Discussion Paper No. B–293

On Short Rate Processes and Their Implications for Term Structure Movements

Erik Schlögl and Daniel Sommer*
University of Bonn, Department of Statistics, Adenauerallee 24–42,
53113 Bonn, Federal Republic of Germany

October 1994

EMail: schloegl@addi.or.uni-bonn.de
sommer@addi.or.uni-bonn.de

*The authors are grateful for helpful discussions with Jørgen Aase Nielsen, Klaus Sandmann, Dieter Sondermann and Alexander Stremme, as well as for helpful comments by seminar participants at Deutsche Bank and at DG Bank, Frankfurt/Main. The usual disclaimers apply. Financial support by Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at the University of Bonn, is gratefully acknowledged. The plots were produced using GNUPLOT 3.5. Document typeset in LaTeX.

Abstract

We compare short rate diffusion models with respect to their implications for term structure movements, the plausiblity of which serves us as a criterion for evaluating the models. Analytically for Gauss–Markov models and numerically for a broader collection of models prevalent in the literature, we isolate the deformations of the term structure generated endogenously. Among other analytical tools we use spread options on the forward rate curve as an aggregate measure of term structure shapes across states.

On the basis of our analysis we conclude that the Ho/Lee model should be discarded, since it cannot explain the emergence of downward sloping term structures, that the introduction of mean reversion is essential in order to generate downward sloping term structures in any substantial proportion, that the models typically favor upward sloping term structures for short maturities and downward sloping term structures for longer maturities, and that there is a surprisingly strong similarity among many of the models prevalent in the literature. A model which allows arbitrary boundaries for the short rate realizations to be fixed exogenously completes our analysis.

JEL Classification: G13

Keywords: arbitrage, term structure of interest rates, spread option pricing

Contents

1 Introduction

In stock option pricing models as a rule only the price processes of the securities underlying the derivative are modelled. When turning to securities whose payments are contingent on interest rates, however, the view changes considerably. No longer is it sufficient for the pricing of an option on a bond to model solely the price process of the particular underlying and that of a so–called reference bond[1]. Rather it is necessary to model the stochastic movements of the entire bond market or, what amounts to the same thing, of the complete term structure of interest rates. The need for this arises from the fact that underlying securities other than zero coupon bonds are liable to have payoffs at a number of dates during their lifetime, and that some derivatives have several payoff dates. Nor is it clear, given the multitude of arbitrage relationships between financial instruments in the "fixed income" markets (money market, bond market, futures, forward rate agreements...), why a hedging strategy should only be carried out in some particular assets whose price processes happen to have been modelled, while strategies in other assets cannot be considered because their price processes are not captured by a given model. However, once a model has been designed in such a way that it in fact encompasses the whole bond market the question naturally arises what such a model implies about the movements of relative prices of the different securities in this market. This view is also expressed in Ho/Lee (1986):

> "[...], when pricing interest rate contingent claims, in most cases we are concerned with how the prices of discount bonds with different maturities move relative to each other."

Since movements of relative prices of different securities are the consequence of movements of the term structure of interest rates the above question can be restated slightly differently as: What do different specifications of stochastic models of the term structure of interest rates imply for its future movements and deformations. This is the first question addressed in this paper.

By raising this question and implying, as we do, that the answer will have bearing on a second question, namely which model, if any, should be regarded as being more plausible than another, we set different priorities in our analysis than Hull/White (1990a), who assert that

> "[it] is important to distinguish between the goal of developing a model that adequately describes term structure movements and the goal of developing a model that adequately values [...] interest rate contingent claims [...]."

We feel somewhat uncomfortable with adopting this view without further analysis of the inherently interesting and distinctive feature of term structure models, namely

[1] Although a number of models have been suggested which do just this, they can no longer be regarded as representing the mainstream in the literature on interest rate derivative securities. For a review of such models see Rady/Sandmann (1994).

their de facto capturing the movements of the entire term structure. In particular, an analysis of possible deformations of the term structure in the models under consideration represents a first step in an evaluation of the fitness of these models for hedging purposes.

Returning to our central question we shall look at the simplest though — in terms of the number of such models that have been suggested in the literature — largest class of term structure models, i.e. models in which only the stochastic process describing the movements of the continuously compounded short rate of interest is exogenously postulated. More specifically, in this paper we shall consider the following specifications

1. The "β–root" models

$$dr_t = (\theta(t) - ar_t)dt + \sigma r_t^{\beta} dW_t$$

   which contain as special cases among others

   (a) the continuous–time Ho/Lee (1986) model

$$dr_t = \theta(t)dt + \sigma dW_t$$

   (b) the generalized[2] Vasicek (1977) model

$$dr_t = (\theta(t) - ar_t)dt + \sigma dW_t$$

   (c) the generalized Brennan/Schwartz (1977) model

$$dr_t = (\theta(t) - ar_t)dt + \sigma r_t dW_t$$

   (d) the generalized Cox/Ingersoll/Ross (1985) model

$$dr_t = (\theta(t) - ar_t)dt + \sigma \sqrt{r_t} dW_t$$

2. Lognormal models

   (a) Black/Derman/Toy (1990)

$$dr_t = r_t \cdot (\theta(t) - a \ln r_t + \frac{1}{2}\sigma^2)dt + \sigma r_t dW_t$$

   (b) Sandmann/Sondermann (1993)

$$dr_t = (1 - e^{-r_t})[(\theta(t) - \frac{1}{2}(1 - e^{-r_t})\sigma^2)dt + \sigma dW_t]$$

3. Completing the specifications above which either allow the short rate to lie in the interval $(-\infty, \infty)$ or in $(0, \infty)$, we also consider a specification which has not been discussed in the literature to date and which confines the short rate realizations to some arbitrary interval (a, b) :

$$dr_t = \left(\frac{c(a + b - 2r_t)}{b - a} + \frac{(r_t - a)(b - r_t)}{b - a}\theta(t) \right) dt + \sigma \frac{(r_t - a)(b - r_t)}{b - a} dW_t$$

In these models the rest of the term structure, namely zero coupon bond prices of different maturities, from which yields and forward rates can be calculated, is then **implicitly** given by taking (conditional) expectations of discounted future payoffs. It is our aim to make explicit the implications for term structure movements embedded in the above specifications of short rate processes. The relevance of this question stems the fact that any user of such models necessarily subscribes to these embedded implications and de facto accepts them as plausible descriptions of future term structure movements.

As we are interested in the deformations of the term structure endogenously generated by the different models under consideration in our analysis we shall, as a general rule, start out with a flat initial term structure. When necessary (in particular when it comes to simulations) we shall also consider initial term structures which are not flat in order to verify that our qualitative results obtained in the case of flat initial term structures carry over to cases where this initial condition is changed.
We concentrate on instantaneous continuously compounded forward rates rather than on zero coupon bond prices or yields because forward rates give the most disaggregate information about the term structure, yields simply being a time average of forward rates. The behaviour of the models will mainly be analysed from three vantage points. **First** we shall look directly at the forward rate curves in the different states of the world, **second** we shall consider the distribution of the slopes of forward rate curves, and **third** we shall use values of European spread options on the forward rate curve as an aggregate measure of its shape. We define the payoff of these options by

$$(1) \qquad [F_{T_0}(x+T_0) - F_{T_0}(y+T_0)]^+ \qquad \text{and} \qquad [F_{T_0}(y+T_0) - F_{T_0}(x+T_0)]^+$$

respectively, where $x > y$ and $F_{T_0}(x+T_0) :=$ forward rate at time T_0 with time to maturity x. Clearly the first option is the more valuable the more the term structure is upward sloping between x and y whereas the second is favoured in the case of downward sloping term structures.
While the first point gives us an idea about how rich the model is in generating different shapes of term structures points two and three address the question in which way a certain model will **typically** change a given term structure, i.e. whether for certain maturities a model will generate more upward or more downward sloping term structures. Intuitively, if there is no marked tendency one way or another one could call the model unbiased. The idea of a model being unbiased between changing the slope of a given term structure in an upward or downward direction will play an important role in our analysis. Having introduced the necessary notation we shall make this idea precise at the end of section 2.1 by stating two concepts of unbiasedness, the first being based on the distribution of the slopes of the forward rate curve, the second on spread option prices.
Of the above models only the Ho/Lee and Vasicek specifications are analytically

[2] We use the term "generalized" in the sense that prior to the seminal work by Ho and Lee (1986), models were not fitted to an initial term structure, while in our study we will always do so. This parallels the usage of the term as introduced by Hull/White (1990a).

tractable. We shall therefore derive analytical results about term structure movements in these models and in the larger class of Gauss/Markov term structure models they belong to. All the models will, however, be analysed numerically with the help of the implementation procedure suggested by Hull/White (1993).

The above approach to comparing term structure models differs from the vast empirical literature on the subject. Some empirical papers focus on how well models can reproduce certain actually observed term structures, e.g. Brown/Schaefer (1994), Chen/Scott (1993) and Stambaugh (1988). Others compare observed prices of derivatives with model prices, e.g. Flesaker (1993), who tests the Ho/Lee model. Chan, Karolyi, Longstaff and Sanders (1992) concentrate on how well a number of short rate processes capture the actual dynamics of the short term interest rate; their results on the β parameter for the aforementioned β–root model specification have recently been subjected to reevaluation in Duffee (1993). Cohen and Heath (1992) take two approaches, for one comparing models on the basis of what likelihood they assign to observed prices and secondly investigating whether deviations between market and model prices can be taken advantage of to acquire wealth. Close in spirit to our approach is a theoretical analysis by Musiela who proves among other things that the support of the distribution of yield curves in one factor Gauss-Markov models of the term structure is a one dimensional manifold and concludes that these models, therefore, provide little scope for hedging against movements of the term structure that may occur in reality.

The rest of the paper is organized as follows. Section 2 contains our analysis of Gauss/Markov term-structure models with a special emphasis on the Ho/Lee and Vasicek specifications. In section 3 we analyse and compare numerically the β–root and lognormal models. In section 4 the "bounded rate" specification is treated. Section 5 concludes our analysis.

2 An Analysis of Gaussian Term Structure Models in Continuous Time

2.1 Model Specification

As is usual we shall assume that there is a complete set of zero bonds with maturities $T \in [0, T^*]$. Immediately placing ourselves in a risk-neutral environment in Gaussian term structure models the instantaneous change in the price at time t of a zero bond maturing at T is then given by the following stochastic differential equation

$$dB(T,t) = r(t)B(T,t)dt + \sigma(T,t)B(T,t)dW_t \tag{2}$$

where W_t is a standard Brownian motion and is the same for all bonds of all maturities. The probability space is given by $(\Omega, \mathcal{F}, \{\mathcal{F}_t; t \geq 0\}, P_W)$ with $\{\mathcal{F}_t; t \geq 0\}$ the augmented natural filtration of the Brownian motion. $\sigma(T,t)$ is the instantaneous volatility of the zero bond maturing at T, which is assumed a deterministic twice continuously differentiable function of T and t, satisfying $\sigma(T,T) \equiv 0$, which means that the value of a zero coupon bond is deterministic at maturity. $r(t)$ is the instantaneous continuously compounded riskless rate of interest prevailing at time t.
Applying Ito's lemma, it can be checked that the solution to this differential equation is given by

$$B(T,t) = D(T)\ \exp\left\{\int_0^t \left(r(s) - \frac{1}{2}\sigma^2(T,s)\right) ds + \int_0^t \sigma(T,s)dW_s\right\} \tag{3}$$

where we have made use of the initial condition $B(T,0) = D(T)$ and $D(T)$ is the price at time zero of a zero bond maturing at T. Requiring the process $(r(t))_{0 \leq t \leq T^*}$ to be such that $B(T,T) \equiv 1 \quad \forall T \in [0,T^*]$ we have

$$\begin{aligned} B(T,t) &= \frac{B(T,t)}{B(t,t)} \\ &= \frac{D(T)}{D(t)} \exp\left\{-\frac{1}{2}\int_0^t \left(\sigma^2(T,s) - \sigma^2(t,s)\right) ds\right\} \cdot \exp\left\{\int_0^t [\sigma(T,s) - \sigma(t,s)]dW_s\right\} \end{aligned}$$

Since we want to concentrate on models where the process of the short rate is Markov the volatility function must necessarily satisfy

$$\frac{\partial \sigma(T,t)}{\partial T} = \hat{\sigma}(t) \exp\left\{-\int_t^T \lambda(s)ds\right\} \tag{4}$$

(For a proof see e.g. El Karoui et al. (1991).)
For this specification of the volatility function the forward rate at time t for maturity T, $F_t(T)$, or, what comes to the same thing, the forward rate at time t with time to maturity $x = T - t$, $F_t(x+t)$, can be expressed in terms of the short rate as

$$\begin{aligned} F_t(T) &= F_t(x+t) \\ &= F_0(T) + \int_0^t (\sigma(T,s) - \sigma(t,s))\frac{\partial \sigma(T,s)}{\partial T}ds + \exp\left\{-\int_t^T \lambda(s)ds\right\}(r(t) - F_0(t)) \end{aligned} \tag{5}$$

which for the Vasicek specification becomes

$$
\begin{aligned}
(6)\quad F_t^{\mathrm{VAS}}(x+t) &= F_0(x+t)+\frac{\sigma^2}{2a^2}(1-\exp\{-2at\})(\exp\{-ax\}-\exp\{-2ax\}) \\
&+ \exp\{-ax\}(r(t)-F_0(t))
\end{aligned}
$$

The continuous time version of Ho/Lee (1986) is obtained as $a \to 0$.
For the reader's convenience in Lemmas 1 and 2 we state and prove some well known results that hold in this framework

Lemma 1

The discounted price process of a zero bond is a martingale with respect to the measure P_W.

(Proof see appendix **??**.)

