Discussion Paper No. B–306

A Systematic Approach to Pricing and Hedging of International Derivatives with Interest Rate Risk

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February 1995

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Financial support by the Deutsche Forschungsgemeinschaft Sonderforschungsbereich 303 at the University of Bonn is gratefully acknowledged. Document typeset in LaTeX.

Abstract

The paper deals with the valuation and hedging of non path-dependent European options on one or several underlyings in a model of an international economy which allows for both interest rate and exchange rate risk. Using martingale theory we provide a unified and easily applicable approach to pricing and hedging Black-Scholes type options on stocks, bonds, forwards, futures, exchange rates and indices. We also cover the pricing and hedging of options to exchange two Black-Scholes type options for one another. The contingent claims may pay off in arbitrary currencies.

Contents

1 Introduction

The present paper deals with the valuation and hedging of non path-dependent European options on one or several underlyings in a model of an international economy which allows for both, interest rate and exchange rate risk. We study options on stocks, bonds, forwards, futures, exchange rates and indices; their payoff may be in any currency and a relatively complex function of one or several underlyings.

There exists meanwhile a huge number of different contracts on various underlyings, and it is easy to construct new payoffs by simply combining in a different manner the elements that make up a certain contract, such that a case by case analysis as has been carried out in the earlier work quoted below is no longer appropriate. Therefore we develop a truly unified method for the pricing and hedging of non path-dependent European options. In order to illustrate the flexibility of our method we deal with a number of examples. However we would like to emphasize that our approach lends itself also to the pricing of many other contracts including Quanto options which are not explicitely considered here.

A satisfactory valuation theory should allow for interest rate risk along with exchange rate and asset-specific risk, because interest-rate-dependent contingent claims are among the most important derivatives in international financial markets. To meet this requirement we have chosen an international economy model similar to the one introduced by Amin and Jarrow [AJ91] as our framework. Their model combines a fully developed stochastic theory of the term structure of interest rates in the sense of [HJM92] with models for the valuation of exchange rate and stock options. However, following the approach of El Karoui et al. [EKR89] [EMV92], we use bond prices instead of forward rates as primitives for the modelling of interest rate risk.

Starting from the concept of a simple derivative — which includes stocks, bonds, forwards, futures, exchange rates and indices — we derive generic option pricing formulas for "simple options" of the Black-Scholes type on simple derivatives. Moreover, to make our results operational we provide a procedure for computing the input parameters for more complex simple derivatives. We then go on to study the pricing of options to exchange one simple option for another. The latter class of derivatives contains for instance options on the maximum of two simple derivatives or spread-options. For certain payoffs such as spread-options where explicit formulas in the Black-Scholes sense do not exist we are still able to give near explicit solutions. We conclude our analysis with a discussion of several numerical methods for evaluating the near closed-form solutions. We conclude our analysis of option pricing with a discussion of several numerical methods for evaluating the near closed-form solutions.

Since pricing formulas are of little practical use without knowledge of the corresponding hedge portfolio we present a systematic approach to deriving hedge strategies. We start by giving hedge portfolios in the underlying simple derivatives and then demonstrate how these simple derivatives can be duplicated by securities for which liquid markets exist. In particular we discuss the hedging of bonds with futures. It is worth noting that in the situations where closed form solutions for the option prices do not exist we are still able to give near explicit solutions for the hedge portfolios without resorting to numerical differentiation.

The pricing of certain types of options belonging to the class of derivatives considered here, and the development of arbitrage free models of international economies have previously been adressed in the literature. Among the early work on correlation dependent options are the papers by Margrabe [Mar78], Stulz [Stu82] and Johnson [Joh87], who are all working in the classical Black-Scholes model. A collection of their results can be found in Rubinstein-Reiner's work on Exotic Opions [Rub90]. Papers on currency options are due to Garman-Kohlhagen [GK83] and Grabbe [Gra83]. More recently Amin and Jarrow [AJ91] have developed the above mentioned arbitrage free model of an international economy. Our treatment of interest rate risk follows the work of El Karoui et al, [EMV92] [EKMV92].

The paper which is closest to our analysis is Jamshidian [Jam93b]. He derives valuation formulas for contingent claims which fall within the class of derivatives we call simple options on simple derivatives. However, his paper does not provide a general procedure for applying his main result to derivatives which are in principle within the scope of his analysis. Moreover, neither this paper nor the related work [Jam93a] contains a systematic approach to the derivation of hedge-portfolios or to the pricing and hedging of options on several underlyings such as our options to exchange two options for one another. Finally Kat and Roozen [KR94] again consider a setting with deterministic interest rates. Using a partial differential equation approach like all the other papers apart from Amin-Jarrow, El Karoui et al. and Jamshidian, they make a step towards presenting a more unified treatment of pricing and hedging non path-dependent options.

The paper is organized as follows:
In section two we present the general N-country model of financial markets. Section three contains the definition of simple derivatives and the pricing and hedging results for simple options on simple derivatives. We then show how to compute the input parameters for more complex simple derivatives and give examples of the application of the results on simple options. Section four contains our results concerning options to exchange simple options. Section five finally concludes.

