \beta_i^0) = \sum_{i \in S} \beta_{iH}$$

must hold. The present setting of the general model contains the examples as studied by Williams (1994).

The analysis of the case with several sellers turns out to be quite tedious. To simplify matters, it is assumed that valuations of sellers are from the same interval $[S_L, S_H]$, those of the buyers from the interval $[B_L, B_H]$, i.e. if $i \in S$ then $\beta_{iL} = S_L$ and $\beta_{iH} = S_H$ whereas if $i \in B$ then $\beta_{iL} = B_L$ and $\beta_{iH} = B_H$. Distribution functions, however, need not be identical.

The *sophisticated case* is characterized by two-sided asymmetric information, i.e. $B_L < B_H$ and $S_L < S_H$. Moreover some buyers at least should receive the good if sellers have the lowest valuations but buyers have the highest valuation, i.e. $S_L < B_H$. Similarly, sellers should keep their good if their valuation is at its maximum while the buyers' valuation is at its minimum, i.e. $B_L < S_H$. Therefore the sophisticated case can be summarized by the following condition:

$$\max[B_L, S_L] < \min[B_H, S_H] \tag{31}$$

Theorem 5 *(sophisticated case)*
In the case of n_s indivisible private goods, the following must be true:

(i) The robust possibility theorem never holds.

(ii) The robust impossibility theorem is valid if and only if either $n_B = n_S$ or $n_B < n_S$ and $S_L \leq B_L$ or $n_B > n_S$ and $B_H \leq S_H$.

Proof:
In order to calculate $\Delta(\beta, \beta^0)$ as defined by (10), some further notation is needed. Notice, if $i \in P(\beta_{iL}, \beta_{-i})$, then $i \in P(\beta)$. Similarly, if $i \in P(\beta)$, then $i \in P(\beta_{iH}, \beta_{-i})$. Taking this into account, the set I of agents can be partitioned as follows:

$S_0 = S_0(\beta) \equiv \{i \in S \mid i \notin P(\beta_{iH}, \beta_{-i})\}$
$S_1 = S_1(\beta) \equiv \{i \in S \mid i \in P(\beta_{iH}, \beta_{-i}), i \notin P(\beta)\}$
$S_2 = S_2(\beta) \equiv \{i \in S \mid i \in P(\beta)\}$
$B_0 = B_0(\beta) \equiv \{j \in B \mid j \notin P(\beta)\}$
$B_1 = B_1(\beta) \equiv \{j \in B \mid j \in P(\beta), j \notin P(\beta_{jL}, \beta_{-j})\}$
$B_2 = B_2(\beta) \equiv \{j \in B \mid j \in P(\beta_{jL}, \beta_{-i})\}$

It then follows that $S_2(\beta) = S \cap P(\beta)$, $B_1(\beta) \cup B_2(\beta) = B \cap P(\beta)$ and, hence, that

$$\#[S_0(\beta) \cup S_1(\beta)] = \#[B_1(\beta) \cup B_2(\beta)]. \tag{32}$$

As a final piece of notation, let

$$M(\beta) \equiv \min\{\beta_i \mid i \in P(\beta)\}$$

$$m(\beta) \equiv \max\{\beta_i \mid i \notin P(\beta)\}$$

be the lowest valuation of agents who receive one unit of the good and the highest valuation of those who do not, respectively. By making use of this notation, it is easily seen that

$$\Delta(\beta,\beta^0) - \Pi^0 =$$
$$\sum_{i\in S_0} W + \sum_{i\in S_1} (\beta_{iH} - M + W) + \sum_{i\in S_2} (\beta_{iH} - \beta_i + W) +$$
$$\sum_{i\in B_0} W + \sum_{i\in B_1} (m - \beta_i + W) + \sum_{i\in B_2} (\beta_{iL} - \beta_i + W) -$$
$$(I-1)W - \sum_{i\in S} \beta_{iH}$$

which, after rearranging terms, is equal to

$$\Delta(\beta,\beta^0) - \Pi^0 = \sum_{i\in B_1} m + \sum_{i\in B_2} \beta_{iL} - \sum_{i\in S_0} \beta_{iH} - \sum_{i\in S_1} M. \tag{33}$$

Notice that, by (32), the first two sums in (33) have as many terms as the second two.