We shall call P_W the *risk neutral measure.* We also introduce the *time* T_0 *forward measure*, $P(T_0)$, proposed by El Karoui/Rochet (1989). This is defined independently of the Gaussian framework as follows

Definition 1

$$
\frac{dP(T_0)}{dP_W}=\frac{B(T_0,T_0)\exp\left\{-\int_0^{T_0} r_t dt\right\}}{D(T_0)}
$$

In the Gaussian term structure model we have

Lemma 2

In the framework of the Gaussian term structure model the time T_0 forward measure $P(T_0)$ is given by

$$
\frac{dP(T_0)}{dP_W}=\exp\left\{-\frac{1}{2}\int_0^{T_0}\sigma^2(T_0,t)dt+\int_0^{T_0}\sigma(T_0,t)dW_t\right\}
$$

(Proof: easy consequence of equation (3) and definition 1)

Now since $Z_t^{T_0}=E\left[\frac{d\ P(T_0)}{d\ P}\middle|\mathcal{F}_t\right]$, $0\le t\le T_0$, is a martingale solving the Doléans equation

$$
(7)\qquad Z_t = 1+\int^t \sigma(T_0,)Z_s dW_s
$$

by Girsanov's theorem it follows that on $[0,T_0]$

$$
(8)\qquad W_t^{T_0}=W_t-\int_0^t\frac{d\langle W,Z\rangle_s}{Z_s}=W_t-\int_0^t\sigma(T_0,s)ds
$$

is a standard Brownian motion under the T_0 forward measure.
Therefore, under the forward measure the forward rate will be distributed as

$$
(9)\qquad F_{T_0}(x+T_0)\sim N(\mu(T_0,x),\sigma^2(T_0,x))
$$

where

$$\begin{aligned}\mu(T_0,x) &= F_0(x+T_0)+\int_0^{T_0}(\sigma(x+T_0,t)-\sigma(T_0,t))\frac{\partial\sigma(x+T_0,t)}{\partial x}dt\\ \sigma^2(T_0,x) &= \int_0^{T_0}\left(\frac{\partial\sigma(x+T_0,t)}{\partial x}\right)^2 dt\end{aligned}$$

In the special case of the Vasicek model, that is $\lambda(u)\equiv a$ these expressions become

$$\mu_{\mathrm{VAS}}(T_0,x) = F_0(T_0+x)+\frac{\sigma^2}{2a^2}\left(1-\exp\{-2aT_0\}\right)\left(\exp\{-ax\}-\exp\{-2ax\}\right) \tag{10}$$

$$\sigma^2_{\mathrm{VAS}}(T_0,x) = \frac{\sigma^2}{2a}\left(1-\exp\{-2aT_0\}\right)\exp\{-2ax\} \tag{11}$$

As already indicated, apart from looking at the forward rate curve itself we shall also be concerned with the distribution of its slope at different times to maturity and with options on the spread between two forward rates. Defining $S_{T_0}(x) := \frac{\partial F_{T_0}(x+T_0)}{\partial x}$, the slope of the time T_0 forward rate curve at time to maturity x we have for the distribution of this random variable

$$S_{T_0}(x)\sim N(\mu'_{\mathrm{VAS}}(T_0,x),\sigma^{2'}_{\mathrm{VAS}}(T_0,x)) \tag{12}$$

where

$$\begin{aligned}\mu'_{\mathrm{VAS}}(T_0,x) &= \frac{\partial F_0(T_0+x)}{\partial x}+\frac{\sigma^2}{2a}\left[\left(1-\exp\{-2aT_0\}\right)\left(2\exp\{-2ax\}-\exp\{-ax\}\right)\right]\\ \sigma^{2'}_{\mathrm{VAS}}(T_0,x) &= \frac{\sigma^2}{2a}\left[\left(1-\exp\{-2aT_0\}\right)\exp\{-2ax\}\right]\end{aligned}$$

Now defining $j := F_{T_0}(x+T_0)-F_{T_0}(y+T_0), x>y$, and using (9) the value of the options on the spread between the forward rates of time to maturity x and y respectively at time T_0 is given by

Proposition 1

$$\begin{aligned}& V_0([F_{T_0}(x+T_0)-F_{T_0}(y+T_0)]^+)\\ =\ & D(T_0)\int_0^\infty j\frac{1}{\sqrt{2\pi\sigma_D^2}}exp\left\{-\frac{1}{2}\left(\frac{j-\mu_D}{\sigma_D}\right)^2\right\}dj\\ =\ & D(T_0)\frac{1}{\sqrt{2\pi\sigma_D^2}}\left[\left[-\sigma_D^2\exp\left\{-\frac{1}{2}\left(\frac{j-\mu_D}{\sigma_D}\right)^2\right\}\right]_0^\infty+\mu_D\int_0^\infty\exp\left\{-\frac{1}{2}\left(\frac{j-\mu_D}{\sigma_D}\right)^2\right\}dj\right]\\ =\ & D(T_0)\left[\mu_D\Phi\left(\frac{\mu_D}{\sigma_D}\right)+\sigma_D\varphi\left(\frac{\mu_D}{\sigma_D}\right)\right]\end{aligned}$$

and

$$\begin{aligned}V_0([F_{T_0}(y+T_0)-F_{T_0}(x+T_0)]^+) &= -D(T_0)\int_{-\infty}^0 j\frac{1}{\sqrt{2\pi\sigma_D^2}}\exp\left\{-\frac{1}{2}\left(\frac{j-\mu_D}{\sigma_D}\right)^2\right\}dj\\ &= -D(T_0)\left[\mu_D\Phi\left(-\frac{\mu_D}{\sigma_D}\right)-\sigma_D\varphi\left(\frac{\mu_D}{\sigma_D}\right)\right]\end{aligned}$$

where Φ is the distribution function of the standard normal distribution, φ is its density and

$$\begin{aligned}
\sigma_D^2 &= \int_0^{T_0} \left(\frac{\partial\sigma(x+T_0,s)}{\partial x} - \frac{\partial\sigma(y+T_0,s)}{\partial y} \right)^2 ds \\
\mu_D &= (F_0(x+T_0) - F_0(y+T_0)) + \int_0^{T_0} \left[(\sigma(x+T_0,s) - \sigma(T_0,s)) \frac{\partial\sigma(x+T_0,s)}{\partial x} \right. \\
&- \left. (\sigma(y+T_0,s) - \sigma(T_0,s)) \frac{\partial\sigma(y+T_0,s)}{\partial y} \right] ds
\end{aligned}$$

In the Vasicek case using (10) and (11) we have for the above expressions

$$\begin{aligned}
(13) \qquad \mu_{D\text{ VAS}} &= (F_0(x+T_0) - F_0(y+T_0)) + \frac{\sigma^2}{2a^2}(1 - \exp\{-2aT_0\}) \\
&\quad [\exp\{-ax\} - \exp\{-2ax\} - [\exp\{-ay\} - \exp\{-2ay\}]] \\
(14) \qquad \sigma^2_{D\text{ VAS}} &= \frac{\sigma^2}{a}(1 - \exp\{-2aT_0\}) \\
&\quad \left[\frac{1}{2}[\exp\{-2ax\} + \exp\{-2ay\}] - \exp\{-a(x+y)\} \right]
\end{aligned}$$

The result for the Ho/Lee model is again obtained for $a \to 0$ which yields $\sigma^2_{D\text{ HL}} = 0$.

We are now ready to introduce our concepts of unbiasedness.

Concept 1

a) Given a flat initial term structure we call a term structure model unbiased between generating upward and downward sloping term structures at time $T_0 \geq 0$ on the interval (a,b), $a \leq b$ if for all times to maturity $x \in (a,b)$

$$P(T_0)\,[S_{T_0}(x) \leq 0] = P(T_0)\,[S_{T_0}(x) \geq 0]$$

b) If a) holds for all times $T_0 \in (\underline{T}, \overline{T})$ for a, b independent of T_0 we say that the model is unbiased on (a,b) and $T_0 \in (\underline{T}, \overline{T})$.

Concept 2

Given a flat initial term structure we call a term structure model unbiased between generating upward and downward sloping term structures at time $T_0 \geq 0$ and time to maturity y for a time to maturity difference c if

$$V_0([F_{T_0}(y+c+T_0) - F_{T_0}(y+T_0)]^+) = V_0([F_{T_0}(y+T_0) - F_{T_0}(y+c+T_0)]^+)$$

Conversely we can say that a model is not unbiased if none of the concepts of unbiasedness is satisfied. If $P(T_0)\,[S_{T_0}(x) < 0] \leq P(T_0)\,[S_{T_0}(x) > 0]$ for some T_0 and x we say that the model is biased towards increasing term structures at time T_0 and time to maturity x according to concept 1. If for some T_0, y and c $V_0([F_{T_0}(y + c + T_0) - F_{T_0}(y + T_0)]^+) \geq V_0([F_{T_0}(y + T_0) - F_{T_0}(y + c + T_0)]^+)$ we say that the model is biased towards increasing term structures at time T_0 time to maturity y and time to maturity difference c according to concept 2. (Similarly for biases towards decreasing term structures.)

2.2 Model Analysis

Being equipped with some useful tools we now turn to a characterization of the term structure movements in this framework. We shall henceforth make

Assumption (A1)

The original term structure is flat.

First looking immediately at the term structures of forward rates themselves we have

Proposition 2

a) Under (A1) term structures in the Ho/Lee model are always increasing. Their slope is deterministic.

b) Under (A1) term structures in the Vasicek model are of three types

   i) monotonically increasing

   ii) monotonically decreasing

   iii) humpshaped, that is they possess an interior maximum.

Proof

a) Follows immediately when letting $a \to 0$ in equation (6) and differentiating with respect to x.

b) From equation (6)

$$S_t(x) = \underbrace{\exp\{-ax\}}_{>0}[a(F_0(t) - r(t)) + \underbrace{\frac{\sigma^2}{2a}(1 - \exp\{-2at\})}_{>0}(2\exp\{-ax\} - 1)]$$

As $2\exp\{-ax\} - 1$ is monotonically decreasing in x and bounded the expression in brackets will either be positive or negative or have a unique zero depending on the value of $F_0(t) - r(t)$. If it has a zero at $\overline{x}$, then $S_t(x) > 0$ for $x < \overline{x}$ and $S_t(x) < 0$ for $x > \overline{x}$. $\square$

Now we turn to spread option prices to check our concept 2 of unbiasedness. Proposition 2 implies that under (A1) $V_0^{\mathrm{HL}}([F_{T_0}(y+T_0) - F_{T_0}(y+c+T_0)]^+) = 0$ and $V_0^{\mathrm{HL}}([F_{T_0}(y+c+T_0) - F_{T_0}(y+T_0)]^+) \geq 0$ in the Ho/Lee model. Hence the Ho/Lee model is nowhere unbiased according to concept 2. In the Vasicek model, again under (A1), we set $x = y + c$, $c > 0$ and without loss of generality $D(T_0) = 1$. Consider

(15) $V_0^{\mathrm{VAS}}([F_{T_0}(x+T_0) - F_{T_0}(y+T_0)]^+) - V_0^{\mathrm{VAS}}([F_{T_0}(x+T_0) - F_{T_0}(y+T_0)]^-) = \mu_{D\ \mathrm{VAS}}$

It is easy to see that $\mu_{D\ \mathrm{VAS}}$ has the following properties:

i) $\lim_{y\to 0} \mu_{D\ \mathrm{VAS}} = \frac{\sigma^2}{2a^2}(1 - \exp\{-2aT_0\})\exp\{-ac\}(1 - \exp\{-ac\}) > 0$

ii) $\lim_{y\to\infty} \mu_{D\ \mathrm{VAS}} = 0$

iii) $\mu_{D\ \mathrm{VAS}} = 0 \quad \Leftrightarrow \quad \overline{y} = \frac{1}{a}\ln(\exp\{-ac\} + 1) > 0$

iv) $\frac{d\mu_{D\ \mathrm{VAS}}}{dy} = 0 \quad \Leftrightarrow \quad \overline{\overline{y}} = \frac{1}{a}\ln(2\exp\{-ac\} + 1) > \frac{1}{a}\ln(\exp\{-ac\} + 1)$

We have thus shown

Proposition 3

For every given time to maturity difference $c > 0$ there exists a unique time to maturity $\overline{y}(a)$ such that

$$V_0^{\mathrm{VAS}}([F_{T_0}(y+c+T_0) - F_{T_0}(y+T_0)]^+) \gtrless V_0^{\mathrm{VAS}}([F_{T_0}(y+c+T_0) - F_{T_0}(y+T_0)]^-)$$

as $y \lessgtr \overline{y}(a)$. Furthermore $\frac{d\overline{y}(a)}{da} < 0$.

Hence for every c there is exactly one time to maturity where the model is unbiased according to concept 2. This time to maturity is independent of T_0.

A similar result is obtained if we consider the slope of the forward rate curve at time T_0, $S_{T_0}(x)$. From equation (12) we have

(16) $$P(T_0)[S_{T_0}(\overline{x}) \leq 0] \leq 0.5 \quad \Leftrightarrow \quad \overline{x} \leq \frac{\ln 2}{a}$$

Hence there is exactly one time to maturity for which the model is unbiased according to concept 1b) where the time interval is $[0, T^*]$. Clearly the Ho/Lee model is again not unbiased according to concept 1.

The results up to now allow us to draw the following conclusions. Using the Ho/Lee model, that is assuming that there is no mean reversion, implies that one necessarily believes that the difference between forward rates of longer maturity and forward rates of shorter maturity can only increase. The model cannot explain the emergence of downward sloping term structures. Introducing mean reversion means that there is a structural bias towards an increase in the difference between the forward rates of two maturities (i.e. towards upward sloping term structures) on the short end of the term structure. The bias is reversed (i.e. favours downward sloping term structures) on the long end. The threshold maturity which marks the boundary between the thus characterized "short" and "long" end of the term structure is determined by the choice of the speed of mean reversion.

Given the obvious importance of the mean reversion coefficient the question arises whether by making this coefficient a function of time, $\lambda(t)$, as is admissible in the general Gauss/Markov framework, one can obtain a model that will be unbiased between upward and downward sloping term structures for an interval of maturities $[\underline{x}, \overline{x}]$ i.e. unbiased according to concept 1a) where the interval is nondegenerate. The negative answer to this question is provided by the following

Proposition 4

> Under (A1) the following holds:
> In the general one–factor Gauss/Markov model there is no C^1 function $\lambda(t) \geq 0$ such that there exists an interval of maturities $[\underline{x}, \overline{x}]$, $\underline{x} \geq 0$ with $P(T_0)[S_{T_0}(x) < 0] = P(T_0)[S_{T_0}(x) > 0] = 0.5 \; \forall x \in [\underline{x}, \overline{x}]$.

