

Discussion Paper No. B-294

Optional decomposition of
supermartingales
and hedging contingent claims
in incomplete security markets

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October 1994

Submitted to Probability Theory and Related Fields

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The research was performed at the University of Bonn. Financial support by the Volkswagen-Stiftung co-operation program and Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at the University of Bonn, is gratefully acknowledged.

Abstract

Let $\mathbf{M}(X)$ be a family of all equivalent local martingale measures for some locally bounded d -dimensional process X , and V be a positive process. Main result of the paper (Theorem 2.1) states that the process V is a supermartingale whatever $Q \in \mathbf{M}(X)$, if and only if this process admits the following decomposition:

$$V_t = V_0 + \int_0^t H_s dX_s - C_t, \quad t \geq 0,$$

where H is an integrand for X , and C is an adapted increasing process. We call such a representation the *optional* because, in contrast to Doob-Meyer decomposition, it generally exists only with an adapted (*optional*) process C .

We apply this decomposition to the problem of hedging European and American style contingent claims in a setting of *incomplete* security markets.

Keywords: Doob–Meyer decomposition, optional decomposition, martingale measure, stochastic integral, semimartingale topology, incomplete market, hedging, options

JEL classification: G13

1 Introduction

The famous Doob-Meyer decomposition states that each positive supermartingale $V = (V_t)_{t \geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geq 0}, P)$ has the following representation:

$$(1.1) \quad V = M - A,$$

where $M = (M_t)_{t \geq 0}$ is a local martingale, $M_0 = V_0$, and $A = (A_t)_{t \geq 0}$ is an increasing process. Moreover, there exists a decomposition involving a *predictable* process A , with Eq.(1.1) being a unique representation in this case.

Now let $\mathbf{Q} = \{Q\}$ be a family of equivalent probability measures on $(\Omega, \mathcal{F}, \mathbf{F})$, and let V be a positive supermartingale with respect to each measure $Q \in \mathbf{Q}$. We are interested in decomposition of the form (1.1), with M being a Q -local martingale for all $Q \in \mathbf{Q}$. Even simple examples show that we can generally hope to obtain such a representation only with an adapted (*optional*) increasing process A . With this notation and following El Karoui and Quenez [7], this decomposition will be referred to as *optional decomposition* of positive supermartingale V with respect to the family of measures \mathbf{Q} .

Note that, in general, such an optional decomposition does not exist. As an example we take \mathbf{Q} to be the family of all equivalent *supermartingale* measures for process $X_t = \pi_t - t$, $t \geq 0$. Here π is a Poisson process under the reference probability measure P . We assume that filtration \mathbf{F} is generated by this Poisson process: $\mathbf{F} = \mathbf{F}^\pi$. Then one can show that, whatever $Q \in \mathbf{Q}$, all local martingales with respect to Q are constants. Therefore the process X which is a Q -supermartingale for all $Q \in \mathbf{Q}$ can not be represented as a difference between a local martingale with respect to all $Q \in \mathbf{Q}$ and an increasing process.

The main result of the paper (Theorem 2.1) states that the optional decomposition exists provided that \mathbf{Q} is a family of all equivalent local martingale measures for some locally bounded d -dimensional process X . More exactly, in this case process V which is a positive supermartingale with respect to each measure $Q \in \mathbf{Q}$, admits the representation as follows:

$$V_t = V_0 + (H \bullet X)_t - C_t, \quad t \geq 0,$$

where H is an integrand for X , $(H \bullet X)_t = \int_0^t H_s dX_s$ is the stochastic integral of H with respect to X , and C is an adapted increasing process.

In finance such a decomposition leads to a convenient *supermartingale* characterization of wealth and consumption portfolios. We apply this characterization to the problem of hedging European and American style contingent claims in a setting of *incomplete* security markets. This enables us to describe the capital evolution for the corresponding minimal hedging portfolios. The results obtained extend the solutions derived in El Karoui and Quenez [7] for the case when X is the solution of a stochastic differential equation governed by Brownian motion. In addition, they can be considered as a “dynamic” version of Theorem 5.7 in Delbaen and Schachermayer [4]. This theorem implies the existence of hedging portfolio with initial wealth equal to the upper bound for arbitrage-free option prices at time $t = 0$. On the contrary, Theorems 3.2 and 3.3 below imply the existence of such a hedging portfolio whose capital at *each time instant* t is equal to the upper bound of arbitrage-free option prices at that time.

The proofs in El Karoui and Quenez [7] are mainly based on Girsanov’s transformation of probability measures. Apparently this approach can not be extended to the general setting under consideration. Instead, we apply the arguments based on the Hahn-Banach theorem. The main source for the results presented here was an important paper of Delbaen and Schachermayer [4].

2 Main results

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space which satisfies the conventional conditions of general theory of stochastic processes, i.e., the filtration \mathbf{F} is right-continuous ($\mathcal{F}_t = \mathcal{F}_{t+}$) and σ -field \mathcal{F}_0 contains all null sets from \mathcal{F} . For simplicity hereinafter the initial σ -field \mathcal{F}_0 is assumed to be the trivial one, i.e., it contains only sets with measure zero or one.

On this filtered probability space, we consider an RCLL (right-continuous with left limits) d -dimensional random process $X = (X^i)_{i \leq d}$. We assume that X is a locally bounded process, i.e., there is a sequence of stopping times $(\tau_n)_{n \geq 1}$ on $(\Omega, \mathcal{F}, \mathbf{F})$ such that the variables τ_n converge to ∞ almost surely as n tends to ∞ , and $|X_t^i| \leq n$ for $t \leq \tau_n$ and $1 \leq i \leq d$.

A probability measure Q is called a *local martingale measure*, if it is

equivalent to P and the process X is a Q -local martingale. By $\mathbf{M}(X)$ we denote the set of all local martingale measures for process X . We suppose that $\mathbf{M}(X)$ is not empty. Because all further concepts are invariant with respect to equivalent changes of measure, hereinafter we assume that $P \in \mathbf{M}(X)$.

The main result of the paper is the following theorem.

Theorem 2.1 (Optional decomposition) *Let $V = (V_t)_{t \geq 0}$ be a positive process. Then V is a supermartingale for each measure $Q \in \mathbf{M}(X)$ if and only if there exist an X -integrable predictable process $H = (H^i)_{1 \leq i \leq d}$ and an adapted increasing process C such that*

$$(2.1) \quad V_t = V_0 + (H \bullet X)_t - C_t, \quad t \geq 0,$$

where $(H \bullet X)_t = \int_0^t H_s dX_s$ is the stochastic integral of H with respect to X .

Because the process $H \bullet X$ mentioned above is uniformly bounded below, it is a local martingale with respect to all measures $Q \in \mathbf{M}(X)$, see Émery [9] and Ansel and Stricker [1]. Therefore, the Eq.(2.1) is indeed an optional decomposition of V with respect to the family $\mathbf{M}(X)$. The proof of Theorem 2.1 see in Section 5.

Theorem 2.1 generalizes the following result of Jacka [10] and Ansel and Stricker [1].

Theorem 2.2 (Jacka, Ansel and Stricker) *Let $M = (M_t)_{t \geq 0}$ be a positive process. Then M is a local martingale for each measure $Q \in \mathbf{M}(X)$ if and only if there exists an X -integrable predictable process $H = (H^i)_{1 \leq i \leq d}$ such that*

$$M_t = M_0 + (H \bullet X)_t \quad t \geq 0.$$

We now consider the question of uniqueness for decomposition (2.1).

We denote by X^c the continuous martingale part of X with respect to P , $\langle X^c, X^c \rangle$ means the quadratic variation of X^c , $[X, X]$ corresponds to the quadratic variation of X , and $[\widetilde{X}, \widetilde{X}]$ designates the compensator of $[X, X]$ with respect to P . For simplicity of notation we formulate the result for the case of $d = 1$.