In order to simplify (33), it proves useful to partition the set of all β into the following four regions:

$$r_1 \equiv \{\beta \mid B_L \leq m(\beta) \text{ and } S_H \leq M(\beta)\}$$
$$r_2 \equiv \{\beta \mid m(\beta) < B_L \text{ and } S_H \leq M(\beta)\}$$
$$r_3 \equiv \{\beta \mid B_L \leq m(\beta) \text{ and } M(\beta) < S_H\}$$
$$r_4 \equiv \{\beta \mid m(\beta) < B_L \text{ and } M(\beta) < S_H\}$$

Claim 1:

$$\Delta(\beta,\beta^0) - \Pi^0 = \begin{cases} \#[B_1(\beta) \cup B_2(\beta)][m(\beta) - S_H] & \text{if } \beta \in r_1 \\ \#B_2(\beta)(B_L - S_H) & \text{if } \beta \in r_2 \\ \#[B_1(\beta) \cup B_2(\beta)][m(\beta) - M(\beta)] & \text{if } \beta \in r_3 \\ \#B_2(\beta)[B_L - M(\beta)] & \text{if } \beta \in r_4 \end{cases} \tag{34}$$

Proof of Claim 1:
Suppose $\beta \in r_1$. If $j \in B_2$ then $M \leq \beta_j$ and $m \leq \beta_{jL} = B_L$ whereas if $i \in S_1$ then $\beta_i \leq m$ and $M \leq \beta_{iH} = S_H$. Since $\beta \in r_1$ it follows that $m = B_L$ and $M = S_H$ and hence, from (33), that

$$\Delta(\beta,\beta^0) - \Pi^0 = \sum_{B_1\cup B_2} m - \sum_{S_0\cup S_1} S_H$$

such that claim 1 holds for $\beta \in r_1$ (c.f. (32)).

Suppose $\beta \in r_2$. If $j \in B_1$ then $M \leq \beta_j$ and $\beta_{jL} = B_L \leq m$ whereas if $i \in S_1$ then, as above, $M = S_H$. Since $\beta \in r_2$ it follows that $B_1 = \varnothing$ and hence from (33) that

$$\Delta(\beta, \beta^0) - \Pi^0 = \sum_{B_2} B_L - \sum_{S_0 \cup S_1} S_H$$

such that claim 1 also holds for $\beta_1 \in r_2$ (c.f. again (32)).

Suppose $\beta \in r_3$. If $j \in B_2$ then, as in the case of r_1, $m = B_L$ whereas if $i \in S_0$ then $\beta_i \leq m$ and $\beta_{iH} = S_H \leq M$. Since $\beta \in r_3$, it follows that $S_0 = \varnothing$ such that

$$\Delta(\beta, \beta^0) - \Pi^0 = \sum_{B_1 \cup B_2} m - \sum_{S_1} M.$$

Claim 1 also holds for $\beta_1 \in r_3$.

Suppose, finally, that $\beta \in r_4$. It follows, as in the case of r_2, that $B_1 = \varnothing$ and, as in the case of r_3, that $S_0 = \varnothing$ and hence that

$$\Delta(\beta, \beta^0) - \Pi^0 = \sum_{B_2} B_L - \sum_{S_1} M.$$

Therefore, claim 1 is shown to hold in all four regions.

Claim 2: There exists β such that $\Delta(\beta, \beta^0) < \Pi^0$ if and only if (31) ist met.

Proof of Claim 2:

Let us, first, assume that $\Delta(\beta, \beta^0) < \Pi^0$. If $\beta \in r_1$ then $B_L \leq m < S_H$, $S_H \leq M$ and $B_1(\beta) \cup B_2(\beta) = B \cap P(\beta) \neq \varnothing$ as follows from (34). Since $B \cap P(\beta) \neq \varnothing$ there exists $j \in B$ such that $M(\beta) \leq \beta_j \leq B_H$. Moreover, since $m < S_H$, it follows that $S_L < S_H$. As a consequence, (31) must hold in this case.

If $\beta \in r_2$ then $m < B_L < S_H \leq M$ and $B_2(\beta) \neq \varnothing$ as follows from (34). Since $B_2(\beta) \neq \varnothing$ there exists $j \in B$ such that $M(\beta) \leq \beta_j \leq B_H$. Moreover, since $m < B_L < S_H$, it follows that $S_L < B_L < S_H$. Therefore (31) must hold in this case as well.