Intuitively, making the mean reversion parameter a function of time can be regarded as considering a sequence of Vasicek models all with different speeds of mean reversion for the short-rate. The effect of this is that there will no longer be a unique time to maturity below which the model will be biased towards increasing term-structures and above which the opposite will hold. However, the above result shows that by mixing across different speeds of mean reversion over time one can still not obtain an unbiased model even if the original term-structure was flat.

Proof

0) The case $\lambda(t) \equiv 0$ need not concern us since this is just the Ho/Lee case.

i) From equation (5) and (9) we have

$$S_{T_0}(x) = \left.\frac{\partial E_{P(T_0)}[F_{T_0}(T)]}{\partial T}\right|_{x+T_0} - \lambda(T) \exp\left\{-\int_{T_0}^{T} \lambda(u)du\right\}(r(T_0) - F_0(T_0))$$

and thus

$$\frac{\partial^2 F_{T_0}(T)}{\partial T \; \partial r(T_0)} = -\lambda(T) \exp\left\{-\int_{T_0}^{T} \lambda(u)du\right\} < 0$$

since $\lambda(\cdot) \not\equiv 0$ by assumption.

As $r(T_0)$ is normally distributed around $F_0(T_0)$ we have:

$$\begin{array}{llll}
\text{if } \left.\frac{\partial E_{P(T_0)}[F_{T_0}(T)]}{\partial T}\right|_{x+T_0} > 0 & \Rightarrow & S_{T_0}(x) > 0 & \forall\, r(T_0) \leq F_0(T_0) \\
 & \Rightarrow & P(T_0)[S_{T_0}(x) > 0] > 0.5 & \\
\text{if } \left.\frac{\partial E_{P(T_0)}[F_{T_0}(T)]}{\partial T}\right|_{x+T_0} < 0 & \Rightarrow & S_{T_0}(x) < 0 & \forall\, r(T_0) \geq F_0(T_0) \\
 & \Rightarrow & P(T_0)[S_{T_0}(x) > 0] < 0.5 &
\end{array}$$

Therefore, due to the strict monotonicity and continuity of the normal distribution function

$$P(T_0)[S_{T_0}(x) < 0] = P(T_0)[S_{T_0}(x) > 0] = 0.5 \qquad \Leftrightarrow \qquad \left.\frac{\partial E_{P(T_0)}[F_{T_0}(T)]}{\partial T}\right|_{x+T_0} = 0$$

Hence

$$\begin{array}{lrcll}
 & P(T_0)[S_{T_0}(x) < 0] = P(T_0)[S_{T_0}(x) > 0] & = & 0.5 & \forall\, x \in [\underline{x}, \bar{x}],\ \underline{x} \geq 0 \\
\Leftrightarrow & \left.\frac{\partial E_{P(T_0)}[F_{T_0}(T)]}{\partial T}\right|_{x+T_0} & = & 0 & \forall\, x \in [\underline{x}, \bar{x}],\ \underline{x} \geq 0 \\
\Leftrightarrow & E_{P(T_0)}[F_{T_0}(T)] & = & \text{const.} & \forall\, x \in [\underline{x}, \bar{x}],\ \underline{x} \geq 0
\end{array}$$

ii) We now proceed to show that there is no C^1 function $\lambda(\cdot) > 0$ such that the above condition can be satisfied. Since we have assumed a flat term structure at time "0" it is sufficient to consider the integral expression in equation (5). Substituting (4) and setting $\Lambda(\cdot)$ the antiderivative of $\lambda(\cdot)$ this can be expressed as

$$\begin{aligned}
&\int_0^{T_0} \hat{\sigma}^2(t) \left[\int_{T_0}^{x+T_0} \exp\{\Lambda(t) - \Lambda(v)\} dv\right] \exp\{\Lambda(t) - \Lambda(x+T_0)\} dt \\
=\ & \underbrace{\exp\{-\Lambda(x+T_0)\} \int_{T_0}^{x+T_0} \exp\{-\Lambda(v)\} dv}_{A(x,T_0)} \cdot \underbrace{\int_0^{T_0} \sigma^2(t) \exp\{2\Lambda(t)\} dt}_{B(T_0)}
\end{aligned}$$

It is obviously sufficient to concentrate on $A(x, T_0)$. We have to show that there is no $\lambda(\cdot) > 0$ such that $A(x, T_0) = \text{const.}\ \forall\, x \in [\underline{x}, \bar{x}],\ \underline{x} \geq 0$. Assume to the contrary that such a function $\lambda(\cdot)$ existed. This would imply

$$\frac{dA(x, T_0)}{dx} = \exp\{-2\Lambda(x+T_0)\} - \lambda(x+T_0) \exp\{-\Lambda(x+T_0)\} \int_{T_0}^{x+T_0} \exp\{-\Lambda(v)\} dv = 0$$

$$(*) \qquad \Leftrightarrow \int_{T_0}^{x+T_0} \exp\{-\Lambda(v)\} dv = \frac{\exp\{-\Lambda(x+T_0)\}}{\lambda(x+T_0)}$$

and using $(*)$

$$\frac{d^2 A(x, T_0)}{dx^2} = -2\lambda(x+T_0) \exp\{-\Lambda(x+T_0)\} - \lambda'(x+T_0) \int_{T_0}^{x+T_0} \exp\{-\Lambda(v)\} dv = 0$$

whence using $\frac{dA(x,T_0)}{dx} = 0$ we get $-2\lambda^2(x+T_0) = \lambda'(x+T_0)$, which is a separable differential equation, and we obtain as the general solution

$$\lambda(t) = \frac{1}{2t-c} \; ,$$

where c is a constant. Since $\lambda(t)$ has to be greater or equal to zero for $t \geq 0$ it follows that $c \leq 0$.

From this we obtain for $\Lambda(t) = \frac{1}{2}\ln(2t-c) + k$, where k is again a constant, which can, however, be neglected if all we are concerned with is whether for this specification of $\lambda(\cdot)$ $\frac{dA(x,T_0)}{dx} = 0$ can hold. Working out $\frac{dA(x,T_0)}{dx}$ for our proposed specification of $\lambda(\cdot)$ we obtain

$$\frac{dA(x,T_0)}{dx} = \frac{\exp\{-\Lambda(x+T_0)\}}{2(x+T_0)-c}\sqrt{2T_0-c} \;>\; 0 \;\forall\; 0 \leq x < \infty, \; c \leq 2T_0$$

Hence we have a contradiction and the proposition is proved. □

3 A Numerical Study

In this section we shall in turn look at the Vasicek (1977), Cox/Ingersoll/Ross (1985 – CIR), Black/Derman/Toy (1990 – BDT), and Sandmann/Sondermann (1993 – SaSo) specifications for the diffusion process for the continuously compounded short rate r, as well as generalizations of CIR in which the exponent of r in the stochastic term of the diffusion is greater $\frac{1}{2}$. Since the analytical tractability of the non–Gaussian model specifications in the literature is very limited, the most efficient way of deriving qualitative results comparable to those in the previous section is to resort to numerical methods.

We calculate discrete approximations of the diffusion processes using the Hull/White (1993) algorithm. This is a forward induction[3] algorithm based on the explicit finite difference method[4], yielding Arrow–Debreu (state) prices for all nodes in the lattice approximating the state space. To get the complete term structure realizations for all nodes, one has to work backwards through the lattice, adding up the one–period state prices along the way[5]. In this manner one can calculate term structure realizations all the way back to period 0. Comparing the period 0 term structure thus calculated with the initial term structure gives us an idea of how exact our approximation is. The maximum deviations of the calculated period 0 term structure from the input term structure, as well as the parameter constellations for each plot, are listed in table D.1.

3.1 The Generalized Vasicek Model

This section serves as an introduction to the different outputs of our numerical study and relates them to the analytical results in section 2. From figure 1, we can read

[3] see Jamshidian (1991)
[4] see Hull/White (1990b)
[5] see appendix C

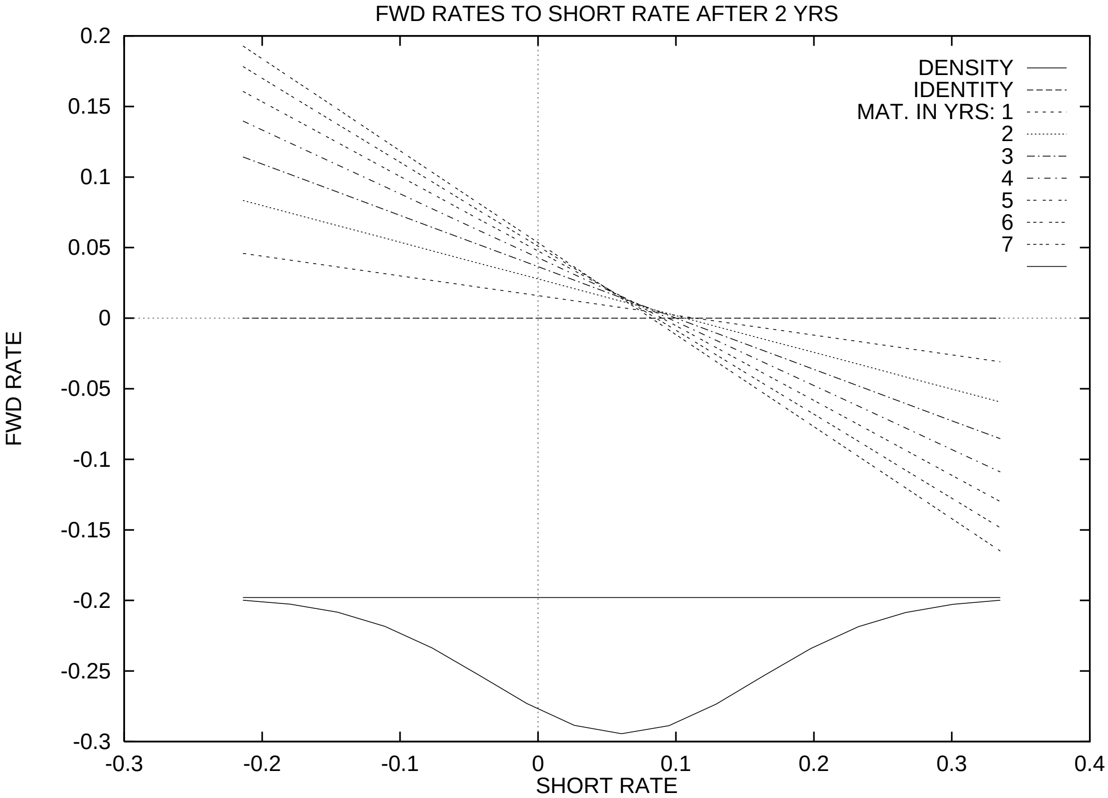


Figure 1: Forward rates in the generalized Vasicek model

off the shapes of term structure realizations in the Vasicek model, as well as the probability mass under the forward measure associated with these realizations. Each of the lines plots the continuously compounded short rate r_t as the state variable on the horizontal axis and the difference between an instantaneous forward rate for a time to maturity y and r_t, i.e. $F_t(y+t) - r_t$, on the vertical axis. In the lower part of the graphic, the density of the short rate under the forward measure is plotted (not to scale).

For Gauss–Markov models forward rates are affine functions of the state variable, as in equation (6). For low short rates the lines for forward rates of longer maturity lie above those of shorter maturity, therefore we have upward sloping term structures. For high short rates the term structures are downward sloping. Also, the lines for forward rates of longer maturities intersect further to the left, so that we get term structures which are upward sloping on the short end and downward sloping on the long end. The distance between any two lines for forward rates of different maturities is an indication of the slope of the forward rate curve on the respective segment. The greater the distance the steeper the forward rate curve.

These properties of the term structure realizations are reflected in figure 2. Furthermore, note the influence of mean reversion. The variance of the forward rates under the forward measure is given in equation (11). Consider the derivative of σ^2_{VAS} with respect to the forward rate maturity x. It becomes immediately clear that for any mean reversion parameter a greater than zero, the variance of instantaneous forward