Lemma 2.1 *Assume that there is a positive predictable process $h = (h_t(\omega))_{t \geq 0}$ such that*

$$[\widetilde{X}, \widetilde{X}]_t(\omega) = \int_0^t h_s(\omega) d \langle X^c, X^c \rangle_s(\omega)$$

almost surely. Then the processes $H \bullet X$ and C in decomposition (2.1) are uniquely defined.

Proof Let H and G be predictable X -integrable processes, and A and B be adapted RCLL processes of bounded variation such that

$$H \bullet X + A = G \bullet X + B := V.$$

We must show that $H \bullet X = G \bullet X$ or, equivalently,

$$(2.2) \quad HI\{|H| \leq N\} \bullet X = GI\{|G| \leq N\} \bullet X,$$

whatever $N \geq 1$.

Because the continuous martingale part of V equals

$$V^c = H \bullet X^c = G \bullet X^c,$$

we deduce that

$$\int_0^\infty (H_s - G_s)^2 d \langle X^c, X^c \rangle = 0.$$

Then the condition of the lemma implies:

$$\int_0^\infty (H_s - G_s)^2 d[\widetilde{X}, \widetilde{X}] = 0,$$

and Eq.(2.2) follows from the Doob inequality for locally square integrable martingales. \square

Remark 2.1 The condition of the lemma is invariant with respect to equivalent change of measure P .

Finally, we state the result that links the “optional” decomposition with “predictable” decomposition of Doob and Meyer.

Let V be a bounded below Q -supermartingale for all $Q \in \mathbf{M}(X)$. We denote by \mathcal{B} the set of predictable increasing RCLL processes B , such that $B_0 = 0$ and $V + B$ is a Q -supermartingale for all $Q \in \mathbf{M}(X)$. We introduce an ordering \prec on \mathcal{B} indicating that A is *less* than B ($A \prec B$) if $B - A$ is an increasing process.

Lemma 2.2 *The maximal element \widehat{B} in the ordered set \mathcal{B} exists and is unique.*

Proof We start with an intermediate claim.

Claim *Let A and B be the elements of \mathcal{B} . Then there exists $C \in \mathcal{B}$ such that $C - A$ and $C - B$ are increasing processes.*

Hahn's decomposition for increasing predictable processes (see Jacod and Shiryaev [12] Chapter I, Proposition 3.13) implies the existence of a predictable process h with values in $\{-1, 1\}$ such that

$$\int_0^t |dA_s - dB_s| = \int_0^t h_s(dA_s - dB_s), \quad t \geq 0.$$

Denote

$$C_t = \frac{1}{2} \left(\int_0^t (1 + h_s) dA_s + \int_0^t (1 - h_s) dB_s \right), \quad t \geq 0.$$

Then the processes $C - A$ and $C - B$ are increasing processes. Moreover, because

$$V + C = \frac{1}{2} \left(\int_0^t (1 + h_s) d(V_s + A_s) + \int_0^t (1 - h_s) d(V_s + B_s) \right), \quad t \geq 0$$

and processes $V + A$ and $V + B$ are supermartingales with respect to all $Q \in \mathbf{M}(X)$, we deduce that $V + C$ is a supermartingale with respect to all $Q \in \mathbf{M}(X)$ and, therefore, $C \in \mathcal{B}$. The claim is proved.

Now let $b = \sup_{B \in \mathcal{B}} EB_\infty$ and $(B^n)_{n \geq 1}$ be a sequence in \mathcal{B} such that the expectations EB_∞^n tend to b as n tends to ∞ . Using the claim, we can construct this sequence so as make the processes $B^{n+1} - B^n$ increasing whatever $n \geq 1$. Then the sequence $(B^n)_{n \geq 1}$ converges to a process \widehat{B} in $[0, +\infty]$ uniformly. It can be easily seen that $\widehat{B} \in \mathcal{B}$. Finally, the claim above and the fact

$$\widehat{B}_\infty = b = \sup_{B \in \mathcal{B}} EB_\infty$$

imply that \widehat{B} is the unique maximal element in \mathcal{B} . \square

Remark 2.2 When $\mathbf{M}(X)$ is a singleton, the process \widehat{B} is exactly the process that appears in Doob–Meyer decomposition.

3 Applications in finance

1. In finance the process $X = (X^i)_{i \leq d}$ is interpreted as a discounted price process of d assets in a security market. As above, we assume that X is a locally bounded RCLL process and that the set $\mathbf{M}(X)$ of local martingale measures for X is not empty. This corresponds to the absence of arbitrage opportunities on the security market, see recent paper of Delbaen and Schachermayer [4] for precise statement.

We remind that a *wealth* and *consumption* portfolio can be described as a triple $\Pi = (v, H, C)$, where v is an *initial wealth* of the portfolio, $H = (H^i)_{i \leq d}$ is a predictable X -integrable process of *numbers* of assets, and $C = (C_t)_{t \geq 0}$ is an adapted increasing right-continuous process of *consumption*. A *capital* process $V = (V_t)_{t \geq 0}$ of portfolio Π equals

$$(3.1) \quad V_t = v + \int_0^t H_s dX_s - C_t, \quad t \geq 0.$$

This equation has a clear economic interpretation: changes in the portfolio wealth are caused by changes in assets prices and by consumption. In particular, when $C \equiv 0$ the Eq.(3.1) means that the portfolio Π is developing in a *self-financing* way. Portfolio Π is called an *admissible* strategy if $V_t \geq 0$, $t \geq 0$. Theorems 2.1 and 2.2 immediately lead to the following important characterization of admissible portfolios.

Theorem 3.1 *Let $(V_t)_{t \geq 0}$ be a positive process. Then*

- (i) *V is a capital of self-financing portfolio if and only if V is a local martingale with respect to all $Q \in \mathbf{M}(X)$.*
- (ii) *V is a capital of wealth and consumption portfolio if and only if V is a supermartingale with respect to all $Q \in \mathbf{M}(X)$.*

2. Let now f be a positive random variable on (Ω, \mathcal{F}) . We interpret f as the value of contingent claim or as the payment of European option with maturity at time $T = \infty$. An admissible strategy Π with wealth V is called the *hedging* portfolio for f if $V_\infty \geq f$. Moreover, hedging portfolio $\hat{\Pi}$ with wealth \hat{V} is called the *minimal hedge* for f if $\hat{V}_t \leq V_t$ almost surely, whatever $t \geq 0$ and hedging strategy Π with wealth V .

The supermartingale property of the wealth V of hedging strategy Π implies that

$$(3.2) \quad V_t \geq \operatorname{ess\,sup}_{Q \in \mathbf{M}(X)} E_Q [V_\infty \mid \mathcal{F}_t] \geq \operatorname{ess\,sup}_{Q \in \mathbf{M}(X)} E_Q [f \mid \mathcal{F}_t], \quad t \geq 0.$$

The following theorem states that the lower bound in (3.2) is achieved and equals to the wealth of the minimal hedge. For continuous price processes this result was proved in the paper of El Karoui and Quenez [7].

Theorem 3.2 *Let f be a positive random variable such that $\sup_{Q \in \mathbf{M}(X)} E_Q f < \infty$. Then the minimal hedging strategy $\hat{\Pi} = (\hat{v}, \hat{H}, \hat{C})$ exists and its wealth \hat{V} equals*

$$\hat{V}_t = \hat{v} + (\hat{H} \bullet X)_t - \hat{C}_t = \operatorname{ess\,sup}_{Q \in \mathbf{M}(X)} E_Q [f \mid \mathcal{F}_t].$$

Proof The proof follows from Theorem 3.1 above and from the fact that process $(\operatorname{ess\,sup}_{Q \in \mathbf{M}(X)} E_Q [f \mid \mathcal{F}_t])_{t \geq 0}$ is a supermartingale for all $Q \in \mathbf{M}(X)$, see Proposition 4.2 in Section 4. \square

If τ is a finite stopping time, i.e. $P(\tau < +\infty) = 1$, and f is a \mathcal{F}_τ -measurable function, then from Theorem 3.2 we deduce that $\hat{V}_\infty = f$. We notice that for general $\mathcal{F} = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$ -measurable claims f this equality does not hold. To demonstrate, we use the following simple example.