Notation:

Throughout the paper we denote the inner product of two vectors X_1, $X_2 \in \mathbb{R}^d$ by $X_1 \cdot X_2$; the norm of a vector will be denoted by $|X| := (X \cdot X)^{\frac{1}{2}}$.

2 An Arbitrage-free Model of an International Economy

In this section we introduce an arbitrage-free model of an international economy that incorporates stochastic interest and exchange rates. This model will serve as our framework for the valuation of derivatives. Our main focus is on the relations between the asset price dynamics in different countries induced by the no-arbitrage condition. These relations were first derived in [AJ91].

2.1 Asset Price Dynamics

We consider N countries indexed by $n \in \{0, \dots, N\}$. Country 0 will be the *domestic* country. The exchange rate between country 0 and country $n \in \{1, \dots, N\}$ will be denoted by e^n, that is e^n units of the domestic currency can be exchanged for one unit of the foreign currency. When working with only two countries we simply talk about the domestic and the foreign country and index them with d and f. The choice of the domestic country is arbitrary and depends on the particular pricing and hedging problem under consideration. We call an asset a *domestic* asset if its payoffs are denominated in currency 0. Notice that every asset whose payoffs are not originally denominated in currency 0 can be transformed into a domestic asset if the payoffs are translated into currency 0 using the appropriate exchange rate.

We assume that in all countries zero-bonds of all maturities $T \in [0, T_F]$ are traded. The zero-bond in country n with maturity date T shall be denoted by $B^n(t,T)$ for $t \in [0,T]$. By assumption $B^n(T,T) \equiv 1 \; \forall\, T, n$. The short rate in country n, r^n, is given by

$$r_t^n = -\frac{\partial}{\partial T}\Big|_{T\,=\,t} \ln B^n(t,T)\,. \tag{1}$$

For an explicit formula deduced from (1) see for instance [EKMV92]. By $\beta_{t,T}^n := \exp(\int_t^T r_s^n\, ds)$ we represent the savings-account of country n.

Apart from zero-bonds we consider other primitive assets such as dividend free stocks. They are denoted by $S^{n,j}$, $0 \le n \le N$, $0 \le j \le i_n$ where $S^{n,j}$ is the price of asset j in country n.

When modelling arbitrage-free asset price dynamics one usually starts from assumptions on the asset prices under the so-called historical probabilities which govern the actual evolution of asset prices. Since we are only interested in the pricing of derivatives by no-arbitrage arguments it is legitimate to model the asset price distribution directly under a domestic risk-neutral measure P. Under such a measure all non-dividend paying domestic assets are martingales after discounting with the domestic savings account. This implies that their mean instantaneous growth rate — in a slight abuse of the usual definition in the sequel simply referred to as their drift — is equal to r^0.

To have a mathematically precise model let there be given a d-dimensional Brownian Motion W on some filtered probability space $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{t \in [0,T_F]}$. We assume that under P the dynamics of the bonds and of the other primitive assets are given by the following stochastic differential equations:

$$\begin{aligned} dB^0(t,T) &= r_t^0 B^0(t,T)dt + \eta^0(t,T)B^0(t,T)dW_t \\ dS_t^{0,j} &= r_t^0 S_t^{0,j} dt + \eta^{0,j}(t) S_t^{0,j} dW_t \end{aligned} \tag{2}$$

and for the foreign assets

$$\begin{aligned} dB^n(t,T) &= \mu^n(t,T)B^n(t,T)dt + \eta^n(t,T)B^n(t,T)dW_t \\ dS_t^{n,j} &= \mu^{n,j}(t) S_t^{n,j} dt + \eta^{n,j}(t) S_t^{n,j} dW_t \end{aligned} \tag{3}$$

Finally the dynamics of the exchange rates are given by

$$de^n(t) = \mu_t^{e^n} e^n(t)dt + \eta^{e^n}(t)e^n(t)dW_t \tag{4}$$

Here $\eta^n(t,T), \eta^{n,j}(t), \eta^{e^n}(t) : [0,T_F] \to \mathbb{R}^d$ are *deterministic* square integrable functions of time. For the bonds we require moreover that $\eta^n(t,T) = 0 \; \forall t \geq T$ and that $\eta^n(t,T)$ is smooth in the second argument. While unnecessary for the arbitrage considerations of the next section the assumption of deterministic dispersion coefficients is essential if we want to obtain explicit pricing formulas. The drift of foreign assets and exchange-rates will be determined from the absence of arbitrage conditions in Proposition 2.1 below.