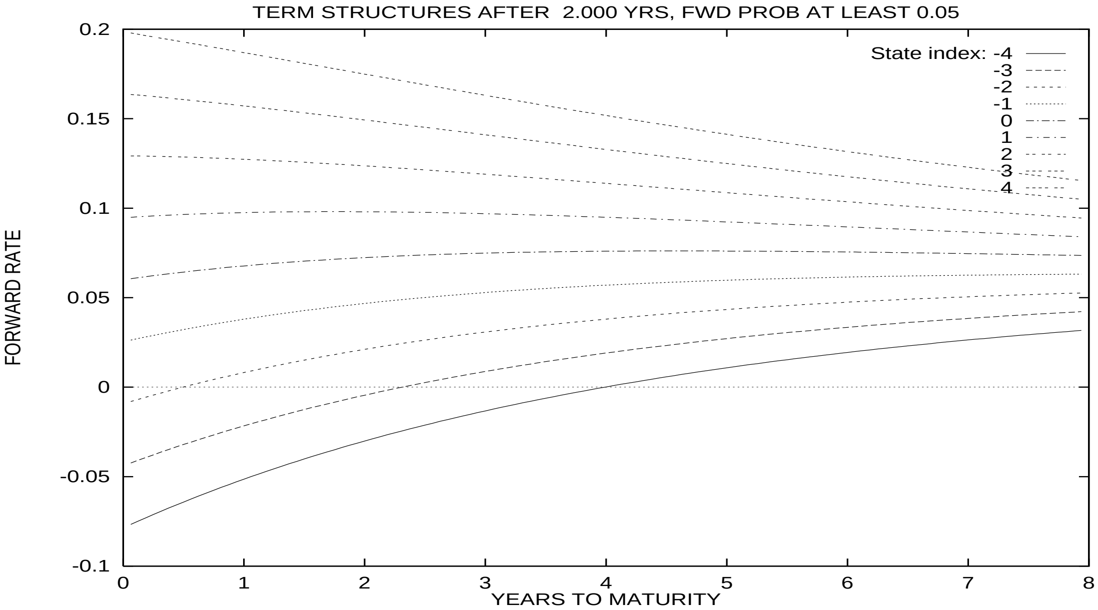


Figure 2: Vasicek term structure realizations

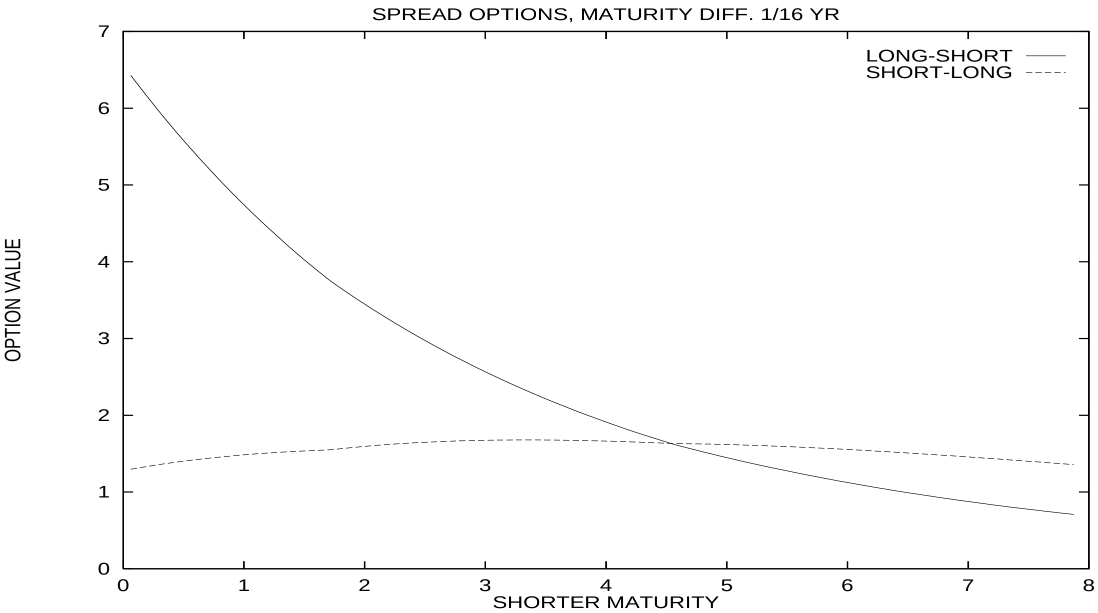


Figure 3: Spread options in the Vasicek model

rates for longer maturities is less than for shorter maturities. Thus the term structures must slope downward in states with a sufficiently[6] high short rate and upward in states with a sufficiently low short rate.

[6]Quantifying "sufficiently" depends on the shape of the initial term structure and on the mean reversion speed a.

In figure 3 we use spread options as an analytical tool to demonstrate the biasedness of the model with respect to the expected (under the forward measure) slope of the term structure. On the vertical axis we plot the price of a contingent claim which pays one dollar for every base point difference between the instantaneous forward rates with time to maturity y years hence and $y+c$ years hence, i.e. either long – short $([F_{T_0}(T_0+y+c) - F_{T_0}(T_0+y)]^+)$ or short – long $([F_{T_0}(T_0+y) - F_{T_0}(T_0+y+c)]^+)$. The first is a bet on upward sloping, the second on downward sloping term structures. On the horizontal axis we plot the shorter time to maturity y, keeping the maturity difference c constant.

Figure 3 demonstrates graphically what we derived analytically in section 2. The initial forward rate curve is flat at 6%. For short times to maturity y, the claim contingent on upward sloping term structures is more valuable, while for longer y this relationship is reversed. When the mean reversion parameter is constant, there is exactly one maturity for which the two curves intersect. For flat initial term structures, the intersection point is determined by the mean reversion parameter only[7]. Thus two effects influence the spread option values as y is increased: It becomes more likely that the term structure is downward sloping at this maturity and the spreads between $F_{T_0}(T_0+y)$ and $F_{T_0}(T_0+y+c)$ become smaller due to the lower volatility of the longer rates. This gives an intuition for why the expected present value of $([F_{T_0}(T_0+y+c) - F_{T_0}(T_0+y)]^+)$ falls monotonically in y, while for some parameter constellations, specifically a sufficiently high speed of mean reversion and a sufficiently high risk parameter σ, there is some $y > 0$ for which the expected present value of $([F_{T_0}(T_0+y) - F_{T_0}(T_0+y+c)]^+)$ is maximal. Increasing σ increases the difference in value between the two spread options on either side of the intersection point: For flat initial term structures, the difference between the spread option prices is proportional to σ^2 (equation (15)). A comparison of figures 3 and 6 illustrates this effect.

3.2 The β–root Process

A whole class of models is based on the specification (Hull/White 1993)

$$dr_t = (\theta(t) - ar_t)dt + \sigma r_t^\beta dW_t \tag{17}$$

This family includes the generalized Vasicek (1977) model for $\beta = 0$, Cox/Ingersoll/Ross (1985) for $\beta = 0.5$, and Brennan/Schwartz (1977) for $\beta = 1$. The most important qualitative difference between the Vasicek specification and those with $\beta \geq 0.5$ is that for these models there are parameter constellations such that there exists a solution to the stochastic differential equation (17) which precludes negative interest rates.[8] Also, in contrast to Vasicek, not every positive initial forward rate curve can be fitted.[9] As for the shapes of the term structures generated by these models, our numerical studies show that there is no qualitative difference from those implied by the Vasicek specification.

[7] see proof of Proposition 3.

[8] Specifications with $\beta \in]0; 0.5[$ do not make sense, because in such a case the solution to (17) is not unique. See Arnold (1973), p. 124.

[9] see Hull/White (1993)

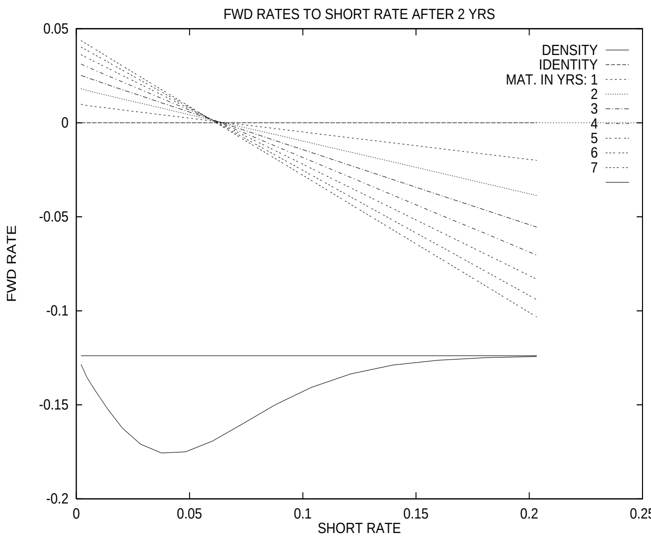


Figure 4: Cox/Ingersoll/Ross

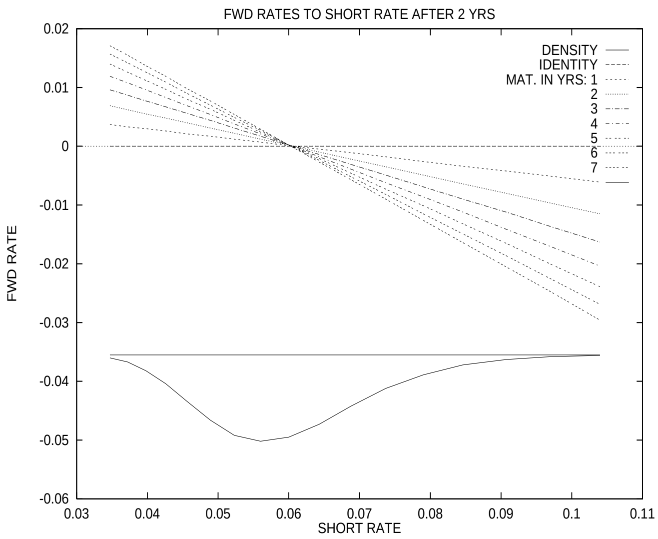


Figure 5: Brennan/Schwartz

As shown in figures 4 and 5, there are three possible term structure shapes: Downward sloping for states in which the short rate is high, upward sloping when the short rate is low, and upward sloping on the short end and downward sloping on the long end for states in the middle.[10]

The distributions of the short rate do differ: It is well known that for $\beta = 0.5$ the short rate is non–central[11] χ^2 and for $\beta = 1$ lognormally distributed. However, this does not have any significant impact on the spread option values. Consider figures 6, 7, 8, and 9. They show the spread option values (as defined in the previous section) for spread options maturing two years hence, for four different pairs of β and σ (see table D.1 for the complete parameter constellations). In order to make the plots comparable, σ was chosen in such a manner that the variance of the short rate in two years' time (as seen from today) is the same in each of the four cases. This leads to nearly identical spread option values for the respective maturities. While it is true that matching variances of the short rate for some fixed horizon does not imply that the short rate variances for other horizons will also be exactly matched, our simulation results as exemplified in table D.2 show that the values of σ required to obtain an exact match of the short rate variances for different time horizons are remarkably stable so that the qualitative result in figures 6 through 9 would carry over to other time horizons with the values of σ unchanged.

We therefore conclude that by specifying different values of β, it is not possible to implement term structure models which differ substantially with respect to their structural implications for future realizations of the shape of the term structure. The qualitative results concerning the development of the term structure shapes over time

[10]For the Cox/Ingersoll/Ross model, the simulations are in keeping with the analytical results about the shape of the yield curve in the original model setting, i.e. with $\theta(t)$ constant over t. See Cox/Ingersoll/Ross (1985), p. 394 and Kan (1992).

[11]see Cox/Ingersoll/Ross (1985), p. 392.

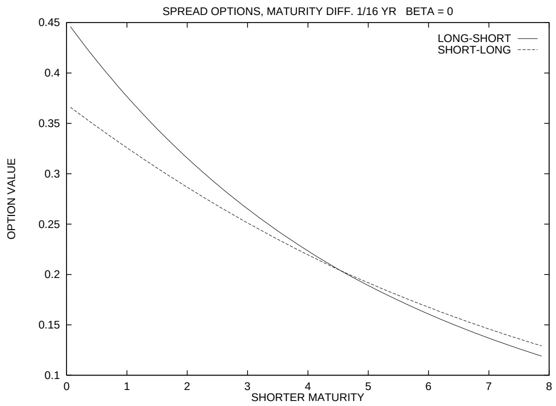


Figure 6: $\beta = 0$

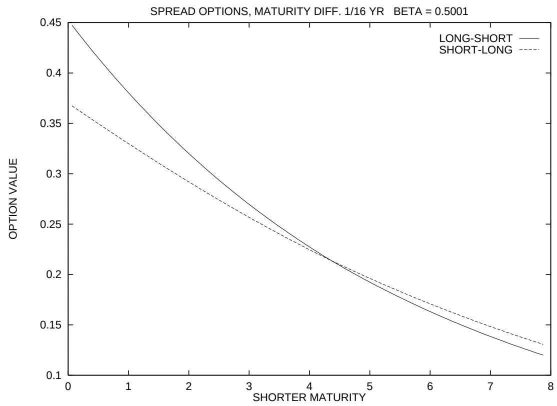


Figure 7: $\beta = 0.5$

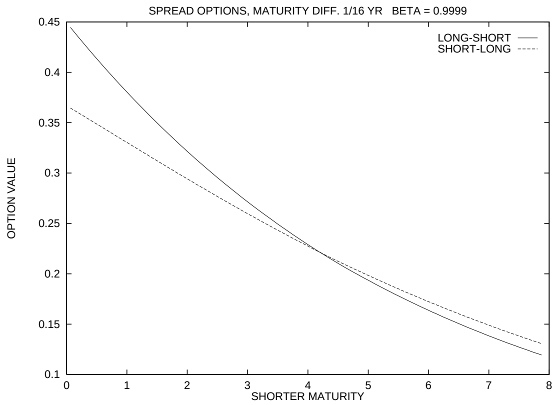


Figure 8: $\beta = 1$

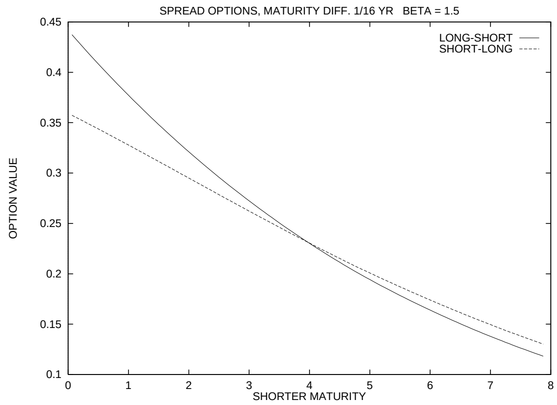


Figure 9: $\beta = 1.5$

derived analytically for the Gauss–Markov case are upheld in our numerical studies for other values of β. In particular we find that given a time to maturity difference c in every model there is only one time to maturity for which the model is unbiased according to concept 2.
It is worth noting that the implicit structural biases identified so far are unaffected by the shape of the initial term structure. By defining "at–the–money" spread options, i.e. options that pay one dollar for every base point difference between the spread between two forward rates at maturity of the option and the respective spread in the initial term structure $[(F_{T_0}(y+c+T_0) - F_{T_0}(y+T_0)) - (F_0(y+c+T_0) - F_0(y+T_0))]^+$, we verified that all these models have a tendency to produce term structures that are more upward sloping on the short and more downward sloping on the long end irrespective of the slope of the initial term structure.

Thus, by focusing on the movements of the whole term structure rather than on the short rate dynamics only, we see the empirical findings of Chan/Karolyi/Longstaff/Sanders (CKLS – 1992) in a different light. For one, while CKLS report that by making the volatility more dependent on the level of the short rate by increasing the exponent β to 1.499 one attains a better empirical fit of the model of short rate dynamics[12], this does not mitigate the structural imbalance in the dynamics of the entire forward rate curve. Secondly, CKLS find that the evidence for mean reversion in the short rate is very weak. This is in contrast to the findings by Chen and Scott (1993) who report strong evidence for mean reversion in the factor reflecting the variability of the level of the short rate in one–, two–, and three–factor models. Our analysis shows that the introduction of a mean reversion parameter has a much stronger effect on the structural features, namely the generation of downward sloping term structures, than the value of β.