Example 3.1 Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be a filtered probability space with a Wiener process W . We set $X \equiv 1$ and define claim f as $f = I(\tau < \infty)$, where stopping time τ equals

$$\tau = \inf \{t \geq 0 : W_t \geq e^t\}.$$

In this case $\mathbf{M}(X)$ is the set of all probability measures equivalent to P . It can be easily seen that $P(\tau < \infty) < 1$, and

$$\hat{V}_t = \operatorname{ess\,sup}_{Q \in \mathbf{M}(X)} Q(\tau < \infty \mid \mathcal{F}_t) = 1, \quad t \geq 0.$$

Therefore, $\hat{V}_\infty = 1 > I(\tau < +\infty)$ with positive probability.

The question of particular interest is to know whether the minimal hedging strategy $\widehat{\Pi}$ is a self-financing portfolio. This is connected with the problem of *attainability* of contingent claims, see papers of Jacka [10], Ansel and Stricker [1] and Delbaen and Schachermayer [5]. From these papers we deduce that the minimal hedge $\widehat{\Pi}$ with wealth \widehat{V} is a self-financing strategy if and only if there is a measure $Q \in \mathbf{M}(X)$ such that \widehat{V} is a Q -uniformly integrable martingale on $[0, +\infty[$.

3. Let now $f = (f_t)_{t \geq 0}$ be an adapted positive process. We interpret f as the reward process of an American type option. The wealth and consumption portfolio $\Pi = (v, H, C)$ with capital process $V = (V_t)_{t \geq 0}$ is called a *hedging* strategy for f if

$$V_t \geq f_t, \quad t \geq 0.$$

The portfolio $\widetilde{\Pi} = (\widetilde{v}, \widetilde{H}, \widetilde{C})$ with capital process $\widetilde{V} = (\widetilde{V}_t)_{t \geq 0}$ is termed the *minimal* hedging portfolio if

$$V_t \geq \widetilde{V}_t \geq f_t,$$

whatever $t \geq 0$ and hedging portfolio Π with capital V .

The following theorem can be considered as generalization of the results of Bensoussan [2] and Karatzas [13] to a setting of incomplete markets.

Denote by \mathcal{M}_t the set of stopping times τ with values in $[t, +\infty[$.

Theorem 3.3 *Let $f = (f_t)_{t \geq 0}$ be an adapted positive process such that*

$$\sup_{\tau \in \mathcal{M}_0} \sup_{Q \in \mathbf{M}(X)} E_Q f_\tau < +\infty.$$

Then the minimal hedging portfolio $\widetilde{\Pi} = (\widetilde{v}, \widetilde{H}, \widetilde{C})$ exists, and its capital at time $t \geq 0$ equals

$$\widetilde{V}_t = \widetilde{v} + (\widetilde{H} \bullet X)_t - \widetilde{C}_t = \operatorname{ess\,sup}_{Q \in \mathbf{M}(X), \tau \in \mathcal{M}_t} E_Q[f_\tau | \mathcal{F}_t].$$

Proof The proof follows from Theorem 3.1 and from the fact that process $(\operatorname{ess\,sup}_{Q \in \mathbf{M}(X), \tau \in \mathcal{M}_t} E_Q[f_\tau | \mathcal{F}_t])_{t \geq 0}$ is a supermartingale whatever $Q \in \mathbf{M}(X)$, see Proposition 4.3 in Section 4. \square

4 Auxiliary facts and results

1. First we recall some facts and definitions from the theory of stochastic integration, for which we refer to Dellacherie and Meyer [6], Protter [16] and Jacod and Shiryaev [12].

Suppose X is a real-valued process; then the *maximal* function $(X)_t^*$ is defined as $\sup_{0 \leq s \leq t} |X_s|$.

Suppose X and Y are semi-martingales; then the *Émery distance* between X and Y equals

$$D(X, Y) = \sup_{|H| \leq 1} \left(\sum_{n \geq 1} 2^{-n} E [\min(|(H \bullet X)_n|, 1)] \right),$$

where sup is taken over the set of all predictable processes H bounded by 1. For this metric the space of semi-martingales is complete, see Émery [8]. The corresponding topology is called a *semi-martingale* or *Émery* topology.

In particular, if A and B are predictable processes of bounded variation, the Émery distance between A and B equals

$$D(A, B) = \sum_{n \geq 1} 2^{-n} E \left[\min \left(\int_0^n |dA_s - dB_s|, 1 \right) \right],$$

where $\int_0^n |dA_s - dB_s|$ is the total variation of $A - B$ on $[0, t]$. This is a consequence of the following Hahn decomposition: there exists a predictable process h with values in $\{-1, +1\}$ such that

$$\int_0^t |dA_s - dB_s| = \int_0^t h_s (dA_s - dB_s), \quad t \geq 0,$$

see Jacod and Shiryaev [12] Chapter I, Proposition 3.13.

Now let H be a predictable process, and X be a semi-martingale. The process H is called *X-integrable* if there exist a local martingale M and a process A of bounded variation such that $X = M + A$ and

1. process $\int_0^t |H_s| |dA_s|$ has a bounded variation,
2. increasing process $(\int H_s^2 d[M, M]_s)^{1/2}$ is locally integrable, where $[M, M]$ is the quadratic variation of the local martingale M .

In this case, $H \bullet A$ is a Lebesgue–Stieltjes integral; the stochastic integral $H \bullet M$ exists as a stochastic integral with respect to a local martingale, and is a local martingale. The stochastic integral $H \bullet X$ equals $H \bullet A + H \bullet M$ and does not depend on a particular choice of M and A .

If a predictable process H is locally bounded, this process is integrable with respect to all semi-martingales. If H is unbounded, then by Theorem 1 in Chou, Meyer and Stricker [3] process H is X -integrable if and only if the sequence $H I\{|H| \leq n\} \bullet X$, $n \geq 1$, converges in semi-martingale topology. Moreover, in this case the limit of the sequence equals $H \bullet X$.

An X -integrable process H is called the *admissible* integrand if there exists a constant a such that $a + (H \bullet X)_t \geq 0$, $t \geq 0$. A counter-example in Émery [9] shows that a stochastic integral with respect to a local martingale can be not local martingale. However, if M is a local martingale and H is an *admissible* integrand for M , then $H \bullet M$ is a local martingale, see Ansel and Stricker [1].

A semi-martingale X is called a *special* semi-martingale if it can be decomposed as $X = M + A$, where M is a local martingale and A is a *predictable* process of bounded variation. Then such a decomposition is unique. For the sequel we need the following proposition on special semi-martingales. For the proof we refer to Chou, Meyer and Stricker [3], where this result was called the Jeulin theorem.

Proposition 4.1 *Let X be a special semi-martingale with a canonical decomposition $X = M + A$, and H be a predictable X -integrable process. Then $H \bullet X$ is a special semi-martingale if and only if*

1. *H is M -integrable in the sense of stochastic integrals of local martingales,*
2. *H is A -integrable in the sense of Lebesgue–Stieltjes integrals.*

In this case, the canonical decomposition of $H \bullet X$ is given as $H \bullet X = H \bullet M + H \bullet A$.

We will also need a technical lemma whose formulation is taken from the paper of Memin [15].

Lemma 4.1 *Let X be a semi-martingale, such that the quadratic variation $[X, X]_\infty^{1/2}$ belongs to $L^p(\Omega, \mathcal{F}, P)$ for $p \geq 1$. Then X is a special semi-martingale, and there exists a universal constant a_p such that*

$$\begin{aligned} \|[A, A]_\infty^{1/2}\|_{L^p} &\leq a_p \|[X, X]_\infty^{1/2}\|_{L^p}, \\ \|[M, M]_\infty^{1/2}\|_{L^p} &\leq (a_p + 1) \|[X, X]_\infty^{1/2}\|_{L^p}, \end{aligned}$$

where $X = M + A$ is a canonical decomposition of X ; in particular, for $p = 2$ we can take $a_2 = 1$.

