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Prediction Error Learning in Univariate Generalized Autoregressive Models

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Abstract

The rational expectations hypothesis is supported if rational expectations are stable with respect to reasonable learning procedures. We consider within a univariate generalized autoregressive model a class of prediction error based learning procedures which contain as a special case the OLS–procedure. We derive a concept of stability, characterize stable and unstable rational expectations equilibria, determine the possible limit outcomes of the learning procedures and show that there is convergence with unit or only positive probability, depending on the respective feedback function which maps the perceived into the true law of motion of the endogenous variable. Finally, in view of the plausibility of the learning procedures, we determine the rate of convergence.

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Introduction

In nearly all economic models the agents' expectations matter and are one important source for the models dynamics. But because expectations are generally unobservable one has to close a model in which expectations matter by using an expectations formation scheme. Clearly, every particular choice of such an expectations formation scheme is ad-hoc and open to criticism. In his famous paper MUTH (1961) introduced the rational expectations hypothesis (REH). He suggested that "expectations, since they are informed predictions of future events, are essentially the same as the predictions of the relevant economic theory" (p. 316). More precisely, he assumed that the agents' expectations coincide with the (mathematical) conditional distributions of the variables to be predicted given the observable history of the model.

The REH seems to be superior to other expectations formation schemes (like the adaptive expectations hypothesis or the classical scheme in which agents use the current value as prediction for the next periods value) since it is consistent with the common view of an agent as *homo economicus*. In calculating rational expectations the complete set of relevant information as specified by the model is used, while other expectation schemes do not incorporate information which is observable, hence available for the agents. In addition, the rational expectations formation scheme, because it is not a constructive scheme, depends of the respective structure of the underlying model. In some models rational expectations coincide with particular adaptive expectations while in other models they can coincide with classical expectations.

Nevertheless, the REH raises a few substantial problems. The main problem is that the REH is extremely demanding in view of agents knowledge and understanding of their economic surrounding. In order to calculate the conditional distribution of the endogenous variables of interest the agents have to know not only the reduced form of the model but also the exact values of its reduced form parameters. The main counter-argument is that economic agents can somehow learn to be rational by repeatedly observing their economic environment and collecting information. Thus rational expectations could be understood as limit outcomes of some 'reasonable' learning procedure. This idea is appealing on the first sight but leads to some problems when investigated formally.

Firstly, one has to specify what 'reasonable' learning procedures are or should be. Most studies concerned with learning to become rational consider Bayesian or classical statistical techniques. Secondly, one has to clarify how sophisticated agents are or should be. Should they take into account that there are other agents which form expectations? Should they take into account that the other agents take into account that and how other agnets form expectations? And so on. At this point the studies fall into two distinct groups. Within the first group it is assumed that agents believe in an auxiliary model which differs from the 'true' model and try to estimate some unknown parameters. This assumption reduces the complexity such that analytical results could be obtained with some effort. The studies of BRAY (1982), BRAY/SAVIN (1986), FOURGEAUD ET AL. (1986), and KOTTMANN (1990), among others, fall into this group. Within the other group it is assumed that agents take into account other agents' expectations. But this complicates the analysis as much that exact mathematical results seem almost impossible to achieve. As one example we quote the study of FRYDMAN (1982)[1]. We want to remark that our study falls into the first group.

The problem of learning to become rational is quite simple if the agents' expectations have no impact on the variables to be predicted. Then the problem reduces to that of consistency of parameter estimates. But if there is a 'forecast feedback', thus if expectations influence the outcome of the variables to be predicted, the classical statistical theory is not able to give a satisfactory answer. The reason for this is that, even under very restrictive conditions, the forecast feedback induces non-stationarities into the models time series. Hence, new techniques have to be developed to prove convergence. Therefore almost all studies are quite technical.

For models in which no lagged endogenous variables appears among the instrument variables used by agents for their predictions KOTTMANN (1990) and MOHR (1990) developed a complete theory based on stochastic approximation results of WALK (1985).

But for models with lagged endogenous variables among the instruments the problem of convergence is not yet solved in a completely satisfactory manner. The first study on this subject was the one of CYERT/DEGROOT (1974), but they confirmed themselves to Monte–Carlo simulations because of the immense analytical difficulties. FOURGEAUD ET AL. (1986) considered the deterministic case of MUTH's cobweb model but their analysis was not fully correct. A breakthrough was achieved by MARCET/SARGENT (1989a,b,c). Their analysis is based on the results of LJUNG (1977) and LJUNG/SÖDERSTRÖM (1983) concerned with the problem of recursive identification. Nevertheless, the results of MARCET/SARGENT are not fully satisfactory. Firstly, the analysis of LJUNG is not always convincing in the eyes of a critical mathematician, although we conjecture that his results are true. Secondly, his convergence analysis uses a certain technical trick, called 'projection facility', which is appropriate in an engineering context but not necessarily in the context of learning procedures. Without using this feature the analysis of MARCET/SARGENT remains a local analysis and the scope of a local convergence analysis is always limited.

KOTTMANN/KULIBERDA (1990) developed for the special case of an univariate autoregressive model with additive expectations a global convergence theory which relies on a convergence result for almost-supermartingales. We take up their approach in this study and investigate the problem of convergence in a more general, but still limited, setting.

In Chapter 1 we introduce the model and a class of learning procedures which contains as a special case the least squares learning procedure. In Chapter 2 we show that these learning procedures have as limit outcomes, if they converge, only rational expectations. In Chapter 3 we develop a concept of stability of rational expectations with respect to a learning procedure and give a simple characterization of stable rational expectations. This concept is a local one but it may already serve to distinguish between rational expectations which could be the limit of a learning procedure or could not. In Chapter 4 we present a global convergence analysis. It turns out that in special cases we can get almost sure convergence while in the general we have only convergence with positive probability less one. In Chapter 5 we determine the rate of convergence of some learning procedures because the rate of convergence is crucial in view of the plausibility of learning procedures.

[1]For a comprehensive overwiew see the monographs of PESARAN (1987) and FRYDMAN/PHELPS (1983)

We conclude this paper with Chapter 6 in which we analyze the effect of learning in MUTH's original cobweb model. This example only serves as an illustration of our ideas. However, we regard our results as a first step towards a complete and satisfactory theory of the effect of learning in general auteregressive models.

Chapter 1

The Model

We consider a generalized autoregressive model in which the current value of the endogenous variable, y_t, depends on the value of the one period lagged endogenous variable, y_{t-1}, and some expectations of future values of the endogenous variable, $y^e_{t+k}, k \geq 0$, held by a homogenous group of agents at time $t-1$. This dependence is given by

$$(1.1) \qquad y_t = F(y_{t-1}, y^e_t, y^e_{t+1}, y^e_{t+2}, \ldots, y^e_{t+n}) + w_t, \qquad t \geq 1,$$

where w_t is a zero mean disturbance term, F is a real valued function which fulfills certain regularity conditions, $y^e_{t+k}, 0 \leq k \leq n$, are the expectations of y_{t+k} held by agents at time $t-1$ and based on the information set $I_{t-1} = \{y_{t-1}, y_{t-2}, \ldots, y_0\}$.

For simplicity of exposition we suppose that only y^e_t matters in (1.1). Suppose furthermore that the agents form their expectations according to an expectations formation scheme given by a sequence of real valued functions $\{g_t\}$, i.e.

$$(1.2) \qquad y^e_t = g_t(y_{t-1}, y_{t-2}, \ldots), \qquad t \geq 1 \quad .$$

Such an expectations formation scheme generates 'rational expectations' (in the sense of Muth (1961)) if the probability distribution of the resulting expectations coincides with the mathematical conditional expectation of the variables of interest. A weaker requirement for expextations to be rational, mostly used in the respective literature, is that the expectations and the first conditional moment of the variable of interest coincide. Such expectations are sometimes called 'wide sense rational expectations'. In our model an expectations formation scheme generates wide sense rational expectations if

$$(1.3) \quad y^e_t = g_t(y_{t-1}, y_{t-2}, \ldots) = F(y_{t-1}, g_t(y_{t-1}, y_{t-2}, \ldots)) = \mathrm{E}[y_t | I_{t-1}] \quad \text{a.s.}$$

for all $t \geq 1$ provided that $\mathrm{E}[w_t | I_{t-1}] = 0$.

We suppose for the moment that there exists an expectations formation scheme which generates rational expectations. If the agents know the function F then they could calculate these rational expectations. If there exists only one solution of (1.3) then the law of motion of the process $\{y_t\}$ is uniquely determined and the system (1.1), (1.2) is in a rational expectations equilibrium. But in general the function F will not be known exactly by the agents, hence their expectations will not be rational, at least on the short-run.

But what happens on the long-run? Is it possible that agents learn to form rational expectations in following a simple and reasonable learning procedure based on observations of the history of $\{y_t\}$?

Before we can answer this question we have to specify how to measure the success of a learning procedure. We thus need a suitable concept of convergence of expectations. Of course, the concept best suited is that of convergence of probability measures, especially if we consider rational expectations in the sense of MUTH. But even if we assume the initial value y_0 and the disturbance process $\{w_t\}$ to have 'nice' probability distributions (e.g. Gaussian distributions) the calculation of the distribution of y_t and y_t^e remains a serious problem because of the complicated dependence structure of these variables. To see this, notice that y_t as well as y_t^e depend on the whole history of the system, $y_{t-1}, y_{t-2}, \ldots, y_0, y_{t-1}^e, y_{t-2}^e, \ldots, y_1^e$. Moreover, this dependence is determined as well by the expectations formation scheme $\{g_t\}$ as by the, possibly nonlinear, function F.

One way to facilitate the analysis is to parameterize the expectations in a suitable manner and then use the induced convergence in the parameter space. This is the way usually followed in the literature concerned with rational or boundedly rational learning models[1]. We join this approach because we believe that it represents a good compromise between simplicity of analysis and realism. In our model the expectations parameter is a real number, hence the analysis will be simple and intuitively appealing.

The second problem consists in specifying a simple and reasonable learning procedure. As PESARAN (1988) points out, "There is no doubt that individuals do learn from their own experience as well as from the experience of others. Generally speaking, learning takes place through two separate but closely connected mechanisms, namely 'repetition', and 'understanding'." (p. 32). Therefore we assume that the agents in our model have in mind a (not necessarily fully specified) model which they believe to describe the situation they face. We thus assume that they understand, at least roughly, how the economic system which they are part of works, but are uncertain of the exact values of some parameters or coefficients. They try to learn these values through repetition, i.e. through a repeated trial-and-error scheme. We model this 'learning by repetition' by a special first order adaptive prediction error mechanism. This mechanism includes as a prominent example the least squares learning procedure.

The following assumption specifies the class of expectations formation schemes we deal with in this paper.

Assumption (A.1) (Behavioural Assumption):
The expectations y_t^e of the value of y_t held by agents at time $t-1$ on the basis of the information set $I_{t-1} = \{y_{t-1}, \ldots, y_0\}$ are given by

$$y_t^e = \theta_{t-1} y_{t-1} \quad , \quad t \geq 1. \tag{1.4}$$

The coefficient θ_t is recursively given by

$$\theta_t - \theta_{t-1} = \gamma_{t-1} \left(\frac{y_t}{y_{t-1}} - \theta_{t-1} \right) \quad , \quad t \geq 1 \tag{1.5}$$

with $\{\gamma_t\}$ a sequence of non-negative real valued random variables adapted at the filtration $\{\mathcal{G}_t\} = \sigma(I_t)$. The sequence of gains[2] $\{\gamma_t\}$ fulfills $\gamma_t \leq 1$ and

$$\sum_{t=0}^{\infty} \gamma_t = \infty \quad \text{and} \quad \sum_{t=0}^{\infty} \frac{\gamma_t^2}{y_t^2} < \infty \quad \text{a.s.} \tag{1.6}$$

[1] See PESARAN (1988) for the methodological background of the concepts of rational and boundedly rational learning, and the related literature. For more recent work see, for example, MARCET/SARGENT (1989a,b,c) and KOTTMANN (1990).

[2] In the sequel we regard the expectations formation scheme as an adaptive on-line estimation algorithm which are well known in engineering and control sciences (see, f.e., BENVENISTE ET AL. (1990), LJUNG/SÖDERSTRÖM (1983)) and for which the weighting coefficients $\{\gamma_t\}$ are called 'gains'.

and $\gamma_t = 0$ whenever $y_t = 0$. The initial value θ_0 is a finite random variable. Such a sequence of gains is called *admissible*.

We assume furthermore that the multi-period expectations are given as $y^e_{t+k} = \theta^{k+1}_{t-1} y_{t-1}$ for all $k \geq 1$.

This assumption seems to be somewhat ad-hoc but can be rationalized as follows: Firstly, we assume that agents believe in a (stationary) law of motion which governs the dynamical behaviour of the process $\{y_t\}$, more specifically, we assume that agents believe in an autoregressive model

$$y_t = \eta y_{t-1} + e_t \tag{1.7}$$

with a constant parameter η and a zero mean disturbance term e_t which is independent of past values of the endogenous variable.[3] The parameter η is not known to the agents and they try to learn about its value by means of a repetitive trial–and–error procedure.

Secondly, we assume that this procedure can be described as a first order adaptive relative prediction error (RPE) mechanism. The relative prediction error at time t (which is observable by agents) is defined as

$$RPE_t = \frac{1}{y_{t-1}}\left(y_t - y^e_t\right) = \frac{1}{y_{t-1}}\left(y_t - \theta_{t-1} y_{t-1}\right) = \frac{y_t}{y_{t-1}} - \theta_{t-1}. \tag{1.8}$$

The coefficient γ_t in (1.5), which we allow to be stochastic, represents the sensitivity of the learning procedure relative to the RPE. A small value of γ_t implies a small adjustment of the expectations coefficient θ_t while a large value implies a large adjustment. The sequence of gains $\{\gamma_t\}$ may be decreasing as well as (asymptotically) constant.

Although the conditions in (1.6) are motivated by the convergence analysis in the following chapters they are reasonable in the context of learning procedures too. The first condition in (1.6) ensures that the sequence of gains do not decrease too fastly towards zero. A decreasing sequence of gains represents the behaviour of agents who believe in the success of their learning procedure. Decreasing gains are equivalent to decreasing weighting of the observed RPE, but in order to learn the exact value of the unknown parameter the weighting of the RPE should not decrease too fastly. Otherwise the agents get too ignorant towards the RPE and the adjustment process converges only because the agents are convinced of the success of their learning. In such a case the resulting limit point of the adjustment process will in general not coincide with the unknown parameter η. An example for a 'good' choice of decreasing gains is $\gamma_t = \frac{1}{t}$ [4].