3.3 The Black/Derman/Toy Model

Black, Derman, and Toy (1990) have constructed a binomial model which avoids the problems that the β–root specifications encounter when fitting the initial term structure. The continuous time equivalent of this model is[13]

$$d \ln r_t = (\theta(t) - a \ln r_t)dt + \sigma dW_t \tag{18}$$

which by Ito's Lemma is transformed to

$$dr_t = r_t \cdot (\theta(t) - a \ln r_t + \frac{1}{2}\sigma^2)dt + r_t \sigma dW_t \tag{19}$$

for the continuously compounded short rate r, where a is the speed of mean reversion. In figure 10, again setting σ so that the variance of the short rate in two years' time is the same as in figures 6 through 9, we note that there is no substantial qualitative difference between this specification and the β–root models. The forward rate curves are merely somewhat flatter, leading to lower spread option prices.

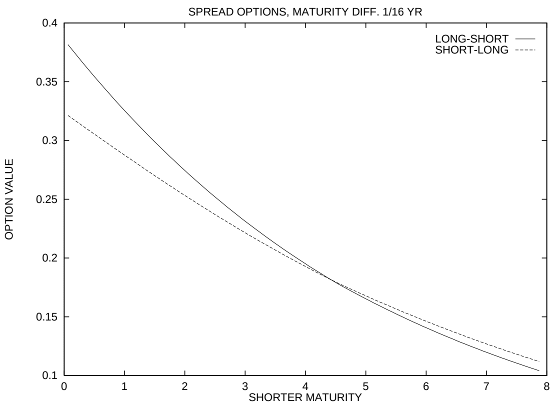


Figure 10: Black/Derman/Toy

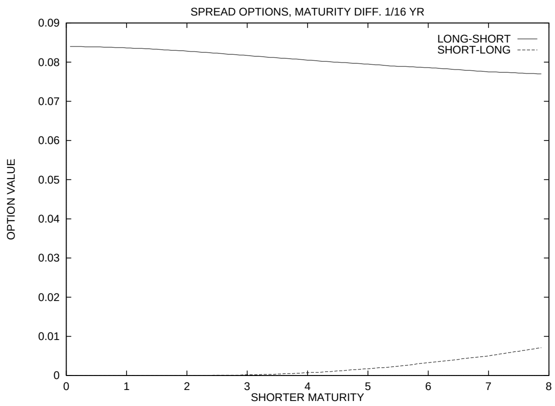


Figure 11: Sandmann/Sondermann

[12]A recent study by Duffee (1993) takes are more comprehensive look at US short–term interest rates. He finds evidence for values of β "anywhere between 0 and 1.5 [...], depending on the short–term rate used and the time period examined."

[13]see Black/Karasinski (1991)

3.4 The Sandmann/Sondermann Model

Sandmann and Sondermann (1993) propose a term structure model in which the actuarial short rate is lognormally distributed[14]. The continuously compounded short rate then follows the diffusion

$$dr_t = (1 - e^{-r_t})[(\theta(t) - \frac{1}{2}(1 - e^{-r_t})\sigma^2)dt + \sigma dW_t] \tag{20}$$

The term that can be viewed as generating a mean reversion effect $\frac{1}{2}(1 - e^{-r})\sigma^2$, is bounded between 0 and $\frac{1}{2}\sigma^2$ for all $r \in \mathbb{R}_+$.[15] For the same variance of the short rate realizations in two years' time as in figure 6 the term structures are therefore flatter and upward sloping term structures carry relatively more weight, as evidenced in figure 11: Spread option prices are much lower and for all maturities the claims contingent on upward sloping term structures are more valuable than those contingent on downward sloping term structures.

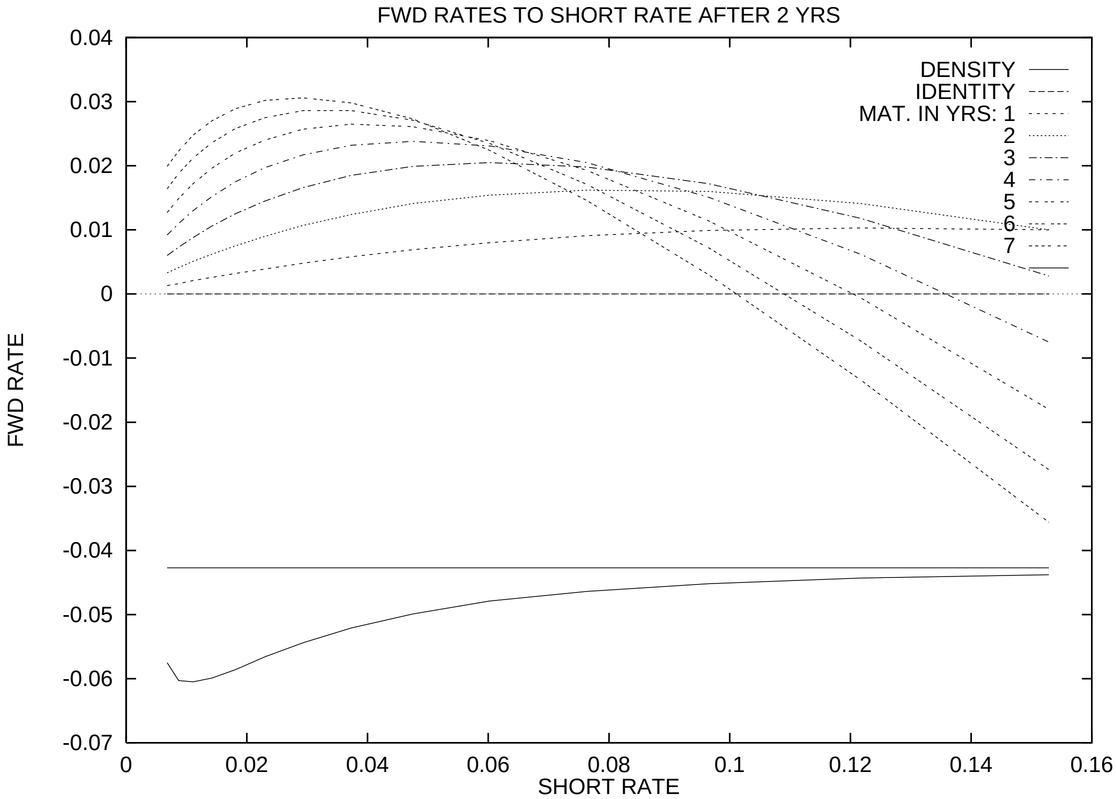


Figure 12: Forward rates in the Sandmann/Sondermann model

However, raising σ sufficiently leads to spread option prices as in figure 14. This graphic is quite similar to figure 3 (the Vasicek case): On the short end upward

[14]An important feature of this specification is that it avoids the problem of infinite expected roll–over returns encountered in the Black/Derman/Toy model or when setting $\beta \geq 0.5$ in the β–root process. See Sandmann/Sondermann (1993) and Hogan/Weintraub (1993).

[15]Negative interest rates are not generated. The problems inherent in the β–root specifications when fitting the initial term structure do not present themselves in this model.

sloping term structure carry more weight while on the long end this relationship is reversed. On the short end only upward sloping term structures carry any substantial probability, as evidenced in figures 12 and 13.

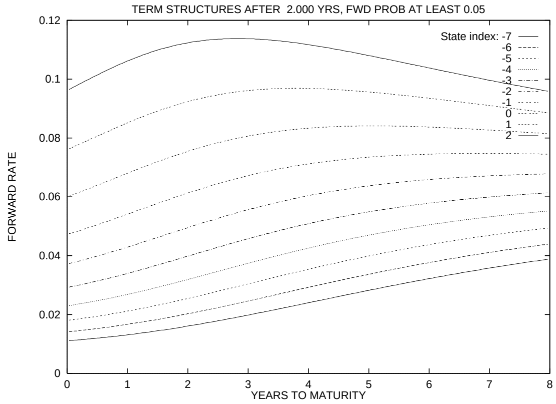


Figure 13: Sandmann/Sondermann term structure realizations

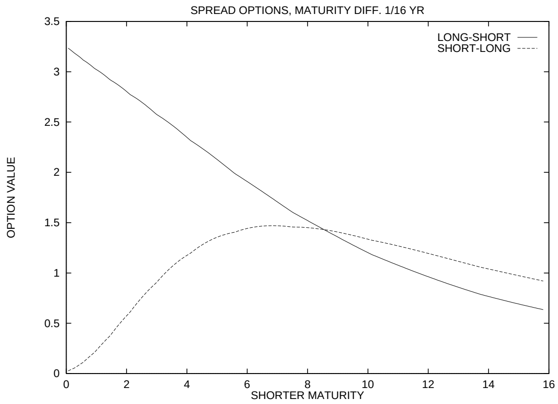


Figure 14: Spread options in the Sandmann/Sondermann model

4 A One Factor Model With Bounded Short Rates

While a lower bound at zero for the realizations of nominal interest rates is necessary in a model designed for a monetary economy in order to avoid violating the no arbitrage condition, there is no such argument for an upper bound.

However, remembering that the nominal interest rate is after all a macroeconomic variable one might want a model to be flexible enough to reflect one's a priori knowledge about the economic environment one is operating in. The knowledge about the institutional set–up and conduct of the monetary policy seems to be particularly pertinent in this context. If e.g. the stability of the value of money is an essential of the monetary policy and institutions are designed in such a way as to make this pledge credible there is reason to believe that interest rates, even in the long run, won't exceed a certain upper bound. Notwithstandingly, they may be rather volatile within a certain range.
These ideas are reflected in the following model

$$(21) \qquad dr_t = \left[\frac{c(a+b-2r_t)}{b-a} - \frac{(b-r_t)(r_t-a)}{b-a}\theta(t)\right] dt + \sigma\frac{(b-r_t)(r_t-a)}{b-a} dW_t$$

where $b > a$ are the upper and lower bounds respectively; $c > 0$ the strength of mean reversion and we assume that a probability space is given as in section 2.

First we show that for this model short rates will indeed remain in the interval (a, b) if the process is started at $r_0 \in (a, b)$. To this end consider

Proposition 5

Let $\theta(t)$ be constant on $[t_1, t_2]$ and $r_{t_1} \in (a, b)$ then the process defined by (21) will remain in (a, b) on $[t_1, t_2]$.

Proof

See appendix A.1.

Given this result considering again the whole time interval $[0, T^*]$ and assuming that there is a sequence of step functions $(\theta_n(t))_{n\in\mathbb{N}}$ approximating $\theta(t)$ we have by Proposition 5 and the continuity of the sample paths that $r_t \in (a, b)$ on $[0, T]$ if $r_0 \in (a, b)$ for any step function $\theta_n(t)$ and hence also for $\theta(t)$ since $(\theta_n(t))_{n\in\mathbb{N}}$ approximates $\theta(t)$.

An immediate consequence of bounded spot rates is given in

Proposition 6

If $r_t \in (a, b) \quad \forall t \in [0, T^*]$ then $F_t(x + t) \in (a, b) \quad \forall t \in [0, T^*], x \leq T^* - t$.

Proof

Suppose to the contrary that there is some $\bar{t} \in [0, T^*]$ for which there is some $\bar{x} \leq T^* - \bar{t}$ such that $F_{\bar{t}}(\bar{x} + \bar{t}) \geq b$. This means that it is possible at time $\bar{t}$ to enter an agreement to lend money at a rate $F_{\bar{t}}(\bar{x} + \bar{t})$ at time $\bar{t} + \bar{x}$. However, this provides an arbitrage opportunity for the lender since at time $\bar{t} + \bar{x}$ he can borrow the required amount at the short rate $r_{\bar{t}+\bar{x}} \in (a, b)$ whereas the rate of return on the forward loan will be $F_{\bar{t}}(\bar{x} + \bar{t}) \geq b > r_{\bar{t}+\bar{x}}$. Reversing the argument shows that $F_t(x + t) > a \ \forall t \in [0, T^*]$, $x \leq T^* - t$. $\square$

Further, in order to make sure that the following simulations make sense, we show

Lemma 3

If $r_0 \in (a, b)$ and $\sup_{t\in[0,T^*]} \theta(t) < \infty$ there exists a pathwise unique strong solution to (21) on $[0, T^*]$ such that $r_t \in (a, b) \ \forall t \in [0, T^*]$.

Proof

See appendix A.2.

To complete the preliminaries notice that the model under consideration can basically be calibrated to any initial forward rate curve that takes values only in (a, b). We give an argument for this assertion in appendix B.