The volatility of asset $S^{n,j}$ is given by $\sigma^{n,j}(t) := |\eta^{n,j}(t)|$. The instantaneous correlations between the assets in our economy are given by[1]

$$\rho(S^{n_1,j_1}, S^{n_2,j_2}) := \frac{\eta^{n_1,j_1} \cdot \eta^{n_2,j_2}}{\sigma^{n_1,j_1}\sigma^{n_2,j_2}}.$$

Only volatilities and instantaneous correlations matter for the pricing of derivatives since they determine the law of the asset prices under the domestic risk neutral measure. In our analysis this is reflected by the fact that only inner products of the volatility coefficients η and hence instantaneous covariances enter the pricing formulas. However, starting with independent Brownian motions and modelling correlations by means of the coefficients η facilitates the use of stochastic calculus. To compute the volatility coefficients η, which would be needed e.g. for the simulation of asset price trajectories, one has to calculate the root of the instantaneous covariance matrix.

2.2 Arbitrage Restrictions on the Drift of Foreign Assets

As mentioned earlier the drift of every domestic asset must equal the domestic interest rate. This leads to the following

[1] Of course similar formulas hold for bonds and exchange rates.

Proposition 2.1 *1. If the model precludes arbitrage, the drift of the exchange rate equals the interest rate differential, i.e.*

$$\mu^{e^n}(t) = r_t^0 - r_t^n \quad \forall 1 \le n \le N.$$

2. Let X^n be an arbitrary asset denominated in the currency of country n which pays no dividends in $[0,T)$ and whose dynamics are as in (3). In an arbitrage-free model the drift $\mu^n(t)$, $t \in [0,T)$ is then given by

$$\mu^n(t) = r_t^n - \eta^{X^n}(t) \cdot \eta^{e^n}(t)\,,$$

that is the risk premium equals minus the "volatility" of the exchange rate.

Remark: If the drift of the assets is chosen as indicated in Proposition 2.1, every discounted domestic asset is a local martingale. Under our assumptions on the volatilities it is moreover square integrable and hence a global martingale, such that our model actually precludes arbitrage opportunities.

As can be seen from statement 2, in general discounted foreign assets are no martingales under the domestic risk-neutral measure; hence this measure must not be used for the valuation of derivatives paying off in foreign currencies.

PROOF: To determine the drift of e^n we consider the domestic asset $Y_t := e^n(t)\beta^n_{0,t}$. It's drift must be equal to r^d. Now we get from Itô's Lemma:

$$dY_t = Y_t(r_t^n + \mu^{e^n}(t))dt + \eta^{e^n}(t)Y_t^n dW_t\,.$$

Hence $r_t^n + \mu^{e^n}(t)$ must equal r^0 which gives $\mu^{e^n}(t) = r_t^0 - r_t^n$.

Now we look at the asset $X^0 := e^n X^n$. Again the drift of X^0 must equal r^n. Itô's Lemma yields

$$\begin{aligned} dX_t^0 &= X_t^n de_t^n + e_t^n dX^n + d\langle X_t^n\,,\, e_t^n\rangle \\ &= X_t^0\left(\mu^n(t) + r_t^0 - r_t^n + \eta^{e^n}(t)\cdot\eta^{X^n}(t)\right)dt + X_t^0\left(\eta^{e^n}(t) + \eta^{X^n}(t)\right)dW_t \end{aligned}$$

Hence the drift of X_t^0 equals $\mu^n(t) + r_t^0 - r_t^n + \eta^{e^n}(t)\cdot\eta^{X^n}(t)$. This expression must equal r^0, such that μ^n is given by $r_t^n - \eta^{X^n}(t)\cdot\eta^{e^n}(t)$.

We further make

Assumption 2.2 *There are d traded domestic assets such that for all $t \in [0, T_F]$ the instantaneous variance-covariance matrix of these assets is strictly positive definite.*

This assumption guarantees that every contingent claim adapted to the filtration generated by the asset prices can be replicated by a dynamic trading strategy in the d assets and the domestic savings-account, see for instance [Duf92, section 6.I]. Hence the price of every contingent claim H at time t is according to [HP81] given by

$$H_t := E^P\left[\left(\beta^0_{t,T}\right)^{-1}\cdot H_T \mid \mathcal{F}_t\right]\,. \tag{5}$$

Moreover the domestic risk-neutral measure is unique. Formula (5) will be the starting point for our valuation results.

3 Pricing and Hedging of Options

3.1 A general valuation theorem

In this section we give a rather general theorem which leads to a unified treatment of the pricing of European options on various underlyings such as foreign and domestic zero-coupon bonds, foreign or domestic stocks or forward and future contracts on foreign and domestic assets. It is similar in spirit to a result by Jamshidian [Jam93b]; however, we feel that our theorem is more easily applicable. As a first step we introduce the class of admissible underlyings.