[3] This assumption is perhaps the most problematic one in our setup because the auxiliary model (1.7) does not include the effect of forecast feedback. The agents in our model are not concerned with what other agents do or what they expect. Because they neglect the effect of forecast feedback they neglect the behavioural uncertainty which exists in our model, at least if there are heterogenous groups of agents.

Nevertheless, the neglect of forecast feedback can be rationalized in some situations, e.g. in a competitive world in which the expectations of a single agent have no impact on (real) outcomes, or in situations in which agents do not know how many and which types of other agents are in the market, or in situations in which it is too costly to gather information about the expectations of all the other agents, or in situations in which agents assume that the aggregate or market expectation is already rational.

[4] Pesaran (1987) points out that "It is interesting that early experimental evidence on the formation of expectations reported by Rotter (1954 pp. 165–83) do in fact support the time–varying adaptive learning rule. In his experiments Rotter observed that the magnitude of revisions in expectations tended to be inversely proportional to the number of trials undertaken by the subjects."(p.41)

The second condition in (1.6) is more difficult to rationalize. Without the emergence of y_t^2 in the sum the two conditions in (1.6) are the so–called 'Dvoretzky–Conditions' which are a kind of natural conditions for the gains of a stochastic approximation procedure. Because we use the relative prediction error for the learning procedure and not the prediction error (PE) $y_t - y_t^e$ the term y_t^2 emerges in the denominator. The use of the RPE instead of the PE is plausible if y_t and y_t^e are large in modulus, thus if the prediction error is large. But if the value of y_{t-1} is small, more precisely, if $|y_{t-1}| < 1$ then the RPE will exceed the PE and this leads possibly to a very large adjustment. We think that it is reasonable to assume that the agents use the adjustment criterion best suited for the situation they face. If the PE is large they will use the RPE for the adjustment of their expectation parameter, otherwise they will use the PE. Although it is possible to model this ability of the agents mathematically the convergence analysis would be very complicated. Instead, we assume implicitly with the second condition in (1.6) that the agents choose their sequence of gains such that a large RPE will not lead to a large adjustment of their expectation parameter, thus if y_t^2 is small then γ_t will be chosen sufficiently small to ensure that the second sum in (1.6) converges. Of course, this reasoning only shows that the conditions (1.6) are not implausible in connection with learning.

Before we proceed we have to show that there exists at least one sequence of gains which fulfills the conditions (1.6). Suppose that agents believe in the auxiliary model (1.7) with a constant parameter η and run an ordinary least squares (OLS) regression to estimate the parameter η. Thus

$$
\begin{aligned}
\theta_t &= \left(\sum_{\tau=0}^{t-1} y_\tau^2\right)^{-1} \sum_{\tau=0}^{t-1} y_\tau y_{\tau+1} \\
&= \left(\sum_{\tau=0}^{t-1} y_\tau^2\right)^{-1} \left(\sum_{\tau=0}^{t-2} y_\tau y_{\tau+1} + y_{t-1} y_t\right) \\
&= \left(\sum_{\tau=0}^{t-1} y_\tau^2\right)^{-1} \left(\left(\sum_{\tau=0}^{t-2} y_\tau^2\right) \theta_{t-1} + y_{t-1} y_t\right) \qquad (1.9) \\
&= \left(\sum_{\tau=0}^{t-1} y_\tau^2\right)^{-1} \left(\left(\sum_{\tau=0}^{t-1} y_\tau^2\right) \theta_{t-1} - y_{t-1}^2 \theta_{t-1} + y_{t-1} y_t\right) \\
&= \theta_{t-1} + \left(\sum_{\tau=0}^{t-1} y_\tau^2\right)^{-1} y_{t-1}^2 \left(\frac{y_t}{y_{t-1}} - \theta_{t-1}\right).
\end{aligned}
$$

In this case we have $\gamma_t = y_t^2 / \sum_0^t y_\tau^2$.

Lemma 1.1:
Let $\{y_t\}$ be a sequence of real numbers with $y_0 \neq 0$. Define

$$
c_t = \frac{y_t^2}{\sum_{\tau=0}^{t} y_\tau^2} \qquad \text{and} \qquad d_t = \frac{y_t}{\sum_{\tau=0}^{t} y_\tau^2}, \qquad (1.10)
$$

then

$$
(a) \sum_{t=0}^{\infty} d_t^2 \;<\; \frac{2}{y_0^2} \;<\; \infty,
$$

$$(b) \sum_{t=0}^{\infty} c_t = \infty \qquad whenever \qquad \sum_{t=0}^{\infty} y_t^2 = \infty.$$

Proof:
(a) See Lemma A.2 of GOODWIN ET AL. (1981) or Lemma 2.2 of KOTTMANN/KULIBERDA (1990).
(b) Suppose that $\sum_0^\infty y_t^2 = \infty$ and $\sum_0^\infty c_t < \infty$. By the Kronecker Lemma we get

$$(1.11) \qquad 1 = (\sum_{t=0}^{T} y_t^2)^{-1} \sum_{t=0}^{T} y_t^2 = (\sum_{t=0}^{T} y_t^2)^{-1} \sum_{t=0}^{T} c_t \sum_{\tau=0}^{t} y_\tau^2 \longrightarrow 0$$

as $T \to \infty$, clearly a contradiction. ■

After we have shown that there exists at least one learning procedure which fulfills Assumption (A.1) we turn to the function F in (1.1).

Assumption (A.2):
There exists a continuous function $f : I\!R \to I\!R$ such that

$$(1.12) \qquad F(y, \theta y, \theta^2 y, \ldots, \theta^{n+1} y) = f(\theta) y \qquad \forall y \in I\!R, \quad \forall \theta \in I\!R.$$

This assumption ensures that if agents form their expectations according to Assumption (A.1) the resulting 'true' model is of a similar kind as their auxiliary model (1.7). Under (A.1) and (A.2) the equation (1.1) turns out to be

$$(1.13) \qquad y_t = f(\theta_{t-1}) y_{t-1} + w_t.$$

Example 1.2 (Additive Expectations):
Suppose that the expectations in (1.1) influence the endogenous variable in an additive way, thus

$$(1.14) \qquad y_t = \phi y_{t-1} + a y_{t+k}^e + w_t \qquad , \qquad t \geq 1,$$

with some $k \geq 0$ and $a, \phi \in I\!R$. Under (A.1) we get $f(\theta) = \phi + a\theta^{k+1}$ which obviously fulfills (A.2).[5]

The given setup allows for the influence of various expectations of different time horizon as long as they are formed by the same expectations formation scheme.

[5] In ZENNER (1992a,b) this model was analyzed extensively as well theoretically as in a Monte–Carlo study (see also KOTTMANN/KULIBERDA (1990)), although with a slightly different expectations formation scheme.

Example 1.3 (Multiple Additive Expectations):

$$y_t = \phi y_{t-1} + \sum_{j=1}^{J} a_j y^e_{t+k_j} + w_t \qquad , \qquad t \geq 1 \tag{1.15}$$

with some $\phi \in I\!R, a_j \in I\!R, k_j \geq 0, j = 0, \ldots, J$. Under (A.1) we have $f(\theta) = \phi + \sum_{j=1}^{J} a_j \theta^{k_j+1}$ which fulfills (A.2).

Comparing (1.13) with (1.7) it is obvious how rational expectations are characterized under (A.1) and (A.2).

Definition 1.4:
Suppose that (A.1), (A.2) hold. We call the expectations $y^e_{t+k} = \theta^{k+1} y_{t-1}$ with $k \geq 0$ *rational expectations* if $f(\theta) = \theta$. If θ^* is a fixed point of the function f then the process $\{y^*_t\}$ evolving according to

$$y^*_t = f(\theta^*) y^*_{t-1} + w_t \qquad , \qquad t \geq 1 \tag{1.16}$$

is called a stationary *rational expectations equilibrium (REE).*

In this paper we confine ourselves to the question whether agents can learn to form stationary rational expectations by some learning procedure. Of course, it is possible to consider also non–stationary rational expectations within our framework. But if we are interested in non–stationary or cyclical rational expectations, especially in convergence towards them, we have to consider other learning procedures because first order adaptive schemes will unlikely detect cyclical structures.[6]

To complete our setup we introduce the following

Assumption (A.3) (Stochastic Assumption):
$\{w_t\}$ is a martingale difference sequence with respect to the canonical filtration $\{\mathcal{F}_t\}_{t\geq 1} = \{\sigma(w_t, w_{t-1}, \ldots)\}_{t\geq 1}$ with

$$\sup_{t\geq 1} \mathrm{E}[w_t^2 | \mathcal{F}_{t-1}] = \sigma^2 < \infty \qquad \text{a.s.,} \tag{1.17}$$

$$\inf_{t\geq 1} \mathrm{E}[w_t^2 | \mathcal{F}_{t-1}] > 0 \qquad \text{a.s.} \tag{1.18}$$

Furthermore we assume that y_0 is a finite random variable.

This assumption is standard for our purposes. It states that $\mathrm{E}[w_t | \mathcal{F}_{t-1}] = 0$ a.s. Thus there is no effect of the disturbances which is forseeable by agents. The conditions (1.17) and (1.18) ensure that there is enough but not too much stochastic fluctuation.

[6] In a Monte–Carlo study ZENNER (1992a,b) found out that the least squares learning procedure is able to produce stable cyclical expectation parameters, but these cyclical parameters were not rational.

Chapter 2

Rationality of Limit Expectations

In this section we determine the possible limit points of the process of expectations parameters $\{\theta_t\}$ given by the expectations formation schemes described in the preceding chapter. Throughout this chapter we suppose that (A.1) and (A.2) hold. The expectations parameters are given recursively by

$$\theta_t = \theta_{t-1} + \gamma_{t-1}(f(\theta_{t-1}) - \theta_{t-1}) + \frac{\gamma_{t-1}}{y_{t-1}} w_t \tag{2.1}$$

with some initial value θ_0. We will call such a recursive updating scheme *adaptive algorithm*, using the terminology of the engineering and control literature. The process $\{y_t\}$, given by

$$y_t = f(\theta_{t-1}) y_{t-1} + w_t \quad , \quad t \geq 1 \tag{2.2}$$

will be called *data process* and the process $\{\theta_t\}$ *estimation process.*

Definition 2.1:
Suppose that the expectations y^e_{t+k} are given as in (A.1), thus $y^e_{t+k} = \theta^{k+1}_{t-1} y_{t-1}$, $k \geq 0$, and that $\mathrm{P}(\theta_t \to \bar{\theta}) > 0$ for some finite fixed value $\bar{\theta}$. Then the expectations

$$\bar{y}^e_{t+k} = \bar{\theta}^{k+1} y_{t-1} \quad , \quad t \geq 1, k \geq 0 \tag{2.3}$$

are called *possible limit expectations* with respect to the respective learning procedure determined by the sequence of gains $\{\gamma_t\}$.
A process $\{\bar{y}_t\}$ which evolves according to

$$\bar{y}_t = f(\bar{\theta}) \bar{y}_{t-1} + w_t \quad , \quad t \geq 1, \tag{2.4}$$

is called a *limit expectations equilibrium (LEE).*

Our aim is to determine the possible limit expectations induced by the learning procedures and to find out whether the resulting limit expectations equilibria are rational expectations equilibria or not. In fact, it turns out that the learning procedures can only produce rational limit expectations.

Theorem 2.2:
Suppose that for a RPE–algorithm as in (A.1) we have $P(\theta_t \to \bar{\theta}) > 0$ with some finite random variable $\bar{\theta}$ Then $f(\bar{\theta}) = \bar{\theta}$ a.s. on the event $[\theta_t \to \bar{\theta}]$.

For the proof we need the following auxiliary result.

Lemma 2.3: (LAI/WEI (1982))
Suppose that $\{w_t\}$ is a martingale difference sequence with respect to a filtration $\{\mathcal{F}_t\}$ and $\sup_{t\geq 1} \mathrm{E}[w_t^2 | \mathcal{F}_{t-1}] < \infty$ a.s. Let $\{\xi_t\}$ be a sequence of predetermined (i.e. $\mathcal{F}_{t-1}$–measurable) random variables, then

(a) $\sum_{t=1}^{T} \xi_t w_t$ *converges a.s. for $T \to \infty$ on the event* $\Big[\sum_{t=1}^{\infty} \xi_t^2 < \infty\Big]$,

(b) $\sum_{t=1}^{T} \xi_t w_t = o\Big(\sum_{t=1}^{T} \xi_t^2\Big)$ *a.s. for $T \to \infty$ on the event* $\Big[\sum_{t=1}^{\infty} \xi_t^2 = \infty\Big]$.

Proof of Theorem 2.2:
Suppose that $\mathrm{P}(\theta_t \to \bar{\theta}, f(\bar{\theta}) \neq \bar{\theta}) > 0$. Without loss of generality we assume that $f(\bar{\theta}) > \bar{\theta}$ a.s.

A summation of (2.1) leads to

$$\theta_T - \theta_1 = \sum_{t=0}^{T-1} \gamma_t (f(\theta_t) - \theta_t) + \sum_{t=0}^{T-1} \frac{\gamma_t}{y_t} w_{t+1}. \tag{2.5}$$

By Lemma 2.3 (a) and Assumption (A.1) the second term on the right-hand side converges a.s. The left–hand term converges by assumption on the event $[\theta_t \to \bar{\theta}]$. Hence the first term on the right–hand side has to converge too on this event. But, as it is easy to see, this term diverges towards infinity because of the continuity of the function f and $\sum_0^\infty \gamma_t = \infty$ a.s. Hence we get a contradiction. ■

The interpretation of Theorem 2.2 is obvious: If agents use an expectations formation scheme which fulfilles Assumption (A.1) they cannot learn to form non–rational expectations. These learning procedures are thus consistent with the aim of 'learning to be rational'. Notice that we have not shown that these learning procedures lead always to rational limit expectations. We have only shown that they cannot lead to non–rational limit expectations.

Chapter 3

Stability of Rational Expectations

In the preceding chapter we have shown that the possible limit points of the learning procedures are fixed points of the function f, thus parameters of rational expectations. In this section we analyze which one of these fixed points (which we will identify with the related rational expectations) are possible limit points of the learning procedures. The rational expectations related to such limit points will be called stable with respect to the respectative learning procedure given by the sequence of gains $\{\gamma_t\}$.

We start with the recursive representation of the estimation process

$$\theta_{t+1} = \theta_t + \gamma_t(f(\theta_t) - \theta_t) + \frac{\gamma_t}{y_t} w_{t+1}. \tag{3.1}$$

It is useful to decompose the right-hand side into a *structural adjustment term*,

$$S_t = \gamma_t(f(\theta_t) - \theta_t), \tag{3.2}$$

and an *accidental adjustment term*,

$$A_t = \frac{\gamma_t}{y_t} w_{t+1}. \tag{3.3}$$

There are two necessary conditions for convergence of $\{\theta_t\}$ towards a fixed point θ^*. On the one hand, the accidental adjustment term has to converge towards zero as $t \to \infty$, on the other hand, the structural adjustment term has to point in the direction of θ^* in some neighborhood of θ^*. In this chapter we are concerned with the second condition.