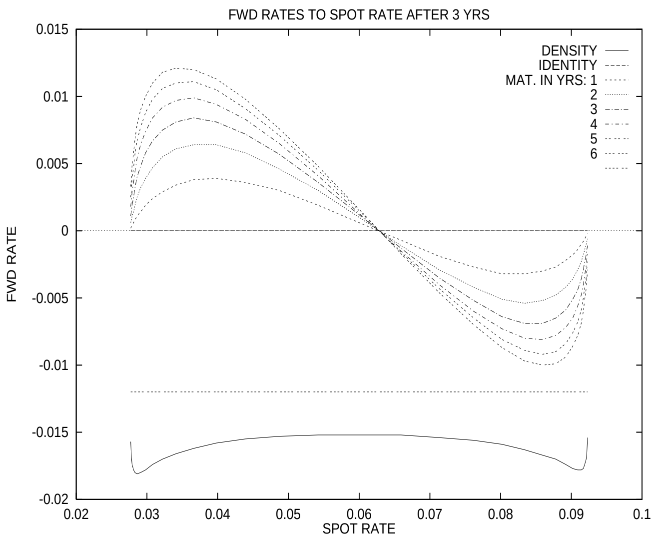


Figure 15: Bounded model, $c = 0.045$

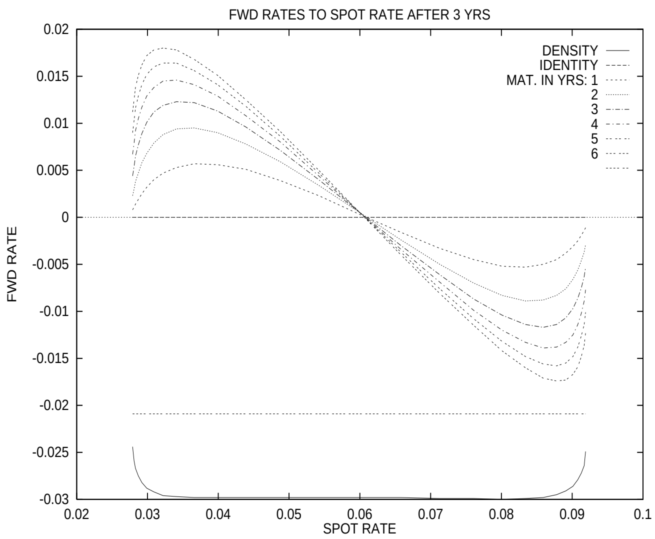


Figure 16: Bounded model, $c = 0.45$

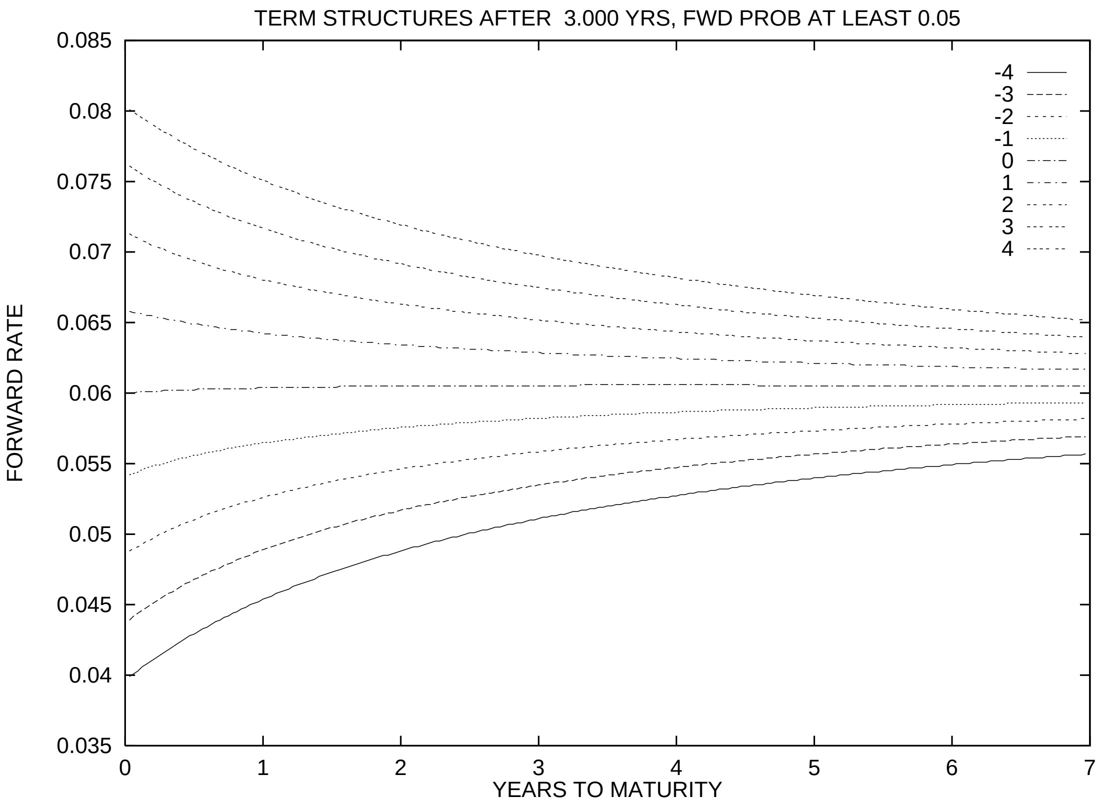


Figure 17: Bounded rate model term structure realizations

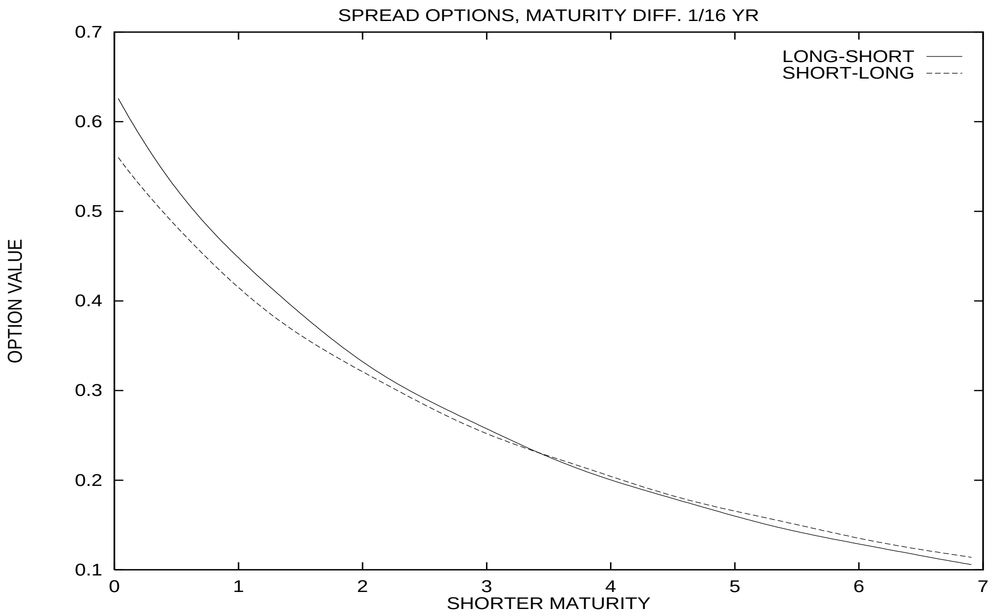


Figure 18: $\sigma = 0.6$ Maximal bias = 10.51%

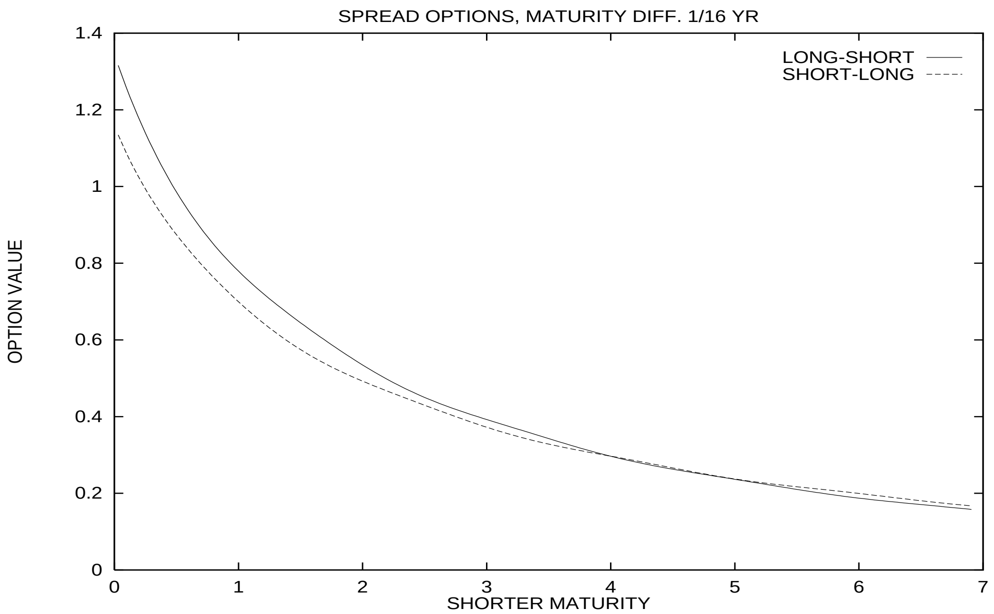


Figure 19: $\sigma = 1.2$ Maximal bias = 13.78%

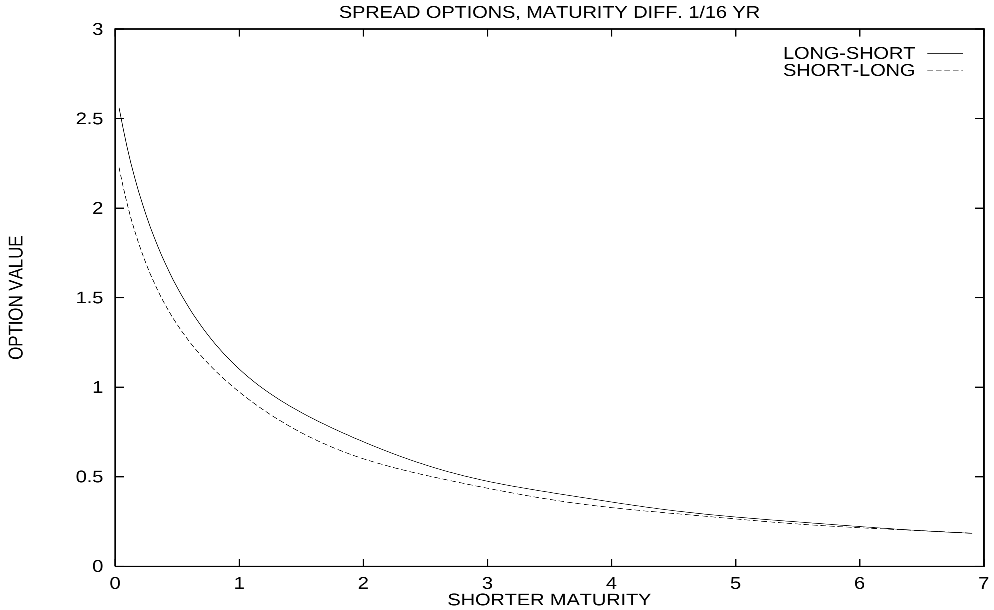


Figure 20: $\sigma = 2.4$ Maximal bias = 13.06%

We start studying the properties of this model by first considering the role of the mean reversion parameter. To this end we assume a flat term structure at 6% and a symmetric interval for the short rate realizations ranging from 2.75% to 9.25%. We set $\sigma = 1.2$ and compare $c = 0.045$ (figure 15) to $c = 0.45$ (figure 16). It can be seen that for weak mean reversion the distribution of the short rate is strongly bimodal whereas this effect is mitigated by increasing the speed of mean reversion. In general, reducing the volatility and/or increasing the speed of mean reversion will lead to unimodal distributions of the short rate. The effect of the speed of mean reversion on endogenously generated term structures are in line with our earlier findings. Weak mean reversion will lead to comparatively flat term structures whereas stronger mean reversion will result in term structures which exhibit more pronounced slopes for the shorter maturities and are rather flat for the long maturities. This is also reflected in figure 17 which shows that strong mean reversion will lead to volatilities decreasing sharply with maturity and hence necessarily to more pronounced slopes of term structures.

Considering the spread options we observe that low speed of mean reversion will again entail a more pronounced bias towards upward sloping term structures whereas increasing the speed of mean reversion will lead to the values of both types of options tracking each other closely across maturities.

It is a distinctive feature of this model that it is the speed of mean reversion, c, rather than the relative size of σ and c which determines whether or not the model is substantially biased towards increasing term structures. This is evidenced by figures 18, 19 and 20, where for a given speed of mean reversion $c = 0.01$ we raise σ from 0.6 to 1.2 and 2.4. The maximal bias towards increasing term structures $[V_0([F_{T_0}(T_0+y+c)-F_{T_0}(T_0+y)]^+)/V_0([F_{T_0}(T_0+y)-F_{T_0}(T_0+y+c)]^+)]-1$ changes from 10.51% to 13.87% and 13.06%. This is a very mild increase when compared to the effects of a similar increase of σ in the models considered in the previous section.

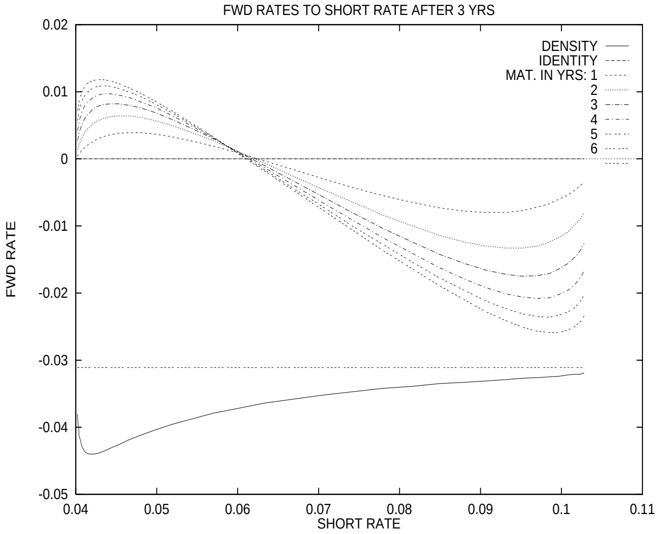


Figure 21: Bounded rates Range (4%; 10.5%)

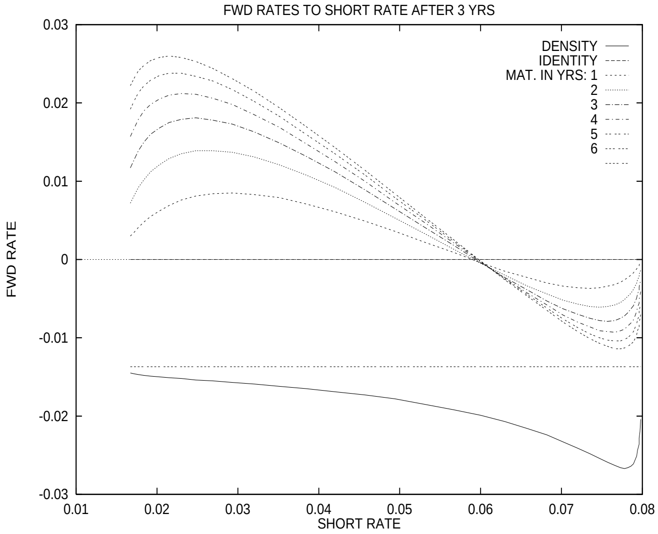


Figure 22: Bounded rates Range (1.5%; 8%)