Definition 3.1 *A domestic contingent-claim X with a single payoff X_T at a certain date T is called a* simple derivative *if its price process*

$$X_t \;=\; E^P\left[\left(\beta^d_{t,T}\right)^{-1}\cdot X_T \mid \mathcal{F}_t\right]$$

admits a representation of the form

$$X_T = X_0\cdot\exp\left(\int_0^t \eta^X_s\,dW_s - \frac{1}{2}\int_0^t |\eta^X_s|^2 ds + \int_0^t r^d_s ds\right) \tag{6}$$

with $X_0 = E^P\left[(\beta_{0,T})^{-1}X_T\right]$ and with deterministic *dispersion coefficients $\eta^X : [0,T]\to\mathbb{R}^d$.*

Remarks:

1. The main restriction made in the definition of a simple derivative is the assumption of η^X being deterministic. In fact, whenever X_T is strictly positive, X_t is always of the form (6) with possibly stochastic "volatility" η^X.
2. Under our assumption on asset price dynamics every primitive domestic asset is a simple derivative, but the class of contingent claims that satisfy Definition 3.1 is much larger as will be shown in Proposition 3.4.

Theorem 3.2 *Let X, Y be simple derivatives and C a European option with maturity date T and payoff $[X_T - Y_T]^+$. Then the price process C_t of the option is given by*

$$C_t \;=\; X_t\mathcal{N}(d_1(t)) - Y_t\mathcal{N}(d_2(t))$$

where $\mathcal{N}$ denotes the one-dimensional normal distribution function, and where $d_1(t)$ and $d_2(t)$ are given by

$$d_1(t) \;=\; \frac{\ln\left(\frac{X_t}{Y_t}\right) + \frac{1}{2}\int_t^T |\eta^X_s - \eta^Y_s|^2 ds}{\sqrt{\int_t^T |\eta^X_s - \eta^Y_s|^2 ds}}$$

$$d_2(t) \;=\; d_1(t) - \sqrt{\int_t^T |\eta^X_s - \eta^Y_s|^2 ds}$$

PROOF: As a first step we give a formulation of the change of numeraire technique introduced in El Karoui/Rochet [EKR89], which is adapted to our situation:

Lemma 3.3 *Define for a simple derivative X_t a new equivalent probability measure Q^X on $\mathcal{F}_T$ by*

$$\frac{dQ^X}{dP} = \frac{X_T \cdot \left(\beta^d_{0,T}\right)^{-1}}{X_0}.$$

1. *Then for every domestic asset Z whose discounted price process $Z_t \cdot \left(\beta^d_{0,t}\right)^{-1}$ is a martingale under P — that is for every asset that pays no dividends in $[0,T)$ — the process Z_t/X_t is a martingale under Q^X. Hence Q^X is the martingale measure corresponding to the numeraire X_t, and we have the transition formula*

$$E^P\left[\left(\beta^d_{t,T}\right)^{-1} \cdot X_T \cdot Z_T \;|\mathcal{F}_t\right] = X_t \cdot E^{Q^X}[Z_T|\mathcal{F}_t] \tag{7}$$

2. *By Girsanov's theorem the d-dimensional process*

$$W^X_t := W_t - \int_0^t \eta^X_s \, ds$$

is a new Brownian Motion under Q^X.

Remark: If $X_t = B^d(t,T)$ the measure Q^X is just the *forward risk adjusted measure* associated with $B^d(t,T)$. This measure is well known in the interest rate literature.

PROOF OF THE LEMMA: Note first that Q^X is actually a probability measure since $X_0 = E^P\left[\left(\beta^t_{0,T}\right)^{-1} \cdot X_T\right]$. To prove the martingale property for Z_t/X_t we recall the "Bayes-rule"

$$E^{Q^X}\left[\frac{Z_T}{X_T} \;\Big|\; \mathcal{F}_t\right] = \frac{1}{E^P\left[\frac{dQ^X}{dP} \;\Big|\; \mathcal{F}_t\right]} \cdot E^P\left[\frac{Z_T}{X_T} \cdot \frac{dQ^X}{dP}\Big|\mathcal{F}_t\right].$$

Hence

$$\begin{aligned} E^{Q^X}\left[\frac{Z_T}{X_T}\Big|\mathcal{F}_t\right] &= \frac{1}{E^P\left[\left(\beta^d_{0,T}\right)^{-1} \cdot \frac{X_T}{X_0}\Big|\mathcal{F}_t\right]} \cdot E^P\left[\frac{Z_T}{X_T} \cdot \left(\beta^d_{0,T}\right)^{-1} \cdot \frac{X_T}{X_0}\Big|\mathcal{F}_t\right] \\ &= \frac{\frac{\left(\beta^d_{0,t}\right)^{-1}}{X_0} \cdot E^P\left[Z_T \cdot \left(\beta^d_{t,T}\right)^{-1} \Big|\mathcal{F}_t\right]}{\frac{\left(\beta^d_{0,t}\right)^{-1}}{X_0} E^P\left[X_T \cdot \left(\beta^d_{t,T}\right)^{-1} \Big|\mathcal{F}_t\right]} \\ &= \frac{Z_t}{X_t}. \end{aligned}$$

The transition formula can be proved by an analogous computation. It follows, however, directly from an economic argument if we notice that the left hand side of (7) is the price of the contingent claim $X_T \cdot Z_T$ at time t expressed as a risk neutral expectation using the domestic savings-account as numeraire, whereas the right hand side gives the same price using the numeraire $(X_t)_{0\le t\le T}$, cf. [HP81].