2. Let now X be a locally bounded martingale, and f be a positive function on (Ω, \mathcal{F}, P) . We denote by $\mathbf{M}(X)$ the set of local martingale measures for X . The following proposition is adapted from the paper of El Karoui and Quenez [7].

Proposition 4.2 *Let f be a positive variable such that $\sup_{Q \in \mathbf{M}(X)} E_Q f < +\infty$. There is an RCLL process $V = (V_t)_{t \geq 0}$ such that*

$$V_t = \operatorname{ess\,sup}_{Q \in \mathbf{M}(X)} E_Q[f \mid \mathcal{F}_t], \quad t \geq 0.$$

The process V is a Q -supermartingale whatever $Q \in \mathbf{M}(X)$.

The proof of Proposition 4.2 is similar to the proof of the following proposition used in the proof of Theorem 3.3. We denote by \mathcal{M}_t the set of stopping times τ with values in $[t, +\infty]$.

Proposition 4.3 *Let $f = (f_t)_{t \geq 0}$ be a positive adapted RCLL process such that*

$$\sup_{Q \in \mathbf{M}(X), \tau \in \mathcal{M}_0} E_Q f_\tau < +\infty$$

There is an RCLL process $V = (V_t)_{t \geq 0}$ such that for all $t \geq 0$

$$V_t = \operatorname{ess\,sup}_{Q \in \mathbf{M}(X), \tau \in \mathcal{M}_t} E_Q[f_\tau \mid \mathcal{F}_t].$$

The process $V = (V_t)_{t \geq 0}$ is a Q -supermartingale whatever $Q \in \mathbf{M}(X)$.

Proof For each time instant t we define variable \tilde{V}_t as

$$(4.1) \quad \tilde{V}_t = \operatorname{ess\,sup}_{Q \in \mathbf{M}(X), \tau \in \mathcal{M}_t} E_Q[f_\tau | \mathcal{F}_t].$$

We have to show that process $\tilde{V} = (\tilde{V}_t)_{t \geq 0}$ is a Q -supermartingale for all $Q \in \mathbf{M}(X)$ and that \tilde{V} admits an RCLL modification.

Let probability measure P be an element of $\mathbf{M}(X)$. By \mathcal{Z}_t we denote the set of processes $z = (z_t)_{t \geq 0}$ such that

1. z is the density process of some measure $Q \in \mathbf{M}(X)$ with respect to P ,
2. $z_s = 1, \quad s \leq t$.

The Eq.(4.1) can be rewritten as

$$\tilde{V}_t = \operatorname{ess\,sup}_{z \in \mathcal{Z}_t, \tau \in \mathcal{M}_t} E[f_\tau z_\tau | \mathcal{F}_t],$$

where E is the expectation with respect to measure P .

Let us fix positive numbers s and t , $s < t$, and show that

$$(4.2) \quad E[\tilde{V}_t | \mathcal{F}_s] = \operatorname{ess\,sup}_{z \in \mathcal{Z}_t, \tau \in \mathcal{M}_t} E[f_\tau z_\tau | \mathcal{F}_s].$$

First we have

$$(4.3) \quad E[\tilde{V}_t | \mathcal{F}_s] = E[\operatorname{ess\,sup}_{z \in \mathcal{Z}_t, \tau \in \mathcal{M}_t} E[f_\tau z_\tau | \mathcal{F}_t] | \mathcal{F}_s] \geq \operatorname{ess\,sup}_{z \in \mathcal{Z}_t, \tau \in \mathcal{M}_t} E[f_\tau z_\tau | \mathcal{F}_s].$$

To prove the reverse inequality we take the sequence $(y^n, \sigma_n)_{n \geq 1}$ in $(\mathcal{Z}_t, \mathcal{M}_t)$ such that

$$\tilde{V}_t = \sup_{n \geq 1} E[f_{\sigma_n} y_{\sigma_n}^n | \mathcal{F}_t].$$

Using this sequence we construct a new sequence $(z^n, \tau_n)_{n \geq 1}$ as follows

$$(z^1, \tau_1) = (y^1, \sigma_1)$$

and for $n \geq 1$

$$(z^{n+1}, \tau_{n+1}) = \begin{cases} (z^n, \tau_n), & \text{if } E[f_{\tau_n} z_{\tau_n}^n | \mathcal{F}_t] \geq E[f_{\sigma_{n+1}} y_{\sigma_{n+1}}^{n+1} | \mathcal{F}_t] \\ (y^{n+1}, \sigma_{n+1}), & \text{if } E[f_{\tau_n} z_{\tau_n}^n | \mathcal{F}_t] < E[f_{\sigma_{n+1}} y_{\sigma_{n+1}}^{n+1} | \mathcal{F}_t] \end{cases}.$$

We have $(z^n, \tau_n)_{n \geq 1} \subseteq (\mathcal{Z}_t, \mathcal{M}_t)$ and

$$E[f_{\tau_n} z_{\tau_n}^n | \mathcal{F}_t] = \max_{k \leq n} E[f_{\sigma_k} y_{\sigma_k}^k | \mathcal{F}_t] \uparrow \tilde{V}_t.$$

Now from the theorem on monotone convergence we deduce

$$\begin{aligned} E[\tilde{V}_t | \mathcal{F}_s] &= E[\lim_{n \rightarrow \infty} E[f_{\tau_n} z_{\tau_n}^n | \mathcal{F}_t] | \mathcal{F}_s] = \lim_{n \rightarrow \infty} E[f_{\tau_n} z_{\tau_n}^n | \mathcal{F}_s] \\ &\leq \text{ess sup}_{z \in \mathcal{Z}_t, \tau \in \mathcal{M}_t} E[f_{\tau} z_{\tau} | \mathcal{F}_s]. \end{aligned}$$

Together with inequality (4.3) this proves Eq.(4.2).

Since $\mathcal{Z}_t \subseteq \mathcal{Z}_s$, $\mathcal{M}_t \subseteq \mathcal{M}_s$ for $s \leq t$, the equality (4.2) implies the supermartingale property for the process \tilde{V} :

$$E[\tilde{V}_t | \mathcal{F}_s] \leq \tilde{V}_s, \quad s \leq t.$$

To finish the proof of Proposition 4.3 we must show that the process \tilde{V} admits an RCLL modification. According to Theorem 3.1 in Liptser and Shiryaev [12], this is the case if and only if the function $(E\tilde{V}_t)_{t \geq 0}$ is right-continuous.

When $s = 0$, the equality (4.2) takes the form

$$(4.4) \quad E\tilde{V}_t = \sup_{z \in \mathcal{Z}_t, \tau \in \mathcal{M}_t} E[f_{\tau} z_{\tau}].$$

Let t , $(t_n)_{n \geq 1}$ be positive numbers such that $t_n \downarrow t$, $n \rightarrow +\infty$, and $t_n < t + 1$, $n \geq 1$. Because \tilde{V} is a supermartingale, we have

$$(4.5) \quad E\tilde{V}_t \geq \lim_{n \rightarrow \infty} E\tilde{V}_{t_n}.$$

To prove the reverse inequality we fix $\epsilon > 0$ and choose stopping time $\sigma = \sigma(\epsilon)$ from \mathcal{M}_t and process $z = z(\epsilon)$ from \mathcal{Z}_t such that

$$(4.6) \quad E\tilde{V}_t < E f_{\sigma} z_{\sigma} + \epsilon \quad \text{and} \quad P(\sigma > t) = 1.$$

This is possible by Eq.(4.4) and right-continuity of the process f .

