DISCUSSION PAPER B–180

A TERM STRUCTURE MODEL AND THE PRICING OF INTEREST RATE DERIVATIVE

PUBLISHED IN:
THE REVIEW OF FUTURES MARKETS, **VOL.12/2, 1993, PP. 391-423.**

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ABSTRACT. The paper developes a general arbitrage free model for the term structure of interest rates. The principal model is formulated in a discrete time structure. It differs substantially from the Ho–Lee–Model (1986) and does not generate negative spot and forward rates. The results for the continuous time limit support this. The probability distribution with finite support is derived for the spot rate return. The model permits the arbitrage free valuation of bond options and interest rate options and produces dynamic portfolio strategies to duplicate these contracts.

INTRODUCTION

The uncertainty of future interest rate movements is a serious aspect to financial decision making. Investment decisions are often very sensitive to changes of the term structure. Therefore the management of interest rate uncertainty is an important subject and it is necessary to analyse financial innovation which are designed to deal with the interest rate risk. Examples of such instruments are put and call options on zero coupon and coupon bonds or direct interest rate options like caps and floors.
The study of options on zero coupon bonds which are special forms of interest rate derivatives was the first important step to analyse these new instruments. C. A. Ball and W. N. Torous (1983) replaced the geometric Brownian motion used by F. Black and M. Scholes by a Brownian bridge process to model the dynamics of a zero coupon bond process. By this they could guarantee the face value for a default free bond at the end of its maturity and were able to derive a closed form solution to European type options over zero coupon bonds. Following the Ball–Torous–Model the principal question of interest rate options appears to be the appropiate modelling of the Brownian bridge process[1].
However, these bond price based models have the disadvantage that in most of them negative spot or forward rates cannot be excluded. Moreover they cannot be easily extended to price options on coupon bonds and by mixing up zero coupon bonds of different maturity one usually leaves the considered class of Brownian bridge processes. But the principal problem of the bond price based method is that this approach is not suitable to describe the whole term structure of interest in an arbitrage free way. As a consequence most of the problems that are inherent to the bond price based models cannot be solved in a satisfactory way. Therefore it is necessary to incorporate the whole term structure of interest rates into one consistent, i.e. intertemporally arbitrage free model.

Date. March 1993. This version: March 23, 1995.
JEL Classification. G13.
Key words and phrases. Arbitrage, Hedging, Interest Rate, Martingale Measure, Option Pricing, Term Structure.
The authors are grateful for helpful discussions with Freddy Delbaen, David Heath and Bjarne Astrup Jensen. Financial support by Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at the University of Bonn, is gratefully acknowledged.

[1]There are many interesting extensions and modfications of the Ball–Torous-Model. Some of them are Bühler (1987), Bühler and Käsler (1989), Locarek and Reißner (1990), Schöbel (1987), Schaefer and Schwartz (1987), etc.

A first solution to this problem was provided by the seminal paper of T. Ho and S.B. Lee (1986) They take the initial term structure as exogeneously given and let the entire curve fluctuate in a discrete time binomial process, thus producing a consistent arbitrage free model of bond prices for all initially given maturities. Unforunately their model produces negative spot and forward rates with positive probability.
The Ho and Lee model has been extended by D. Heath, R. Jarrow and A. Morton in a series of papers (1987, 1988, 1989) in three significant ways. First they take forward rates instead of bond prices as the basic building blocks for their analysis. This seems a more natural approach to modelling stochastic interest rate movements and facilitates the estimation of the parameters for the stochatic processes involved. Second, they extend Ho and Lee to incorporate continuous trading. In particular, they show that with continuous trading the valuation formula becomes independent of the so–called "pseudo" probabilites. Third, they extend the one–factor model of Ho and Lee to include multiple random factors. Their work give important insight into the basics of term structure models both from a theoretical and a practical point of view.
In the present paper we develope an alternative binomial model of the term structure, in which the basic building blocks are not the forward rates but the spot rates. In contrast to the Ho–Lee–Model our risk measure is the volatility of the spot rate and not of the return. Furthermore, negative spot rates and forward rates are not generated within the model. The model permits the arbitrage free valuation of interest rate options and produces dynamic hedge strategies to duplicate these options, where it is possible to choose from several equivalent strategies.
Although our model is driven only by the process of the short term interest rates it can incorporate practically any term structure with time dependent volatility, since the volatility may depend both on time and state. Our model is akin to the Binomial models proposed by Courtadon and Weintraub (1989) and Black, Derman and Toy (1990) which were developed independently. But as Jensen and Nielsen (1990) have shown the Courtadon–Weintraub model is not arbitrage free. We go beyond Black, Derman and Toy by proving the existence of a unique positive short rate process for any term structure and martingale measure. The only requirement beeing that today's forward rates are positive. This result implies that our model is arbitrage–free and complete in the sense of Harrison and Kreps (1979). Hence any interest rate dependent contingent claim, including caps, swaptions, option on both zero and coupon bonds, whether European or American, can be uniquely priced by arbitrage. We also go beyond Black, Derman and Toy by providing limit results for the discrete interest rate process (section 3). This results show that the model has an implicite mean reversion property in expectation.
In a recent paper Jensen and Nielsen (1991) have studied the generale structure of Binomial Lattice Models for bonds. They also provide an excellent survey on these types of models which were initiated by the seminal paper of Ho and Lee (1986). As they have shown our model, as well as the Black, Derman and Toy–Model, falls into their class of monotone binomial lattice models.
The paper is organised as follows: Section 1 contains the general implications of the no arbitrage requirement for a model of the term structure of interest rates. The resulting discrete term structure model is discussed in section 2, while section 3 analyses the limit aspects of the model. Given this model of the term structure of interest, the pricing of different types of interest rate derivatives is investigated in section 4. Section 5 contains some concluding remarks.

1. Arbitrage Implications

The no arbitrage requirement together with non negativity of forward rates does impose several restrictions on zero coupon and coupon bond price processes. As a first consequence the following four conditions have to be fulfilled.

1. Neglecting the default risk any bond price process has a non–stochastic terminal value at the end of its maturity.

2. During the lifetime of a bond its value cannot exceed his terminal value plus the outstanding coupon payments.
3. The value of any zero coupon bond is restricted from above by the value of an identical zero coupon bond with shorter maturity.
4. The value of any coupon bond must be equal to a portfolio of zero coupon bonds with face value and maturity corresponding to the coupon payments.

To fulfill these principal requirements would impose an enormous amount of boundary conditions on a bond price based approach.

One way to overcome these difficulties, following the analysis of J. M. Harrison and D. M. Kreps (1979)[2] would be to consider the set of probability measures under which the bond price processes are martingales. It turns out, that any price system consistent with the no arbitrage condition corresponds exactly to one equivalent martingale measure. Therefore if this set is empty there are arbitrage opportunities. If there exists only one equivalent martingale measure then the arbitrage consistent price system is unique. Furthermore the market is complete in this case and any derivative security is redundant with respect to the existing securities.

The consequences of the probability theoretical characterisation of price systems consitent with no arbitrage can best be seen if one assumes only one source of uncertainty, i.e. a one–factor–model. There are mainly two approaches by which the no arbitrage condition can close such a model. First one can specify two price processes in such a way that they define a unique martingale measure. In continuous time models this is usually done by two differential equations for zero coupon bonds of different maturities with respect to one Brownian motion process. Then any other security depending on the Brownian motion process and with maturity in the considered time range is redundant and can be priced by arbitrage. In other words there is no freedom in specifying additional zero bond price processes.

The second approach is to use the martingale feature to describe the suitable class of price processes. Suppose the instantaneous spot rate process $(r_t)_t$ is given by a stochastic differential equation

$$dr = \mu(t,r)dt + \sigma(t,r)dW \tag{1}$$

where W is a Brownian motion and the functions $\mu(...)$ and $\sigma(...)$ do fulfill the Lipschitz–and growth conditions such that a solution of (1.1) exist[3]. The price process of a zero coupon bond $\{B(t,T)\}_{t\epsilon[0,T]}$ with maturity date T then entirely depends on the spot rate process. Therefore the stochastic differential equation of the zero coupon bond can be derived by Itô's Lemma[4]

$$dB = \left(B_t + \mu(t,r)B_r + \frac{1}{2}\sigma^2(t,r)B_{rr}\right)dt + \sigma(t,r)B_r dW. \tag{2}$$

The no arbitrage condition implies that the excess return per unit risk of any zero coupon bond does not depend on the maturity date of this particular bond[5]. Therefore there exists a function $\lambda(r,t)$ independent

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[2] See also Sandmann (1988) for an application of Harrison an Kreps to the case of an interest rate market .