Definition 3.1:
Let θ^* be a fixed point of the function f. We call the largest open connected set

$$D_{\theta^*} \subset \left\{ (\theta, \gamma) \in I\!R \times I\!R_0^+ \,\middle|\, \left((1-\gamma)\theta + \gamma f(\theta) - \theta^*\right)^2 \leq (\theta - \theta^*)^2 \right\} \tag{3.4}$$

such that $(\theta^*, \cdot) \in D_{\theta^*}$ the *attracting region* of θ^*.

It is easy to see that the attracting region D_{θ^*} consists of those pairs (θ_t, γ_t) which lead to a structural adjustment in the direction of θ^*. Moreover, in the deterministic case, thus if $w_t \equiv 0$ for all $t \geq 1$, $(\theta_t, \gamma_t) \in D_{\theta^*}$ implies that $(\theta_{t+1} - \theta^*)^2 \leq (\theta_t - \theta^*)^2$.

Now let $\{\gamma_t\}$ be an admissible sequence of gains. In Chapter 1 we imposed no restrictions on the asymptotic behaviour of $\{\gamma_t\}$, i.e. we allowed $\{\gamma_t\}$ to be decreasing as well as constant, convergent as well as fluctuating in $[0, 1]$. But at this point we have to distinguish several cases.

Case (a): *Convergent sequences.* These squences are allowed to be deterministic as well as stochastic, but the stochastic sequences have to converge a.s. to some fixed limit point $\gamma^* \in (0, 1)$.

Case (b): *Stochastic sequences with path–dependent limit points.* As already mentioned in Chapter 1 we allow the agents to skip between the RPE– and the PE–algorithm in some sense. Now we want to carry out this feature in detail. Recall that the agents believe in the first order autoregressive model

$$y_t = \eta y_{t-1} + e_t \tag{3.5}$$

and try to learn about the unknown parameter η while the true model is given by

$$y_t = f(\theta_{t-1}) y_{t-1} + w_t. \tag{3.6}$$

Unlike as in almost all literature concerned with learning to form rational expectations we do not restrict ourselves to stable data processes, more precisely, to the case of $|f(\theta)| < 1$ for all θ. We allow f to have arbitrary range within the real numbers, hence we do not exclude the case of an unstable data process $\{y_t\}$. If $|f(\theta^*)| > 1$ then the rational expectations equilibrium related to θ^* itself is unstable in the sense that the data process grows approximately geometrically in modulus. We assume that the agents are able to recognize if the time series they observe is generated by a stable or by an unstable process (although, because of the forecast feedback, not by an AR(1)–process). An unstable time series in this sense is characterized by large absolute values of the data process $\{y_t\}$ and, as already mentioned, it is then reasonable for the agents to use the RPE–algorithm. We allow the agents to use different sequences of gains if they observe an unstable time series or a stable one, for example $\gamma_t = 1/t$ if the time series is stable and $\gamma_t \equiv \gamma^* \in (0, 1)$ if the time series is unstable. But we assume that all of these sequences converge.[1] In order to simplify the analysis we assume that, with the exception of the OLS–algorithm, there is only one limit point γ_s^* related to stable data processes and one limit point γ_u^* related to unstable data processes.

Case (c): *Fluctuating stochastic sequences.* These sequences do not necessarily converge, they fluctuate within some fixed (thus not path–depended) range $[\gamma_{min}, \gamma_{max}] \subset [0, 1]$ in a not further specified manner. We assume that $\gamma_{max} = \limsup_t \gamma_t$ a.s.

Now we can introduce our concept of stability of rational expectations with respect to a learning procedure.

[1] The concept of sequences with path–dependend limit points is motivated by the study of the OLS–algorithm. This algorithm has the property that the sequence of gains decreases towards zero if the data process is stable, otherwise it converges towards some positive constant which depends on the fixpoint θ^*.

Definition 3.2:
Let $\{\gamma_t\}$ be an admissible sequence of gains and θ^* a fixpoint of the function f.

(a) Let $\{\gamma_t\}$ be a convergent sequence with $\gamma^* = \lim_t \gamma_t$. Then the fixed point θ^* is called *stable with respect to* $\{\gamma_t\}$ if the attracting region of θ^* is non-trivial in the sense that there exist some $\epsilon > 0, \delta > 0$ such that

$$(\theta^* - \epsilon, \theta^* + \epsilon) \times (\gamma^* - \delta, \gamma^* + \delta) \subset D_{\theta^*} \tag{3.7}$$

if $\gamma^* > 0$ and

$$(\theta^* - \epsilon, \theta^* + \epsilon) \times (0, \delta) \subset D_{\theta^*} \tag{3.8}$$

if $\gamma^* = 0$. Otherwise θ^* is called *unstable.*

(b) Let $\{\gamma_t\}$ be a stochastic sequence of gains with path–dependent limit points γ_s^* and γ_u^*. Then the fixed point θ^* of f is called *stable with respect to* $\{\gamma_t\}$ if (3.7), (3.8), resp. are fulfilled with

$$\gamma^* = \begin{cases} \gamma_u^* & \text{if } |\theta^*| > 1, \\ \gamma_s^* & \text{if } |\theta^*| \leq 1. \end{cases} \tag{3.9}$$

Otherwise it is called *unstable.*

For the OLS–algorithm, characterized by the stochastic sequence of gains $\gamma_t = y_t^2 / \sum_0^t y_\tau^2$, the value $\gamma^* = \gamma^*(\theta^*)$ is defined as

$$\gamma^* = \begin{cases} \frac{\theta^{*2}-1}{\theta^{*2}} & \text{if } |\theta^*| > 1, \\ 0 & \text{if } |\theta^*| \leq 1. \end{cases} \tag{3.10}$$

A fixed point θ^* which is stable with respect to the OLS–algorithm is also called *OLS–stable.*

(c) Let $\{\gamma_t\}$ be a fluctuating stochastic sequence. Then the fixed point θ^* of f is called *stable with respect to* $\{\gamma_t\}$ if there exists some $\epsilon > 0$ such that the attracting region D_{θ^*} of θ^* contains $(\theta^* - \epsilon, \theta^* + \epsilon) \times [\gamma_{min}, \gamma_{max}]$.

It is possible to characterize the stable fixed points by a simple criterion. To do so we need the following

Assumption (A.4):
The function f is differentiable and f' is continuous at the fixed points θ^* of f.

Theorem 3.3:
Suppose that (A.4) holds. Let $\{\gamma_t\}$ be an admissible sequence of gains. A fixed point θ^* of the function f is stable with respect to the learning procedure given by $\{\gamma_t\}$, iff

(i) $1 - \frac{2}{\gamma^*} < f'(\theta^*) < 1$ *if* $\gamma^* > 0$, *and* $f'(\theta^*) < 1$ *if* $\gamma^* = 0$, *or*

(ii) $f'(\theta^*) = 1$ *and $f'(\cdot)$ has a local maximum at θ^*, or*

(iii) $f'(\theta^*) = 1 - \frac{2}{\gamma^*}$, $\gamma^* \neq 0$ *and $f'(\cdot)$ has a local minimum at θ^*,*

with $\gamma^* = \lim_t \gamma_t$ for convergent sequences $\{\gamma_t\}$, $\gamma^* = \gamma^*(\theta^*)$ for sequences $\{\gamma_t\}$ with path–dependent limit points, and $\gamma^* = \gamma_{max}$ for fluctuating sequences $\{\gamma_t\}$.

Proof:
Consider the function

$$
\begin{aligned}
g(\theta,\gamma) &= (1-\gamma)\theta + \gamma f(\theta) - \theta^* \\
&= (1-\gamma)(\theta-\theta^*) + \gamma(f(\theta)-\theta^*).
\end{aligned} \tag{3.11}
$$

with $\gamma > 0$. By the mean value theorem of differential calculus there exists some value ξ between θ and θ^* such that

$$
\begin{aligned}
g(\theta,\gamma) &= (1-\gamma)(\theta-\theta^*) + \gamma f'(\xi)(\theta-\theta^*) \\
&= [1-\gamma(1-f'(\xi))](\theta-\theta^*).
\end{aligned} \tag{3.12}
$$

Hence

$$
\begin{aligned}
(\theta,\gamma) \in D_{\theta^*} &\iff g(\theta,\gamma)^2 \leq (\theta-\theta^*)^2 \\
&\iff |1-\gamma(1-f'(\xi))| \leq 1 \\
&\iff 1-\frac{2}{\gamma} \leq f'(\xi) \leq 1.
\end{aligned} \tag{3.13}
$$

Because $g(\cdot,\cdot)$ is continuous in θ and in γ (at least from above in γ) and $f'(\cdot)$ is continuous at θ^*, it is easy to see that (i)–(iii) are sufficient conditions for stability. Notice that $\tilde{\gamma} < \gamma$ implies $1 - 2/\tilde{\gamma} < 1 - 2/\gamma$, hence $(\theta,\tilde{\gamma}) \in D_{\theta^*}$ if $(\theta,\gamma) \in D_{\theta^*}$. The fixed point θ^* is thus stable with respect to a fluctuating sequence of gains if it is stable with respect to a (convergent) sequence of gains with limit point γ_{max}.

To see that (i)–(iii) are necessary conditions assume, firstly, that $f'(\theta^*) > 1$. This implies that there is some $\epsilon > 0$ such that

$$
f(\theta) > \theta \qquad \forall \theta \in (\theta^*, \theta^* + \epsilon). \tag{3.14}
$$

This implies for $\theta \in (\theta^*, \theta^* + \epsilon)$

$$
(1-\gamma)\theta + \gamma f(\theta) - \theta^* > \theta - \theta^* > 0 \qquad \forall \gamma > 0, \tag{3.15}
$$

hence $(\theta,\gamma) \notin D_{\theta^*} \forall \gamma > 0$ and $\forall \theta \in (\theta^*, \theta^* + \epsilon)$. The attracting region D_{θ^*} is thus trivial in the sense of Definition 3.2.

Secondly, assume that $f'(\theta^*) = 1$ and (ii) does not hold. If $f'(\cdot)$ is increasing at θ or $f'(\cdot)$ has a local minimum in θ^* the same reasoning as above shows that D_{θ^*} is trivial. If $f'(\cdot)$ is decreasing at θ^* there is some $\epsilon > 0$ such that

$$
f(\theta) < \theta \qquad \forall \theta \in (\theta^* - \epsilon, \theta^*) \tag{3.16}
$$

and by this

$$
(1-\gamma)\theta + \gamma f(\theta) - \theta^* < \theta - \theta^* < 0 \qquad \forall \gamma > 0 \quad \forall \theta \in (\theta^* - \epsilon, \theta^*). \tag{3.17}
$$

Again, D_{θ^*} is trivial.

The remaining case, when $f'(\theta^*) = 1 - \frac{2}{\gamma^*}$ and (iii) is violated, can be handled analogously. ■

The concept of stability of rational expectations with respect to a learning procedure presented in this chapter should be treated with care. Stability in this sense does not mean that a stable REE is *always* the limit outcome of the underlying learning procedure. Of course, it may, but it *need not.* Whether or not there is convergence of the learning procedure towards such a REE with positive or unit probability depends also on the initial value θ_0 of the estimation process, the respective path of the disturbance process $\{w_t\}$, and the shape of the function f. Therefore, stability in the sense defined above is only a *local property*. If the estimation process $\{\theta_t\}$ gets sufficiently close to θ^* and θ^* is stable then the structural adjustment term points in direction of θ^*. On the other hand, if θ^* is unstable the structural adjustment term points away from θ^* in every sufficiently small neighborhood of θ^*, hence convergence of $\{\theta_t\}$ towards θ^* is impossible.

Stability of rational expectations could be interpreted economically in the following way. Suppose the whole system is in a stable REE and is pertubated slightly, for example by a slight change of the function f, then agents have a chance of finding back to the (new) REE by using the respective learning procedure.

Theorem 3.3 has another nice interpretation. We can understand the function f (which was first used by MARCET/SARGENT (1989a,b,c) in this way) as a function which transforms the perceived law of motion into the actual (or true) law of motion. It is intuitively appealing that if this function is very sensible with respect to changes of belief, i.e. if $f'(\theta^*) > 1$, it is impossible for the agents to learn the fixed point of this function by a simple trial–and–error mechanism. The reason for this is that in case of $f'(\theta^*) > 1$ the forecast feedback destabilizes the whole system. For example, a small overestimation of θ^* will likely induce a realisation of the data process which in turn gives rise to larger and larger overestimation in the following periods, and so on. Both, the data process $\{y_t\}$ and the estimation process $\{\theta_t\}$ explode with increasing speed in such a situation.

Therefore it is a little surprising that in case of learning procedures with decreasing gains the value of $f'(\theta^*)$ might be arbitrary small and the fixed point is still stable[2]. Clearly, this property hinges upon $f'(\theta^*) < 0$ as well as upon the decreasing gains and is easily understood graphically together with (3.1).

There exist other concepts of stability of rational expectations with respect to a learning procedure in the related literature. Mostly a REE is called stable if there is (real time) convergence of the respective learning procedure towards it. But there exist also virtual concepts in which the expectations formation scheme is iteratively applied onto only one data point. More exactly and translated into our setup, the expectations formation scheme is applied onto a fixed value of y_t and the resulting expectations, together with, again, y_t are the input for the next application of the expectations formation scheme. If this procedure leads to the REE in the limit then it is called stable. Of course, this is not real time convergence and does not take into account the sometimes unpleasant feature of forecast feedback. The studies of DECANIO (1979) and EVANS (1983), for example, use this virtual stability concept.

[2] This result is new for generalized autoregressive models with forecast feedback under a learning procedure. MARCET/SARGENT (1989a,b,c) who analyzed autoregressive models showed OLS–stability only under a condition which corresponds to $|f'(\theta^*)| < 1$ in our model. Nevertheless, CYERT/DEGROOT (1974) observed this feature within their Monte–Carlo study.

Chapter 4

Convergence Results

In this chapter we show some global convergence results. Our main tool is the following convergence theorem for almost–supermartingales.

Proposition 4.1: (Robbins/Siegmund (1971))
Suppose that $\{z_t\}, \{\alpha_t\}, \{\beta_t\}$ and $\{\eta_t\}, t \in \mathbb{N}$ are sequences of non–negative random variables, adapted to a filtration $\{\mathcal{F}_t\}$ such that

$$\mathrm{E}[z_{t+1}|\mathcal{F}_t] \leq (1+\alpha_t)z_t + \beta_t - \eta_t \qquad a.s. \tag{4.1}$$

Then on the event $[\sum_1^\infty \alpha_t < \infty, \sum_1^\infty \beta_t < \infty]$ $\lim_{t\to\infty} z_t$ exists and is finite a.s. and $\sum_1^\infty \eta_t < \infty$ a.s.