If $\beta \in r_3$ then $B_L \leq m < M < S_H$ and $B_1(\beta) \cup B_2(\beta) \neq \varnothing$ such that there must exist $j \in B \cap P(\beta)$ for which it holds that $M(\beta) \leq \beta_j \leq B_H$. Since $M < S_H$ it follows that $S_L \leq M \leq B_H$. If it were true that $S_L = B_H = M$ then, for $j \in B \cap P(\beta)$, $\beta_j = B_H$ and, hence, $m(\beta) = B_H$ which contradicts $m(\beta) < M(\beta)$ .Therefore $\mathrm{S}_L < B_H$, and (31) must hold in this case as well.

Finally, if $\beta \in r_4$ then $m < B_L < M < S_H$ and $B_2(\beta) \neq \varnothing$. Therefore, as before, there exists $j \in B$ such that $M \leq \beta_j \leq B_H$. Therefore, since $S_L \leq M$ is always true, it follows that $S_L \leq B_H$. If it were true that $S_L = B_H = M$ then $B_L < B_H = S_L < S_H$ such that $m < B_L$ would not be feasible. Therefore $S_L < B_H$ as was to be shown such that, indeed, if β exists such that $\Delta(\beta, \beta^0) < \Pi^0$ then (31) must hold.

The other way round, let us assume that (31) holds. It is then easy to find β such that

$$\max[B_L, S_L] < m(\beta) < M(\beta) < \min[B_H, S_H]$$

and such that $B \cap P(\beta) \neq \varnothing$. It follows that β must be in region r_3 such that (34) implies that $\Delta(\beta, \beta^0) - \Pi^0 < 0$. Claim 2 is fully established.
Claim 3: There exists β such that $\Delta(\beta, \beta^0) > \Pi^0$ if and only if

$$\begin{array}{lll} \text{either} & \text{(a)} \quad S_H < B_H & \text{and} \quad n_B > n_S \\ \text{or} & \text{(b)} \quad S_H < B_L & \text{and} \quad n_B \leq n_S \\ \text{or} & \text{(c)} \quad S_L < B_L, S_L < S_H & \text{and} \quad n_B < n_S \end{array}$$

Proof of Claim 3: If there exists β in r_1 such that $\Delta(\beta, \beta^0) > \Pi^0$ then $B_L \leq m$, $S_H < m \leq M$ and $B \cap P(\beta) \neq \varnothing$. Since $S_H < m$ it follows that $n_B > n_S$ and $S_H < B_H$, i.e. (a). Moreover, if (a) holds then there exists β in r_1 such that $\Delta(\beta, \beta^0) > \Pi^0$.

If there exists β in r_2 such that $\Delta(\beta, \beta^0) > \Pi^0$. Then $m < B_L$, $S_H \leq M$, $S_H < B_L$ and $B_2 \neq \varnothing$ hence $S_H \leq M \leq B_H$. It follows that $S_L \leq S_H < B_L \leq B_H$ and, since $m < B_L$, that $n_B \leq n_S$ such that (b) holds. Conversely, if (b) holds then $S_L \leq S_H < B_L \leq B_H$ and $n_B \leq n_S$. If $n_B = n_S$ then $B_L \leq M \leq B_H$ and $S_L \leq m \leq S_H$ such that there exists β in r_2 as claimed. If $n_B < n_S$ then $S_L \leq m \leq M \leq S_H$ whereas $M = S_H$ is feasible such that β in r_2 does exist in this case as well.

There never exists β in r_3 with the desired property. If β in r_4, finally, such that $\Delta(\beta, \beta^0) > \Pi^0$ then $m < B_L$, $m \leq M < B_L$ and $M < S_H$. Since $M < B_L$ it follows that $n_B < n_S$. Moreover, since $S_L \leq M$ always holds, it follows that $S_L < B_L$, and since $M < S_H$ it must be true that $S_L < S_H$ such that (b) holds indeed. Conversely, if (b) holds then $M < B_L$ must always be true. Moreover, since $S_L < S_H$ and $n_B < n_S$ it is feasible that $S_L < m < M < B_L$ and $M < S_H$. Hence r_4 is not empty and, since $P(\beta) \cap B = B$, it follows from (34) that β must exist as claimed. Claim 3 is established.