To prove statement 2 note that by assumption dQ^X/dP is given by

$$\frac{dQ^X}{dP} = \exp\left(\int_0^t \eta_s^X dW_s - \frac{1}{2}\int_0^t |\eta_s^X|^2 ds\right).$$

Hence the claim follows from Girsanovs theorem.

Using this Lemma the rest of the proof is rather straightforward: According to (5) the price of the option is given by

$$\begin{aligned} C_t &= E^P\left[\left(\beta_{t,T}^d\right)^{-1}[X_T - Y_T]^+ \,\Big|\, \mathcal{F}_t\right] \\ &= E^P\left[\left(\beta_{t,T}^d\right)^{-1} X_T \cdot 1_{\left\{\frac{Y_T}{X_T}<1\right\}} \,\Big|\, \mathcal{F}_t\right] - E^P\left[\left(\beta_{t,T}^d\right)^{-1} Y_T \cdot 1_{\left\{\frac{X_T}{Y_T}>1\right\}} \,\Big|\, \mathcal{F}_t\right] \\ &= X_t \cdot E^{Q^X}\left[1_{\left\{\frac{Y_T}{X_T}<1\right\}} \,\Big|\, \mathcal{F}_t\right] - Y_t \cdot E^{Q^Y}\left[1_{\left\{\frac{X_T}{Y_T}>1\right\}} \,\Big|\, \mathcal{F}_t\right] \end{aligned}$$

The last line follows from Lemma 3.3 if we take once X_t and once Y_t as numeraire. Now we get under Q^X for $\frac{Y_T}{X_T}$

$$\frac{Y_T}{X_T} = \frac{Y_t}{X_t} \cdot \exp\left(\int_t^T (\eta_s^Y - \eta_s^X) dW_s^X - \frac{1}{2}\int_t^T |\eta_s^Y - \eta_s^X|^2 ds\right)$$

Hence

$$\begin{aligned} Q^X\left[\frac{Y_T}{X_T} < 1 \,\Big|\, \mathcal{F}_t\right] &= Q^X\left[\ln Y_T - \ln X_T < 0 \mid \mathcal{F}_t\right] \\ &= Q^X\left[\frac{\int_t^T (\eta_s^Y - \eta_s^X) dW_s^X}{\sqrt{\int_t^T |\eta_s^Y - \eta_s^X|^2 ds}} < \frac{\ln X_T - \ln Y_T + \frac{1}{2}\int_t^T |\eta_s^Y - \eta_s^X|^2 ds}{\sqrt{\int_t^T |\eta_s^Y - \eta_s^X|^2 ds}}\right] \end{aligned}$$

Since η^X and η^Y are deterministic, $\int_t^T (\eta_s^Y - \eta_s^X) dW_s^X / \sqrt{\int_t^T |\eta_s^Y - \eta_s^X|^2 ds}$ is a standard normally distributed random variable so that

$$Q^X\left[\frac{Y_T}{X_T} < 1 \,\Big|\, \mathcal{F}_t\right] = \mathcal{N}(d_1).$$

Analogously we get

$$Q^Y\left[\frac{X_T}{Y_T} > 1 \,\Big|\, \mathcal{F}_t\right] = \mathcal{N}(d_2)$$

and the theorem follows.

3.2 Examples

Now we want to consider a number of examples which illustrate flexibility and generality of Theorem 3.2. The following Proposition gives a method for computing the initial value and the volatility coefficients of the simple derivative corresponding to a certain payoff also in cases where these parameters cannot be read directly from the asset price dynamics.

Proposition 3.4 *Let X_T be given by*

$$X_T = \gamma^X \cdot \exp\left(\int_0^T \sum_{n=0}^N \alpha^{n,X} r_s^n \, ds + \int_0^T \mu_s^X \, ds + \int_0^T v_s^X \, dW_s \right)$$

where $\gamma^X \in \mathbb{R}^{++}$, $\alpha^{n,X} \in \mathbb{Z} \; \forall n$, and where $\mu^X : [0,T] \to \mathbb{R}$, $v^x : [0,T] \to \mathbb{R}^d$ are deterministic *functions. Then there is a $X_0 \in \mathbb{R}^{++}$ and a* deterministic *function $\eta^X : [0,T] \to \mathbb{R}^d$ such that X_T is a simple derivative of the form*

$$X_T = X_0 \cdot \exp\left(\int_0^t \eta_s^X dW_s - \frac{1}{2} \int_0^t |\eta_s^X|^2 ds + \int_0^t r_s^d ds \right) .$$