To exclude some pathological cases we introduce the following

Assumption (A.5)
The function f possesses only isolated fixed points, i.e. for every fixed point θ^* there exists some $\epsilon > 0$ such that $f(\theta) \neq \theta \quad \forall \theta \in (\theta^* - \epsilon, \theta^* + \epsilon)$.

The first theorem shows that the attracting region D_{θ^*} as defined in the preceding chapter has, indeed, a property usually attributed to attracting regions or attractors in the theory of differential equations. If the estimation process $\{\theta_t\}$, more exactly, the *augmented estimation process* $\{(\theta_t, \gamma_t)\}$, stays in the attracting region after some finite time then θ_t converges towards θ^*. Because D_{θ^*} was defined as open and we want to exclude convergence towards a boundary point we consider the following subsets

$$D^\alpha_{\theta^*} = \{(\theta,\gamma) \in D_{\theta^*} \mid |\theta - \eta| \geq \alpha \quad \forall (\eta, \cdot) \in D^c_{\theta^*}\} \tag{4.2}$$

with $\alpha > 0$ and $D^c_{\theta^*}$ the complement of D_{θ^*}.

Theorem 4.2:
Suppose that (A.1)–(A.3) and (A.5) hold. Let $\{\gamma_t\}$ be an admissible sequence of gains and θ^* a stable fixed point of the function f with respect to $\{\gamma_t\}$. Then for every $\alpha > 0$

$$\theta_t \longrightarrow \theta^* \qquad a.s.\ on \qquad [(\theta_t, \gamma_t) \in D^\alpha_{\theta^*} \text{ for all but finitely many } t \geq 0]. \tag{4.3}$$

Proof:
The estimation process $\{\theta_t\}$ possesses the recursive representation

$$\theta_{t+1} - \theta^* = (1-\gamma_t)\theta_t + \gamma_t f(\theta_t) - \theta^* + \frac{\gamma_t}{y_t} w_{t+1}. \tag{4.4}$$

Squaring and taking conditional expectations leads to

$$\mathrm{E}[(\theta_{t+1} - \theta^*)^2 | \mathcal{F}_t] = [(1-\gamma_t)\theta_t + \gamma_t f(\theta_t) - \theta^*]^2 + \left(\frac{\gamma_t}{y_t}\right)^2 \mathrm{E}[w_{t+1}^2 | \mathcal{F}_t]. \tag{4.5}$$

Define now $z_t = (\theta_t - \theta^*)^2, \beta_t = \sigma^2\gamma_t^2/y_t^2,\ \eta_t \equiv 0$ and α_t according to

$$\alpha_t = \begin{cases} 0 & \text{if } (\theta_t, \gamma_t) \in D_{\theta^*} \\ \dfrac{[(1-\gamma_t)\theta_t + \gamma_t f(\theta_t) - \theta^*]^2}{(\theta_t - \theta^*)^2} - 1 & \text{if } (\theta_t, \gamma_t) \notin D_{\theta^*}, \end{cases} \tag{4.6}$$

then we have

$$\mathrm{E}[z_{t+1} | \mathcal{F}_t] \le (1+\alpha_t) z_t + \left(\frac{\gamma_t}{y_t}\right)^2 \sigma^2. \tag{4.7}$$

By Proposition 4.1 we have a.s. convergence of $\{z_t\}$ on the event $[\sum_0^\infty \alpha_t < \infty, \sum_0^\infty \beta_t < \infty]$. Because $\{\gamma_t\}$ is an admissible sequence of gains $\sum_0^\infty \beta_t < \infty$ a.s. The event $[\sum_0^\infty \alpha_t < \infty]$ contains the event $[\alpha_t = 0$ for all but finitely many $t \ge 0] = [(\theta_t, \gamma_t) \notin D_{\theta^*}^\alpha$ for only finitely many $t \ge 0]$.

By Theorem 2.2 we know that the only possible limit points of the estimation process are the fixed points of f. If f possesses only one fixed point θ^* the proof is complete. Otherwise we have to ensure that the estimation process $\{\theta_t\}$ does not converge towards another fixed point $\bar\theta$. The following lemma shows that the attracting regions of different fixed points are disjunct. By construction $\{\theta_t\}$ can only converge towards an interior point of D_{θ^*} if $\theta_t \in D_{\theta^*}^\alpha$ for all but finitely many $t \ge 1$. Hence $\{\theta_t\}$ converges a.s. towards θ^* on the event $\sum_0^\infty \alpha_t < \infty$. ■

Lemma 4.3:
Suppose that (A.3) and (A.5) hold. Let θ_1^* and θ_2^* be two distinct fixed points of the function f. Then $D_{\theta_1^*} \cap D_{\theta_2^*} = \emptyset$.

Proof:
The fixed points θ^* of f are given graphically as the intersection points of the graph of f with the 45^o–ray. Because of (A.5) and continuity of f there are alternately up- and down-crossing intersection points. For the up-crossing intersection points θ^* we have $f'(\theta^*) > 1$, hence these fixed points are unstable and $D_{\theta^*} = \emptyset$.

Suppose now that $\theta_1^* < \theta_3^*$ are two stable fixed points. Then there is another fixed point $\theta_2^* \in (\theta_1^*, \theta_3^*)$ which is unstable. Because the attracting regions D_{θ^*} are defined as open $f'(\theta_2^*) > 1$ implies that $(\theta_2^*, \gamma) \notin D_{\theta_1^*}$ and $(\theta_2^*, \gamma) \notin D_{\theta_3^*}$ for all $\gamma > 0$. But this implies that $D_{\theta_1^*} \cap D_{\theta_3^*} = \emptyset$ because the attracting regions are defined as connected. ■

Notice that the scope of Theorem 4.2 is limited. It is a satisfactory result only if the function f fulfills a global regularity condition like

$$(f(\theta) - \theta^*)^2 \le (\theta - \theta^*)^2 \qquad \forall \theta \in I\!R. \tag{4.8}$$

Corollary 4.4:
Suppose that (A.1)–(A.3) and (A.5) hold. Let θ^* be a stable fixed point of the function f with respect to an admissible sequence of gains $\{\gamma_t\}$ such that (4.8) holds. Then $\theta_t \rightarrow \theta^*$ a.s.

Proof:
It is easy to see that (4.8) implies $\mathbb{R} \times (0,1) \subset D_{\theta^*}$ and $D^{\alpha}_{\theta^*} = D_{\theta^*}$. ■

Example 4.5: (Additive Expectations)
Consider the linear model with additive expectations one period ahead

$$y_{t+1} = \phi y_t + a y^e_{t+1} + w_{t+1} \tag{4.9}$$

with $a \neq 0$ and $\phi \neq 0$. Thus $f(\theta) = \phi + a\theta$ and f has a unique fixed point $\theta^* = \frac{\phi}{1-a}$ if $a \neq 1$. Then (4.8) holds whenever $-1 \leq a < 1$, hence $\theta_t \rightarrow \theta^*$ a.s. (This result was already achieved by Kottmann/Kuliberda (1990) for $|a| < 1$.)

For a learning procedure given by a sequence of gains which converges towards zero a.s. we can get a stronger result.

Proposition 4.6:
Suppose that (A.1)–(A.3) and (A.5) hold. Let θ^* be a stable fixed point of the function f with respect to an admissible sequence of gains $\{\gamma_t\}$ for which $\lim_t \gamma_t = 0$ a.s. If

$$M \leq f'(\theta) \leq 1 \qquad \forall \theta \in \mathbb{R} \tag{4.10}$$

for some negative constant M then $\theta_t \rightarrow \theta^*$ a.s.

Proof:
As in the proof of Theorem 4.2 we have

$$\mathrm{E}\left[(\theta_{t+1} - \theta^*)^2 \,|\mathcal{F}_t\right] \leq [1 - \gamma_t(1 - f'(\xi_t))]^2(\theta_t - \theta^*)^2 + \frac{\gamma_t^2}{y_t^2}\sigma^2. \tag{4.11}$$

We define $\alpha_t = \max\{[1 - \gamma_t(1 - f'(\xi_t))]^2 - 1, 0\}$. Because of (4.10) we have

$$1 - \gamma_t(1 - M) \leq 1 - \gamma_t(1 - f'(\xi_t)) \leq 1, \tag{4.12}$$

and because $\gamma_t \rightarrow 0$ a.s. we can conclude that $\alpha_t = 0$ for all but finitely many $t \geq 0$. Then Proposition 4.1 gives the desired result. ■

Now consider the example of additive expectations two periods ahead. The data process is thus given by

$$y_{t+1} = \phi y_t + a y^e_{t+2} + w_{t+1} \qquad , \qquad t \geq 0 \tag{4.13}$$

It is easily seen that $f(\theta) = \phi + a\theta^2$ and the condition (4.8) is not fulfilled. In such a situation Theorem 4.2 is not very helpful because it states only 'that everything is fine if nothing goes wrong'. The estimation process converges towards θ^* if the augmented estimation process $\{(\theta_t, \gamma_t)\}$ remains inside the attracting region D_{θ^*} after some finite time (supposed that θ^* is stable). But does the estimation process actually stays in this region forever after a transient phase, at least with positive probability? The next result gives a first answer.

Theorem 4.7:
Suppose that (A.1)–(A.3) and (A.5) hold and $\{\gamma_t\}$ is an admissible sequence of gains such that $\mathrm{E}[\sum_0^\infty \gamma_t^2/y_t^2] < \infty$. Let θ^* be a stable fixed point of the function f with respect to $\{\gamma_t\}$ with $|f'(\theta^*)| < 1$. Then

$$\mathrm{P}\left[(\theta_t, \gamma_t) \in D_{\theta^*} \text{ for all but finitely many } t \geq 1\right] > 0 \tag{4.14}$$

provided that $(\theta_0, 1) \in D_{\theta^*}$.

Proof:
The main idea of the proof is quite simple. We know that inside the attracting region D_{θ^*} the structural adjustment term points in the direction of θ^*, hence, if $\{\theta_t\}$ has reached the attracting region it will stay there unless the accidental adjustment term forces $\{\theta_t\}$ to leave it. We show that this happens only with a probability less one. Nevertheless the technique used is somewhat involved. We proceed in a step-by-step manner.

1. Let $B_1 \subset I\!R$ be the largest connected set such that

$$(f(\theta) - \theta^*)^2 \leq (\theta - \theta^*)^2 \qquad \text{and} \qquad f(\theta) \in B_1 \quad \forall \theta \in B_1. \tag{4.15}$$

Because of $|f'(\theta^*)| < 1$ the set B_1 is not empty. Notice that (4.15) implies

$$\left[(1-\gamma)\theta + \gamma f(\theta) - \theta^*\right]^2 \leq (\theta - \theta^*)^2 \qquad \forall \theta \in B_1, \forall \gamma \in [0,1]. \tag{4.16}$$

It is obvious that $B_1 \times [0,1] \subset D_{\theta^*}$, hence $[(\theta_t, \gamma_t) \notin D_{\theta^*}] \subset [\theta_t \notin B_1]$. If B_1 is not bounded then $D_{\theta^*} = I\!R \times [0,1]$ and $\mathrm{P}(\theta_t \notin B_1) = 0$. Therefore we assume B_1 as bounded, hence $B_1 = [b_l, b_r]$ with $-\infty < b_l < b_r < \infty$. It is thus sufficient to show that

$$\mathrm{P}(\theta_t \in B_1 \text{ for all but finitely many } t \geq 1) > 0. \tag{4.17}$$

2. Now fix some $T \in I\!N$, then (4.17) is equivalent to

$$\mathrm{P}(\theta_{T+t} \in B_1 \text{ for all but finitely many } t \geq 1) > 0. \tag{4.18}$$

We denote the event in (4.18) by $\mathcal{A}$. Let $B_0 = [b_l + \epsilon, b_r - \epsilon]$ with some appropriate $\epsilon > 0$ such that $\emptyset \neq B_0 \subset B_1$ and $f(\theta) \in B_0$ if $\theta \in B_0$. Such a set B_0 exists, again because of $|f'(\theta^*)| < 1$. Then

$$\mathrm{P}(\mathcal{A}) = \mathrm{P}(\theta_T \in B_0)\mathrm{P}(\mathcal{A} \,|\, \theta_T \in B_0) + \mathrm{P}(\theta_T \notin B_0)\mathrm{P}(\mathcal{A} \,|\, \theta_T \notin B_0). \tag{4.19}$$

Hence it is sufficient to show $\mathrm{P}(\theta_T \in B_0) > 0$ and $\mathrm{P}(\mathcal{A} \,|\, \theta_T \in B_0) > 0$.