The proof of theorem 5 easily follows from the above claims. Claim 2 implies (i). Claims 2 and 3 jointly imply (ii) because condition (b) in claim 3 does not occur in the sophisticated case, whereas $S_L < S_H$ in (b) always holds in the sophisticated case. Theorem 5 is fully established. ■

It also follows from the above proof that the robust possibility theorem holds for all cases which are not sophisticated. In fact, if (31) is violated then it follows from claim 2 in the above proof that no β exists for which $\Delta(\beta, \beta^0) < \Pi^0$.

The proof of theorem 5, while elementary, is tedious. But the result reaches beyond the existing literature. In the sophisticated case where the robust possibility theorem can never be valid, the robust impossibility theorem is always valid if the numbers of sellers and buyers are equal or if there are fewer buyers than sellers but the lowest valuation of sellers does not exceed the lowest valuation of buyers. The robust impossibility theorem is also valid if there are more buyers than sellers but the highest valuation of buyers does not exceed the highest valuation of sellers. In particular, if the valuation of buyers and sellers are drawn from the same interval, i.e. if $B_L = S_L$ and $B_H = S_H$ then, no matter what the number of buyers and sellers, the robust impossibility theorem always has to be valid. If, in addition, the density functions of all buyers are identical and those of all sellers are identical, too, the above result has been established by WILLIAMS (1994). Notice, however, that theorem 5 covers more general cases. In particular, it does not require identical density functions.

To conclude this section let us generalize the result of CRAMTON ET AL. (1987) (c.f. section 4) to the case including several sellers. By assumption, it is possible to specify ex ante which fraction x_i^0 of a good agent i initially should own. Since there are n_S units of the good available, the constraint for the default decision is $\sum x_i^0 = n_S$. If the agents specify the default decision x^0 properly then the following possibility theorem is valid.

Theorem 6 *There exists a default decision $x^0 \in [0,1]^n$, $\sum x_i^0 = n_S$ such that*

$$\Delta(\beta, \beta^0) \geq \sum \pi_i(x^0, \beta_i^0)$$

holds for all β. At this default decision, the possibility theorem must be valid.

Proof:

The proof is similar to the one of theorem 4. It is again sufficient to deal with the case where all valuations are drawn from the same interval $[B_L, B_H]$.

Moreover, also as in theorem 4, a default option x^0 can be shown to exist such that the participation constraint of all agents is binding at the same value B^0 of the information parameter. It follows that $\Pi^0 = \sum \pi_i(x^0, \beta_i^0) = n_S B_0$.

In order to calculate $\Delta(\beta, \beta^0)$, let us partition the set I of all agents as follows:

$$\begin{array}{lll} I_0 = & I_0(\beta) = & \{i \mid i \notin P(\beta), \quad i \notin P(B^0, \beta_{-i})\} \\ I_1 = & I_1(\beta) = & \{i \mid i \in P(\beta), \quad i \notin P(B^0, \beta_{-i})\} \\ I_2 = & I_2(\beta) = & \{i \mid i \notin P(\beta), \quad i \in P(B^0, \beta_{-i})\} \\ I_3 = & I_3(\beta) = & \{i \mid i \in P(\beta), \quad i \in P(B^0, \beta_{-i})\} \end{array}$$

Similarly as in the proof of (33), it can be shown that

$$\Delta(\beta, \beta^0) - \Pi^0 = \sum_{I_1} m + \sum_{I_2 \cup I_3} B^0 - n_S B^0 - \sum_{I_2} M. \tag{35}$$

Two cases must be distinguished. If, first, $M(\beta) \leq B^0$. Then, if $i \in I_1$ it follows that $M \leq \beta_i$ but $B^0 \leq m$ and, hence, that $m = M = B^0$. If $i \in I_2$ then $\beta_i \leq m$ but $M \leq B^0$. Therefore (35) implies that

$$\Delta(\beta, \beta^0) - \Pi^0 \geq \sum_{I_1} B^0 + \sum_{I_2 \cup I_3} B^0 - n_S B^0 - \sum_{I_2} B^0 = 0.$$

If, second, $B^0 < M(\beta)$ then $I_2 = \varnothing$. Moreover, if $i \in I_1$ then $M \leq \beta_i$ but $B^0 \leq m$. It then follows from (35) that

$$\Delta(\beta, \beta^0) - \Pi^0 \geq \sum_{I_1} B^0 + \sum_{I_3} B^0 - n_S B^0 = 0$$

as was to be shown. ■

Notice, if all distribution functions are identical then, for the default decision $x_i^0 = n_S/n$ $(i = 1, ..., n)$, the participation constraints of all agents are binding at the same information parameter such that equal shares lead to the possibility theorem. In more general cases, however, unequal shares may be needed to ensure the possibility theorem.