PROOF: Since $B^n(T,T) = 1$ we get

$$\begin{aligned}
X_T &= X_T \cdot B^0(T,T) \cdot \prod_{n=0}^N B^n(T,T)^{-\alpha^{n,X}} \\
&= \gamma^X \cdot B^0(T,T) \cdot \prod_{n=0}^N B^n(0,T)^{-\alpha^{n,X}} \\
&\quad \cdot \exp\left[\int_0^T r_s^0 \, ds + \int_0^T \left(\mu_s^X + \frac{1}{2} \sum_{n=0}^N \alpha^{n,X} |\eta^n(s,T)|^2 - \frac{1}{2} |\eta^0(s,T)|^2 \right) ds \right] \\
&\quad \cdot \exp\left[\int_0^T \left(v_s^X - \sum_{n=0}^N \alpha^{n,X} \eta^n(s,T) + \eta^0(s,T) \right) dW_s \right]
\end{aligned}$$

If we now define

$$\begin{aligned}
\eta_s^X &:= v_s^X - \sum_{n=0}^N \alpha^{n,X} \eta^n(s,T) + \eta^0(s,T) \qquad \text{and} \\
X_0 &:= \gamma^X \prod_{n=0}^N B^n(0,T)^{-\alpha^{n,X}} B^0(0,T) \\
&\quad \cdot \exp\left(\int_0^T \mu_s^X + \frac{1}{2} \sum_{n=0}^N \alpha^{n,X} |\eta^n(s,T)|^2 - \frac{1}{2} |\eta^0(s,T)|^2 + \frac{1}{2} |\eta_s^X|^2 \, ds \right)
\end{aligned}$$

we get $X_T = X_0 \exp\left(\int_0^T \left(r_s^0 - \frac{1}{2} |\eta_s^X|^2 \right) \, ds + \int_0^T \eta_s^X \, dW_s \right)$.

3.2.1 Currency Options

The payoff of a plain vanilla currency option equals $[e_T - K]^+$. Define the simple derivatives $X_t := e_t B^f(t,T)$ and $Y_t := KB^d(t,T)$; their parameters can be read from the asset price dynamics and are given by $X_0 = e_0 B^f(0,T)$, $\eta^X(t) = \eta^e(t) + \eta^f(t,T)$ and $Y_0 = KB^d(0,T)$, $\eta^Y(t) = \eta^d(t,T)$, respectively. Since $B^d(T,T) = B^f(T,T) = 1$ the options payoff equals $[X_T - Y_T]^+$, and its price can be computed by means of Theorem 3.2.

3.2.2 Currency Converted Options

Option to Exchange one Asset for the Other: The payoff is given by $[N_1 e_T^{n_1} S_T^{n_1,j_1} - N_2 e_T^{n_2} S_T^{n_2,j_2}]^+$ To apply Theorem 3.2 we define the simple derivatives $X_T := N_1 e_T^{n_1} S_T^{n_1,j_1}$ and $Y_T := N_2 e_T^{n_2} S_T^{n_2,j_2}$. Again initial value and volatility of X and Y can be read from the asset price dynamics. We get $X_0 = N_1 e_0^{n_1} S_0^{n_1,j_1}$, $Y_0 = N_2 e_0^{n_2} S_0^{n_2,j_2}$, $\eta_s^X = \eta_s^{e^1} + \eta_s^{S^1}$ and $\eta_s^Y = \eta_s^{e^2} + \eta_s^{S^2}$.

Foreign Asset/ Domestic Strike–Option: The pay-off equals $[e_T S_T^f - K]^+$. Set $X_T := e_T S_T^f$ and notice that this is a simple derivative with $X_t = e_t S_t^f$ and $\eta^X = \eta^e + \eta^{S^f}$. Next set $Y_T := K \cdot B^d(T,T)$ and apply Theorem 3.2 to these simple derivatives.

Domestic Asset/ Foreign Strike–Option: Here the payoff is given by $[S_T^d - e_T K]^+$, where K is in foreign currency. To apply Theorem 3.2 we use the simple derivatives $X_t := S_t^d$ and $Y_t = Ke_t B^f(t,T)$.

3.2.3 Options on the Product of two Assets

In this case we have the following payoff:

$$\left[N_1 e_T^{n_1} S_T^{n_1,j_1} \cdot N_2 e_T^{n_2} S_T^{n_2,j_1} - K\right]^+$$

Here we refer to Proposition 3.4 to price the contract:
Set $X_T := N_1 e_T^{n_1} S_T^{n_1,j_1} \cdot N_2 e_T^{n_2} S_T^{n_2,j_1}$ and notice that

$$\begin{aligned}
&\sum_i \alpha^{i,X} r_s^i = 2r_s^0, \quad \mu_s^X = \frac{1}{2}|\eta_s^{e^{n_1}} + \eta_s^{S^{n_1,j_1}}|^2 + \frac{1}{2}|\eta_e^{e^{n_2}} + \eta_s^{S^{n_2,j_2}}|^2, \\
&v_s^X = \eta_s^{e^{n_1}} + \eta_s^{S^{n_1,j_1}} + \eta_s^{e^{n_2}} + \eta_s^{S^{n_2,j_2}}.
\end{aligned}$$