3. Because of

$$\begin{aligned}
&\mathrm{P}(\theta_{T+t} \in B_1 \text{for all but finitely many } t \geq 1 \,|\, \theta_T \in B_0) \\
&\quad \geq \mathrm{P}(\theta_{T+t} \in B_1 \forall t \geq 1 \,|\, \theta_T \in B_0) \\
&\quad = 1 - \mathrm{P}(\theta_{T+t} \notin B_1 \text{ for some } t \geq 0 \,|\, \theta_T \in B_0) \\
&\quad = 1 - \mathrm{P}\left(\bigcup_{N \geq 1} [\theta_{T+k} \notin B_1 \text{ for some } 1 \leq k \leq N] \,\middle|\, \theta_T \in B_0 \right) \\
&\quad = 1 - \lim_{N \to \infty} \mathrm{P}(\theta_{T+k} \notin B_1 \text{ for some } 1 \leq k \leq N \,|\, \theta_T \in B_0)
\end{aligned} \tag{4.20}$$

it is sufficient to show that

$$\mathrm{P}(\theta_{T+k} \notin B_1 \text{ for some } 1 \leq k \leq N \,|\, \theta_T \in B_0) < \delta < 1 \tag{4.21}$$

uniformly in N with some $\delta \in (0,1)$.
4. Now consider the estimation process

$$\begin{aligned} \theta_{t+1} &= (1-\gamma_t)\theta_t + \gamma_t f(\theta_t) + \frac{\gamma_t}{y_t} w_{t+1} \\ &= \theta_t + S_t + A_t \end{aligned} \tag{4.22}$$

with S_t the structural and A_t the accidental adjustment term. Now $\theta_t \in B_i$ implies $f(\theta_t) \in B_i$, hence $\theta_t + S_t = (1-\gamma_t)\theta_t + \gamma_t f(\theta_t) \in B_i$ by construction of B_i, $i = 0, 1$. The estimation process $\{\theta_t\}$ thus leaves the sets B_i only if it is forced to do by the accidental adjustment term. More precisely

$$\left[\theta_{T+k} \notin B_1 \text{ for some } 1 \le k \le N \;\middle|\; \theta_T \in B_0\right] \subset \left[\sup_{1\le k\le N} \left| \sum_{t=T}^{T+N-1} A_t \right| > \epsilon \right]. \tag{4.23}$$

5. Notice that A_t is a martingale difference sequence with respect to the filtration $\{\mathcal{F}_t\}$ because of

$$\begin{aligned} \mathrm{E}[A_{t+1}|\mathcal{F}_t] &= \mathrm{E}\left[\frac{\gamma_t}{y_t} w_{t+1}|\mathcal{F}_t\right] \\ &= \frac{\gamma_t}{y_t}\mathrm{E}\left[w_{t+1}|\mathcal{F}_t\right] \\ &= 0. \end{aligned} \tag{4.24}$$

Hence the sequence $\{Q_n^T\}_{n\ge 1}$ with $Q_n^T = \sum_{t=T}^{T+n-1} A_{t+1}$ is a martingale with respect to the shifted filtration $\{\tilde{\mathcal{F}}_n\}_{n\ge1} = \{\mathcal{F}_{T+n}\}_{n\ge1}$ and $\{|Q_n^T|\}$ is a (non-negative) submartingale. The Chebyshev inequality and Doob's maximal inequality give

$$\begin{aligned} \mathrm{P}\left(\sup_{1\le k\le N} |Q_k^T| > \epsilon\right) &\le \frac{1}{\epsilon^2}\mathrm{E}\left[\sup_{1\le k\le N}|Q_n^T|\right]^2 \\ &\le \frac{4}{\epsilon^2}\mathrm{E}\left[|Q_N^T|^2\right] \\ &= \frac{4}{\epsilon^2}\mathrm{E}\left[\sum_{t=T}^{T+N-1} \frac{\gamma_t^2}{y_t^2} w_{t+1}^2\right]. \end{aligned} \tag{4.25}$$

The equality holds because the cross terms disappear as can be seen as follows. Let $s < t$ then

$$\begin{aligned} \mathrm{E}\left[\frac{\gamma_s}{y_s} w_{s+1} \frac{\gamma_t}{y_t} w_{t+1}\right] &= \mathrm{E}\left[\mathrm{E}\left[\frac{\gamma_s}{y_s}\frac{\gamma_t}{y_t} w_{s+1} w_{t+1} \middle| \mathcal{F}_t\right]\right] \\ &= \mathrm{E}\left[\frac{\gamma_s}{y_s}\frac{\gamma_t}{y_t} w_{s+1} \mathrm{E}\left[w_{t+1}\,|\mathcal{F}_t\right]\right] \\ &= 0. \end{aligned} \tag{4.26}$$

We proceed in calculating

$$\begin{aligned} \frac{4}{\epsilon^2}\mathrm{E}\left[\sum_{t=T}^{T+N-1} \frac{\gamma_t^2}{y_t^2} w_{t+1}^2\right] &= \frac{4}{\epsilon^2}\sum_{t=T}^{T+N-1} \mathrm{E}\left[\frac{\gamma_t^2}{y_t^2} w_{t+1}^2\right] \\ &= \frac{4}{\epsilon^2}\sum_{t=T}^{T+N-1} \mathrm{E}\left[\mathrm{E}\left[\frac{\gamma_t^{2\,2}}{y_t^2} w_{t+1}^2 \middle| \mathcal{F}_t\right]\right] \\ &\le \frac{4\sigma^2}{\epsilon^2}\sum_{t=T}^{T+N-1} \mathrm{E}\left[\frac{\gamma_t^2}{y_t^2}\right] \end{aligned} \tag{4.27}$$

$$
\begin{aligned}
&= \frac{4\sigma^2}{\epsilon^2}\mathrm{E}\left[\sum_{t=T}^{T+N-1}\frac{\gamma_t^2}{y_t^2}\right] \\
&\leq \frac{4\sigma^2}{\epsilon^2}\mathrm{E}\left[\sum_{t=T}^{\infty}\frac{\gamma_t^2}{y_t^2}\right]
\end{aligned}
$$

and this holds uniformly in N.

6. By assumption the last expectation in (4.27) is finite. By the monotone convergence theorem we get

$$
\lim_{T\to\infty} \mathrm{E}\left[\sum_{t=T}^{\infty}\frac{\gamma_t^2}{y_t^2}\right] = 0, \tag{4.28}
$$

hence for $\delta = \frac{\epsilon^2}{8\sigma^2}$ there exists a number $T = T(\delta)$ such that

$$
\mathrm{E}\left[\sum_{t=S}^{\infty}\frac{\gamma_t^2}{y_t^2}\right] < \delta \qquad \forall S \geq T(\delta). \tag{4.29}
$$

This implies that for all $S \geq T$

$$
\mathrm{P}\left(\sup_{1\leq k\leq N} |Q_k^S| > \epsilon\right) \leq \frac{4\sigma^2}{\epsilon^2}\frac{\epsilon^2}{8\sigma^2} = \frac{1}{2}, \tag{4.30}
$$

hence (now with $T = T(\delta)$)

$$
\mathrm{P}(\theta_{T+k} \in B_1 \forall k \geq 1 | \theta_T \in B_0) > 0. \tag{4.31}
$$

7. Because T is a finite number it can be shown that $P(\theta_T \in B_0) > 0$. (Because $\{\theta_t\}$ starts with $\theta_0 \in B_1$ and the algorithm shows a tendency to stay in B_1 we only have to show that the disturbance terms $w_1, \ldots, w_{T-1}$ do not force $\{\theta_t\}$ to leave B_1 with positive probability.) But this finally implies $\mathrm{P}(\theta_t \in B_1$ for all but finitely many $t \geq 1) > 0$. ■

Because the same result holds if we replace D_{θ^*} by $D_{\theta^*}^{\alpha}$ with $\alpha > 0$ sufficiently small Theorem 4.7 together with Theorem 4.2 give

Corollary 4.8:
Suppose that (A.1)–(A.3) and (A.5) hold. Let $\{\gamma_t\}$ be an admissible sequence of gains with $\mathrm{E}\left[\sum_1^\infty \frac{\gamma_t^2}{y_t^2}\right] < \infty$ and θ^* a stable fixed point of f with respect to $\{\gamma_t\}$ such that $|f'(\theta^*)| < 1$. Then $\mathrm{P}(\theta_t \to \theta^*) > 0$ provided that $(\theta_0, 1) \in D_{\theta^*}$.

Now it remains to show that in the situation of Theorem 4.7 there is convergence only with a probability less than one. Without further assumptions on the shape of the function f or on the disturbance terms this seems difficult to realize. Instead we give a few remarks to enlight what can happen in particular cases.

1. Suppose that the function f is U–shaped with two distinct intersection points $\theta_l^* < \theta_r^*$ with the 45^o–ray. Clearly $f'(\theta_r^*) > 1$, hence this fixed point is unstable with respect to every admissible sequence of gains and the structural adjustment in every neighborhood of θ_r^* points away from θ_r^*.

If θ_l^* is stable then the attracting region of θ_l^* is bounded above in the θ–coordinate by θ_r^* for every $\gamma \in (0,1)$. With the same technique as in the proof of Theorem 4.7 it is possible to show that

$$\mathrm{P}(\theta_{T+t} > \theta_r^* \quad \forall t \geq 1 | \theta_T > \theta_r^*) > 0. \tag{4.32}$$

If we can ensure that, for example, $\mathrm{P}(\theta_l^* < \theta_T < \theta_r^*) > 0$ then we get $\mathrm{P}(\theta_t \to \infty) > 0$.

2. More generally, whenever there is a connected set C such that $\theta \in C$ implies $f(\theta) \in C$ the estimation process shows the tendency to stay in C and only suitable accidental adjustment terms can force $\{\theta_t\}$ to leave this set. In such a situation we can use the technique as above to show that the estimation process actually stays in C with positive probability.

3. If we allow the disturbance terms to be unbounded, for example if we assume $\{w_t\}$ to be a sequence of independent Gaussian distributed random variables, we do not need any initial condition for θ_0 because only one large realisation of the disturbance term is sufficient for that the estimation process reaches or leaves a bounded set C with positive probability.

4. If there are two or more stable fixed points it is possible that the estimation process moves from one attracting region into anotherone with positive probability. We conjecture that this happens only finite times but we do not know how to show this.

5. The probability by which the estimation process stays in a particular region depends on the shape of the function f, which determines the size of the attracting regions, but also on the long–term behaviour of the data process $\{y_t\}$. If the data process is unstable, thus geometrically increasing in modulus, then the influence of the accidental adjustment term decreases rapidely. Various Monte–Carlo runs, reported in ZENNER (1992b), have shown that in such cases the transient phase in which the estimations process eventually leaves the attracting region after starting there is relatively short — say about 10 periods — and the estimation process converges quite rapidely.

   On the other hand, if the data process is stable, for example if the estimation process stays in a region which implies $|f(\theta_t)| \leq 1$, the transient phase is relatively long. We faced situations in which the path of the estimation process seemed to stabilyze but exploded suddenly after more than 300 periods, due only to some 'unpleasant' disturbances.

   Of course, the behaviour of the estimation process depends also on the initial value θ_0. An initial value far–away from the attracting region reduces the probability by which converge occurs.

   All these findings are only of a qualitative nature but it seems to be impossible to calculate the exact probability values even in particular cases.

At this point we have to come back to the problem of an admissible choice of the sequence of gains. We started our investigations with the OLS–algorithm because of two reasons. Firstly, if we assume that the agents believe in a first order autoregressive model it is 'rational' for them to estimate the unknown parameter by least squares.[1] In addition, least squares estimaton is perhaps the only estimation technique known also by non–statisticians and it is implemented in almost all

[1] Notice that the OLS–esitimator is the best linear unbiased estimator for a large class of linear regression models.

statistical software program packages. Hence, if we assume that the agents learning procedure can be modeled by a parameter estimation technique the OLS–procedure is the first choice. Secondly, the OLS–estimation technique is the one best known and understood from the mathematical point of view. Again, it is reasonable to start the investigation with the OLS–procedure.

On the other hand, the recursive representation of the OLS–estimators leads in a quite natural way to the investigation of more general first order adaptive estimation schemes which can be understood as RPE– or PE–algorithms. But to ensure that these schemes have similar analytical properties it is sometimes necessary to impose restrictions onto the admissible choice of the sequence of gains which are not always easy to verify. For example, the condition for the sequence of gains in Theorem 4.7, $\mathrm{E}[\sum_0^\infty \gamma_t^2/y_t^2] < \infty$, is fulfilled for the gains of the OLS–procedure whenever $y_0^2 > \mu > 0$ a.s. for some μ (see Lemma 1.1).

As we have seen it is a serious problem to show global stability of rational expectations with respect to a learning procedure. To do so, we need global regularity assumptions for the function f. These assumptions have to ensure that an 'explosion' of the estimation process is impossible[2]. Such assumptions have to ensure that at least $\lim_{\theta \to +/-\infty} |f(\theta)/\theta| < 1$. Thus if f is a polynomial there is always the risk of explosion.

There exists another possibility to avoid the problem of explosion used by Marcet/Sargent (1989a,b,c). They used a 'projection facility' due to Ljung(1977) which supresses the risk of explosion. Within their setup agents ignore values of their (OLS-) estimation process which lie outside an exogenously and a-priori given set. Instead of using these implausible estimates they choose in a not further specified manner values in a 'most likely region' around θ^* for their expectations formation and, after that, continue with their learning procedure as before. In this way Marcet/Sargent got a.s. convergence of the estimation process.

Such a modelling is appropriate in an engineering context in which the physical system described by the data process $\{y_t\}$ is fairly well understood and the range in which the unknown parameter θ^* lies is known or physically determined. But in the context of learning agents this modelling is not very plausible. If we assume that agents follow a quite mechanical learning rule it is paradoxial that agents reject the outcomes of their learning rule but *do not* reject the learning rule which produces these outcomes. In addition, agents have to know the 'most likely region' more or less exactly for that the projection facility works[3].

But the analysis of Marcet/Sargent has another shortcoming because it excludes unstable, thus geometrically increasing data processes[4]. Again, this restriction stems from the engineering context for which Ljung's theory was developed. In engineering applications growing systems are quite rare but in economics, especially in macroeconomics, they are not.

---

[2] By explosion we mean divergence of the estimation process caused by the feedback mechanism. For example, if the function f is shaped such that overestimation of the rational expectations parameter θ^*, $\theta_t > \theta^*$, leads to underestimation of the actual law of motion, $\theta_t < f(\theta_t)$, such a feedback to cause explosion is likely to occur.

[3] Notice that the 'most likely region' is the attracting region of a system of differential equations which describe the asymptotic behaviour of the estimation process. For details see Chang et al. (1991)

[4] The same shortcoming appears also in the analysis of Bray/Savin (1986), Fourgeaud et al. (1986), Kottmann (1990), and many others because all these studies base their convergence analysis on some kind of law of large numbers.

Chapter 5

Rate of Convergence

In this section we want to determine the rate of convergence of the estimation process. The rate of convergence is not only of interest as a purely theoretical result but also in view of the plausibility of the learning produres. Within our setup we suppose that agents use the same learning and prediction rule for a long time, in fact, forever. But this is plausible only if agents have a good reason not to change this rule, in particular, only if they can recognize a success in learning. Because agents do not know the true parameter θ^* of the model, they can not observe the success of their learning procedure directly. They can only observe whether their estimates θ_t stabilize or not. If they can recognize convergence by looking at the history of their estimation process they have an incentive to keep on using this rule, if not they will likely look for another one.

It turns out that the rate of convergence depends as well on the respective learning procedure as on the long-term behaviour of the data process $\{y_t\}$. If this process explodes the rate of convergence is much higher than in case of a stable data process. This phenomenon is very similar to the superefficiency of the least squares estimator in the case of an exploding autoregression.

Theorem 5.1:
Suppose that (A.1)–(A.4) hold. Let θ^* be a fixed point of the function f with respect to an admissible sequence of gains $\{\gamma_t\}$. Let $\{\pi_t\}$ be a sequence of positive random variables such that on the event $[\theta_t \to \theta^*]$

$$\text{(5.1)} \qquad \frac{\pi_{t+1}^2}{\pi_t^2}\left[(1-\gamma_t)(\theta_t-\theta^*)+\gamma_t(f(\theta_t)-\theta^*)\right]^2 \le 1+\alpha_t \qquad a.s.$$

with $\{\alpha_t\}$ a sequence of non–negative random variables with $\sum_1^\infty \alpha_t < \infty$ a.s. on $[\theta_t \to \theta^*]$ and

$$\text{(5.2)} \qquad \sum_{t=1}^{\infty} \frac{\gamma_t^2}{y_t^2}\pi_{t+1}^2 < \infty \qquad a.s.$$

Then $\theta_t - \theta^* = O(\pi_t^{-1})$ a.s. on the event $[\theta_t \to \theta^*]$.