6 The First Order Approach

As a final application of the method propagated by the present paper, let us consider a set of agents $I = \{1, ..., n\}$ facing a divisible public decision

$x \in [0, \infty]$. Profit functions $\pi_i(x, \beta_i)$ are assumed to be differentiable in quantity and information parameter. It is further assumed that profits are concave functions of quantity, i.e. $\pi_{ixx} = \partial^2 \pi_i / \partial x^2 < 0$ and that the single crossing properly holds, i.e. $\pi_{ix\beta} = \partial^2 \pi_i / \partial x \partial \beta_i > 0$. Finally it is assumed that $\pi_i(0, \beta_i) \equiv 0$ for all $\beta_i \in [\beta_{iL}, \beta_{iH}]$. It then follows by integration that $\pi_{i\beta} = \partial \pi_i(x, \beta_i)/\partial \beta_i > 0$ for $x > 0$. The ex post efficient surplus $W(\beta) = \sum \pi_i(x(\beta), \beta_i)$ is characterized by first order conditions

$$\sum_{i \in I} \pi_{ix}(x(\beta), \beta_i) = 0. \tag{36}$$

The envelope theorem then yields

$$\partial W(\beta)/\partial \beta_i = \pi_{i\beta}(x(\beta), \beta_i) \geq 0$$

and, hence,

$$\partial^2 W(\beta)/\partial \beta_i \partial \beta_j = \pi_{i\beta x} \partial x / \partial \beta_j$$

must hold for all $j \neq i$. Differentiating first order condition (36) leads, by making use of the single crossing property, to the conclusion that $\partial x / \partial \beta_j \geq 0$ for all $j \in I$. The following line summarizes these findings:

$$\frac{\partial W(\beta)}{\partial \beta_i} \geq 0 \text{ and, for } i \neq j, \frac{\partial W(\beta)}{\partial \beta_i \partial \beta_j} \geq 0 \tag{37}$$

Moreover, if the outside option is assumed to be $x^0 = 0$ then, by (37), the participation constraints of all agents are easily seen to be binding at the lowest information parameter, i.e. $\beta_i^0 = \beta_{iL}$ for all $i \in I$. It follows that

$$W(\beta_L) \geq \Pi^0 = \sum \pi_i(x^0 = 0, \beta_{iL}) = 0. \tag{38}$$

Theorem 7 *If (37) and (38) hold, as would be the case in the above setting of a divisible public decision, then the following must be true:*

(i) The robust possibility theorem is valid if and only if $\Delta(\beta_H, \beta_L) \geq 0$.

(ii) The robust impossibility theorem is valid if and only if $W(\beta_L) = 0$.

Proof:
Since $\Delta(\beta, \beta_L) = \sum W(\beta_{iL}, \beta_{-i}) - (n-1)W(\beta)$ it follows that

$$\frac{\partial \Delta(\beta, \beta_L)}{\partial \beta_j} = \sum_{i \neq j} \left[\frac{\partial W(\beta_{iL}, \beta_{-i})}{\partial \beta_j} - \frac{\partial W(\beta)}{\partial \beta_j} \right].$$

Since $\beta_{iL} \leq \beta_i$, it follows from (37) that

$$\partial \Delta(\beta, \beta_L)/\partial \beta_j \leq 0. \tag{39}$$

Moreover, $\Pi^0 = 0$ by (38) and $\Delta(\beta_L, \beta_L) = W(\beta_L)$ by definition. Therefore, the theorem easily follows from monotonicity condition (39). ■

This result generalizes findings of McAfee (1991) who, however, deals with a setting of two agents only where private and public decisions cannot be distinguished. Notice, in the case of a divisible decision, it is quite likely that, even at the lowest information parameters, the ex post efficient surplus will be strictly positive, i.e. $W(\beta_L) > 0$. In this case the robust impossibility theorem can never be valid. Moreover, under the assumptions of the theorem, the robust possibility theorem is valid if and only if the condition

$$\Delta(\beta_H, \beta_L) = \sum_{i \in I} W(\beta_{iL}, \beta_{-iH}) - (n-1)W(\beta_H) \geq 0$$

is satisfied. Therefore, in the case of a divisible public decision, there is scope for a robust possibility theorem even if the ex post efficient decision truly depends on the realization of the information parameters (sophisticated case).