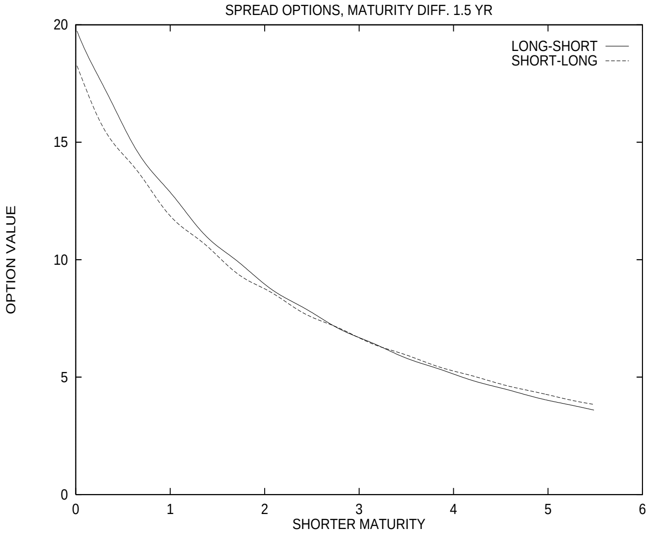


Figure 23: Bounded rates Range (4%; 10.5%)

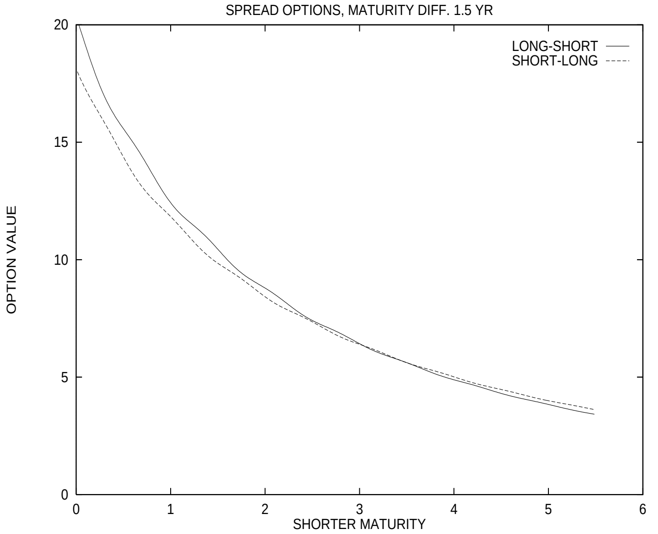


Figure 24: Bounded rates Range (1.5%; 8%)

We now turn to the asymmetric situation where the original term structure is still at 6% but the intervals are now from 4% to 10.5% (figures 21 and 23) and from 1.5% to 8% (figures 22 and 24) respectively. The mean reversion is 0.45 and $\sigma = 1.2$. The distributions of the short rate are now right (figure 23) and left (figure 24) skewed respectively, which is intuitive since under the forward measure the expected value of the short rate has to equal 6% in this framework.

The remarkable result, however, is that disregarding numerical effects the spread option values indicate that this model is fairly little biased between generating upward and downward sloping term structures across all maturities.

5 Conclusion

In this paper we have reviewed a number of specifications of diffusion processes for the short term interest rate, all but one of which have been suggested in the literature. Our aim has been twofold. The first was to establish which typical features of future movements of the term structure a user of these models implicitly accepts. The second was to judge on the plausibility of the term structure models analyzed. Our main results are

- The continuous time Ho/Lee model will always generate upward sloping term structures. A user of this model can never expect an upward sloping term structure to become downward sloping. The other models are very similar in that they can all generate monotonically increasing and decreasing as well as hump-shaped term structures. Some peculiarities are, however, worth noting. In the Sandmann/Sondermann model monotonically decreasing term structures are fairly unimportant while this is true for hump-shaped term structures in the bounded rate model.

- The introduction of mean reversion is essential if a model of the kind considered before is to generate in a significant proportion term structures which are downward sloping over some maturity range. Without mean reversion all the models considered are biased towards upward sloping term structures. Introducing mean reversion will lead to more steeply sloped term structures for short maturities.

- The generalized Vasicek, β–root, and Black/Derman/Toy models are biased towards upward sloping term structures for short maturities and downward sloping term structures for long maturities. This is also true of the Sandmann/Sondermann model; however, the fact that the mean reversion of this specification is bounded and cannot be increased exogenously implies a stronger bias towards upward sloping term structures than in the models where this bias can be mitigated by increasing the mean reversion parameter.

- In the generalized Vasicek, β–root, and Black/Derman/Toy models, the model–immanent bias towards either upward or downward sloping term structures (concept 2) increases with σ. The maturity where the bias switches from upward to downward sloping term structures is reduced as the speed of mean reversion is increased.

- For comparable values of σ the spread option values obtained in the generalized Vasicek, β–root, and Black/Derman/Toy models do not differ substantially.

Hence the implicit structural biases (concept 2) concerning the generation of future term structures are very similar in these models.

- In the bounded short rate model the bias towards both upward and downward sloping term structures can be reduced considerably across all maturities, as compared to the other models. This result continues to hold if σ is increased for a given speed of mean reversion.

Returning to our second question as to which of the models analysed can be regarded as being the most plausible in view of the above analysis the following picture emerges. Apart from the Ho/Lee model which can be dismissed as being implausible since it only generates increasing term structures there is a great similarity between the models considered, both concerning the shapes of term structures that can be generated and the biasedness towards increasing and decreasing term structures. Arguably the typical pattern of the models being biased towards increasing term structures for short maturities and decreasing term structures for longer maturities is hardly convincing since it is not clear why the bias of a model should be a function of time to maturity. A bias towards increasing term structures is not preferable to any other bias even if one regards upward sloping term structures as the "normal" case. For a model with a bias towards upward sloping term structures will put ever more weight on ever more upward sloping term structures, even if the initial term structure is already increasing. Therefore it would seem that a model would be more plausible than others if it allowed for the emergence of many shapes of term structures while exhibiting as small a bias as possible. The model with bounded realizations of the short rate goes some way in this direction.

We see the analysis of the qualitative implications of various one–factor models conducted in this paper as a platform from which to embark on two lines of further research: For one it would be interesting to see how the introduction of a second stochastic factor influences the results contained herein. Secondly, these results may prove useful when evaluating term structure models empirically, as we have identified some model–immanent restrictions on the term structure movements.

A Proofs

A.1 Proof of Proposition 5

By Lemma 6.3 in Karlin/Taylor it is sufficient to show that

$$\begin{aligned}\lim_{x\to a}\int_{x_0}^{x} S(\xi)d\xi &= -\infty\\ \lim_{x\to b}\int_{x_0}^{x} S(\xi)d\xi &= +\infty\end{aligned}$$

where $x_0 \in (a,b)$ is an arbitrary constant and

$$S(\xi) := \exp\{-\int_{\xi_0}^{\xi}[2\mu(\eta)/\sigma^2(\eta)]d\eta\}$$

$\xi_0 \in (a,b)$ is an arbitrary constant and $\mu(\eta)$ and $\sigma(\eta)$ are the drift and diffusion coefficients respectively of the process in equation (21).

(i) Let us first consider

$$\begin{aligned}&\int_{\xi_0}^{\xi}\frac{2\theta}{\sigma^2}(b-a)\frac{1}{(b-\eta)(\eta-a)}d\eta\\ =\ &\frac{2\theta}{\sigma^2}\left[\int_{\xi_0}^{\xi}\frac{1}{b-\eta}d\eta+\int_{\xi_0}^{\xi}\frac{1}{\eta-a}d\eta\right]\quad\text{(using partial fractions)}\\ =\ &\frac{2\theta}{\sigma^2}\left[\ln\left(\frac{\xi-a}{b-\xi}\right)+\ln\left(\frac{b-\xi_0}{\xi_0-a}\right)\right]\\ =\ &\frac{2\theta}{\sigma^2}\ln\frac{\xi-a}{b-\xi}\end{aligned}$$

where ξ_0 was chosen to be $\frac{a+b}{2} \in (a,b)$.

(ii) Now consider

$$\begin{aligned}&\int_{\xi_0}^{\xi}\frac{2c}{\sigma^2}(b-a)\frac{a+b-2\eta}{(b-\eta)^2(\eta-a)^2}d\eta\\ =\ &\frac{2c}{\sigma^2}\left[\int_{\xi_0}^{\xi}\frac{1}{(\eta-a)^2}d\eta-\int_{\xi_0}^{\xi}\frac{1}{(b-\eta)^2}d\eta\right]\quad\text{(using partial fractions)}\\ =\ &\frac{2c}{\sigma^2}\left[\frac{4}{b-a}-\left(\frac{1}{\xi-a}+\frac{1}{b-\xi}\right)\right]\quad\text{,where } \xi_0=\frac{b+a}{2}\end{aligned}$$

(iii) This yields

$$\int_{x_0}^{x}S(\xi)d\xi=\int_{x_0}^{x}\left(\frac{\xi-a}{b-\xi}\right)^{\frac{2\theta}{\sigma^2}}\exp\left\{\frac{2c}{\sigma^2}\left(\frac{1}{\xi-a}+\frac{1}{b-\xi}\right)\right\}kd\xi$$

where

$$k = \exp\left\{\frac{-8c}{\sigma^2(b-a)}\right\}$$

Since the exponential function grows faster than any polynomial as its argument goes to infinity, for every $\theta \in \mathbb{R}$ there exists some $\bar{x}_0(\theta) < b$ such that $S(\xi) > \frac{k}{b-\xi}, \quad \xi \in (\bar{x}_0, b)$. Since x_0 is arbitrary we can assume $x_0 \geq \bar{x}_0(\theta)$. We then have

$$\lim_{x\to b}\int_{\bar{x}_0}^{x} S(\xi)d\xi \geq \lim_{x\to b}\int_{\bar{x}_0}^{x}\frac{k}{b-\xi}d\xi = \lim_{x\to b} k[\ln(b-\bar{x}_0) - \ln(b-x)] = \infty$$

Similarly there exists $\bar{\bar{x}}_0$ such that $S(\xi) > \frac{k}{\xi-a}, \quad \xi \in (a, \bar{\bar{x}}_0)$. Hence for $x \leq \bar{\bar{x}}_0$

$$\lim_{x\to a}\int_{\bar{\bar{x}}_0}^{x} S(\xi)d\xi \leq \lim_{x\to a}\int_{\bar{\bar{x}}_0}^{x}\frac{k}{\xi-a}d\xi = \lim_{x\to a} k[\ln(x-a) - \ln(\bar{\bar{x}}_0 - a)] = -\infty$$

A.2 Proof of Lemma 3

It is sufficient to check the Lipschitz and growth conditions for the drift and diffusion coefficients.

(i) the growth condition is trivially satisfied since the diffusion coefficient and

$$\frac{c(a+b)-2r)}{b-a} - \frac{(b-r)(r-a)}{b-a}\sup_{t\in[0,T]}\theta(t)$$

are continuous functions of r on $[a, b]$.

(ii) Lipschitz condition

$$\begin{aligned}
&\left|\left(\frac{c(a+b-2x)}{b-a} - \frac{(b-x)(x-a)}{b-a}\theta(t)\right) - \left(\frac{c(a+b-2y)}{b-a} - \frac{(b-y)(y-a)}{b-a}\theta(t)\right)\right| \\
&\leq \frac{1}{b-a}\left|\Big(2c + |\theta_t|(b-a)\Big)|y-x|\right| \\
&\leq \sup_{t\in[0,T^*]}\frac{1}{b-a}\Big(2c + |\theta(t)|(b-a)\Big)|y-x| \\
&= k|y-x|
\end{aligned}$$

similarly for the diffusion coefficient.

B Fitting the Bounded Short Rate Model to an Initial Term Structure

We consider the question in how far the model in (21) can be calibrated to a given original term structure. To this end apply the transformation

$$x = f(r) = \ln\frac{r-a}{b-r}$$

to (21) to obtain

$$dx_t = \left[\frac{(a+b-2r_t)\cdot c}{(b-r_t)(r_t-a)} - \frac{1}{2}\frac{a+b-2r_t}{b-a}\sigma^2 - \theta(t)\right] dt + \sigma dW_t \tag{22}$$

A discrete time approximation to (22) is given by

$$\Delta x_t = \left[\frac{(a+b-2f^{-1}(x_t))\cdot c}{(b-f^{-1}(x_t))(f^{-1}(x_t)-a)} - \frac{1}{2}\frac{a+b-2f^{-1}(x_t)}{b-a}\sigma^2 - \theta(t)\right] \Delta t + \sigma\epsilon_t\sqrt{\Delta t}$$

where

$$\begin{aligned}
\epsilon_t &\sim \text{i.i.d. } N(0,1) \\
t &= j\cdot\Delta t \\
j &= 0,1,\ldots\frac{T}{\Delta t} \\
f^{-1}(x) &= a + \frac{b-a}{1+\exp\{-x\}}
\end{aligned}$$

We assume that the original term structure is such that the forward rates are all in (a,b). We thus have $r_0 = F_0(0) \in (a,b)$. Now we consider the zero bond maturing at time $2\Delta t$. There must exist a $\theta(0)$ such that

$$D(2\Delta t) = \exp\{-\sum_{i=0}^{1} F_0(i)\Delta t\} \stackrel{!}{=} E\left[\exp\left\{(-r_0 - f^{-1}(x_0+\Delta x_0))\Delta t\right\}\right]$$

$$\Leftrightarrow \exp\{-F_0(1)\Delta t\} \stackrel{!}{=} E[\exp\{-f^{-1}(x_0+\Delta x_0)\Delta t\}]$$