Proof:
The estimation process $\{\theta_t\}$ is recursively given by

$$\theta_{t+1} - \theta^* = [1 - \gamma_t (1 - f'(\xi_t))] (\theta_t - \theta^*) + \frac{\gamma_t}{y_t} w_{t+1} \tag{5.3}$$

with ξ_t between θ_t and θ^* given by the mean value theorem of differential calculus. Squaring and taking conditional expectations leads to

$$\mathrm{E}[(\theta_{t+1} - \theta^*)^2 | \mathcal{F}_t] \leq [1 - \gamma_t (1 - f'(\xi_t))]^2 (\theta_t - \theta^*)^2 + \sigma^2 \frac{\gamma_t^2}{y_t^2} \tag{5.4}$$

and multiplication with π_{t+1}^2 to

$$\begin{aligned} \mathrm{E}[\pi_{t+1}^2(\theta_{t+1} - \theta^*)^2 | \mathcal{F}_t] &\leq \frac{\pi_{t+1}^2}{\pi_t^2} [1 - \gamma_t (1 - f'(\xi_t))]^2 \pi_t^2(\theta_t - \theta^*)^2 + \sigma^2 \frac{\gamma_t^2}{y_t^2} \pi_{t+1}^2 \\ &\leq (1 + \alpha_t) \pi_t^2 (\theta_t - \theta^*)^2 + \sigma^2 \frac{\gamma_t^2}{y_t^2} \pi_{t+1}^2. \end{aligned} \tag{5.5}$$

By Proposition 4.1 $\{\pi_t^2(\theta_t - \theta^*)^2\}$ converges a.s. towards some finite limit on the event $[\theta_t \to \theta^*]$, hence $\theta_t - \theta^* = O(\pi_t^{-1})$. ■

Now we are looking for squences $\{\pi_t\}$ such that (5.1), (5.2) are fulfilled for a given sequence of gains $\{\gamma_t\}$ and a given fixed point θ^*. Of course, the admissible choice of $\{\pi_t\}$ depends as well on $\{\gamma_t\}$ as on θ^*. We will distinguish between $|\theta^*| > 1$ and $|\theta^*| \leq 1$, and between sequences $\{\gamma_t\}$ with $\gamma_t \to \gamma^* > 0$ and $\gamma_t \to 0$. We begin with $|\theta^*| > 1$, the case of an exploding data process $\{y_t\}$.

Lemma 5.2:
Suppose that (A.1)–(A.4) hold. Let θ^* be a fixed point of f which is stable with respect to the admissible sequence of gains $\{\gamma_t\}$ such that $|\theta^*| > 1$. Then $\rho^{2t} = O(y_t^2)$ a.s. as $t \to \infty$ on the event $[\theta_t \to \theta^*]$ for every ρ such that $1 < |\rho| < |\theta^*|$.

Proof:
We will only present the main idea of the proof without working out all the details. Define $\tilde{y}_t := \rho^{-t} y_t$ with some fixed ρ such that $1 < |\rho| < |\theta^*|$. Then we have

$$\tilde{y}_{t+1} = \frac{f(\theta_t)}{\rho} \tilde{y}_t + \rho^{-(t+1)} w_{t+1}. \tag{5.6}$$

Because $|\rho| > 1$ the scale of the disturbance terms $\tilde{w}_t := \rho^{-t} w_t$ decreases geometrically. Hence, for large t we have

$$\tilde{y}_{t+1} \approx \frac{f(\theta_t)}{\rho} \tilde{y}_t. \tag{5.7}$$

Now consider a trajectory $\{\tilde{y}_t(\omega)\}$ with $\omega \in [\theta_t \to \theta^*]$. Because $|\rho| < |\theta^*| = |f(\theta^*)|$ and continuity of f there exists a finite number $N = N(\omega)$ such that

$$\left| \frac{f(\theta_t)}{\rho} \right| > 1 + \epsilon \qquad \forall t \geq N \tag{5.8}$$

with some $\epsilon > 0$. The trajectory $\{\tilde{y}_t(\omega)\}$ thus explodes geometrically as $t \to \infty$. Hence $\rho^{2t} = O(y_t^2(\omega))$ as $t \to \infty$. ■

Theorem 5.3:
Suppose that (A.1)–(A.4) hold. Let θ^* be a fixed point of f with $|\theta^*| > 1$ which is stable with respect to the admissible sequence of gains $\{\gamma_t\}$ with $\gamma_t \to \gamma^*$ a.s. and $\gamma^* > 0$. Then on the event $[\theta_t \to \theta^*]$

$$(5.9) \qquad \theta_t - \theta^* = O\left(\frac{t^{\frac{1}{2}+\delta}}{\rho^t}\right) \qquad a.s.$$

for all $\delta > 0$ and all $\rho > 1$ such that

$$(5.10) \qquad \rho < \min\left\{|\theta^*|, \frac{1}{|1-\gamma^*(1-f'(\theta^*))|}\right\}$$

provided that $1 - \frac{2}{\gamma^*} < f'(\theta^*) < 1$.

Proof:
Because $1-\frac{2}{\gamma^*} < f'(\theta^*) < 1$ is equivalent to $|1-\gamma^*(1-f'(\theta^*))| < 1$ there are values for ρ which fulfill (5.10).

Now set $\pi_t^2 = \rho^{2t}/t^{1+2\delta}$. Then

$$(5.11) \qquad \begin{aligned} &\frac{\pi_{t+1}^2}{\pi_t^2}\left[(1-\gamma_t)(\theta_t-\theta^*) + \gamma_t(f(\theta_t)-\theta^*)\right]^2 \\ &\qquad = \rho^2\left(\tfrac{t}{t+1}\right)^{1+2\delta}\left[1-\gamma_t(1-f'(\xi_t))\right]^2 \end{aligned}$$

with ξ_t between θ_t and θ^* given by the mean value theorem of differential calculus. Now let $\omega \in [\theta_t \to \theta^*, \gamma_t \to \gamma^*]$. The following considerations apply for such an ω. Because $\gamma_t(\omega) \to \gamma^*$ and $\theta_t(\omega) \to \theta^*$ and $f'(\cdot)$ is continuous in θ^* the term $1-\gamma_t(1-f'(\xi_t))(\omega)$ converges towards $1-\gamma^*(1-f'(\theta^*))$. Because of (5.10) there exists a finite number $N = N(\omega)$ such that

$$(5.12) \qquad \rho^2\left[1-\gamma_t(1-f'(\xi_t))\right]^2 \le 1 \qquad \forall t \ge N.$$

Define $\alpha_t = \alpha_t(\omega) = 0$ if $t \ge N$ and in an appropriate manner such that (5.1) holds if $t < N$. Because $N(\omega)$ is finite for all $\omega \in [\theta_t \to \theta^*]$ we have for these ω $\sum_{t=1}^{\infty}\alpha_t(\omega) < \infty$. Hence (5.1) is fulfilled on the event $[\theta_t \to \theta^*]$.

To verify (5.2) we consider

$$(5.13) \qquad \sum_{t=1}^{T}\frac{\gamma_t^2}{y_t^2}\pi_{t+1}^2 \le \sum_{t=1}^{T}\frac{\rho^{2(t+1)}}{y_t^2}\frac{1}{(t+1)^{1+2\delta}}1_{[y_t\neq 0]}.$$

Notice that because $\gamma_t = 0$ whenever $y_t = 0$ every summand is finite. By Lemma 5.2 we know that $\rho^{2t} = O(y_t^2)$ on $[\theta_t \to \theta^*]$ as $t \to \infty$, hence the first quotient under the right-hand sum is bounded pathwise by a finite constant. The whole sum thus converges on $[\theta_t \to \theta^*]$. ■

Now we consider the case $|\theta^*| > 1$ and $\gamma_t \to 0$. Because $1-\gamma_t(1-f'(\xi_t)) \to 1$ whenever $\theta_t \to \theta^*$ the choice $\{\pi_t\}$ of Theorem 5.3 does not fulfill (5.1). Hence, we have to look for another sequence $\{\pi_t\}$ which increases with lower speed.

Theorem 5.4:
Suppose that (A.1)–(A.4) hold. Let θ^* be a fixed point of f which is stable with respect to an admissible sequence of gains $\{\gamma_t\}$ with $\gamma_t \to 0$ a.s. and $|\theta^*| > 1$. Then on the event $[\theta_t \to \theta^*]$

$$\theta_t - \theta^* = O\left(\exp\left(-2c\sum_{s=1}^{t}\gamma_s\right)\right) \qquad a.s. \tag{5.14}$$

for all $c < 1 - f'(\theta^*)$ provided that $f'(\theta^*) < 1$.

Proof :
In order to simplify the notation we assume that $\theta_t \to \theta^*$ a.s. Now we fix two values c and c' such that $0 < c \leq c' < 1 - f'(\theta^*)$. Because $f'(\cdot)$ is continuous in θ^* we have $1 - \gamma_t(1 - f'(\xi_t)) \uparrow 1$. Hence there exists a (path–dependend) finite number $N_1 = N_1(\omega, c')$ such that

$$0 \leq 1 - \gamma_t(1 - f'(\xi_t)) \leq 1 - c'\gamma_t \qquad \forall t \geq N_1. \tag{5.15}$$

We define $\pi_{t+1}^2 = \prod_{s=1}^{t}(1 - c\gamma_t)^{-2}$ for $t > 1$ and $\pi_1 = C_0$ with some finite constant $C_0 > 0$. Because $\gamma_t \to 0$ there exists another finite number $N_2 = N_2(\omega, c)$ such that

$$(1 - c\gamma_t)^{-2} \geq 1 \qquad \forall t \geq N_2. \tag{5.16}$$

Hence

$$\begin{aligned} \frac{\pi_{t+1}^2}{\pi_t^2}(1 - \gamma_t(1 - f'(\xi_t)))^2 &= (1 - c\gamma_t)^{-2}(1 - \gamma_t(1 - f'(\xi_t)))^2 \\ &\leq \frac{(1 - c'\gamma_t)^2}{(1 - c\gamma_t)^2} \\ &\leq 1 \qquad \forall t \geq \max\{N_1, N_2\}. \end{aligned} \tag{5.17}$$

In order to verify condition (5.1) we define $\alpha_t = 0$ if $t \geq \max\{N_1, N_2\}$ and otherwise in an appropriate manner such that (5.1) holds. Because $N_1(\omega)$ and $N_2(\omega)$ are finite for almost all ω we have $\sum_1^\infty \alpha_t < \infty$ a.s. Thus (5.1) is verified.

To verify (5.2) we fix two values ρ, ρ' such that $1 < \rho < \rho' < |\theta^*|$. As before we can conclude that there exists a (path–dependent) finite number $N_3(\omega, \rho)$ such that

$$1 > (1 - c\gamma_t)^2 > \frac{1}{\rho} \qquad \forall t \geq N_3. \tag{5.18}$$

Hence for $t \geq N_3$ we have

$$\begin{aligned} \pi_{t+1}^2 &= \prod_{s=1}^{t}(1 - c\gamma_s)^{-2} \\ &= \prod_{s=1}^{N_3}(1 - c\gamma_s)^{-2} \prod_{N_3+1}^{t}(1 - c\gamma_s)^{-2} \\ &< C(N_3, \rho)\rho^{2t} \end{aligned} \tag{5.19}$$

with a finite constant $C(N_3, \rho)$. Then

$$\begin{aligned} \sum_{t=1}^{\infty}\frac{\gamma_t^2}{y_t^2}\pi_{t+1}^2 &\leq C(N_3, \rho)\sum_{t=1}^{\infty}\frac{\gamma_t^2}{y_t^2}\rho^{2t} \\ &\leq C(N_3, \rho)\sum_{t=1}^{\infty}\frac{\rho'^{2t}}{y_t^2}\left(\frac{\rho}{\rho'}\right)^{2t}, \end{aligned} \tag{5.20}$$

and by Lemma 5.2 we can conclude that this sum is finite a.s. Hence (5.2) is verified.

To finish the proof notice that

$$
\begin{aligned}
\pi_{t+1}^{-1} &= \prod_{s=1}^{t}(1-c\gamma_s)^2 \\
&= \exp\left(\log\left(\prod_{s=1}^{t}(1-c\gamma_s)^2\right)\right) \\
&\le \exp\left(2\sum_{s=N_4}^{t}\log(1-c\gamma_s)\right) \\
&\le \exp\left(-2c\sum_{s=N_4}^{t}\gamma_s\right)
\end{aligned}
\tag{5.21}
$$

with $N_4 = N_4(\omega)$ such that $1 - c\gamma_t > 0$ for all $t \ge N_4$. Because N_4 is finite and $\sum_1^\infty \gamma_s = \infty$ it does not matter that the sum in (5.21) starts at N_4, hence $\pi_{t+1}^{-1} = O(\exp(-2c\sum_1^t \gamma_s))$ a.s. ■

It is easy to show that the condition (5.10) in Theorem 5.3 can be rewritten as

$$
\rho^2 < \begin{cases} [1-\gamma^*(1-f'(\theta^*))]^{-1} & \text{if } f'(\theta^*) > 0 \\ \theta^{*2} & \text{if } f'(\theta^*) \le 0. \end{cases}
\tag{5.22}
$$

By this we can recognize how the slope of the function f at θ^* determines the rate of convergence.

Consider first the boundary case $f'(\theta^*) = 0$. Assume for the moment that that f is constant, thus $f(\theta) = \theta^*$ for all $\theta \in I\!R$. Then we have a system without forecast feedback so that the problem of convergence of the learning process reduces the problem of consistency of the OLS–estimator in an autoregressive model. Lai and Wei studied this problem extensively in a more general setting (see for example Lai/Wei (1982), Lai/Wei (1983), Wei (1987), Lai (1989)). They found out that in the explosive (or geometrically increasing) case the rate of convergence of the least squares estimator is of order $O(t^{1/2+\delta}/\theta^{*t})$. This result is slightly stronger than Theorem 5.3 but only for the reason that we have to include into our considerations a time-varying coefficent $f(\theta_t)$ which is (almost) never equal to θ^*. This implies that the rate of growth of our data process $\{y_t\}$ is not constant (although nearly constant). Especially if θ_t^2 converges to θ^{*2} from below $\rho^2 < \theta^{*2}$ can not be sharpend.