[3] The conditions on $\mu(t,r)$ and $\sigma(t,r)$ are:

a) $\mu,\sigma : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ measurable functions.

b) $\exists\, K_1 > 0$, such that the Lipschitz–condition holds, i.e. $\forall t \in [0,T];\quad x,g \in \mathbb{R}$

$$\mid \mu(t,x) - \mu(t,y) \mid \; + \; \mid \sigma(t,x) - \sigma(t,y) \mid \; \leq K_1 \; \mid x - y \mid \; .$$

c) $\exists\, K_2 > 0$, such that the growth condition is specified, i.e. $\forall t \in [0,T],\quad x \in \mathbb{R}$

$$\mid \mu(t,x) \mid^2 \; + \; \mid \sigma(t,x) \mid^2 \leq K_2(1 + \; \mid x \mid^2) \; .$$

[4] See for example Friedman (1969)

[5] See for example Heath, Jarrow and Morton (1986).

of T, such that at any $t < T$

$$\frac{E_t[dB(t,T,r)] - r_t B(t,T,r)dt}{\sqrt{V_t[dB(t,T,r)]}} = \lambda(r,t) \qquad \forall T. \tag{3}$$

From (1.2) and (1.3) the bond price dynamics are determinated by the differential equation

$$B_t + \Big(\mu(t,r) - \sigma(t,r)\cdot\lambda(r,t)\Big)B_r + \frac{1}{2}\sigma^2(t,r)B_{rr} - rB = 0 \tag{4}$$

with the boundary condition

$$B(T,T,r) = \text{face value} = 1.$$

The unique solution for the Feyman-Kac-Equation (1.4)[6] is given by

$$B(t,T,r) \;=\; E[\exp\{-\int_t^T \tilde{r}_s ds\} \mid \tilde{r}_t] \tag{5}$$

where

$$d\tilde{r} \;=\; dr - \lambda(r,t)\cdot\sigma(r,t)dt.$$

If the risk primium per unit risk $\lambda(r,t)$ is equal to zero (1.3) takes the form of the local expection hypothesis [7] , which is the only form of expectation hypothesis compatible with the no arbitrage requirement. In general the risk premia $\lambda(r,t)$ may not be equal to zero. We know from the no arbitrage condition and completeness of the market structure that there exists a unique martingale measure. This measure depends on the risk premium function $\lambda(r,t)$. Under the equivalent martingale measure the lokal expectation hypothesis is fulfilled. Therefore equation (1.5) can be written as

$$\begin{aligned} B(t,T,r) &= E\left[\exp\left\{-\int_t^T \tilde{r}_s ds\right\}\middle| r_t\right] \\ &= E_P\left[\exp\left\{-\int_t^T r_s ds\right\}\middle| r_t\right] \end{aligned} \tag{6}$$

where P is the equivalent martingale measure such that the lokal expectation hypothesis

$$E_P[dB(t,T,r) \mid t] - rBdt \;=\; 0 \tag{7}$$

is fulfilled. The change of measure from the original probability distribution Q to the equivalent martingal measure P depends on the risk premium and is determinated by the Radon–Nikodym density function

$$\frac{dP}{dQ} = \rho(t) = \exp\int_t^T \lambda(r,s)dW(s) - \frac{1}{2}\int_t^T \lambda(r,s)^2 ds \tag{8}$$

The no arbitrage conditions guarantee the existence of such a transformation from the original model with risk premium to a model without risk premium where the price processes are determinated under the local expectation hypothesis. If the original market model is complete, the transformation is unique. Therefore the local expectation hypothesis is not an additional assumption for those models which assume a complete market structure.

2. Term Structure of Interest

The goal of this section is to present an arbitrage free discrete term structure model which has enough flexibility to explain the different features of interest rate movements and allows the valuation of a rich class of interest rate derivatives. In a discrete time setting the most simple way to model uncertain price movements is the binomial process. The first binomial term structure model which excludes arbitrage opportunities was given by T. Ho and Sang–Bin–Lee (1986). Their starting point is, that at time $t = 0$ the prices of zero coupon bonds for any maturity of the discrete time scale are known. Then the idea is to

[6] See Friedman (1969) and Duffie (1988).

[7] See Ingersoll (1987, chapter 18)

model the price movements of all zero coupon bonds in one step by introducing two maturity dependent pertubation functions $h(t)$ and $h^*(t)$. In principle the term structure at time t_1 equals the original one at time t_0 multiplied either by the pertubation function h with probability π (relative up–movement of the term structure) or by the pertubation function h^* with probability $1-\pi$ (relative down–movement of the term structure). With the assumption of the path independence of price movements of all zero coupon bond prices Ho and Lee are able to give an explicit form of the pertubation function h and h^* dependent of the measure π. Furthermore they show that the volatility of the one period return is the appropriate risk measure which enters into the pertubation functions.
The probability π of the up– and down– movements of the term structure has to be estimated from the data and it is not obvious how sensitive the model depends on π. Second, the construction principle does allow the zero coupon bond price to exceed his face value and therefore generates negative spot and forward rates. Especially for the pricing of interest rate options this can lead to serious mispricing. If one cuts off such undesired interest rate pathes by an exogenous boundary condition the resulting volatility of the model does no longer correspond to the input data.
Therefore it seems to us of some importance to remodel the term structure of interest in order to avoid some of the disadvantages of the Ho–Lee–Model. The principal assumptions of the underlying market structure are quite standard.

- There exists a discrete set of trading dates $\underline{\underline{T}} = \{0 = t_0 < \ldots < t_N = T\}$. For simplicity they are chosen to be equidistant, that is $\Delta t = t_{i+1} - t_i$ for all $i = 0, \ldots, N-1$
- There are no transaction costs or taxes.
- There is no default risk.
- As trading can only occur at trading date $t_i \in \underline{\underline{T}}$ we assume that at time $t_0 = 0$ the prices of all zero coupon bonds $B(t_0, t_i)$ with maturity $t_i \in \underline{\underline{T}}$ are known. We interpret this as full price information at time t_0.

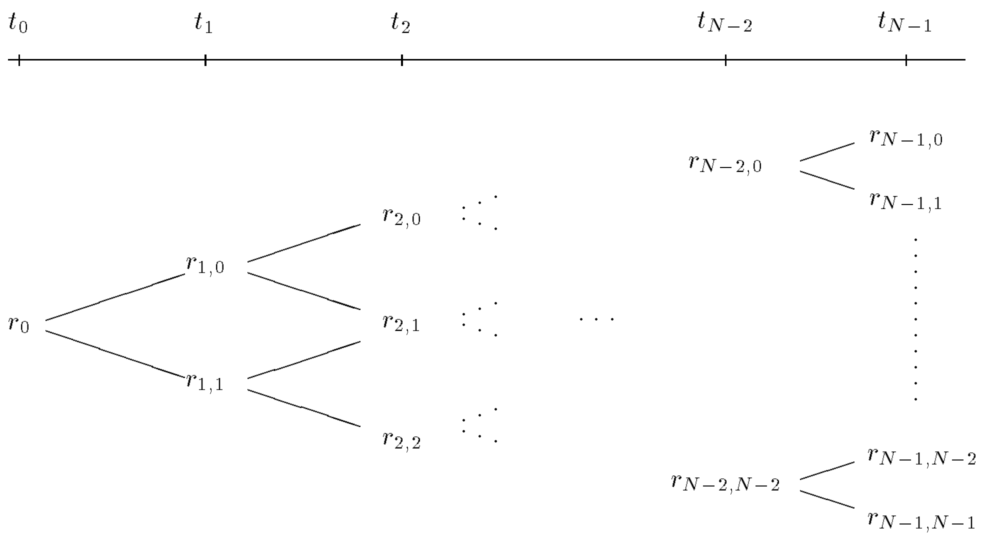

Figure 2.1: Binomial spot rate process

For the stochastic process of the spot rate process $\{r_i\}_{i=0}^{N-1}$ we construct a path independent binomial process with the following features:

- The current market information, i.e. the observed zero bond prices, have to be reflected.
- Negative spot rates are not generated within the model.

- The zero coupon bond price processes are derived under the local expectation hypothesis. As shown in section 1 this assumption is equivalent to consider a risk premium function $\lambda(r,t)$ which is independent of the bonds maturity.

At time t_0 the initial spot rate for the first period is r_0. At time t_1 the possible spot rate realisations for the next period are $r_{1,0}$ resp. $r_{1,1}$. If the transition probability of the first period $p(t_1) \neq 0$ and the local volatility of the spot rate $\sigma(t_1)$ are known then the following relationship gives us more information about the binomial interest rate process:

$$\begin{aligned} & & \sigma^2(t_1) &:= V[\log r_{1,\cdot}|r_0] = p(t_1)\Big(1-p(t_1)\Big)\left[\log\frac{r_{1,0}}{r_{1,1}}\right]^2 \\ (9) \qquad & \Leftrightarrow & r_{1,0} &:= r_{1,1}\exp\left\{\frac{\sigma(t_1)}{\sqrt{p(t_1)\Big(1-p(t_1)\Big)}}\right\} \\ & & &= r_{1,1}\, g\Big(\sigma(t_1), p(t_1)\Big) \end{aligned}$$

The pertubation function $g(\sigma,p) = \exp\left\{\frac{\sigma}{\sqrt{p(1-p)}}\right\}$ is continuous on $\mathbb{R}_{>0}\times]0,1[$, bounded below by 1, and, for any $\sigma > 0$, has a minimum at $p = 1/2$. The function $g(.,.)$ replaces in a way the pertubation function h and h^* of the Ho–Lee–Model. In contrast to the Ho–Lee–Model $g(.,.)$ defines the relation between different spot rates and not between returns. Therefore the risk measure is the local volatility of the spot rate.

Suppose for example that the local volatility of the spot rate and the transition probability are both constant over time. Due to the path independence of the binomial spot rate process the following relation holds:

$$(10) \qquad r_{n,j} = r_{n,n}\cdot g(\sigma,p)^{n-j} \qquad \forall j = 0,\dots,n; \quad n = 0,\dots,N-1$$

From the binomial process of the spot rate the price process of the two period zero coupon bond with face value one is given by figure 2.2

$$\begin{array}{llll} t_0 & & t_1 & t_2 \\ & p \nearrow & B^{down}(t_1,t_2) = \dfrac{1}{1+r_{1,0}} = \dfrac{1}{1+r_{1,1}g(\sigma,p)} & —\ 1 \\ B(t_0,t_2) & & & \\ & 1-p \searrow & B^{up}(t_1,t_2) = \dfrac{1}{1+r_{1,1}} & —\ 1 \end{array}$$

Figure 2.2: Price process of the two period zero coupon bond.