Then the application of Proposition 3.4 yields after some calculations

$$\begin{aligned}
X_0 &= \frac{N_1 N_2 e_0^{n_1} S_0^{n_1,j_1} e_0^{n_2} S_0^{n_2,j_2}}{B^d(0,T)} \exp\left\{ \int_0^T \eta_s^{e^{n_1}} \left(\eta_s^{e^{n_2}} + \eta_s^{S^{n_2,j_2}}\right) + \eta_s^{S^{n_1,j_1}} \left(\eta_s^{e^{n_2}} + \eta_s^{S^{n_2,j_2}}\right) \right. \\
&\quad \left. -\eta_s^{B^d} \left(\eta_s^{e^{n_1}} + \eta_s^{S^{n_1,j_1}} + \eta_s^{e^{n_2}} + \eta_s^{S^{n_2,j_2}}\right) + |\eta_s^{B^d}|^2 ds \right\} \\
\eta_s^X &= \eta_s^{e^{n_1}} + \eta_s^{S^{n_1,j_1}} + \eta_s^{e^{n_2}} + \eta_s^{S^{n_2,j_2}} - \eta_s^{B^d}.
\end{aligned}$$

The simple derivative corresponding to the exercice price is $Y_T := KB^d(T,T)$.

Remark: Clearly this approach accomodates the case where S_T is replaced by some bond. It is also straightforward to obtain a pricing formula for options on the quotient of two assets (Quanto options).

Notice that there is a big difference between product options and currency converted options which is due to the fact that $(e_t S_t^t)_{0\le t\le T}$ is the price process of a domestic asset which is is not true for $\left(e_t^{e^{n_1}} S_t^{n_1,j_1} \cdot e_t^{e^{n_2}} S_t^{n_2,j_2}\right)_{0\le t\le T}$.

3.2.4 Guaranteed Exchange rate Options

The payoff of this derivative equals $[\bar{e} S_T^f - \bar{e} K]^+$, where S^f is a foreign asset and $\bar{e}$ some predetermined exchange rate. Again we have to apply Proposition 3.4 to find the parameters of the simple derivative X corresponding to $\bar{e} S_T^f$. We have

$$\sum_i \alpha^{i,X} r_s^i = r_s^f, \qquad \mu_s^X = -\frac{1}{2}|\eta^{S^f}|^2, \qquad v_s^X = \eta^{S^f} .$$

Applying Proposition 3.4 yields

$$\begin{aligned} X_0 &= \bar{e} B^d(0,T) \frac{S_0^f}{B^f(0,t)} \exp\left\{ \int_0^T |\eta_s^{B^f}|^2 + \eta_s^{S^f} \cdot \eta_s^{B^d} + \eta_s^e \cdot \eta_s^{B^f} - \right. \\ &\quad \left. \eta_s^{S^f} \cdot \eta_s^{B^f} - \eta_s^{B^f} \cdot \eta_s^{B^d} - \eta_s^{S^f} \cdot \eta_s^e ds \right\} \\ \eta_s^X &= \eta^{S^f} + \eta^{B^d} - \eta^{B^f} \end{aligned}$$

Next set $Y_T := \bar{e} K \cdot B^d(T,T)$ and apply Theorem 3.2.

Remark: Due to the fact that the issuer of this options assumes the exchange rate risk there are a number of correlations which enter the pricing formula that do not appear in the formula for an option on foreign assets without guaranteed exchange rate feature. It is, however, worth noting that taking interest rate risk into account does not necessarily raise the price of a GER–Option over that of a GER–Option in a model with deterministic interest rates since e.g. $\eta^e \cdot \eta^{B^f}$, the covariance between exchange rate and foreign bonds will typically be negative while $\eta^{S^f} \cdot \eta^{B^f}$, that is the covariance between foreign bonds and stocks will typically be positive. Both effects lead to a reduction in the option price while the direct effect of stochastic interest rates, namely $|\eta^{B^f}|^2$ unambiguously raises the option price.

3.2.5 Options on Forwards and Futures

Options on forwards: Assume that $(\tilde{X}_t)_{0\le t\le \bar{T}}$ is the price process of a domestic simple derivative. Consider two points in time T_1 and T_2 with $0 < T_1 \le T_2 \le \bar{T}$. The payoff in T_1 of an option with maturity date T_1 on a forward contract on $\tilde{X}_t$ with maturity date T_2 is given by $\left[\tilde{X}_{T_1} - B(T_1,T_2)\cdot K\right]^+$. Its price immediately follows from Theorem 3.2, if we use the simple derivatives $\tilde{X}_t$ and $KB^d(t,T_2)$.