Now for $n \geq 1$ we define stopping time $\sigma_n \in \mathcal{M}_{t_n}$ and process $z^n \in \mathcal{Z}_{t_n}$ as

$$\sigma_n = \begin{cases} \sigma, & \sigma \geq t_n \\ t + 1, & \sigma < t_n \end{cases}, \quad z^n = \begin{cases} z_t / z_{t_n}, & \sigma \geq t_n \text{ and } t \geq t_n \\ 1, & \sigma < t_n \text{ or } t < t_n \end{cases}.$$

We have $\sigma_n \rightarrow \sigma$ and $z_{\sigma_n}^n \rightarrow z_\sigma$ almost surely as n tends to ∞ . Now we deduce from Fatou's lemma and (4.4) and (4.6) that

$$E\tilde{V}_t \leq \lim_{n \rightarrow \infty} E f_{\sigma_n} z_{\sigma_n}^n + \epsilon \leq \lim_{n \rightarrow \infty} E\tilde{V}_{t_n} + \epsilon.$$

Because ϵ is an arbitrary positive number and by eq.(4.5) we have that the function $(E\tilde{V}_t)_{t \geq 0}$ is right-continuous. This completes the proof of Proposition 4.3. \square

3. The next proposition is a slight modification of Theorem 5.7 in Delbaen and Schachermayer [4] and we only sketch the proof. We use the notations of Proposition 4.2.

Proposition 4.4 *Let τ and σ be stopping times on $(\Omega, \mathcal{F}, \mathbf{F}, P)$ such that $\tau \leq \sigma$, and f be a bounded \mathcal{F}_σ -measurable random variable. Denote $V_\tau = \text{ess sup}_{Q \in \mathbf{M}(X)} E_Q[f \mid \mathcal{F}_\tau]$. There is an admissible integrand H such that*

$$(H \bullet X)_t = 0, \quad t \leq \tau, \quad \text{and} \quad V_\tau + (H \bullet X)_\sigma \geq f.$$

Proof For simplicity we consider the case $\tau = 0$ and $\sigma = \infty$.

Following Delbaen and Schachermayer [4] we define the sets:

$$\begin{aligned} K_0 &= \{V_0 + (H \bullet X)_\infty : H \text{ is an admissible integrand}\}, \\ C_0 &= K_0 - L_+^0, \\ C &= C_0 \bigcap L^\infty, \end{aligned}$$

where L_+^0 and L^∞ are the sets of positive and bounded random functions on (Ω, \mathcal{F}, P) respectively.

We need to prove that $f \in C$. We proceed on a well known path. By Theorem 4.2 in Delbaen and Schachermayer [4] (see also remark after corollary 1.2) the set C is $\sigma(L^\infty, L^1)$ -closed. Therefore, if $f \notin C$, by the Hahn–Banach theorem there is a signed measure $R \in L^1$ such that

$$\sup_{g \in C} E_R g < E_R f.$$

This inequality and construction of the set C imply that R is positive measure such that $E_R(H \bullet X)_\infty = 0$, if H and $-H$ are admissible processes. If we

normalize the measure R as $R(\Omega) = 1$, we obtain that R is an absolutely continuous probability such that X is a local R -martingale. Therefore for each $\epsilon > 0$ measure $R^\epsilon = \epsilon P + (1 - \epsilon)R$ belongs to $\mathbf{M}(X)$. We easily deduce that

$$\sup_{g \in C} E_{R^\epsilon} g = V_0 = \sup_{Q \in \mathbf{M}(X)} E_Q f \geq E_{R^\epsilon} f.$$

Therefore

$$E_R f = \lim_{\epsilon \rightarrow 0} E_{R^\epsilon} f \leq V_0 \leq \sup_{g \in C} E_R g$$

and we come to a contradiction. \square

4. Finally we prove a technical lemma to be used later in several occasions. This lemma can be considered as an extension of Lemma A 1.1 from Delbaen and Schachermayer [4] to the setting of increasing processes.

Suppose \mathcal{A} is a family of random processes; then the notation $B \in \text{conv } \mathcal{A}$ means that the process B is a finite convex combination of elements in \mathcal{A} .

Lemma 4.2 *Let $(A^n)_{n \geq 1}$ be a sequence of positive increasing adapted processes on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$. There exist a sequence $B^n \in \text{conv}(A^n, A^{n+1}, \dots)$, $n \geq 1$, and a $[0, +\infty]$ -valued increasing process B such that*

$$B_t = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} B_{t+\epsilon}^n = \lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} B_{t+\epsilon}^n$$

If there are $T > 0$ and $\epsilon > 0$ such that for all n : $P(A_T^n > \epsilon) > \epsilon$, then $P(B_T > 0) > 0$.

Proof Let $(t_i)_{i \geq 1}$ be a dense subset of $[0, +\infty[$. Application of Lemma A 1.1 from Delbaen and Schachermayer [4] and diagonalization procedure results in a sequence $B^n \in \text{conv}(A^n, A^{n+1}, \dots)$, $n \geq 1$, such that for all $i \geq 1$ the sequence $(B_{t_i}^n)_{n \geq 1}$ converges almost surely to a $[0, +\infty]$ -valued variable B'_{t_i} .

We now define process $B = (B_t)_{t \geq 0}$ as

$$B_t = \inf_{t_i > t} B'_{t_i}.$$

It can be easily seen that the sequence $(B^n)_{n \geq 1}$ and the process B are the processes required by the lemma.

Finally, if $P(A_T^n > \epsilon) > \epsilon$ then

$$E(B_T \wedge 1) \geq \liminf_{n \rightarrow \infty} (E B_T^n \wedge 1) \geq \liminf_{n \rightarrow \infty} (E A_T^n \wedge 1) > \epsilon^2$$

and the result follows. \square

5 Proof of the main theorem

We start with two auxiliary lemmas.

We are in the setting of Theorem 2.1. Let V be a positive supermartingale for all $Q \in \mathbf{M}(X)$. By \mathcal{C} we denote the set of increasing processes C such that $C_0 = 0$ and the process $V + C$ is a supermartingale for all $Q \in \mathbf{M}(X)$. We introduce an order relation \prec on \mathcal{C} saying that C_1 is *less* than C_2 ($C_1 \prec C_2$) if $C_2 - C_1$ is an increasing process.

Lemma 5.1 *There exists the maximal element \widehat{C} on the ordered set \mathcal{C} .*

Proof Kuratowski's lemma (see Kelley [14] th.25, page 33) implies the existence of a maximal chain $\widetilde{\mathcal{C}} \in \mathcal{C}$. Denote

$$a = \sup_{C \in \widetilde{\mathcal{C}}} EC_\infty.$$

If $C \in \mathcal{C}$, the process $V + C$ is a P -supermartingale, and hence

$$EC_\infty \leq E(V_\infty + C_\infty) \leq V_0.$$

It follows that $a < +\infty$.

Now we find an *ordered* sequence $(C^n)_{n \geq 1}$ in $\widetilde{\mathcal{C}}$ such that expectations EC_∞^n tend to a as n tend to ∞ , and define the process \widehat{C} as the limit: $C^n \uparrow \widehat{C}$, $n \rightarrow \infty$. Notice that the convergence here is uniform on $[0, +\infty]$.

It is easy to see that $\widehat{C} \in \mathcal{C}$. Moreover, since $\widetilde{\mathcal{C}}$ is the maximal chain in \mathcal{C} , the process \widehat{C} is the maximal element of \mathcal{C} iff it is the maximal element of $\widetilde{\mathcal{C}}$. Let $C \in \widetilde{\mathcal{C}}$. Because all elements of $\widetilde{\mathcal{C}}$ are comparable between each other, there are two possibilities:

1. $C \prec C^{n_0}$ for some n_0 and then $C \prec \widehat{C}$,
2. $C^n \prec C$ for all $n \geq 1$ and then $\widehat{C} \prec C$. However the theorem on monotone convergence implies that

$$E\widehat{C}_\infty = \lim_{n \rightarrow \infty} EC_\infty^n = a \geq EC_\infty.$$

Therefore $\widehat{C}_\infty = C_\infty$ and hence $\widehat{C} = C$.

The proof of the Lemma 5.1 is finished. \square

Let \widehat{C} be the maximal element of \mathcal{C} given by Lemma 5.1. Define the process $U = (U_t)_{t \geq 0}$ by

$$U_t = V_t + \widehat{C}_t, \quad t \geq 0.$$

Let also T be a positive number, and \mathcal{T} be a dense subset of $[0, T]$.