Under the local expectation hypothesis the value at time t_0 must be equal to the expected value of the discounted face value. Therefore

$$\begin{aligned} B(t_0,t_2) &= \frac{1}{1+r_0}\left[\frac{p}{1+r_{1,0}} + \frac{1-p}{1+r_{1,1}}\right] \\ (11) \qquad & \\ &= \frac{1}{1+r_0}\left[\frac{p}{1+r_{1,1}g(\sigma,p)} + \frac{1-p}{1+r_{1,1}}\right] \end{aligned}$$

For fixed $p \in]0,1[$ and $\sigma \geq 0$ there exists a unique positive solution for $r_{1,1}$ which solves (2.3) iff $B(t_0,t_2) < B(t_0,t_1) = \frac{1}{1+r_0}$. Given the solution for $r_{1,1}$ and $r_{1,0}$ we can solve the recursion for the next period under the assumption of the path independence of the spot rate process. The t_0 value of a three period zero coupon bond under the local expectation hypothesis is equal to its expected discounted payoff

$$\begin{aligned} B(t_0,t_3) &= E_p\left[\prod_{n=0}^{2} \frac{1}{1+r_{n,\cdot}} \middle| r_0\right] \\ &= \frac{1}{1+r_0}\left[\frac{p}{1+r_{1,0}}\left(\frac{p}{1+r_{2,2}\, g^2(\sigma,p)} + \frac{1-p}{1+r_{2,2}\, g(\sigma,p)}\right)\right. \\ &\quad \left. + \frac{1-p}{1+r_{1,1}}\left(\frac{p}{1+r_{2,2}\, g(\sigma,p)} + \frac{1-p}{1+r_{2,2}}\right)\right] \end{aligned} \tag{12}$$

The following theorem guarantees the existence and uniqueness of a non negative path independent binomial spot rate process under weak conditions. Especially the local volatility and the transition probability are allowed to be both time and path dependent.

Theorem 2.1. Let A denote the set of time and state tupels, i.e. $A := \{(n,j) \mid t_n \in \underline{\underline{T}} \setminus \{t_0\}, j = 0,\ldots,n\}$ and let $\sigma(n,j)$ denote the local volatility at time $t_n \in \underline{\underline{T}}$ and $p(n,j)$ the transition probability from time t_{n-1} to t_n, given the spot rate at time t_{n-1} is equal to $r_{n-1,j}$. If the zero coupon bond prices at time t_0 are strictly decreasing with maturity, i.e.

$$B(t_0,t_1) > B(t_0,t_2) > \ldots > B(t_0,t_N)$$

then for any volatility function

$$\sigma : A \longrightarrow \mathbb{R}_{\geq 0}$$

and any transition probability function

$$p : A \longrightarrow]0,1[$$

there exists a unique non negative, path independent binomial spot rate process, such that the local expectation hypothesis for the zero coupon bonds is satisfied.

Proof: Appendix

For a flexible class of transition probabilities $p()$ and volatility functions $\sigma()$ Theorem 2.1 guarantees the existence of a binomial spot rate process. Under the local expectation hypothesis the zero coupon bond price processes are then determined.

The transition probability seems to be a purely exogenuous and restrictive factor of the model. As in the original Ho–Lee–Model one could come to the conclusion that the transition probability has to be estimated from the data. In fact this is not the only possible interpretation. As for the binomial option pricing model of Cox and Rubinstein (1985) and the continuous time Black–Scholes Model (1973) the role of the transition probability has to be associated with the corresponding limit model.

Section 3 analyses the special case where the transition probability of the discrete time model is constant. As a result the instantaneous spot rate is lognormal distributed in the limit. It is shown that in this case the valuation formulas for interest rate contingent claims and the trading strategies become independent of the transition probability with the trading interval going to zero, an observation already made by Heath, Jarrow and Morton (1989). In general, the choice of a specific transition probability for the discrete time model corresponds to the choice of a spcific continuous time limit model. For example Ball (1989) has reformulated a binomial Ehrenfest–Model which corresponds to the mean reverting square root process for the spot rate analysed by Cox, Ross and Ingersoll (1985 a, b). Thus by using another class of transition probabilities the binomial model of Theroem 2.1 can be used to explain and study different continuous time models.

3. Continuous Time Consideration

In this section we want to look at a special limit result for the interest rate process $\{r_i\}_{i=0}^{N-1}$ if the length of the time period goes to zero. Let $\underline{\underline{T}} = \{0 = t_0 < \ldots < t_N\}$ be the original set of trading dates, then a refinement of order $n \in \mathbb{N}$ is defined by

$$\underline{\underline{T}}(n) = \{0 = t'_0 < t'_1 < \ldots < t'_{N \cdot n}\} \tag{13}$$

with $t'_{j+1} - t'_j = \frac{\Delta t}{n}$ and $t'_{i \cdot n} = t_i \in \underline{\underline{T}}$; $t'_0 = t_0$; $\Delta t = t'_{i \cdot n} - t'_{(i-1)n}$.
If the zero coupon bond prices with respect to the refinement of the set of trading dates $\underline{\underline{T}}(n)$ are known and decreasing with maturity then Theorem 2.1 is satisfied. Otherwise we could for example compute the additional bond prices by

$$\begin{aligned} B(t'_0, t'_{i \cdot n+j}) &:= B(t_0, t_i)\, [f_0(t_{i-1}, t_i) + 1]^{-j\frac{\Delta t}{n}} \\ &= B(t_i, t_{i+1})\, [f_0(t_{i-1}, t_i) + 1]^{\Delta t \frac{(n-j)}{n}} \quad \forall j = 0, \ldots, n \end{aligned} \tag{14}$$

where $f_0(t_{i-1}, t_i) = \frac{B_0(t_0, t_{i-1})}{B_0(t_0, t_i)} - 1$ is the forward rate at time t_0. The additional bond prices defined by (3.2) obviously satisfy the monotonicity requirement if this is true for the original ones. Therefore Theorem 2.1 can be applied.
The limit result for the interest rate process in general depends on the function of the transition probability $p(.,.)$ and the local volatility function $\sigma(.,.)$. We want to restrict our view to the case of a constant transition probability $p \in]0, 1[$ and a exclusively time dependent volatility function. Under these additional assumptions the following limit theorem for the interest rate process can be derived:

Theorem 3.1. Let $p \in]0, 1[$ be the constant transition probability and $\sigma : \underline{\underline{T}} \rightarrow \mathbb{R}_{\geq 0}$ the volatility function, so that $\sigma^2(t_i)$ is proportional to the length of the time intervall with

$$\sigma^2(t_i) = h(t_i) \cdot \Delta t$$

were $h(.)$ converges against a bounded function on $[0, T]$. Then

$$\left\{ x_{t_i} = \log \frac{r_i}{r_{i-1}} \right\}_{t_i \in \underline{\underline{T}}(n)}$$

satisfies the Central Limit Theorem.

Proof: Appendix

The appropriate limit model for the discrete interest rate process of section two is therefore given by

$$\frac{dr}{r} = \mu(t)dt + \sigma(t)dW \tag{15}$$

if the transition probability p is constant and the discrete volatility function $\sigma(\ )$ fulfills the conditions of theorem 3.1.[8]. The drift function $\mu(\ )$ is determinated by the local expectation hypothesis for the zero coupon bonds at time t_0,i.e. $\mu(t)$ has been chosen such that

$$B(t_0, t, r_0) = E\left[\exp\{-\int_o^t r(s, \mu(s), \sigma(s))ds\} \middle| r_0 \right] \tag{16}$$

[8] By changing the transition probability other stochastic processes than the geometric Brownian Motion can be approximated in the limit by the binomial spot rate process. Using for example the idea of Ball (1989) another possible limit model for the spot rate is the square root process

$$dr = \mu(t)r\ dt + \sigma(t)\sqrt{r}\ dW.$$

For a discussion of the resulting term structure model see Cox, Ross and Ingersoll (1985a, b). Furthermore Hull and White (1990) have studied different continuous time models for the spot rate.

holds. Under the local expectation hypothesis the risk premium $\lambda(r,t)$ (1.3) is equal to zero and the dynamics of the zero coupon bond prices are given by the stochastic differential equation (1.4)[9]

$$
\begin{aligned}
B_t \quad & +\mu(t)\cdot rB_r + \frac{1}{2}\sigma^2(t)\cdot r^2 B_{rr} - rB = 0 \\
B \quad & (t,t,r) = 1 \qquad\qquad \forall t \in [0,T].
\end{aligned} \tag{17}
$$

From (3.3) we can derive the process of the instantanous discount factor $\left\{y_t = \frac{1}{1+r_t}\right\}$ at time t where r_t is lognormally distributed. The stochastic differential equation for the instantanous discount factor is given by

$$
dy = y(1-y)[(1-y)\sigma^2(t) - \mu(t)]dt - y(1-y)\sigma(t)dW; \quad y_0 = \frac{1}{1+r_0}. \tag{18}
$$

In 1949 N.L. Johnson studied a special system of distribution functions which are generated by translation of a normal distributed variable. Among others he considered the following transformation

$$
\begin{aligned}
y &= a + \frac{b-a}{1+\exp\{-(z\sigma+\eta)\}} \qquad a < b \\
\Leftrightarrow z &= -\frac{\eta}{\sigma} + \frac{1}{\sigma}\log\left(\frac{y-a}{b-y}\right) \qquad a < y < b \quad \eta, \sigma > 0
\end{aligned} \tag{19}
$$

where z is a $N(0,1)$ distributed variable. The idea is to find a flexible transformation of a $N(0,1)$ variable, so that the resulting density function has support $[a,b]$. For $[a,b] = [0,1]$ and

$$
\begin{aligned}
\sigma^2 &= \int_0^t \sigma^2(s)ds \\
-\eta &= \log r_0 + \int_0^t \mu(s) - \frac{1}{2}\sigma^2(s)ds
\end{aligned} \tag{20}
$$

this describes exactly the transformation for the discount factor y_t at time t. The probability density function of the discount function y_t is therefore given by

$$
\begin{aligned}
\rho\Big(y; r, \mu(.), \sigma(\,), t\Big) &= \frac{1}{\sqrt{2\pi\int_0^t \sigma^2(s)ds}}\ \frac{1}{y(1-y)} \\
&\quad \cdot \exp\left\{ -\frac{\left(\log\frac{1-y}{y} - \log r - \int_0^t \mu(s) - \frac{1}{2}\sigma^2(s)ds\right)^2}{2\int_0^t \sigma^2(s)ds} \right\} \\
&\quad 0 \le y \le 1
\end{aligned} \tag{21}
$$

Figure 3.1 shows some typically pattern of the density function $\rho(.)$, where we assume σ to be constant and $\int_0^t \mu(s)ds$ equal to $\log f_0(t)/r_0$, $f_0(t)$ is the instantaneous forward rate.
Furthermore the transformation (3.7) induces that the median of the probability distribution given by $\rho(y,\dots)$ is equal to

$$
y_{0,5}(t, r_0) = \left[1 + r_0 \exp\left\{\int_0^t \mu(s) - \frac{1}{2}\sigma^2(s)ds\right\}\right]^{-1} \tag{22}
$$

[9] Contrary to the elliptic partial differential equation obtained in the Cox–Ingersoll–Ross–Model, the partial differential equation (3.5) is parabolic and we do not know whether an explicite solution exists.