We should also point out that, in the case of private goods, the monotonicity condition (39) may be violated. In fact, for the indivisible decision cases of sections 4 and 5, the monotonicity constraint typically fails to hold if more than two parties are involved. This is the reason why the proofs of theorems 3 and 5 are more complicated than those of theorems 1 and 7.

Nevertheless the first order approach may be of use beyond the case of public decisions. Let x^0 be the outside option and let the participation constraints of agent i be binding at β_i^0 (see (7)). For simplicity, let us assume that β_i^0 does not depend on the prior distribution of information. Moreover, let $\beta_i^{\min}$ and $\beta_i^{\max}$ be solutions of the following problems:

$$\beta^{\min} \in \arg\min_{\beta} \ \Delta(\beta, \beta^0), \beta^{\max} \in \ \arg\max_{\beta} \ \Delta(\beta, \beta^0)$$

To find the above solutions, use of the first order condition could possibly be made. The following theorem must hold.

Theorem 8 *Let* $\Pi^0 = \sum \pi_i(x^0, \beta_i^0)$. *Then:*

(i) The robust possibility theorem is valid if and only if $\Delta(\beta^{\min}, \beta^0) \geq \Pi^0$

(ii) The robust impossibility thoerem is valid if and only if $\Delta(\beta^{\max}, \beta^0) \leq \Pi^0$ *and there exists* β *such that* $\Delta(\beta, \beta^0) < \Pi^0$.

(iii) In all other cases, no robust theorem can be valid.

7 Concluding Remarks

In this paper, the robust possibility theorem is said to be valid if, for all prior distributions of private information, a balanced mechanism exists which agents voluntarily participate at and which leads to the ex post efficient decision. The robust impossibility theorem is said to be valid if, for no prior distribution of private information, the ex post efficient decision can be the result of voluntary trade. In the case of indivisible decisions (sophisticated cases only), robust theorems can never be valid whereas, in the case of divisible decisions, there is scope for robust possibility theorems while robust impossibility theorems are unlikely to hold. In the case of an indivisible public decision, the robust impossibility theorem is always valid whereas, in the case of an indivisible private decision, there is scope for a possibility theorem but not for a robust possibility theorem.

If the robust impossibility theorem is valid the ex post efficient solution cannot be reached by a balanced mechanism allowing for voluntary participation. A double oral auction would be such a mechanism. Any computerized experimentation of the double oral auction gives rise to a mechanism that can be described by a game form such as (3). SADRIEH (1998) has worked out the details of such a game form. Under complete information, he shows that the efficient decision has to be reached in Nash equilibrium. Moreover, he presents some evidence that a static experimental repetition of a game of incomplete information leads, as a behavioral prediction, to the Nash equilibrium of the corresponding game with complete information. If this is true even in settings where the robust impossibility theorem is valid then the efficiency must result from the boundedly rational behavior of real subjects.

In a framework of fully rational behavior, the theoretical results of Sadrieh cannot generally be extended to settings where individual valuations are private information. The impossibility theorem or even the robust impossibility theorem is too likely to be valid if commodities are indivisible.

A similar argument holds for market solutions where the role of the Walrasian auctioneer and the strategy sets of households are explicitely modeled as a game form (3). Take any parameter configuration for which the robust impossibility theorem has been identified to hold then the Walrasian equilibrium cannot generally be reached, at least not if the auctioneer does not know individual valuations. In other words, to reach the Walrasian equilibrium, informational requirements are substantial. In fact, since some robust impossibility theorems hold for private as well as for public decisions, Walrasian equilibrium can be equally demanding in terms of informational requirements as Lindahl equilibrium would be. The crucial question seems to be, not, whether private or public goods are at stake but, rather, whether the robust possibility theorem or robust impossibility theorem is valid.

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