Now consider the case with $f'(\theta^*) > 0$. Recall that

$$
\theta_{t+1} - \theta^* = (1-\gamma_t)(\theta_t - \theta^*) + \gamma_t(f(\theta_t) - \theta^*) + \frac{\gamma_t}{y_t} w_{t+1},
\tag{5.23}
$$

while in case of $f(\theta) \equiv \theta^*$ we have

$$
\theta_{t+1} - \theta^* = (1-\gamma_t)(\theta_t - \theta^*) + \frac{\gamma_t}{y_t} w_{t+1}.
\tag{5.24}
$$

Now it is apparent how the slope of the function f at θ^* comes into the picture. If $f'(\theta^*) > 0$ then, supposed that θ_t is sufficiently close to θ^*, $f(\theta_t) - \theta^*$ has the same sign as $\theta_t - \theta^*$ so that the structural adjustment in (5.24) is 'better' than the one in (5.23). Especially if $f'(\theta^*) \approx 1$ we have $\theta_{t+1} - \theta^* \approx \theta_t - \theta^* + (\gamma_t/y_t)w_{t+1}$ and the rate of convergence is very low. On the other hand, if $f'(\theta^*) < 0$ then $f(\theta_t) - \theta^*$

and $\theta_t - \theta^*$ have opposite sign so that the structural adjustment in (5.23) is 'better' than the one in (5.24), again supposed that θ_t is sufficiently close to θ^*. In this case the rate of convergence of the esitmators given by (5.23) will be larger than stated in Theorem 5.3, but we have made no attempt to calculate it explicitly.

Now we consider the case $|\theta^*| \leq 1$. This case is much more complicated because the asymptotic behaviour of the data process $\{y_t\}$ is not as simple as in the case of $|\theta^*| > 1$. In that case the $\{y_t\}$ increases geometrically on $[\theta_t \to \theta^*]$, hence, for large t, the dynamical system $(\{y_t\}, \{\theta_t\})$ is 'almost deterministic' and the long-term behaviour of the trajectories is easy to determine. But in the case of $|\theta^*| \leq 1$ the system remains stochastic and the trajectories are erratic.

Firstly, we consider an admissible sequence of gains $\{\gamma_t\}$ with $\gamma_t \to \gamma^*$ with $\gamma^* > 0$. For (5.2) to hold it is necessary that

$$\sum_{t=1}^{\infty} \frac{1}{y_t^2} < \infty \qquad \text{a.s.} \tag{5.25}$$

on the event $[\theta_t \to \theta^*]$. For first order autoregressive processes with a constant coefficient $|\theta^*| < 1$ it is well known that $\sum_1^T y_t^2 = O(T)$. With the aid of the Kronecker Lemma it is then easy to show that $\sum_1^\infty y_t^{-2} = \infty$ a.s. We conjecture that the same holds also for autoregressive processes with a time-varying but convergent coefficient. Nevertheless, this does not imply that $\mathrm{P}(\theta_t \to \theta^*) = 0$ for the learning procedures with $\gamma_t \to \gamma^* > 0$ and $|\theta^*| < 1$.[1]

For $|\theta^*| = 1$ it is well known that $y_t^2 = O(t \log\log t)$ a.s. for autoregressive processes with constant coefficients (see, f.e., Lai/Wei (1985)). Again, it can be shown that $\sum_1^\infty y_t^{-2} = \infty$ a.s. For this reason we do not consider further learning procedures given by a sequence of gains $\{\gamma_t\}$ with $\gamma_t \to \gamma^* > 0$ in combination with $|\theta^*| \leq 1$.

Now let $\{\gamma_t\}$ be a sequence of gains with $\gamma_t \to 0$ a.s. and θ^* a fixed point of f with $|\theta^*| \leq 1$. In order to determine the rate of convergence $\{\theta_t\}$ with the aid of Theorem 5.1 we have to find a sequence $\{\pi_t\}$ which fulfills (5.1) and (5.2). The condition (5.1) is fulfilled by the same sequence $\{\pi_t\}$ as in the proof of Theorem 5.4. Hence, the problem consists in finding a sequence $\{\pi_t\}$ which increases sufficiently slow to fulfill (5.2). This problem seems hard to solve for general $\{\gamma_t\}$ in a satisfoctory way. We present a solution for a special case, the OLS–algorithm. In order to do so we need some more auxiliary results.

Lemma 5.5:
Suppose that (A.1)–(A.4) hold. Let θ^* be an OLS–stable fixed point of the function f with $|\theta^*| \leq 1$. Then on the event $[\theta_t \to \theta^*]$

(i) $$\frac{y_t^2}{\sum_{\tau=1}^t y_\tau^2} \longrightarrow 0 \qquad a.s.$$

(ii) $$\frac{\log(\sum_{\tau=1}^t y_\tau^2)}{\log t} \longrightarrow c \qquad a.s.\ with\ c = \begin{cases} 1 & if\ |\theta^*| < 1 \\ 2 & if\ |\theta^*| = 1. \end{cases}$$

[1] If we impose additional conditions on the disturbance terms $\{w_t\}$, more precisely, if we assume $\{w_t\}$ to be normed regular in the sense of Marcinkiewicz and Zygmund (see, for example, Stout (1974), p. 80), then $\sum_1^\infty y_t^{-2} = \infty$ implies $\mathrm{P}(\theta_t \to \theta^*) = 0$. Notice that every i.i.d. sequence $\{w_t\}$ with $\mathrm{E}w_t = 0$ and $\mathrm{E}w_t^2 = 1$ which is bounded or uniformly integrable is a normed regular martingale difference sequence in the sense of Marcinkiewicz and Zygmund.

(iii) $\liminf_t \frac{1}{t}\sum_{\tau=1}^{t} y_\tau^2 > 0$ *a.s.*

This result is well-known for autoregressive processes with constant coefficients (see, f.e., Wei (1987)) and it can be shown with some effort to hold also for autoregressive processes with time-varying but convergent coefficients. We omit a formal proof.

Lemma 5.6: (Lai/Wei (1982))
For every sequence $\{y_t\}_{t\geq 1}$ of real numbers with $y_1^2 > 0$

$$\sum_{t=1}^{T} \frac{y_t^2}{\sum_{\tau=1}^{t} y_\tau^2} = O\left(\log\left(\sum_{t=1}^{T} y_t^2\right)\right). \tag{5.26}$$

Lemma 5.7: (Lai/Wei (1982))
Suppose that $\{w_t\}$ is a martingale difference sequence with respect to a filtration $\{\mathcal{F}_t\}$ and $\sup_t \mathrm{E}[\, w_{t+1}^2 \,|\, \mathcal{F}_t\,] < \infty$, and $\{u_t\}$ is a sequence of predetermined (i.e $\mathcal{F}_{t-1}$-measurable) random variables. Then

$$\sum_{t=1}^{T} |u_t| w_t^2 = o\left(\left(\sum_{t=1}^{T} |u_t|\right)^{1+\delta}\right) \qquad a.s.\ on \qquad \left[\sum_{t=1}^{\infty} |u_t| = \infty\right] \tag{5.27}$$

for all $\delta > 0$. If, in addition, $\sup_t \mathrm{E}[|w_{t+1}|^\alpha|\mathcal{F}_t] < \infty$ for some $\alpha > 2$ then

$$\limsup_T \frac{\sum_{t=1}^{T} |u_t| w_t^2}{\sum_{t=1}^{T} |u_t|} < \infty \qquad a.s.\ on \qquad \left[\sup_t |u_t| < \infty, \sum_{t=1}^{\infty} |u_t| = \infty\right] \tag{5.28}$$

The next result is a refinement of Proposition 4.1.

Proposition 5.8: (Lai (1989))
Let $\{w_t\}$ be a martingale difference sequence with respect to a filtration $\{\mathcal{F}_t\}$ such that $\sup_t \mathrm{E}[w_{t+1}^2|\mathcal{F}_t] < \infty$ a.s. Let α_t, β_t, ν_t and z_t be non-negative $\mathcal{F}_t$-measurable random variables such that $\sum_1^\infty \alpha_t < \infty$ a.s., and let u_t be $\mathcal{F}_t$-measurable. Suppose that

$$z_{t+1} \leq (1+\alpha_t) z_t + \beta_{t+1} - \nu_{t+1} + u_t w_{t+1} \qquad a.s. \tag{5.29}$$

Then

(i) $\{z_t\}$ converges a.s. and $\sum_0^\infty \mathrm{E}[\nu_{t+1}|\mathcal{F}_t] < \infty$ a.s. on the event $[\sum_0^\infty \mathrm{E}[\beta_{t+1}|\mathcal{F}_t] < \infty]$.

(ii) For every $\delta > 0$, $max(z_t, \sum_1^t \nu_s) = O\left(\sum_1^t \beta_s + (\sum_1^{t-1} u_s^2)^{1/2+\delta}\right)$ a.s.

Now we can formulate the result which determines the rate of convergence for the OLS–algorithm in the stable case.

Theorem 5.9:
Suppose that (A.1)–(A.4) hold. Let θ^* be an OLS–stable fixed point of the function f with $|\theta^*| \le 1$. If $-\infty < f'(\theta^*) < \frac{1}{2}$ then

$$\theta_t - \theta^* = o\left(\frac{(\log t)^{1+\delta}}{t}\right) \qquad a.s.\ as\ t \to \infty \tag{5.30}$$

on the event $[\theta_t \to \theta^*]$.

Proof:
Notice that the OLS–algorithm is given by the sequence of gains $\{\gamma_t\}$ with

$$\gamma_t = \frac{y_t^2}{\sum_{\tau=1}^t y_\tau^2}. \tag{5.31}$$

We want to use Proposition 5.8. Define $z_{t+1} := (\sum_1^t y_\tau^2)(\theta_t - \theta^*)^2$, then

$$\begin{aligned}
z_{t+1} &= (\sum_{\tau=1}^t y_\tau^2)\Big[[1-\gamma_t(1-f'(\xi_t))]^2(\theta_t-\theta^*)^2 \\
&\qquad + 2[1-\gamma_t(1-f'(\xi_t))](\theta_t-\theta^*)\frac{\gamma_t}{y_t}w_{t+1} + \frac{\gamma_t^2}{y_t^2}w_{t+1}^2\Big] \\
&= (\sum_{\tau=1}^t y_\tau^2)\left[1-2\gamma_t(1-f'(\xi_t)) + \gamma_t^2(1-f'(\xi_t))^2\right](\theta_t-\theta^*)^2 \\
&\qquad + 2(\sum_{\tau=1}^t y_\tau^2)\left[1-\gamma_t(1-f'(\xi_t))\right](\theta_t-\theta^*)\frac{\gamma_t}{y_t}w_{t+1} + (\sum_{\tau=1}^t y_\tau^2)\frac{\gamma_t^2}{y_t^2}w_{t+1}^2 \qquad (5.32) \\
&= z_t + y_t^2(\theta_t-\theta^*)^2 - 2y_t^2(1-f'(\xi_t))(\theta_t-\theta^*)^2 + y_t^2\gamma_t(1-f'(\xi_t))^2(\theta_t-\theta^*)^2 \\
&\qquad + 2y_t\left[1-\gamma_t(1-f'(\xi_t))\right](\theta_t-\theta^*)w_{t+1} + \gamma_t^2 w_{t+1}^2 \\
&= z_t - y_t^2\left[1-\gamma_t(1-f'(\xi_t)) - 1 - \gamma_t(1-f'(\xi_t))^2\right](\theta_t-\theta^*)^2 \\
&\qquad + 2y_t\left[1-\gamma_t(1-f'(\xi_t))\right](\theta_t-\theta^*)w_{t+1} + \gamma_t^2 w_{t+1}^2
\end{aligned}$$

We define

$$\begin{aligned}
\tilde\beta_{t+1} &= \gamma_t^2 w_{t+1}^2 \\
u_t &= 2y_t\left[1-\gamma_t(1-f'(\xi_t))\right](\theta_t-\theta^*) \\
\tilde\nu_{t+1} &= y_t^2\left[2(1-f'(\xi_t)) - 1 - \gamma_t(1-f'(\xi_t))^2\right](\theta_t-\theta^*)^2.
\end{aligned} \tag{5.33}$$

We thus have

$$z_{t+1} = z_t + \tilde\beta_{t+1} - \tilde\nu_{t+1} + u_t w_{t+1}. \tag{5.34}$$

This representation does not yet fit into Proposition 5.8 because we can not ensure that $\tilde\nu_t$ is non-negative for all t. But in the limit it is as will be shown.

Define $\nu_t = \tilde\nu_t 1_{[\tilde\nu_t \ge 0]}$ and $\beta_t = \tilde\beta_t - \tilde\nu_t 1_{[\tilde\nu_t < 0]}$. We want to show that for almost all $\omega \in [\theta_t - \theta^*]$ ν_t and $\tilde\nu_t$ differ only for finitely many $t \ge 1$. Let $\epsilon = \frac{1}{2} - f'(\theta^*) > 0$

and fix $\omega \in [\theta_t \to \theta^*]$. By continuity there exists a finite number $N_1 = N_1(\omega)$ such that

$$f'(\xi_t) \leq \frac{1}{2} - \frac{\epsilon}{2} \qquad \forall t \geq N_1. \tag{5.35}$$

This implies that $1 - 2f'(\xi_t) \geq \epsilon$ for all $t \geq N_1$.

On the other hand, because $\gamma_t \to 0$, there exists for almost all $\omega \in [\theta_t \to \theta^*]$ another finite number $N_2 = N_2(\omega)$ such that

$$\gamma_t(1 - f'(\xi_t))^2 \leq \frac{\epsilon}{2} \qquad \forall t \geq N_2. \tag{5.36}$$

For $t \geq \max\{N_1, N_2\}$ we thus have

$$\begin{aligned} \tilde{\nu}_{t+1} &= y_t^2 \left[1 - 2f'(\xi_t) - \gamma_t(1 - f'(\xi_t))^2\right] (\theta_t - \theta^*)^2 \\ &\geq y_t^2[\epsilon - \epsilon/2](\theta_t - \theta^*)^2 \\ &= \frac{\epsilon}{2} y_t^2 (\theta_t - \theta^*)^2. \end{aligned} \tag{5.37}$$

This implies that for almost all $\omega \in [\theta_t \to \theta^*]$ there are only finitely many $t \geq 1$ such that $\tilde{\nu}_t \neq \nu_t$ and $\tilde{\beta}_t \neq \beta_t$.

We have the representation

$$z_{t+1} = z_t + \beta_{t+1} - \nu_{t+1} + u_t w_{t+1}. \tag{5.38}$$

With Lemma 5.6 and Lemma 5.7 we get

$$\sum_{t=1}^{T} \beta_t = O(\sum_{t=1}^{T} \tilde{\beta}_t) = o\left(\log\left(\sum_{\tau=1}^{t} y_\tau^2\right)^{1+\delta}\right) \qquad \text{a.s.} \tag{5.39}$$

on $[\theta_t \to \theta^*]$ for every $\delta > 0$.