The expected value of the discount factor y_t at time t equals

$$
\begin{aligned}
(23) \qquad E[y_t \mid r_0] &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\Pi \int_0^t \sigma^2(s)ds}} \quad \frac{1}{1+e^y} \\
&\quad \cdot \exp\left\{ - \frac{\left(y - \log r - \int_0^t \mu(s) - \frac{1}{2}\sigma^2(s)ds\right)^2}{2\int_0^t \sigma^2(s)ds} \right\} dy
\end{aligned}
$$

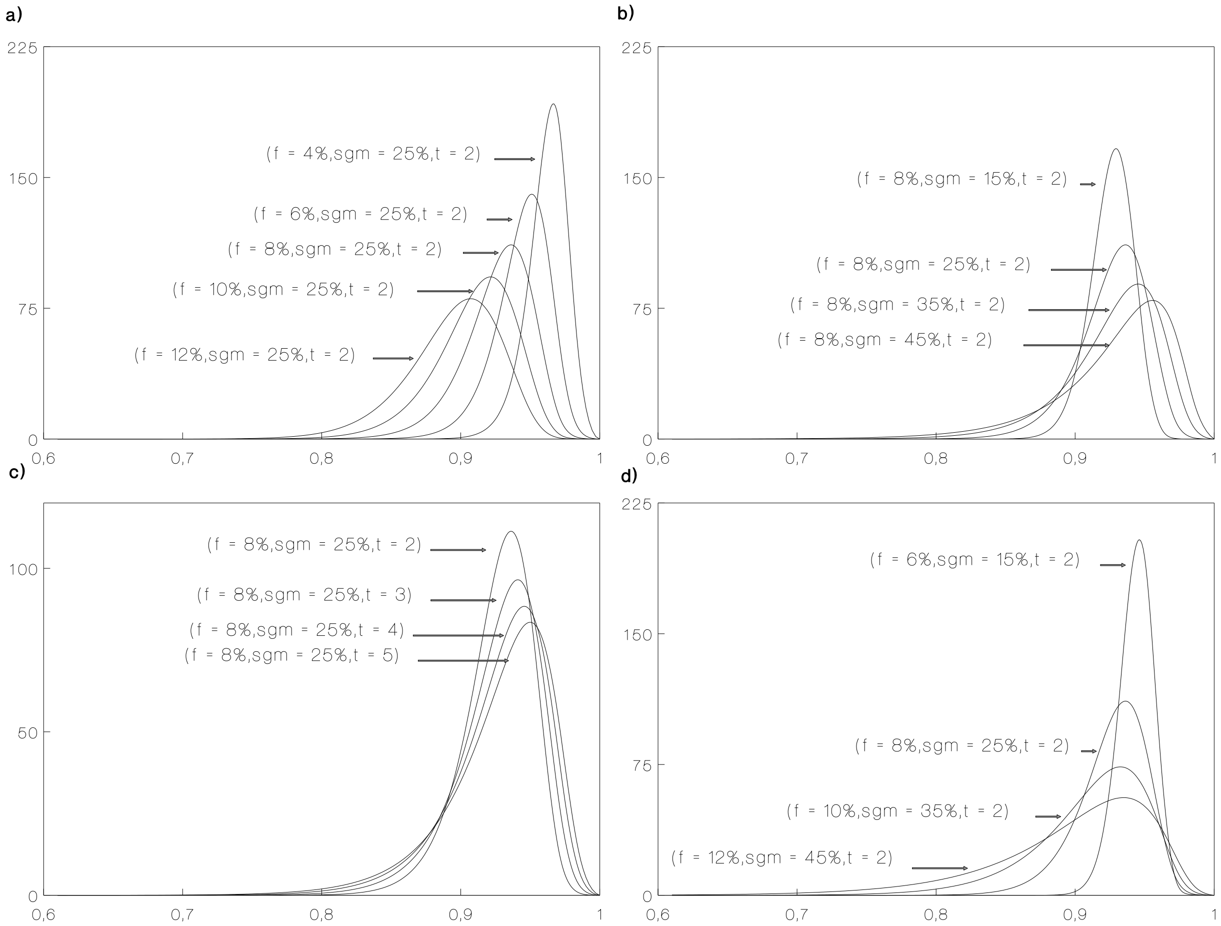

Figure 3.1: Density function $\rho(y, r, f, \sigma, t)$

a) fixed σ and t and different forward rate f. b) fixed f and t and different volatility σ.

c) fixed σ and f and different time period t. d) fixed t and different forward rate f and volatility σ.

N.L. Johnson (1949) proposes a numerical method to compute the moments of $\{y_t\}$.

Table 3.1 shows some examples of the expectation and standard error of the discount function $\{y_t\}$ where $\log r_0 + \int_0^t \mu(s)ds = \log f$.

However, with Jensen's Inequality and some easy calculations we get the following boundary condition (for the proof see the appendix)

$$
(24) \qquad \frac{1}{1 + r_0 \exp\left\{\int_0^t \mu(s)ds\right\}} \leq E[y_t \mid r_0] \leq \frac{1}{1 + r_0 \exp\left\{\int_0^t \mu(s) - \sigma^2(s)ds\right\}}
$$

Therefore the expected discount factor at time t is not far away from the discount factor obtained by the expected spot rate at time t which is

$$E[r_t \mid r_0] = r_0 \exp\left\{\int_0^t \mu(s)ds\right\} \tag{25}$$

Table 3.1:

Forward rate f	Expectation $E[y_t]$				Standard error $\sigma(y_t)$			
	$\sigma_r =$				$\sigma_r =$			
	0.15	0.25	0.35	0.45	0.15	0.25	0.35	0.45
0.01	0.99010	0.99011	0.99013	0.99015	0.00211	0.00357	0.00512	0.00681
0.02	0.98041	0.98044	0.98050	0.98057	0.00412	0.00696	0.00996	0.01316
0.03	0.97091	0.97098	0.97110	0.97127	0.00605	0.01020	0.01454	0.01910
0.04	0.96160	0.96173	0.96192	0.96221	0.00790	0.01330	0.01888	0.02468
0.05	0.95248	0.95266	0.95296	0.95338	0.00967	0.01625	0.02301	0.02992
0.06	0.94353	0.94379	0.94420	0.94478	0.01138	0.01908	0.02692	0.03487
0.07	0.93476	0.93510	0.93563	0.93640	0.01301	0.02178	0.03065	0.03954
0.08	0.92616	0.92658	0.92726	0.92821	0.01458	0.02437	0.03420	0.04396
0.09	0.91772	0.91824	0.91906	0.92022	0.01609	0.02685	0.03758	0.04815
0.10	0.90943	0.91006	0.91104	0.91241	0.01754	0.02922	0.04080	0.05213
0.11	0.90130	0.90204	0.90318	0.90478	0.01893	0.03149	0.04388	0.05591
0.12	0.89332	0.89417	0.89549	0.89731	0.02027	0.03367	0.04683	0.05951
0.13	0.88549	0.88645	0.88795	0.89001	0.02155	0.03576	0.04964	0.06293
0.14	0.87779	0.87888	0.88056	0.88286	0.02279	0.03776	0.05233	0.06620
0.15	0.87024	0.87145	0.87331	0.87586	0.02398	0.03969	0.05490	0.06932

Equation (3.12) shows the mean reversion property of the discount function. The expected discount factor is bounded from both sides and under certainty both bounds are equal. To qualify the boundary conditions (3.12) one could assume that the unbiased expectation hypothesis would be satisfied. As Ingersoll (1987, chapter 18) has shown, this form of an expectation hypothesis is not arbitrage free. Nevertheless using the unbiased expectation hypothesis

$$E[r_t \mid r_0] = f(0,t) \quad \text{(instantaneous forward rate)} \tag{26}$$

yields for the drift of the spot rate process

$$\begin{aligned} \int_0^t \mu(s)ds &= \log\frac{f(0,t)}{r_0} \\ \Leftrightarrow \qquad \mu(t) &= \frac{\partial \log f(0,t)}{\partial t} + c \end{aligned} \tag{27}$$

Therefore the instantaneous drift of the spot rate is given by the shape of the actual continously compounded forward rate. Furthermore the boundary condition (3.12) can be reformulated

$$\frac{1}{1+f(0,t)} \leq E[y_t \mid r_0] \leq \frac{1}{1+f(0,t)\exp\left\{-\int_0^t \sigma^2(s)ds\right\}} \tag{28}$$

As a consequence the zero coupon prices predicted within the model by the local expectation hypothesis are bounded from below by those predicted under the unbiased expectations hypothesis. As already mentioned the unbiased expectation hypothesis (3.14) is not arbitrage free and therefore (3.15) and (3.16) are not exactly satisfied. But if there is no uncertainty ($\sigma() \equiv 0$) about future spot rates the local and the unbiased expectation hypothesis coincide as can also be seen from relation (3.12). Thus the difference between the cumulated drift $\int_0^t \mu(s)ds$ and $\log \frac{f(0,t)}{r_0}$ decreases with decreasing volatility. The same observation is true for the boundary condition and therefore both relationships (3.15) and (3.16) can be regarded as approximations.