Now the random variable under the E–operator has support $S = [\exp\{-b\Delta t\}, \exp\{-a\Delta t\}]$ and $\exp\{-F_0(1)\Delta t\} \in \text{int } S$ by assumption. Varying $\theta(0) \in \mathbb{R}$ the expectation on the right hand side can be moved to any value in the interior of S. Since f is strictly monotonic in θ there is exactly one $\theta(0)$ such that the above equation is satisfied. Moreover $r_1 = f^{-1}(x_0 + \Delta x_0) \in (a,b)$ since $f^{-1}: \mathbb{R} \to (a,b)$.
Assuming that the function $\theta(\cdot)$ has been constructed up to time $(j-1)\Delta t$ to fit the original term structure we now consider

$$\begin{aligned}
D((j+1)\Delta t) &= \exp\left\{-\sum_{i=0}^{j} F_0(i)\Delta t\right\} \\
&\stackrel{!}{=} E\left[\exp\left\{-\sum_{k=0}^{j-1} f^{-1}\left(x_0 + \sum_{i=0}^{k}\Delta x_i\right)\Delta t\right\}\exp\left\{-f^{-1}\left(x_0+\sum_{i=0}^{j}\Delta x_i\right)\Delta t\right\}\right]
\end{aligned}$$

$$\Longleftrightarrow$$

$$\exp\{-F_0(j)\Delta t\} \stackrel{!}{=}$$

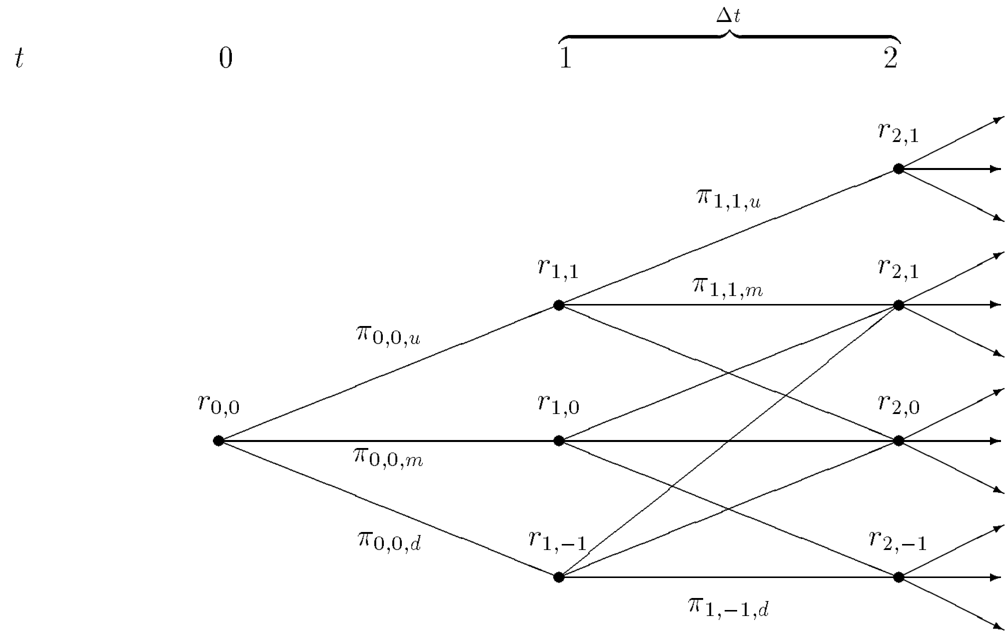
$$E\left[\exp\left\{-\left[\sum_{h=0}^{j-1} f^{-1}\left(x_0+\sum_{i=0}^{h}\Delta x_i\right) + \sum_{i=0}^{j-1}F_0(i)\right]\Delta t\right\}\exp\left\{-f^{-1}\left(x_0+\sum_{i=0}^{j}\Delta x_i\right)\Delta t\right\}\right]$$

Now varying $\theta(j-1) \in \mathbb{R}$ the expectation can take any value between $\exp\{-b\Delta t\}$ and $\exp\{-a\Delta t\}$ and the argument proceeds as before[16].

[16]Obviously if $\Delta t \to 0$ additional assumptions on the smoothness of the original term structure are needed to guarantee that the function $\theta(\cdot)$ remains finite.

C Calculating Term Structures From the Hull/White (1993) Algorithm Output

Consider the following case



The one–period zero bond price for state i in period n is given by

$$B_{n,i}(1) = \exp\{-\Delta t r_{n,i}\}$$

Let $\pi_{n,i,d}$, $\pi_{n,i,m}$, $\pi_{n,i,u}$ be the one–period state prices of the down–, middle–, and up–state, as seen from state i in period n. The price of a k–period zero bond in (n, i) is the weighted sum of $(k-1)$–period zero bond prices in the three states attainable from (n, i). For state -1 in period 1:

$$B_{1,-1}(k) = \pi_{1,-1,d}B_{2,-1}(k-1) + \pi_{1,-1,m}B_{2,0}(k-1) + \pi_{1,-1,u}B_{2,1}(k-1)$$

If the model has been calculated for N periods, then by backward induction for each state (n, i) we can determine the values of all zero bonds maturing in $k \in \{1, \ldots, N-n\}$ periods, where n runs from $N-1$ to 0.

D Tables

D.1 Parameter Constellations

The initial curve of instantaneous forward rates is flat at 6% for all plots.

Figure	Model	Time[17]	β	MR	σ	Range[18]	Ref.[19]	Dev.[20]
1	Vasicek	2 yrs.	0	0.15	0.08	$(-\infty;\infty)$	16	0.12500
2	Vasicek	2 yrs.	0	0.15	0.08	$(-\infty;\infty)$	16	0.12500
3	Vasicek	2 yrs.	0	0.15	0.08	$(-\infty;\infty)$	16	0.12500
4	CIR	2 yrs.	0.5	0.15	0.12	$[0;\infty)$	16	0.01468
5	Brennan/Schwartz	2 yrs.	1	0.15	0.16	$(0;\infty)$	16	0.00180
6	Vasicek	2 yrs.	0	0.15	0.01	$(-\infty;\infty)$	16	0.00195
7	CIR	2 yrs.	0.5	0.15	0.0408914	$[0;\infty)$	16	0.00193
8	Brennan/Schwartz	2 yrs.	1	0.15	0.1649986	$(0;\infty)$	16	0.00192
9	β–root	2 yrs.	1.5	0.15	0.6578632	$(0;\infty)$	16	0.00146
10	BDT	2 yrs.	na	0.15	0.1436448	$(0;\infty)$	16	0.00145
11	SaSo	2 yrs.	na	na	0.1483548	$(0;\infty)$	16	0.00147
12	SaSo	2 yrs.	na	na	0.8	$(0;\infty)$	32	0.01358
13	SaSo	2 yrs.	na	na	0.8	$(0;\infty)$	32	0.01358
14	SaSo	2 yrs.	na	na	0.8	$(0;\infty)$	32	0.01358
15	bounded rate	3 yrs.	na	0.045	1.2	(2.75%; 9.25%)	32	0.30175
16	bounded rate	3 yrs.	na	0.45	1.2	(2.75%; 9.25%)	32	0.27803
17	bounded rate	3 yrs.	na	0.45	1.2	(2.75%; 9.25%)	32	0.27803
18	bounded rate	3 yrs.	na	0.45	0.6	(2.75%; 9.25%)	32	0.08242
19	bounded rate	3 yrs.	na	0.45	1.2	(2.75%; 9.25%)	32	0.27803
20	bounded rate	3 yrs.	na	0.45	2.4	(2.75%; 9.25%)	32	0.71588
21	bounded rate	3 yrs.	na	0.45	1.2	(4%; 10.5%)	80	3.24455
22	bounded rate	3 yrs.	na	0.45	1.2	(1.5%; 8%)	80	3.29063
23	bounded rate	3 yrs.	na	0.45	1.2	(4%; 10.5%)	80	3.24455
24	bounded rate	3 yrs.	na	0.45	1.2	(1.5%; 8%)	80	3.29063

[17]The point in time for which the possible term structure realizations are plotted. For options, the maturity of the option.

[18]Range of possible short rate realizations.

[19]Refinement: the discretization of the time line in periods per year.

[20]The maximum deviations of the calculated period 0 term structure from the input term structure (in base points). See section 3.

D.2 Matching the Variances of the Short Rate for Different Horizons

β–root specification, speed of mean reversion $a = 0.15$, refinement 16 p.a., initial forward rate curve flat at 6% (instantaneous) p.a.

Horizon (years)	σ for $\beta = 0$	σ for $\beta = 1.5$
1	0.01	0.6669764
2	0.01	0.6557919
5	0.01	0.6337317
8	0.01	0.6305960
9	0.01	0.6345709

D.3 Diffusion Processes for the Short Rate

Listed below are the diffusion process specifications considered in this paper, as well as the transformations into processes with constant instantaneous standard deviation. These transformations are needed when implementing the Hull/White (1993) algorithm.

Short rate	Transformation	Transformed process drift	Reference
$dr = (\theta(t) - ar)dt + \sigma r^{\beta} dW$	$x = \frac{1}{1-\beta} r^{1-\beta}$	$(\theta(t) - ar)r^{-\beta} - \frac{1}{2}\beta\sigma^2 r^{\beta-1}$	Ho/Lee (1986) for $\beta = 0$, $a = 0$
			Vasicek (1977) for $\beta = 0$
			CIR (1985) for $\beta = 0.5$
	$x = \ln r$		Brennan/Schwartz (1977) for $\beta = 1$
$dr = r \cdot (\theta(t) - a \ln r + \frac{1}{2}\sigma^2)dt + \sigma r dW$	$x = \ln r$	$\theta(t) - a \ln r$	Black/Derman/Toy (1990)
$dr = (1 - e^{-r})[(\theta(t) - \frac{1}{2}(1 - e^{-r})\sigma^2)dt + \sigma dW]$	$x = \ln \frac{1-e^{-r}}{e^{-r}}$	$\theta(t) - \frac{1}{2}\sigma^2$	Sandmann/Sondermann (1993b)
$dr = (\frac{c(a+b-2r)}{b-a} + \frac{(r-a)(b-r)}{b-a}\theta(t))dt + \sigma \frac{(r-a)(b-r)}{b-a} dW$	$x = \ln \frac{r-a}{b-r}$	$\frac{a+b-2r)\cdot c}{(b-r)(r-a)} - \frac{1}{2}\frac{a+b-2r}{b-a}\sigma^2 - \theta(t)$	—

References

Arnold, L. (1973) *Stochastische Differentialgleichungen,* R. Oldenbourg Verlag.

Black, F., E. Derman and W. Toy (1990) A One–Factor Model of Interest Rates and Its Application to Treasury Bond Options; *Financial Analysts Journal,* January–February 1990, 33–39.

Black, F. and P. Karasinski (1991) Bond and Option Pricing when Short Rates are Lognormal; *Financial Analysts Journal,* July–August 1991, 52–59.

Brace, A. and M. Musiela (1994) A Multifactor Gauss Markov Implementation of Heath, Jarrow, Morton; working paper, School of Mathematics, UNSW, Australia

Brennan, M.J. and E.S. Schwartz (1977) Savings Bonds, Retractable Bonds and Callable Bonds; *Journal of Financial Economics,* 5(1) March 1977, 67-88.

Brown, R.H. and S.M. Schaefer (1994) The Term Structure of Real Interest Rates and the Cox, Ingersoll, and Ross Model; *Journal of Financial Economics,* 35(1) February 1994, 3–42.

Chan, K.C., Karolyi, G., Longstaff, F.A. and A.B. Sanders (1992) An Empirical Comparison of Alternative Models of the Short–Term Interest Rate; The Journal of Finance, 47(3) July 1992, 1209–1227.

Chen, R.–R. and L. Scott (1993) Maximum Likelihood Estimation for a Multifactor Equilibrium Model of the Term Structure of Interest Rates; *The Journal of Fixed Income,* 3(3) December 1993, 14–31.

Cohen, H. and D. Heath (1992) A New Method of Testing Pricing Models As Applied to Forward Interest Rate Models; working paper 92–16, Federal Reserve Bank of Atlanta, November 1992.

Cox, J.C., J.E. Ingersoll and S.A. Ross (1985) A Theory of the Term Structure of Interest Rates; *Econometrica,* 53(2) March 1985, 385–467.

Duffee, G.R. (1993) On the Relation Between the Level and Volatility of Short–term Interest Rates: A Comment on Chan, Karolyi, Longstaff, and Sanders; working paper, Federal Reserve Board, May 1993.

El Karoui, N., C. Lepage, R. Myneni, N. Roseau and R. Viswanathan (1991) The Valuation and Hedging of Contingent Claims with Markovian Interest Rates; working paper, Université de Paris 6.

Flesaker, B. (1993) Testing the Heath–Jarrow–Morton/Ho–Lee Model of Interest Rate Contingent Claims Pricing; *Journal of Financial and Quantitative Analysis,* 28(4) December 1993, 483–495.

Ho, T.S.Y. and S.–B. Lee (1986) Term Structure Movements and Pricing of Interest Rate Contingent Claims; *The Journal of Finance,* 41 December 1986, 1011–1029.

Hogan, M. and K. Weintraub (1993) The Lognormal Interest Rate Model and Eurodollar Futures; working paper, Citibank New York.

Hull, J. and A. White (1990a) Pricing Interest Rate Derivative Securities; *Review of Financial Studies,* 3(4) 1990.

Hull, J. and A. White (1990b) Valuing Derivative Securities Using the Explicit Finite Difference Method; *Journal of Financial and Quantitative Analysis,* 25(1) March 1990, 87–100.

Hull, J. and A. White (1993) One–Factor Interest–Rate Models and the Valuation of Interest–Rate Derivative Securities; *Journal of Financial and Quantitative Analysis,* 28(2) June 1993, 235–254.

Jamshidian, F. (1990) Bond Option Evaluation in the Gaussian Interest Rate Model; Merrill Lynch Working Paper.

Jamshidian, F. (1991) Forward Induction and Construction of Yield Curve Diffusion Models; *The Journal of Fixed Income,* 62–74.

Kan, R. (1992) Shape of the Yield Curve Under CIR Single Factor Model: A Note; working paper, University of Chicago, June 1992.

Karlin, S. and H. Taylor (1981) *A Second Course in Stochastic Processes.* Academic Press.

Musiela, M. (undated) Stochastic PDEs and Term Structure Models; working paper, School of Mathematics, UNSW, Australia

Rady, S. and K. Sandmann (1994) The Direct Approach to Debt Option Pricing; *The Review of Futures Markets,* 13(2), 461–514.

Sandmann, K. and D. Sondermann (1993a) A Term Structure Model and the Pricing of Interest Rate Derivatives; *The Review of Futures Markets,* 12(2), 391–423.

Sandmann, K. and D. Sondermann (1993b) On the Stability of Lognormal Interest Rate Models; discussion paper B–263, University of Bonn, December 1993.

Stambaugh, R.F. (1988) The Information in Forward Rates: Implications for Models of the Term Structure; *Journal of Financial Economics,* 21(1) May 1988, 41–70.

Vasicek, O.A. (1977) An Equilibrium Characteriztion of the Term Structure; *Journal of Financial Economics,* 5 November 1977, 177–188.