Lemma 5.2 *Let $(G^n)_{n \geq 1}$ be a sequence of admissible integrands, and $(A^n)_{n \geq 1}$ be a sequence of adapted increasing processes such that $A_0^n = 0$, $n \geq 1$, and*

$$U_0 + (G^n \bullet X)_t \geq -a, \quad t \in [0, T], \quad n \geq 1 \quad \text{for some } a \geq 0$$

and

$$\limsup_{n \rightarrow \infty} |U_0 + (G^n \bullet X)_t - A_t^n - U_t| = 0, \quad t \in \mathcal{T},$$

where process U and set \mathcal{T} are defined before the formulation of the lemma.

Then

- 1) variables A_T^n tend to 0 in probability as n tends to ∞
- 2) maximal functions $(U_0 + G^n \bullet X - U)_T^*$ tend to 0 in probability as n tends to ∞ .

Proof 1) Assume that there exist an increasing sequence $(n_k)_{k \geq 1}$ and a positive number ϵ such that $P(A_T^{n_k} > \epsilon) > \epsilon$, $k \geq 1$. Then Lemma 4.2 implies the existence of a sequence $B^k \in \text{conv}(A^{n_k}, A^{n_{k+1}}, \dots)$, $k \geq 1$ and of increasing adapted process B such that $P(B_T > 0) > 0$ and

$$B_t = \lim_{\delta \rightarrow 0} \limsup_{k \rightarrow \infty} B_{t+\delta}^k = \lim_{\delta \rightarrow 0} \liminf_{k \rightarrow \infty} B_{t+\delta}^k, \quad t \geq 0.$$

We come to a contradiction with the maximality of \widehat{C} , if show that $B_0 = 0$ and the process $U + B$ is a supermartingale on interval $[0, T]$ for all $Q \in \mathbf{M}(X)$.

By H^k we denote the convex combination of $(G^{n_k}, G^{n_{k+1}}, \dots)$ obtained with the same weights as $B^k \in \text{conv}(A^{n_k}, A^{n_{k+1}}, \dots)$. We have

$$\limsup_{k \rightarrow \infty} |U_0 + (H^k \bullet X)_t - B_t^k - U_t| = 0, \quad t \in \mathcal{T}$$

and

$$U_0 + (H^k \bullet X)_t \geq -a, \quad t \in [0, T], \quad k \geq 1.$$

Now we fix $t \geq 0$ and define a sequence $(t_l)_{l \geq 1}$ in \mathcal{T} such that $t_l \downarrow t$, $l \rightarrow \infty$. We deduce

$$U_t + B_t = \lim_{l \rightarrow \infty} \liminf_{k \rightarrow \infty} (U_{t_l} + B_{t_l}^k) = U_0 + \lim_{l \rightarrow \infty} \liminf_{k \rightarrow \infty} (H^k \bullet X)_{t_l}.$$

If $t = 0$, Fatou's lemma and the supermartingale property of $H^k \bullet X$ imply that

$$\begin{aligned} B_0 &= EB_0 = E \lim_{l \rightarrow \infty} \liminf_{k \rightarrow \infty} (H^k \bullet X)_{t_l} \\ &\leq \liminf_{l \rightarrow \infty} \liminf_{k \rightarrow \infty} E(H^k \bullet X)_{t_l} \leq 0. \end{aligned}$$

Since $B \geq 0$, it follows that $B_0 = 0$.

Further, for $t > 0$, $s \in \mathcal{T}$, $s < t$ and $Q \in \mathbf{M}(X)$ Fatou's lemma and the supermartingale property of $H^n \bullet X$ imply

$$\begin{aligned} E_Q[U_t + B_t | \mathcal{F}_s] &\leq U_0 + \liminf_{l \rightarrow \infty} \liminf_{k \rightarrow \infty} E_Q[(H^k \bullet X)_{t_l} | \mathcal{F}_s] \\ &\leq U_0 + \liminf_{k \rightarrow \infty} (H^k \bullet X)_s \\ &= \liminf_{k \rightarrow \infty} (U_s + B_s^k) \leq U_s + B_s. \end{aligned}$$

Because the set \mathcal{T} is dense in $[0, T]$, we deduce that the process $U + B$ is a supermartingale on $[0, T]$ for all $Q \in \mathbf{M}(X)$ and come to the contradiction. The first assertion of the lemma is proved.

2) We must prove that maximal functions $((G^n - G^m) \bullet X)_T^*$ tend to 0 in probability as n and m tend to ∞ . If this was not the case, we could find two increasing sequences $(i_k, j_k)_{k \geq 1}$ and a positive number ϵ such that

$$P\left(\sup_{0 \leq t \leq T} ((G^{i_k} - G^{j_k}) \bullet X)_t \geq \epsilon\right) > \epsilon, \quad k \geq 1.$$

We define stopping time τ_k as

$$\tau_k = \inf \{t \geq 0 : ((G^{i_k} - G^{j_k}) \bullet X)_t \geq \epsilon\}.$$

We have

$$\left\{ \omega : \sup_{t \leq T} ((G^{i_k} - G^{j_k}) \bullet X)_t \geq \epsilon \right\} = \{\omega : \tau_k(\omega) \leq T\}.$$

Therefore

$$(5.1) \quad P(\tau_k \leq T) \geq \epsilon, \quad k \geq 1.$$

Now for $k \geq 1$ we define integrand L^k and increasing process C^k as

$$\begin{aligned} L^k &= G^{i_k} I\{t \leq \tau_k\} + G^{j_k} I\{t > \tau_k\}, \\ C^k &= ((G^{i_k} - G^{j_k}) \bullet X)_{\tau_k} I_{[\tau_k, +\infty[}. \end{aligned}$$

We have

$$(L^k \bullet X)_t - C_t^k = (G^{i_k} \bullet X)_t I_{[0, \tau_k[} + (G^{j_k} \bullet X)_t I_{[\tau_k, +\infty[}.$$

It follows that

$$U_0 + (L^k \bullet X)_t \geq -a, \quad t \in [0, T], \quad k \geq 1.$$

Part 1) of the lemma implies that variables $A_T^{i_k}$ and $A_T^{j_k}$ tend to 0 in probability as k tends to ∞ . Passing to a subsequence we can suppose that this convergence holds almost surely. Then

$$\limsup_{k \rightarrow \infty} (|U_0 + (G^{i_k} \bullet X)_t - U_t| + |U_0 + (G^{j_k} \bullet X)_t - U_t|) = 0, \quad t \in \mathcal{T}$$

and therefore

$$\limsup_{k \rightarrow \infty} |U_0 + (L^k \bullet X)_t - C_t^k - U_t| = 0, \quad t \in \mathcal{T}.$$

Now we deduce from the part 1) of the lemma that variables C_T^k tend to 0 in probability as k tends to ∞ . But

$$C_T^k \geq \epsilon I_{\{\tau_k \leq T\}}$$

and we come to the contradiction with (5.1). \square

Proof (of Theorem 2.1) Let \widehat{C} be a maximal element of \mathcal{C} given by Lemma 5.1. By $U = (U_t)_{t \geq 0}$ we denote process

$$U_t = V_t + \widehat{C}_t, \quad t \geq 0.$$

To prove Theorem 2.1 we have to construct an admissible integrand L such that

$$U_t = U_0 + (L \bullet X)_t, \quad t \geq 0.$$

Notice that it is sufficient to prove this representation only for any finite time interval $[0, T]$. Indeed, if there are admissible integrands L^n such that

$$U_t = U_0 + (L^n \bullet X)_t, \quad 0 \leq t \leq n,$$

then we can define the desired integrand L as

$$L = \sum_{n=1}^{+\infty} L^n I\{n-1 \leq t < n\}.$$

In the sequel we consider the case when t belongs to $[0, 1]$. To make the proof more readable we divide it on a number of steps.