4. Interest Rate Derivatives

In general an interest rate derivative or interest rate contingent claim H is defined by a k–dimensional random variable with

$$H = (H_{t'_i}, \dots, H_{t'_k}) : \Omega \rightarrow \mathbb{R}^k \tag{29}$$

where each $H_{t'_i} : \Omega \rightarrow \mathbb{R}$ is measurable with respect to the information given by the term structure model at time t'_i. Concerning the term structure model in section 2 the information at time t_i is represented by the σ–algebra $\Im_{t'_i}$ generated by the historical and present prices of all zero coupon bonds and the spot rate up to time t'_i. The set $\{t'_1, \dots, t'_k\}$ is then the set of payment dates of the contingent claim and must be contained in $\underline{\underline{T}}$.

The first example of such interest rate contingent claims are European bond options on either zero coupon or coupon bonds. The second example consists of direct interest rate options, namely caps and floors. However, the class of interest rate derivatives defined by (4.1) is much larger and include also American options as well as e.g. look–back options where the payment at time t'_i may depend on some historical realisation.

More precisely the payoff of a European call option with exercise price E and maturity $t \in \underline{\underline{T}}$ on a zero coupon or coupon bond B is defined by

$$[B(t) - E]^+ := \max\Big\{B(t) - E, 0\Big\} \tag{30}$$

respectively for a put option we have

$$[E - B(t)]^+ := \max\Big\{E - B(t), 0\Big\} \tag{31}$$

For zero coupon bonds we know from Merton's (1973) analysis that a call option is always more worth alive than dead. Therefore an American type call option, i.e. premature exercise is possible, has the same price as an European call option. For a put option or, if the underlying security is a coupon bond, this result is not transferable. However, the payoff of an American type option at some time t' less than the maturity date t is equal to the maximum of the future expected discounted payoff and the exercise payoff at time t'.

Direct interest rate options like caps and floor are somehow special. Their properties are closely related to those of European call and put options. In principal the protection against upward movements of the interest rate is called a cap and the protection against downward movements a floor. The underlying security of these contracts is the interest rate, for example 3 or 6 month Libor. Let r_{t_i} be the interest rate at time $t_i \in \underline{\underline{T}}$ for the time period $[t_i, t_{i+1}]$. The payoff of a cap with cap level L and face value V at time t_{i+1}, i.e. at the end of the period is the equal to

$$V\,[r_{t_i} - L]^+ \tag{32}$$

Similarly to a floor the payoff at time t_{i+1} is equal to

$$V\,[L - r_{t_i}]^+ \tag{33}$$

In general caps and floor are characterised by the following instruments:

- the underlying interest rate r; e.g. LIBOR
- The set of comparing dates (tenor of the contract)
- The set of payment dates (frequency of the contract)
- The level L
- The face value V

For simplicity we assume that the set of comparing dates is contained in $\underline{\underline{T}} \setminus \{t_N\}$ and the payment will take place always at the end of the time interval. Clearly if r is denoted in yearly interest rates the payment has to be adjusted with respect to the contract tenor. From (4.4) resp. (4.5) we can discount the payment to the beginning of the time period:

$$V\left[1 - \frac{1+L}{1+r_{t_i}}\right]^+ = V(1+L)\left[\frac{1}{1+L} - \frac{1}{1+r_{t_i}}\right]^+ \tag{34}$$

Therefore the discounted cap payment at time t_i is equal to $V(1+L)$ times the payment of a European put option with exercise price $\frac{1}{1+L}$, expiration date t_i where the underlying security is a zero coupon bond with maturity date t_{i+1} and face value 1. Therefore the arbitrage price of a cap must be equal to the value of a portfolio of European put options on zero coupon bonds with different maturity.

$$\mathrm{Cap}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V] = V(1+L)\sum_{i=0}^{N-1} \mathrm{Put}_E\left[B(t_{i+1}), \frac{1}{1+L}, t_i\right] \tag{35}$$

For the floor the same argument leads to

$$\mathrm{Floor}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V] = V(1+L)\sum_{i=0}^{N-1} \mathrm{Call}_E\left[B(t_{i+1}), \frac{1}{1+L}, t_i\right] \tag{36}$$

From the characterisation of cap and floor contracts as a portfolio of European bond options it is obvious that the arbitrage pricing of these contracts requires a term structure model.
Furthermore cap and floor contracts are convex functions of the level L and the underlying interest rate r[10]. In analogy to the put–call–parity the cap–floor–parity is given by

$$\mathrm{Cap}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V] = \mathrm{Floor}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V] + \mathrm{Swap}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V] \tag{37}$$

were $\mathrm{Swap}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V]$ denotes an interest rate swap with fixed rate L, floating rate r, face value V and comparing frequency $\underline{\underline{T}} \setminus \{t_N\}$. The cash flow of a swap at time $t_{i+1} \in \underline{\underline{T}}$ is therefore equal to $V\,[r_{t_i} - L]$. The arbitrage price of such an interest rate swap is equal to

$$\mathrm{Swap}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V] = V \cdot \left(1 - B(t_0, t_N) - L\sum_{i=1}^{N} B(t_0, t_i)\right) \tag{38}$$

Finally we have the following boundary condition for the arbitrage prices of cap and floors:

$$\begin{aligned}
\max\{0, \mathrm{Swap}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V]\} &\le \mathrm{Cap}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V] \le V[1 - B(t_0, t_N)] \\
\max\{0, -\mathrm{Swap}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V]\} &\le \mathrm{Floor}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V] \le V \cdot L\sum_{i=1}^{N} B(t_0, t_i) \\
\mathrm{Swap}[r, \overline{L}, \underline{\underline{T}} \setminus \{t_N\}, V]\} &\le \mathrm{Cap}[r, L_1, \underline{\underline{T}} \setminus \{t_N\}, V] - \mathrm{Floor}[r, L_2, \underline{\underline{T}} \setminus \{t_N\}, V] \\
&\le \mathrm{Swap}[r, \underline{L}, \underline{\underline{T}} \setminus \{t_N\}, V] \\
\text{with} \quad \overline{L} = \max\{L_1, L_2\} \quad &, \quad \underline{L} = \min\{L_1, L_2\}
\end{aligned} \tag{39}$$

In the framework of the term structure model (section two) bond options, interest rate options and more complicated contracts such as swaptions can be priced by arbitrage. Since the presented model describe a complete market structure the martingal measure is unique and we have the following pricing relation:

[10]The proofs of the distribution free properties of cap and floor contracts are not complicated. Some of the proofs are given in the appendix.

Theorem 4.1. The arbitrage price of an interest rate derivative H given by (4.1) is equal to the sum of the expected discounted payoff under the martingal measure P defined by the family of the transition probabilities, i.e.

$$\overline{H}(t'_1, \ldots, t'_k) = \sum_{l=1}^{k} E_p \left[\frac{H_{t'_l}}{\prod_{j=0}^{l-1}(1 + r_j, \cdot)} \right] \tag{40}$$

where $\{t'_1, \ldots, t'_k\} \subset \underline{\underline{T}}$.

Proof. The proof is given in the appendix and implements self–financing dynamic portfolio strategies which duplicate the payoff of the interest rate derivative H.

Example 4.1. We consider three different portfolio strategies to duplicate the payoff of a cap contract. The market data and model assumptions are:

interest rate	:	per anno		
maturity of the cap	:	3 years		
level	:	4%		
comparing date	:	every year	=	$\{t_0, t_2, t_4\}$
payment date	:	end of the year	=	$\{t_2, t_4, t_6\}$
hedge frequency	:	every 6 month	=	$\{t_0, t_1, \ldots, t_5\}$
face value	:	100$		
zero coupon prices	:	$B(t_0, t_1) = .9806$,		$B(t_0, t_2)$=.9615, $B(t_0, t_3)$=.9406
		$B(t_0, t_4) = .9200$,		$B(t_0, t_5)$=.8977, $B(t_0, t_6)$=.8759

For the term structure model we assume the local volatiltiy per period to be 17.86% and the transition probability to be .5 .