To determine the order of $\sum_1^T u_t^2$ notice that because of

$$\begin{aligned} u_t^2 &\leq 4y_t^2(\theta_t - \theta^*)^2 \\ &= \frac{8}{\epsilon}\frac{\epsilon}{2} y_t^2 (\theta_t - \theta^*)^2 \\ &\leq \frac{8}{\epsilon}\nu_{t+1} \end{aligned} \tag{5.40}$$

for t sifficiently large we have $(\sum_{t=1}^{T-1} u_t^2)^{1/2+\delta} = O(\sum_{t=1}^{T} \nu_t)^{1/2+\delta}$ as $t \to \infty$.

By Proposition 5.8 we thus have

$$\max\left\{z_t, \sum_{t=1}^{T} \nu_t\right\} = O\left(\sum_{t=1}^{T} \beta_t\right) + O\left(\sum_{t=1}^{T} \nu_t\right)^{2/3} \qquad \text{a.s.} \tag{5.41}$$

hence

$$\sum_{t=1}^{T} \nu_t = O\left(\sum_{t=1}^{T} \beta_t\right) \quad \text{a.s.} \quad \text{and} \quad z_t = O\left(\sum_{t=1}^{T} \beta_t\right) \quad \text{a.s.} \tag{5.42}$$

With Lemma 5.5 and Lemma 5.6 we can conclude thst

$$z_t = o\left(\left(\log \sum_1^T y_t^2\right)^{1+\delta}\right) = o\left((\log T)^{1+\delta}\right) \qquad \text{a.s} \tag{5.43}$$

Again with Lemma 5.5 we get

$$(\theta_t - \theta^*)^2 = o\left(\frac{(\log t)^{1+\delta}}{t}\right) \qquad \text{a.s.} \ \blacksquare \tag{5.44}$$

Corollary 5.10:
Suppose that the assumptions of Theorem 5.9 are fulfilled and, additionaly,

$$\sup_t \mathrm{E}[|w_{t+1}|^\alpha | \mathcal{F}_t] < \infty \qquad a.s. \tag{5.45}$$

for some $\alpha > 2$*. Then* $(\theta_t - \theta^*)^2 = O\left(\frac{\log t}{t}\right)$ *a.s.*

It is a remarkable fact that the rate of convergence determined by Theorem 5.9 and Corollary 5.10 is exactly the same as the one determined by LAI/WEI (1982) for the OLS–estimator within an stable autoregressive model with constant coefficients. Of course, our setup contains the constant coefficient model as a special case (set $f(\theta) \equiv \theta^*$), but it is interesting that we can increase $f'(\theta^*)$ up to $1/2$ without getting a lower rate of convergence. We conjecture that the rate of convergence for the case $f'(\theta^*) < 0$ is higher than the one determined by Theorem 5.9 but we are not able to show this. The main problem consists in finding a suitable term μ_t such that $\sum_1^T \beta_t = O(\mu_t)$ when we define $z_t = \eta_t(\theta_t - \theta^*)^2$ with $\eta_t \neq \sum_1^t y_\tau^2$. For the choice $\eta_t = \sum_1^t y_\tau^2$ Lemma 5.6 gives such a suitable term. (Notice that this choice also fits into the construction used in the proof of Theorem 5.4. If we can choose $c = 1$, thus if $f'(\theta^*) < 0$, then $\pi_{t+1}^2 = C \sum_1^t y_t^2$ with some constant C.)

For general sequence of gains the problem is even harder because one has to find a suitable sequence η_t such that the calculations as in the proof of Theorem 5.9 remain trackable.

Chapter 6

An Example

Now we want to give a simple economic example. Although autoregressive relationships between economic variables are quite usual it is difficult to find a non-trivial example which fits exactly into our setup. We need an economic model which can be described by one univariate equation using only lagged endogenous variables, expectations of the current endogenous variable and disturbance terms. Clearly, this kind of model is quite rare in economics. Nevertheless, the original model of MUTH (1961) is one of this kind.

The Model
Consider the model of an isolated market with a fixed production lag of a commodity which cannot be stored. The market equations take the form

$$\begin{aligned} C_t &= -\beta p_t && \text{(Demand)} \\ P_t &= \alpha p_t^e + u_t && \text{(Supply)} \\ P_t &= C_t && \text{(Market equilibrium)} \end{aligned} \tag{6.1}$$

where

- P_t represents the number of units produced in a period lasting as long as the production lag,
- C_t is the amount consumed,
- p_t is the market price in the tth period,
- p_t^e is the market price expected to prevail during the tth period on the basis of information available through the $(t-1)'$st period,
- u_t is an error term — representing, say, variations in yields due to weather, and
- $\alpha > 0, \beta > 0$.

Notice that all the variables are deviations from equilibrium values.

The quantity variables may be eliminated from (6.1) to give

$$p_t = -\frac{\alpha}{\beta} p_t^e - \frac{1}{\beta} u_t. \tag{6.2}$$

We assume that $\{u_t\}$ is a white noise process. Then, provided that $\alpha/\beta \neq -1$, the expectations are rational if and only if $p_t^e = 0$.

MUTH considered, beside rational expectations, also other expectations formation schemes and derived related stability conditions for the price process in terms of α and β. For example, the so-called classical formation scheme $p_t^e = p_{t-1}$ leads to the stability condition $\alpha < \beta$. (This is the well known cobweb situation.)

But MUTH was not concerned explicitely with learning to form rational expectations. It took some time until CYERT/DEGROOT (1974) introduced a learning procedure into the model. More precisely, within the model which they called "an inconsistent model" they assumed that the firms (or the agents) believe in the model

$$p_{t+1} = \theta p_t + v_{t+1} \tag{6.3}$$

with an unknown parameter θ. The firms assign a prior distribution to the value of the parameter, and after the price in each period is observed, the firms update this distribution in accordance with Bayes's theorem. Supposed that the firms assume the error terms v_t to have a normal distribution the Bayes's learning procedure coincides with the OLS–procedure. Because "an analytic study of the convergence or divergence of this process appears to be difficult" (p. 529) they confined their study to some Monte–Carlo runs. With the results of the previous chapters we are now able to study this model analytically.

Stability of Rational Expectations
The function f is given as

$$f(\theta) = -\frac{\alpha}{\beta}\theta \qquad \forall \theta \in \mathbb{R} \tag{6.4}$$

and posesses the unique fixed point $\theta^* = 0$. Because

$$f'(\theta^*) = -\frac{\alpha}{\beta} \tag{6.5}$$

we can conclude that the 'classical' stability condition $\alpha < \beta$ leads to $|f'(\theta^*)| < 1$, hence $\theta^* = 0$ is stable with respect to every admissible sequence of gains $\{\gamma_t\}$. Moreover, $\alpha \leq \beta$ implies

$$\Big((1-\gamma)\theta + \gamma f(\theta)\Big)^2 \leq \theta^2 \qquad \forall \theta \in \mathbb{R} \quad \forall \gamma \in [0,1], \tag{6.6}$$

hence the attracting region D_{θ^*} of θ^* is $\mathbb{R} \times [0,1]$.

Now let $1 < \alpha/\beta < \infty$. Then $\theta^* = 0$ is stable with respect to every admissible sequence of gains $\{\gamma_t\}$ with $\gamma_t \rightarrow 0$. Some elementary calculations lead to

$$\gamma \leq \frac{2\beta}{\alpha+\beta} \quad \Longrightarrow \quad \Big((1-\gamma)\theta + \gamma f(\theta)\Big)^2 \leq \theta^2 \qquad \forall \theta \in \mathbb{R}. \tag{6.7}$$

Hence $\mathbb{R} \times [0, \frac{2\beta}{\alpha+\beta}] \subset D_{\theta^*}$. Notice that $\theta^* = 0$ is also stable with respect to the OLS–algoritm.

Convergence of the Learning Procedures
Firstly, consider the case $\alpha \leq \beta$. Because $D_{\theta^*} = \mathbb{R} \times [0,1]$ in that case Corollary 4.4 implies that $\theta_t \rightarrow \theta^*$ a.s. Thus under the stability conditions of the classical cobweb model every admissible learning procedure leads with probability one to rational limit expectations.

Now consider the case $1 < \alpha/\beta < \infty$. In that case we have $\mathbb{R} \times [0, \frac{2\beta}{\alpha+\beta}] \subset D_{\theta^*}$. Then $\gamma_t \rightarrow 0$ implies that

$$\mathrm{P}[(\theta_t, \gamma_t) \in D_{\theta^*} \text{ for all but finitely many t } \geq 1] = 1. \tag{6.8}$$

By Theorem 4.2 we get $\theta_t \longrightarrow \theta^*$ a.s. Thus all learning procedures given by an admissible sequence of gains $\{\gamma_t\}$ with $\gamma_t \longrightarrow 0$ a.s. lead to rational limit expectations as long as $\alpha/\beta < \infty$, thus for all plausible parameter configurations.

For the OLS–procedure the situation is not as simple because in case of $\alpha/\beta > 1$ we can not ensure that $\gamma_t \longrightarrow 0$ a.s. Notice that in that case

$$(f(\theta) - \theta^*)^2 > (\theta - \theta^*)^2 \qquad \forall \theta \in I\!R. \tag{6.9}$$

Because the signs of $f(\theta) - \theta^*$ and $\theta - \theta^*$ differ this does not necessarily imply that the estimation process $\{\theta_t\}$ always explodes, but the risk of explosion is always present. Notice that

$$\begin{aligned} \theta_{t+1} &= \theta_t + \frac{p_t^2}{\sum_{\tau=1}^t p_\tau^2}[f(\theta_t) - \theta_t] + \frac{p_t}{\sum_{\tau=1}^t p_\tau^2} w_{t+1} \\ &= \left[1 - \frac{p_t^2}{\sum_{\tau=1}^t p_\tau^2}\left(\frac{\alpha}{\beta} + 1\right)\right]\theta_t + \frac{p_t}{\sum_{\tau=1}^t p_\tau^2} w_{t+1} \end{aligned} \tag{6.10}$$

with $w_t = -\frac{1}{\beta}u_t$. Hence it depends on the evolution of the size of $p_t^2/\sum_1^t p_\tau^2$ whether $\{\theta_t\}$ converges to $\theta^* = 0$ or not. If we start with a large initial value θ_0 this may lead to an price process which increases in modulus. This in turn leads to large values of $p_t^2/\sum_1^t p_\tau^2$, hence to large adjustments of the estimation process, which in turn leads to an increasing rate of growth of the price process which in turn ... Thus both processes explode. In fact, CYERT/DEGROOT noticed that as the value of α/β is increased beyond 1 the model produced an increased number of explosive solutions. We thus have for the case $\alpha/\beta > 1$ convergence of the OLS–procedure only with probability less than one.

An Extended Model

Now we extend the model by introducing another additional type of firms into it, more exactly, we assume that one fraction, say a_2 of the producing firms are learning about the unknown hypothetical parameter as before while the other fraction, say a_1 uses the 'classical' expectations formation scheme.[1] The market expectation is thus given as

$$p_t^e = a_1 p_{t-1} + a_2\theta_t p_{t-1} \tag{6.11}$$

with $a_1, a_2 > 0, a_1 + a_2 = 1$ and $\{\theta_t\}$ according to an admissible learning procedure. For this model the function f takes the form

$$f(\theta) = -\frac{\alpha}{\beta}(a_1 + a_2\theta) \qquad , \theta \in I\!R, \tag{6.12}$$

which possesses the unique fixed point

$$\theta^* = -\frac{a_1\alpha}{\beta + a_2\alpha} \tag{6.13}$$

[1] As MUTH (1961) remarked, cobweb phenomena do occasionally appear in reality, for example in cattle and hog markets. So we can think of two types of farmers, the one believing in a steady state world regardless what has happened before, and the other, a little more clever, tries to learn about the underlying law of motion of the price process out of the observed history.

Of course, we can imagine an even more sophisticated type of farmers which takes into account that price expectations matter in the market and tries to learn about other farmers expectations and not only about a hypothetical parameter. But introducing this kind of behaviour into the model leads to substantial difficulties because expectations are in general not observable. Even if there is an institution which reports the last periods' market expectations to the agents there remains an identification problems as FRYDMAN (1982) has shown. In addition, there arises the problem of infinite regress of expectations because such a sophisticated farmer has to take into account the other sophisticated farmers expectations about what the other sophisticated farmers expectations are, and so on. Again, see FRYDMAN (1982) or FRYDMAN/PHELPS (1983) for a detailed discussion of this kind of problem.

with

$$f'(\theta^*) = -a_2 \frac{\alpha}{\beta}. \tag{6.14}$$

If $\alpha/\beta \leq 1/a_2$ then $|f'(\theta^*)| \leq 1$ and θ^* is stable with respect to every admissible learning procedure and we we can conclude as before that $\theta_t \longrightarrow \theta^*$ a.s.

Notice that the range of parameter values α, β for which $|f'(\theta^*)| \leq 1$ grows as a_2 decreases. The firms with a 'classical' expectations formation scheme have thus a stabilyzing effect in the sense that, no matter of the values of α and β, every admissible learning procedure converges towards rational expectations as long as the fraction of 'classical' firm is sufficiently large to ensure $\alpha/\beta \leq 1/a_2$.

Notice furthermore that if the learning firms expectations are rational, thus if $\theta_t = -\frac{a_1\alpha}{\beta + a_2\alpha}$, then they coincide with the aggregate market expectation while the expectations of the 'classical' firm are far apart from being rational. The 'classical' firms thus do not participate in the success of learning of the other firms.

Now suppose that $\alpha/\beta > 1/a_2$. Because

$$\theta_{t+1} - \theta^* = \left[1 - \gamma_t \left(1 + a_2 \frac{\alpha}{\beta}\right)\right] (\theta_t - \theta^*) + \frac{\gamma_t}{p_t} w_{t+1} \tag{6.15}$$

we have to deal with the same problem as before, the risk of explosion for the OLS–algorithm. If $|\theta^*| \leq 1$, thus if the rational expectations equilibrium is an stable autoregressive process, we can say nothing more than before. A large initial value θ_0 may cause the sequence of gains $\{\gamma_t\}$ to stabilyze on a level such that the term in brackets is larger than one in modulus. Hence the estimation process explodes.

Now let $|\theta^*| > 1$ and define according to Chapter 3 $\gamma^* = \frac{\theta^{*2} - 1}{\theta^{*2}}$. Then it can be shown by elementary calculations that always

$$1 - \frac{2}{\gamma^*} < f'(\theta^*) < 1. \tag{6.16}$$

The fixed point θ^* is thus OLS–stable. But as before we can only conclude that $\mathrm{P}(\theta_t \longrightarrow \theta^*) > 0$ and the risk of explosion is present as in the case of $|\theta^*| \leq 1$.

On the other hand, exactly as in the original model, we have $\theta_t \longrightarrow \theta^*$ a.s. for every admissible sequence of gains with $\gamma_t \longrightarrow 0$ a.s.

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