Step 1 *There are admissible integrands $(K^n)_{n \geq 1}$ such that*

$$(5.2) \quad \limsup_{n \rightarrow \infty} (U_0 + (K^n \bullet X)_s - U_s)_1^* = 0,$$

$$(5.3) \quad U_0 + (K^n \bullet X)_t \geq 0, \quad t \in [0, 1], \quad n \geq 1$$

$$(5.4) \quad \sup_{n \geq 1} [K^n \bullet X, K^n \bullet X]_1 < +\infty,$$

where all relations hold almost surely.

Proof First we construct the sequence of admissible integrands $(G^n)_{n \geq 1}$ such that

$$(5.5) \quad \limsup_{n \rightarrow \infty} (U_0 + (G^n \bullet X)_s - U_s)_2^* = 0,$$

$$(5.6) \quad U_0 + (G^n \bullet X)_t \geq 0, \quad t \in [0, 2], \quad n \geq 1.$$

The desired sequence $(K^n)_{n \geq 1}$ will be obtained later as the sequence of appropriate convex combinations of $(G^n)_{n \geq 1}$.

By $\mathcal{T}(n)$ we denote the set of numbers of the form $i \cdot 2^{-n}$, $0 \leq i \leq 2^{n+1}$. We have that $\mathcal{T}(n) \subseteq \mathcal{T}(n+1)$ and that the limiting set $\mathcal{T}(\infty) = \bigcup_{n \geq 1} \mathcal{T}(n)$ is dense in $[0, 2]$. For $n \geq 1$ we define process $U^n = (U_t^n)_{t \geq 0}$ as $U_t^n = \min(U_t, n)$. It is clear that U^n is a Q -supermartingale for all $Q \in \mathbf{M}(X)$.

For $n \geq 1$ and $0 \leq i \leq 2^{n+1} - 1$ Proposition 4.4 implies the existence of an admissible integrand G^{ni} such that

$$(G^{ni} \bullet X)_t = 0, \quad t \leq i \cdot 2^{-n}$$

and

$$U_{i \cdot 2^{-n}}^n + (G^{ni} \bullet X)_{(i+1) \cdot 2^{-n}} \geq U_{(i+1) \cdot 2^{-n}}^n.$$

Now we define integrand G^n and increasing process A^n as

$$\begin{aligned} G^n &= \sum_{i=0}^{2^{n+1}-1} G^{ni} I\{i \cdot 2^{-n} < t \leq (i+1) \cdot 2^{-n}\}, \\ A^n &= \sum_{i=0}^{2^{n+1}-1} \left(U_{i \cdot 2^{-n}}^n + (G^{ni} \bullet X)_{(i+1) \cdot 2^{-n}} - U_{(i+1) \cdot 2^{-n}}^n \right) I_{[(i+1) \cdot 2^{-n}, +\infty[}. \end{aligned}$$

We have

$$(5.7) \quad U_0^n + (G^n \bullet X)_{i \cdot 2^{-n}} - A_{i \cdot 2^{-n}}^n = U_{i \cdot 2^{-n}}^n, \quad 0 \leq i \leq 2^{n+1}.$$

The process G^n as a finite sum of admissible integrands is an admissible integrand. Therefore $G^n \bullet X$ is a supermartingale. Since

$$U_0^n + (G^n \bullet X)_2 = U_2^n + A_2^n \geq 0,$$

we deduce that $U_0^n + (G^n \bullet X)_t \geq 0$, $t \in [0, 2]$. This proves inequality (5.6).

Further, because maximal functions $(U^n - U)_2^*$ tend to 0 almost surely as n tends to ∞ , we deduce from (5.7) that

$$\limsup_{n \rightarrow \infty} \sup_{t \in \mathcal{T}(n)} |U_0^n + (G^n \bullet X)_t - A_t^n - U_t| = 0.$$

Now part 2) of Lemma 5.2 implies the convergence of maximal functions $(U_0 + (G^n \bullet X)_s - U_s)_2^*$ to 0 in probability. Passing, if necessary, to a subsequence, we can suppose that this convergence holds almost surely. It follows that the Eq.(5.5) for the sequence $(G^n)_{n \geq 1}$ takes place.

Since $(U)_2^* < +\infty$, we deduce from Eq.(5.5) that

$$\sup_{n \geq 1} (G^n \bullet X)_2^* < +\infty.$$

Therefore, the probability of stopping time

$$\sigma_m = \inf_{n \geq 1} \inf \{t \geq 0 : |(G^n \bullet X)_t| \geq m\}$$

being below 2 tends to 0 as m tends to ∞ . Accounting for the supermartingale property of $G^n \bullet X$ we obtain that

$$E [U_0 + (G^n \bullet X)_{\sigma_m}^*] \leq m + E [U_0 + (G^n \bullet X)_{\sigma_m}] \leq m + U_0.$$

Now the Davis inequality implies the existence of constant $D_m < +\infty$ such that

$$E [G^n \bullet X, G^n \bullet X]_{\sigma_m}^{1/2} \leq D_m.$$

According to Lemma 4.2 there are a sequence of increasing processes $C^n \in \text{conv} \left([G^n \bullet X, G^n \bullet X]^{1/2}, [G^{n+1} \bullet X, G^{n+1} \bullet X]^{1/2}, \dots \right)$, $n \geq 1$, and an increasing process C such that

$$C_t = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} C_{t+\epsilon}^n = \lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} C_{t+\epsilon}^n, \quad t \geq 0.$$

From Fatou's lemma we deduce that

$$EC_1 I(\sigma_m \geq 2) \leq \liminf_{n \rightarrow \infty} EC_{\sigma_m}^n I(\sigma_m \geq 2) \leq D_m.$$

Since $\lim_{m \rightarrow \infty} P(\sigma_m \geq 2) = 1$, we have that $C_1 < +\infty$ almost surely and therefore $\sup_{n \geq 1} C_1^n < +\infty$.

Now we define the desired sequence $(K^n)_{n \geq 1}$ as the sequence of convex combinations of $(G^n)_{n \geq 1}$ obtained in much the same way as the sequence $(C^n)_{n \geq 1}$ was obtained from $\left([G^n \bullet X, G^n \bullet X]^{1/2} \right)_{n \geq 1}$. Now the proof of Step 1 follows from (5.5), (5.6), and the ‘‘Minkowski’’ inequality: $[K^n \bullet X, K^n \bullet X]_1^{1/2} \leq C_1^n$ (see Dellacherie and Meyer [6] Chapter VII, Eq.(54.1)). The proof of Step 1 is finished. \square

If the sequence $([K^n \bullet X, K^n \bullet X])_{n \geq 1}$ is bounded not only in probability but in L_Q^2 -norm for some measure $Q \in \bar{\mathbf{M}}(X)$:

$$\sup_{n \geq 1} E_Q [K^n \bullet X, K^n \bullet X]_1 < +\infty,$$

then by standard arguments we can find a sequence $L^n \in \text{conv}(K^n, K^{n+1}, \dots)$, $n \geq 1$, such that martingales $(L^n \bullet X)_{n \geq 1}$ converge in the space $\mathcal{M}^2(Q, [0, 1])$ of square integrable martingales. The limiting martingale has the form $L \bullet X$ for some integrand L (see, for example, Jacod [11] Chapter 4), and we finish.

Under more weak condition (5.4), the construction of desirable integrand L is much more complex. We proceed on a similar way as in Delbaen and Schachermayer [4].

Denote $\rho = \sup_{n \geq 1} [K^n \bullet X, K^n \bullet X]_1$, and define probability R on (Ω, \mathcal{F}) such that

$$\frac{dR}{dP} = \frac{e^{-\rho}}{Ee^{-\rho}}.$$

Then the inequality (5.4) in Step 1 implies that $R \sim P$.