We consider three different hedging strategies based either on a zero coupon bond, a coupon bond with yearly coupon of 4.5% or an interest rate swap with swap level 4.5%. All instruments have a maturity equal to t_6 and face value 100. The easiest way to describe the whole structure is to use the binomial tree (figure 4.1). Therefore at each knot r denotes the interest rate per anno, d the product of the face value and the cap payment at the end of the period and PV the present value of the contract at the beginning of the period. The notation of the portfolio strategie is:

Portfoliostrategie: Zero Coupon Bond (P1)

ZB : price of the zero coupon bond
ΔZB : hedgeratio in percent
MB : amount of money invested at the spot rate

Portfoliostrategie: Coupon Bond (P2)

CB : price of the coupon bond
ΔCB : hedgeratio in percent
MC : amount of money invested at the spot rate

Portfoliostrategie: Swap (P3)

S : price of the interest rate swap
ΔS : hedgeratio in percent
MS : amount of money invested at the spot rate

trading date: t_4

	r	2,3576	3,3575	4,7814	6,8093	9,6973
	d	0,0000	0,0000	0,7814	2,8093	5,6973
	PV	0,0000	0,0000	0,7458	2,6302	5,1936
	ZB	97,6967	96,7516	95,4368	93,6248	91,1600
	CB	102,0931	101,1054	99,7314	97,8379	95,2622
	S	-2,0931	-1,1054	0,2686	2,1621	4,7378

trading date: t_3

	r	2,5462	3,6261	5,1640	7,3542
	d	0,0000	0,0000	0,0000	0,00000
	PV	0,0000	0,3663	1,6460	3,7755
	ZB	96,0095	94,3979	92,1806	89,1716
(P1)	Δ ZB	0,0000	-0,5672	-1,0400	-1,0400
	MB	0,0000	53,9075	97,5139	96,5140
	CB	104,7738	103,0664	100,71646	97,5275
(P2)	Δ CB	0,0000	-0,5428	-0,9952	-0,9952
	MC	0,0000	59,3069	101,8810	100,8364
	S	-1,5793	-0,4110	1,1851	3,3297
(P3)	Δ S	0,0000	0,5428	0,9952	0,9952
	MS	0,0000	0,5894	0,4666	0,4618

trading date: t_2

	r	3,0788	4,3846	6,2441
	d	0,0000	0,3846	2,2441
	PV	0,1804	1,3532	4,7422
	ZB	93,7712	91,3090	87,9712
(P1)	Δ ZB	-0,2273	-0,5772	-0,7077
	MB	21,4933	54,0522	67,0012
	CB	102,3564	99,7288	96,1653
(P2)	Δ CB	-0,2145	-0,5447	-0,6677
	MC	22,1401	55,6726	68,9503
	S	-2,3589	0,2682	3,8317
(P3)	Δ S	0,3135	0,8018	0,9930
	MS	0,9200	1,1382	0,9374

trading date: t_1

	r	3,3031	4,7040
	d	0,0000	0,0000
	PV	0,7544	2,9784
	ZB	91,0486	87,6033
(P1)	Δ ZB	-0,4663	-1,0153
	MB	44,1243	91,9241
	CB	103,8415	100,1193
(P2)	Δ CB	-0,4464	-0,9510
	MC	47,1039	98,1947
	S	-1,0285	2,0034
(P3)	Δ S	0,4464	0,9510
	MS	1,2136	1,0732

trading date: t_0

	r	4,0000
	d	0,0000
	PV[11]	1,8302
	ZB	87,5913
(P1)	Δ ZB	-0,6455
	MB	58,3727
	CB	100,0000
(P2)	Δ CB	-0,5975
	MC	61,5795
	S	-0,0028
(P3)	Δ S	0,7335
	MS	1,8322

Figure 4.1: Payoff pattern and portfolio strategies for a cap contract.

[11] The arbitrage price of the cap is equal to 1,8302% of the face value.

Finally with respect to the limit results of section 3 we want to compare the impact of more frequent trading with a change of the transition probability for the cap contract of example 4.1 . Tabels 4.1 to 4.3 gives the initial positions for the hedge strategies for zero coupon, coupon bonds and swaps, as well as the arbitrage price and the hedge strategies. As the limit result suggests the influence of the transition probability vanishes if the trading frequency is augmented.

Tabel 4.1: Portfolio Strategy Zero Coupon Bond (P1)

Hedge Tenor		Transition Probability (p) 0.3	0.4	0.5	0.6	0.7
1/2 Year	PV	1.83503	1.75159	1.83022	1.81936	1.83202
	Δ ZB	-0.61114	-0.64004	-0.64553	-0.676910	-0.70452
	MB	55.36562	57.81324	58.37315	61.11074	63.54228
1/4 Year	PV	1.80575	1.77102	1.81121	1.80685	1.77541
	Δ ZB	-0.56884	-0.59371	-0.59922	-0.61758	-0.63198
	MB	51.63136	53.77490	54.29784	55.90105	57.13150
Monthly	PV	1.79312	1.79397	1.79211	1.78743	1.76734
	Δ ZB	-0.55308	-0.56235	-0.57049	-0.57911	-0.58000
	MB	50.23775	51.05089	51.76225	52.51219	53.44329
1/2 Month	PV	1.78139	1.78795	1.78395	1.78657	1.78428
	Δ ZB	-0.55272	-0-55791	-0.56390	-0.56859	-0.57491
	MB	50.19454	50.65638	51.17694	51.59018	52.14130
Weekly	PV	1.78790	1.78426	1.78618	1.78369	1.78300
	Δ ZB	-0.55168	-0.55609	-0.55949	-0.56329	-0.56737
	MB	50.11024	50.49251	50.79222	51.12308	51.47965

For example the maximal difference of the cap contract is less than 0,09% of the face value for a hedge tenor equal to 1/2 and less than 0,005% if the hedge is adjusted weekly.

Tabel 4.2: Portfolio Strategy Coupon Bond (P2)

Hedge Tenor		Transition Probability (p) 0.3	0.4	0.5	0.6	0.7
1/2 Year	PV	1.83503	1.75159	1.83022	1.81936	1.83202
	Δ CB	-0.56576	-0.59246	-0.59750	-0.62649	-0.65198
	MC	58.41064	60.99737	61.57997	64.46817	67.02993
1/4 Year	PV	1.80575	1.77102	1.81121	1.80685	1.77541
	Δ CB	-0.52418	-0.54706	-0.55210	-0.56897	-0.58219
	MC	54.22381	56.47669	57.02096	58.70349	59.99429
Monthly	PV	1.79312	1.79397	1.79211	1.78743	1.76734
	Δ CB	-0.50827	-0.51677	-0.52423	-0.53212	-0.54207
	MC	52.62012	53.47069	54.21477	54.99933	55.97417
1/2 Month	PV	1.78139	1.78795	1.78395	1.78657	1.78428
	Δ CB	-0.50761	-0.51236	-0.51785	-0.52214	-0.52792
	MC	52.54234	53.02443	53.56875	54.00016	54.57605
Weekly	PV	1.78790	1.78426	1.78618	1.78369	1.78300
	Δ CB	-0.50648	-0.51051	-0.51362	-0.51710	-0.52083
	MC	52.43586	52.83543	53.14833	53.49406	53.86644

Tabel 4.3: Portfolio Strategy Swap (P3)

Hedge Tenor		Transition Probability (p)				
		0.3	0.4	0.5	0.6	0.7
1/2 Year	PV	1.83503	1.75159	1.83022	1.81936	1.83202
	Δ S	.69342	.72677	.73354	.76980	.80198
	MS	1.83668	1.75350	1.83225	1.82164	1.83461
1/4 Year	PV	1.80575	1.77102	1.81121	1.80685	1.77541
	Δ S	.70237	.73373	.74115	.76452	.78317
	MS	1.80843	1.77394	1.81434	1.81026	1.77909
Monthly	PV	1.79312	1.79397	1.79211	1.78743	1.76734
	Δ S	.72042	.73300	.74406	.75576	.77049
	MS	1.79666	1.79776	1.79600	1.79142	1.77151
1/2 Month	PV	1.78139	1.78795	1.78395	1.78657	1.78428
	Δ S	.72951	.73676	.74500	.75152	.76029
	MS	1.78522	1.79203	1.78816	1.79064	1.78862
Weekly	PV	1.78790	1.78426	1.78618	1.78369	1.78300
	Δ S	.73336	.73949	.74422	.74955	.75525
	MS	1.79177	1.78836	1.79011	1.78792	1.78731

For example the maximal price difference for the considered cap is less then 0.004% of the face value if trading is done weekly.

Conclusion

For a model of a pure interest rate dependent market bond price based models are not flexible enough to satisfy the first object. Especially for the analysis of direct interest rate options like caps and floors this approach leads to serious problems. Therefore it was necessary to model the term structure of interest rates in an arbitrage free way. The presented discrete term structure model does not permit arbitrage opportunities and is flexible enough to analyse the different aspects of interest rates. Furthermore a large class of probability distributions satisfies the existence conditions of our model. For a special subclass of probability distributions the corresponding continuous time model is characterised.
The presented term structure model describes a complete market structure. Within this structure interest rate derivatives like bond options or caps and floors can be priced under arbitrage. Furthermore it is possible to implement portfolio strategies to duplicate the payoff of such derivatives. In contrast to option pricing on stocks it is possible to choose among several equivalent strategies, which is certainly important for portfolio managers.

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Appendix

Proof of Theorem 2.1.

a) The proof is given by induction. For simplicity we first consider the case were $\sigma(.,.)$ and $p(.,.)$ are constant parameters of the model. For $N=2$ the local expectation hypothesis is satisfied if

$$B(t_0,t_2) = \frac{1}{1+r_0}\left[\frac{p}{1+r_{1,1}g(\sigma,p)} + \frac{1-p}{1+r_{1,1}}\right] =: k(r_{1,1})$$

Since $g(\sigma,p)=\exp\left\{\frac{\sigma}{\sqrt{p(1-p)}}\right\} > 1$ the function $k()$ is continuous and decreasing in r. For $r=0$ we have

$$k(0) = \frac{1}{1+r_0} = B(t_0,t_1)$$

Therefore there exists a unique $r>0$ which is a solution under the expectation hypothesis if $B(t_0,t_1) > B(t_0,t_2) > 0$.

Suppose now the statement is true for N periods. For the $N+1$ period let $B(t_0,t_{N+1}) > 0$ be the price of a zero coupon bond which pays one unit at time t_{N+1}. The binomial spot at rate process at time t_N is then given by

$$r_{N,j} = r_{N,N}[g(\sigma,p)]^{N-j} \qquad \text{for } j=0,\ldots,N$$

Under the local expectation hypothesis $B(t_0,t_{N+1})$ must be equal to the expected value of the discounted face value. We can describe the pathes of the spot rate by

$$K(N) = \left\{i=(0,i_1,\ldots,i_N)\in\{0\}\times\{0,1\}^N\right\}$$

Define by

$$\begin{array}{lcll} S(n,i) & := & \sum_{j=0}^{n} i_j & \hat{=} \text{ the number of up–movements of the } i\in K(N) \text{ at time } t_n, n\le N \\ n-S(n,i) & & & \hat{=} \text{ the number of down–movements of the path } i\in K(N) \text{ at time } t_n, n\le N. \end{array}$$

Furthermore set $x := r_{N,N}$ and $r_{0,0} := r_0$ then x is a solution of

$$\begin{aligned} B(t_0,t_{N+1}) &= \sum_{i\in K(N)}\left[p^{S(N,i)}(1-p)^{N-S(N,i)}\frac{1}{\prod_{j=0}^{N-1}\left(1+r_{j,j-S(j,i)}\right)}\right. \\ &\qquad \left.\cdot\frac{1}{1+xg(\sigma,p)^{S(N,i)}}\right] \end{aligned}$$

The function

$$\begin{aligned} G(x,t_{N+1}) &= \sum_{i\in K(N)}\left[p^{S(N,i)}(1-p)^{N-S(N,i)}\frac{1}{\prod_{j=0}^{N-1}\left(1+r_{j,j-S(j,i)}\right)}\right. \\ &\qquad \left.\cdot\frac{1}{1+xg(\sigma,p)^{S(N,i)}}\right] - B(t_o,t_{N+1}) \end{aligned}$$

is strictly decreasing and continuous for $x\ge 0$ with

$$\begin{aligned} \lim_{x\to\infty} G(x,t_{N+1}) &= -B(t_0,t_{N+1}) < 0 \\ G(0,t_{N+1}) &= B(t_0,t_N) - B(t_0,t_{N+1}) > 0 \end{aligned}$$

b) For the general case the transition probability and local volatility are time and spot rate dependent in a predictable way. However the path independence of the spot rate process gives us

$$r_{n,j} = r_{n,n} \cdot \prod_{i=j}^{n-1} g\Big(\sigma(n,i); p(n-1,i)\Big) \qquad \text{for} \quad j = 0, \dots, n-1$$
$$n = 1, \dots, N$$

where $\sigma(n,i)$ is the local volatility at time t_n if the spot rate from t_{n-1} to t_n is equal to $r_{n-1,i}$

$p(n-1,i)$ is the transition probability from t_{n-1} to t_n if the spot rate is $r_{n-1,i}$.

As before we can define a function $G(x, t_{N+1})$

$$\begin{aligned} G(x,t_{N+1}) \quad := \quad & \sum_{i \in K(N)} \Bigg[\prod_{n=0}^{N} p\Big(n; S(n,i)\Big)^{i_n} \Big(1 - p\Big(n; S(n,i)\Big)\Big)^{1-i_n} \\ & \cdot \frac{1}{\prod_{n=0}^{N-1} \left(1 + r_{n,n-S(n,i)}\right)} \\ & \cdot \frac{1}{1 + x \cdot \prod_{j=N-S(N,i)}^{N-1} g\Big(\sigma(N,j); p(N-1,j)\Big)} \Bigg] \\ & - B(t_0, t_{N+1}) \end{aligned}$$

were $\prod_{j=N}^{N-1} g\Big(\sigma(.,.), p(.,.)\Big) = 1$. The function $G(x, t_{N+1})$ has the same feature as $G(.,.)$ in part a. Namely continuous and decreasing in $x \geq 0$.

$$\begin{aligned} \lim_{x \to \infty} G(x, t_{N+1}) &= -B(t_0, t_{N+1}) < 0 \\ G(0, t_{N+1}) &= B(t_0, t_N) - B(t_{N+1}) > 0 \Leftrightarrow B(t_0, t_N) > B(t_0, t_{N+1}) \end{aligned} \qquad \square$$

Proof of Theorem 3.1. Let $\underline{\underline{T}}(n) = \{t'_0 < t'_1 < \ldots < t'_{N \cdot n} = T\}$ be a refinement of the set of trading dates with $\Delta t' = \frac{T}{N \cdot n}$. The binomial structure of the interest rate process implies that

$$r_{i,j} = r_{i,i} \cdot \exp\left\{ \frac{\sigma(t'_i)}{\sqrt{p(1-p)}} \right\} = r_{i,i} \cdot g\Big(\sigma(t'_i), p\Big) \quad \forall \quad i = 0, \dots, N \cdot n \ , \ j = 0, \dots, i$$

were $r_{i,i} > 0$ and depends on the yield curve at time t'_0. The expectation and variance of $x_{t'_i}$ is given by:

$$\begin{aligned} \eta_i \quad &= \quad E_p \left[\log \frac{r_i}{r_{i-1}} \right] \\ &= \quad \sum_{j=0}^{i-1} \binom{i-1}{j} (1-p)^j p^{i-1-j} \Bigg[p \left(\log \frac{r_{i,i}}{r_{i-1,i-1}} + \log g\Big(\sigma(t'_i), p\Big) \right. \\ & \qquad \left. + (i-1) \log g\Big(\sigma(t'_i) - \sigma(t'_{i-1}), p\Big) \right) + (1-p) \left(\log \left(\frac{r_{i,i}}{r_{i-1,i-1}} \right) \right. \\ & \qquad \left. + (i-1) \log g\Big(\sigma(t'_i) - \sigma(t'_{i-1}), p\Big) \right) \Bigg] \\ &= \quad \log \frac{r_{i,i}}{r_{i-1,i-1}} + p \cdot \log g\Big(\sigma(t'_i), p\Big) + (i-1) \log g\Big(\sigma(t'_i) - \sigma(t'_{i-1}), p\Big) \\ s_i^2 \quad &= \quad V_p \left[\log \frac{r_i}{r_0} \right] = \sum_{j=1}^{i} \sigma^2(t'_j) = \sum_{j=1}^{i} h(t'_j) \Delta t' \overset{n \to \infty}{\longrightarrow} \int_0^{t_i} h(t) dt < \infty \end{aligned}$$

It remains to show that the Ljapunoff-condition (Bauer(1978) 268f.) is fulfilled, i.e. there exists at least one $\delta > 1$ such that $\lim_{n\to\infty} \frac{1}{s^{2+\delta}_{N\cdot n}} \sum_{i=1}^{n\cdot N} E_p\left[\left|\log\frac{r_i}{r_{i-1}} - \eta_i\right|^{2+\delta}\right] = 0$

For $\delta = 1$ we have:

$$E_p \left[\left|\log\frac{r_i}{r_{i-1}} - \eta_i\right|^{3}\right] = \sum_{j=0}^{i-1}\binom{i-1}{j}(1-p)^j p^{i-1-j}$$
$$\cdot\left[p\left(\left|\log\frac{r_{i,i}}{r_{i-1,i-1}} + \log g\Big(\sigma(t'_i),p\Big) - \log\frac{r_{i,i}}{r_{i-1,i-1}} - p \quad \log g\Big(\sigma(t'_i),p\Big)\right|^3\right)\right.$$
$$\left.+(1-p)\left(\left|\log\frac{r_{i,i}}{r_{i-1,i-1}} - \log\frac{r_{i,i}}{r_{i-1,i-1}} - p \quad \log g\Big(\sigma(t'_i),p\Big)\right|^3\right)\right]$$
$$= \Big(\log g\Big(\sigma(t'_i,p)\Big)\Big)^3\Big(p(1-p)^3 + (1-p)p^3\Big)$$
$$= \sigma(t'_i)^3\frac{(1-p)^2+p^2}{\sqrt{p(1-p)}}$$
$$\Rightarrow \frac{1}{s^3_{N\cdot n}} \sum_{i=1}^{n\cdot N} E_p\left[\left|\log\frac{r_i}{r_{i-1}} - \eta_i\right|^3\right] = \frac{1}{\left(\sum_{i=1}^{N\cdot n}\sigma^2(t'_i)\right)^{\frac{3}{2}}}\sum_{i=1}^{N\cdot n}\sigma^3(t'_i)\frac{(1-p)^2+p^2}{\sqrt{p(1-p)}}$$
$$= \frac{1}{\left(\sum_{i=1}^{N\cdot n} h(t'_i)\Delta t'\right)^{\frac{3}{2}}}\cdot\sum_{i=1}^{N\cdot n} h^{\frac{3}{2}}(t'_i)\cdot\Delta^{\frac{3}{2}}t'\cdot\frac{(1-p)^2+p^2}{\sqrt{p(1-p)}}$$
$$= \frac{\sqrt{\Delta t'}}{\left(\sum_{i=1}^{N\cdot n} h(t'_i)\Delta t'\right)^{\frac{3}{2}}}\sum_{i=1}^{N\cdot n} h^{\frac{3}{2}}(t'_i)\Delta t'\frac{(1-p)^2+p^2}{\sqrt{p(1-p)}} \overset{n\to\infty}{\longrightarrow} 0$$

because $\lim_{n\to\infty}\sum_{i=1}^{n\cdot N} h(t'_i)\Delta t' = \int_0^T h(t)dt < \infty$, $\lim_{n\to\infty}\sum_{i=1}^{N\cdot n} h^{\frac{3}{2}}(t'_i)\Delta t \le T\cdot\max_{t\in[0,T]} h(t)^{\frac{3}{2}} < \infty$ □