The process X is a locally bounded P -martingale. Therefore, it is a special semi-martingale with respect to measure R . Let $X = M + A$ be the canonical decomposition of X , where M is an R -local martingale and A is a process of R -integrable variation. Because the definition of stochastic integral is invariant with respect to equivalent changes of measure, the stochastic integral $K^n \bullet X$ exists on $(\Omega, \mathcal{F}, \mathbf{F}, R)$ and is a semi-martingale. Since

$$(5.8) \quad E_R \sup_{n \geq 1} [K^n \bullet X, K^n \bullet X]_1 < +\infty,$$

Lemma 4.1 and Proposition 4.1 imply that $K^n \bullet X$ is a special semi-martingale on $[0, 1]$ with canonical decomposition

$$(K^n \bullet X)_t = (K^n \bullet M)_t + (K^n \bullet A)_t, \quad 0 \leq t \leq 1.$$

Step 2 *There is a sequence $L^n \in \text{conv}(K^n, K^{n+1}, \dots)$, $n \geq 1$, such that the sequence $(L^n \bullet M)_{n \geq 1}$ converges in semi-martingale topology on $[0, 1]$.*

Proof Lemma 4.1 and Eq.(5.8) imply that $\sup_{n \geq 1} E_R [K^n \bullet M, K^n \bullet M]_1 < +\infty$. Therefore the sequence $(K^n \bullet M)_{n \geq 1}$ is bounded in the space $\mathcal{M}^2(R, [0, 1])$ of square integrable martingales. Hence there is a sequence $L^n \in \text{conv}(K^n, K^{n+1}, \dots)$, $n \geq 1$, such that R -martingales $(L^n \bullet M)_{n \geq 1}$ converge in $\mathcal{M}^2(R, [0, 1])$ and therefore converge also in the semi-martingale topology on $[0, 1]$. \square

Step 3 *The sequence $(L^n \bullet A)_{n \geq 1}$, where integrands $(L^n)_{n \geq 1}$ are given in Step 2, converges in semi-martingale topology on $[0, 1]$.*

Proof The proof of Step 3 follows the same lines as the proof of Lemma 4.11 in Delbaen and Schachermayer [4]. We have to show that variances $\int_0^1 |L_t^n - L_t^m| |dA_t|$ tend to 0 as n and m tend to ∞ . If this was not the case,

we could find two increasing sequences $(i_n, j_n)_{n \geq 1}$ and a number $\epsilon > 0$ such that $P(A_1^n > \epsilon) > \epsilon$, $n \geq 1$, where $A_t^n = \frac{1}{2} \int_0^t |L_s^{i_n} - L_s^{j_n}| |dA_s|$.

Hahn's decomposition for increasing predictable processes implies the existence of predictable processes h^n with values in $\{-1, 1\}$ such that

$$A_t^n = \frac{1}{2} \int_0^t h^n (L^{i_n} - L^{j_n}) dA.$$

We define integrand G^n as

$$G^n = \frac{1}{2} (L^{i_n} + L^{j_n} + h^n (L^{i_n} - L^{j_n})).$$

We have

$$G^n \bullet X = \frac{1}{2} (L^{i_n} + L^{j_n}) \bullet X + A^n.$$

Because L^n is a convex combination of integrands (K^n, K^{n+1}, \dots) given in Step 1, we deduce that

$$(5.9) \quad \limsup_{n \rightarrow \infty} (U_0 + G^n \bullet X - A^n - U)_1^* = 0.$$

By construction of h^n and because

$$\begin{aligned} (G^n - L^{i_n}) \bullet A &= \frac{1}{2} (h^n - 1) (L^{i_n} - L^{j_n}) \bullet A, \\ (G^n - L^{j_n}) \bullet A &= \frac{1}{2} (h^n + 1) (L^{i_n} - L^{j_n}) \bullet A, \end{aligned}$$

we deduce that processes $(G^n - L^{i_n}) \bullet A$ and $(G^n - L^{j_n}) \bullet A$ are the increasing processes. Moreover, since

$$(G^n - L^{i_n}) \bullet M = \frac{1}{2} (h^n - 1) (L^{i_n} - L^{j_n}) \bullet M$$

and because processes $(L^{i_n} - L^{j_n}) \bullet M$ tend to 0 in semi-martingale topology on $[0, 1]$, we deduce that maximal functions $((G^n - L^{i_n}) \bullet M)_1^*$ tend to 0 in probability. The same holds for $((G^n - L^{j_n}) \bullet M)_1^*$. Taking, if necessary, a subsequence, we can suppose that these maximal functions converge almost surely. Then stopping times τ_n defined as

$$\tau_n = \inf_{m \geq n} \inf \{t \geq 0 : (G^m \bullet M)_t < \max((L^{i_m} \bullet M)_t, (L^{j_m} \bullet M)_t) - 1\}$$

form an increasing sequence such that

$$(5.10) \quad \lim_{n \rightarrow \infty} I(\tau_n < 1) = 0.$$

For $t < \tau_n$ and $m \geq n$ we have

$$\begin{aligned} (G^m \bullet X)_t &= (G^m \bullet A)_t + (G^m \bullet M)_t \geq \max((L^{im} \bullet A)_t, (L^{jm} \bullet A)_t) \\ &+ \max((L^{im} \bullet M)_t, (L^{jm} \bullet M)_t) - 1 \geq \max((L^{im} \bullet X)_t, (L^{jm} \bullet X)_t) - 1. \end{aligned}$$

At time τ_n the jump $\Delta(G^m \bullet X)$ is either $\Delta(L^{im} \bullet X)$ or $\Delta(L^{jm} \bullet X)$, and hence the inequality

$$(G^m \bullet X)_t \geq \max((L^{im} \bullet X)_t, (L^{jm} \bullet X)_t) - 1$$

holds for $t \leq \tau_n$ and $m \geq n$. Because L^n is a convex combination of the variables (K^n, K^{n+1}, \dots) given in Step 1, we deduce that

$$U_0 + (G^m \bullet X)_t \geq -1, \quad 0 \leq t \leq \tau_n, \quad m \geq n.$$

Now we define integrand H^n as $H^n = G^n I_{[0, \tau_n]}$. We obtain that H^n is an admissible integrand on $[0, 1]$:

$$U_0 + (H^n \bullet X)_t \geq -1, \quad 0 \leq t \leq 1.$$

Moreover, Eqs. (5.9) and (5.10) imply that

$$(5.11) \quad \limsup_{n \rightarrow \infty} (U_0 + (H^n \bullet X)_s - A_s^n - U_s)_1^* = 0.$$

Now Lemma 5.2 implies that variables A_1^n tend to 0 in probability as n tend to ∞ , and we come to the contradiction. The proof of Step 3 is finished. \square

Now we are able to finish the proof of Theorem 2.1. Steps 2 and 3 imply the existence of a sequence of admissible integrands $(L^n)_{n \geq 1}$ such that the sequences $(L^n \bullet M)_{n \geq 1}$ and $(L^n \bullet A)_{n \geq 1}$ converge in semi-martingale topology on $[0, 1]$. Therefore, the sequence of stochastic integrals $(L^n \bullet X)_{n \geq 1}$ also converges in the semi-martingale topology on $[0, 1]$. Now Memin's theorem (see Memin [15]) implies the existence of predictable process L such that integrals $L^n \bullet X$ converge to $L \bullet X$ in the semi-martingale topology on $[0, 1]$. In particular, maximal functions $(L^n \bullet X - L \bullet X)_1^*$ tend to 0 in probability as

n tends to ∞ . Because L^n is a convex combination of the variables $(K^m)_{m \geq n}$ given in Step 1, we deduce that

$$U_t = U_0 + (L \bullet X)_t, \quad 0 \leq t \leq 1$$

and finish the proof of Theorem 2.1. \square

Acknowledgements I would like to thank M.V. Bondarenko, H. Föllmer, A.A. Gushchin, Y.M. Kabanov, M. Schäl and A.N. Shiryaev for remarks and discussions. It is a pleasure to thank Nicole El Karoui for “small” lecture on a subject during her short visit at the University of Bonn. Special thanks go to W. Schachermayer for discussions and for sending me his joint paper with F. Delbaen [4].

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