Proof of the inequality (3.12). For simplicity set

$$\overline{\mu} := \log r_0 + \int_0^t \mu(s) - \frac{1}{2}\sigma^2(s)ds \quad , \quad \overline{\sigma}^2 := \int_0^t \sigma^2(s)ds$$

a)

$$\begin{aligned}
\Rightarrow E[y_t] &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\Pi}\,\overline{\sigma}}\frac{1}{1+e^y}\exp\left\{-\frac{(y-\overline{\mu})^2}{2\overline{\sigma}^2}\right\}dy\\
&= 1-\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\Pi}\,\overline{\sigma}}\frac{e^y}{1+e^y}\exp\left\{-\frac{(y-\overline{\mu})^2}{2\overline{\sigma}^2}\right\}dy\\
&= 1-\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\Pi}\,\overline{\sigma}}\frac{1}{1+e^y}\exp\left\{-\frac{\Big(y-(\overline{\mu}+\overline{\sigma}^2)\Big)^2}{2\overline{\sigma}^2}+\overline{\mu}+\frac{1}{2}\overline{\sigma}^2\right\}dy\\
&= 1-e^{\overline{\mu}+\frac{1}{2}\overline{\sigma}^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\Pi}\,\overline{\sigma}}\,\frac{1}{1+e^{y+\overline{\sigma}^2}}\exp\left\{-\frac{(y-\overline{\mu})^2}{2\overline{\sigma}^2}\right\}dy\\
&= 1-e^{\overline{\mu}-\frac{1}{2}\overline{\sigma}^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\Pi}\,\overline{\sigma}}\frac{e^{\overline{\sigma}^2}}{1+e^{y+\overline{\sigma}^2}}\exp\left\{-\frac{(y-\overline{\mu})^2}{2\overline{\sigma}^2}\right\}dy\\
&\le 1-e^{\overline{\mu}-\frac{1}{2}\overline{\sigma}^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\Pi}\,\overline{\sigma}}\frac{1}{1+e^y}\exp\left\{-\frac{(y-\overline{\mu})^2}{2\overline{\sigma}^2}\right\}dy\\
\Rightarrow E[y_t] &\le \frac{1}{1+e^{\overline{\mu}-\frac{1}{2}\overline{\sigma}^2}} = \frac{1}{1+r_0\exp\{\int_0^t\mu(s)-\sigma^2(s)ds\}}
\end{aligned}$$

b) Since $g(r) = \frac{1}{1+r}$ is convex on $[0, +\infty[$ we know from Jensens Inequality

$$E[y_t] = E\left[\frac{1}{1+r_t}\right] \geq \frac{1}{1+E[r_t]} = \frac{1}{1 + r_0 \exp\left\{\int_0^t \mu(s)ds\right\}}$$

$\square$

Proposition. Cap and floor contracts are convex functions of the level L and the underlying interest rate r.

Proof.

a) Level L
Let $t_i \in \underline{\underline{T}}$ a payment date of the cap resp. floor contract. For $\lambda \in]0,1[$ with $L = \lambda L_1 + (1 - \lambda)L_2, \quad 0 < L_1 < L_2$ the cap payment with level L satisfies always

$$\begin{aligned} V[r-L]^+ &= V[r - (\lambda L_1 + (1-\lambda)L_2)]^+ \\ &\leq \lambda V[r - L_1]^+ + (1-\lambda)V[r - L_2]^+ \end{aligned}$$

resp. for the floor payment with level L

$$\begin{aligned} V[L-r]^+ &= V[(\lambda L_1 + (1-\lambda)L_2) - r]^+ \\ &\leq \lambda V[L_1 - r]^+ + (1-\lambda)V[L_2 - r]^+ \end{aligned}$$

b) The result for the underlying interest rate is prooved by the same argument.

$\square$

Proof of (4.9) (Cap – Floor – Parity). Consider the following portfolio at time t_0

sell	Cap $[r, L, \underline{\underline{T}} \setminus \{t_N\}, V]$
buy	Floor $[r, L, \underline{\underline{T}} \setminus \{t_N\}, V]$

then at any time $t_i \in \underline{\underline{T}} \setminus \{t_0\}$ the cash flow of the portfolio is equal to $V[L - r_{t_{i-1}}]$ which defines just the cash flow of a swap contract.

$\square$

Proof of (4.10). Consider the following portfolio strategy which obviously duplicates the cash flow of an interest rate swap:
Sell a coupon bond with face value V and coupon payment L equal to the fix rate at any date $t_i \in \underline{\underline{T}} \setminus \{t_o\}$
Invest the face value V at the underlying interest rate. At time t_i the cash flow is equal to

$$V(1 + r_0) - V \cdot L = V[r_0 - L] + V$$

which equals the swap payment plus the face value. Then reinvest the face value at the underlying interest rate and so on. At the last period t_N this leads to

$$V(1 + r_{t_{N-1}}) - V(1 + L) = V[r_{t_{N-1}} - L]$$

$\square$

Proof of (4.11) (Boundary condition for cap and floor contracts). From the monotonicity with respect to the level L and the cap – floor parity we have

$$\begin{aligned}\max\{0, \mathrm{Swap}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V]\} &\le \mathrm{Cap}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V] \\ &\le \mathrm{Cap}[r, 0, \underline{\underline{T}} \setminus \{t_N\}, V]\end{aligned}$$

The cash flow of a cap contract with level $L = 0$ is equal to the interest earned by investing the face value at each period. Therefore

$$\mathrm{Cap}[r, 0, \underline{\underline{T}} \setminus \{t_N\}, V] = V - VB(t_0, t_N)$$

The arbitrage prices of cap and floor contracts are non negative. Together with the cap – floor – parity this leads to:

$$\max\left\{0, -\mathrm{Swap}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V]\right\} \le \mathrm{Floor}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V]$$

Furthermore the cash flow of a floor contract at any time $t_i \in \underline{\underline{T}} \setminus \{t_0\}$ is bounded above by

$$V[L - r_{t_{i-1}}]^+ \le V \cdot L$$

which gives us

$$\mathrm{Floor}[r, L, \underline{\underline{T}} \setminus \{t_N\}, V] \le \sum_{i=1}^{N} V \cdot L \; B(t_0, t_i)$$

Finally the last statement results from the monotonicity of the interest rate swap with respect to the swap level L.

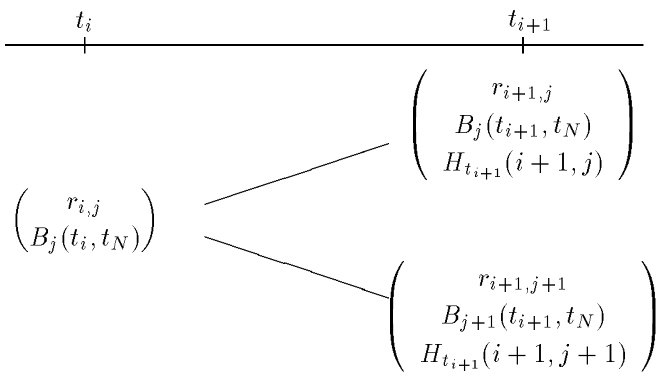
□

Proof of Theorem 4.1. With respect to the complete market structure and the results of Harrison and Kreps (1979) we have only to demonstrate that every self–financing portfolio strategy which duplicates the payoff of a given interest rate derivative has the same initial price (4.12).

1. Let $H = (H_{t'_i}, \ldots, H_{t'_k})$ be an interest rate derivative with $\{t'_i, \ldots, t'_k\} \subseteq \underline{\underline{T}}$. First we consider a portfolio strategy using a zero–coupon bond with maturity t_N and the spot rate $\{r\}$. Under the assumptions of the term structure model the price process of the zero–coupon bond is determined by the local expectation hypothesis

$$B_j(t_i, t_N) = \frac{E_{p(i,j)}[B \cdot (t_{i+1}, t_N) | r_{i,j}]}{1 + r_{i,j}}$$

for all knots (i, j) of the binomial model with the terminal condition $B_j(t_N, t_N) = 1$ for all $j = 0, \ldots, N$. With respect to these two instruments the market situation at time t_k can be described by



Suppose t_{i+1} is the last payment date of the interest rate derivative, then the resulting cash flow would be given by $H_{t_{i+1}}(., .)$. As the interest rate process and the zero coupon bond process span the whole market structure at time t_{i+1} and the payment is adapted to the information structure, there exists a portfolio at time t_i which duplicates $H_{t_{i+1}}$. The portfolio is given by

$$\begin{aligned} \Delta &= \frac{H_{t_{i+1}}(i+1,j) - H_{t_{i+1}}(i+1,j+1)}{B_j(t_{i+1},t_N) - B_{j+1}(t_{i+1},t_N)} \qquad \text{(hedge ratio)} \\ K &= \frac{1}{1+r_{i,j}}\left[\frac{B_j(t_{i+1},t_N)H_{t_{i+1}}(i+1,j+1) - B_{j+1}(t_{i+1},t_N)H_{t_{i+1}}(i+1,j)}{B_j(t_{i+1},t_N) - B_{j+1}(t_{i+1},t_N)}\right] \end{aligned}$$

Under the no arbitrage condition the value $\overline{H}$ at time t_i of the payment $H_{t_{i+1}}$ must be equal to the value of the portfolio. Under the local expectation hypothesis of the zero coupon bond this can be reformulated to

$$\begin{aligned} \overline{H} &= \Delta B_j(t_i,t_N) + K \\ &= \frac{1}{1+r_{i,j}} E_{p(i,j)}[H_{t_{i+1}}(.,.)|r_{i,j}; B_j(t_i,t_N)] \end{aligned}$$

By induction this lead to the statement with respect to the choosen instruments.

2. From the assumptions of the model the cash flow of any other security, e.g. zero–coupon bonds with shorter maturity, coupon bonds, swaps etc. can be interpreted as an interest rate derivative. Therefore a self financing portfolio strategy using the zero coupon bond $B(t_N)$ and the interest rate exists for each of these instruments. The initial value of this strategy is equal to the expected discounted payoff. This means that any interest rate depending security within the model can be replaced by a self–financing portfolio strategy of the first two instruments. As a conclusion the arbitrage price do not depend on the choice of the instruments.

□

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