Options on futures: If $(\tilde{X}_t)_{0\le t\le \bar{T}}$ is the price process of a domestic asset the futures price at time t of a futures contract on $\tilde{X}$ with maturity T_2 equals $\tilde{X}_t^f := E^P[\tilde{X}_{T_2} \mid \mathcal{F}_t]$. A European option on this futures contract with maturity date $T_1 < T_2$ and strike K has payoff $[\tilde{X}_T^f - K]^+$. Assume that $(\tilde{X}_t)_{0\le t\le \bar{T}}$ is a simple derivative. Then the contingent claim X with the single payoff $X_{T_1} = \tilde{X}_{T_1}^f$ is a simple derivative, too.

$$
\begin{aligned}
\tilde{X}_{T_1}^f &= E^P\left[\tilde{X}_{T_2} \,\middle|\, \mathcal{F}_{T_1}\right] = E^P\left[\tilde{X}_{T_2}\cdot\left(B^d(T_2,T_2)\right)^{-1} \,\middle|\, \mathcal{F}_{T_1}\right] \\
&= E^P\left[\frac{\tilde{X}_0}{B^d(0,T_2)}\cdot\exp\left(-\frac{1}{2}\int_0^{T_2}|\eta^{\tilde{X}}|^2 - |\eta^d(t,T_2)|^2 dt + \int_0^{T_2}\eta^{\tilde{X}} - \eta^d(t,T_2)dW_t\right)\middle|\mathcal{F}_{T_1}\right] \\
&= \underbrace{\frac{\tilde{X}_0}{B^d(0,T_2)}\exp\left(\int_0^{T_2}|\eta^d(t,T_2)|^2 - \eta^{\tilde{X}}\cdot\eta^d(t,T_2)dt\right)}_{=:\,\gamma^X} \\
&\quad\cdot\exp\left(\int_0^{T_1}\eta^{\tilde{X}} - \eta^d(t,T_2)dW_t - \frac{1}{2}\int_0^{T_1}|\eta^{\tilde{X}} - \eta^d(t,T_2)|^2 dt\right)
\end{aligned}
$$

We may now apply Proposition 3.4 with $\mu_t^X = -\frac{1}{2}|\eta^{\tilde{X}} - \eta^d(t,T_2)|^2$, $v^X = \eta^{\tilde{X}} - \eta^d(t,T_2)$, and γ^X as above.

3.2.6 Options on Indices

In order to apply Theorem 3.2 to options on indices we must model them by

$$
dI_t \;=\; r_t I_t dt + I_t \eta_t^I dW_t \tag{8}
$$

where $\eta^I : [0,T] \to \mathbb{R}^n$ is deterministic. This approach is without problems if the index is dividend protected and a geometric mean of simple derivatives. If the index is, however, given by an arithmetic mean of such assets i.e. if we have $I_t = \sum_{i=1}^m \alpha^i S_t^i$ then the volatility of the index will be stochastic and is given by

$$
\sigma_t^I = \frac{\sqrt{\sum_{j=1}^d\left(\sum_{i=1}^m \alpha^i S_t^i \eta_t^{i,j}\right)^2}}{\sum_{i=1}^m \alpha^i S_t^i}.
$$

However, for the purposes of option pricing the above model may still be a good approximation within the framework of the general model. Consider e.g. an option on the index maturing at T i.e. a payoff of the form $[I_T - K]^+$. As we are dealing with stochastic interest-rates it is convenient to look at the time T forward price of the index given by $I_T^F := I_T/B^d(t,T)$. The volatility of this derivative is given by

$$
\sigma_t^{I^F} = \frac{\sqrt{\sum_{j=1}^d\left(\sum_{i=1}^m \frac{\alpha^i S_t^i}{B^d(t,T)}\left(\eta_t^{i,j} - \eta^{d,j}(t,T)\right)\right)^2}}{\sum_{i=1}^m \alpha^i S_t^i / B^d(t,T)} = \frac{\sqrt{\sum_{j=1}^d\left(\sum_{i=1}^m \alpha^i (S^F)_t^i \tilde{\eta}_t^{i,j}\right)^2}}{\sum_{i=1}^m \alpha^i (S^F)_t^i} \tag{9}
$$

Now by Theorem 3 in [EKJP94] we know that if $\sigma_t^{I^F}$ is bounded above by a deterministic function η_t^I then we may use the model in (8) to obtain an upper bound for the price of the option $[I_T - K]^+$ together with a hedging strategy in I and $B(\cdot, T)$ that will always provide a terminal payoff greater or equal to $[I_T - K]^+$, even though in the framework of our general model the index does in fact not evolve according to (8). Fortunately we have the following

Lemma 3.5 *Let $(\tilde{\sigma}_t^i)^2 = \sum_{j=1}^d \left(\eta_t^{i,j} - \eta^{d,j}(t,T)\right)^2 = \sum_{j=1}^d \left(\tilde{\eta}_t^{i,j}\right)^2$ be the volatility of the time T forward price of the simple derivative i entering the index (e.g. the volatility of the forward price of some stock). Then $\sigma_t^{I^F} \leq \max_i \tilde{\sigma}_t^i$*

Proof: We have the following estimate:

$$
\begin{aligned}
&\left(\sum_{j=1}^d \left(\sum_{i=1}^m \alpha^i (S^F)_t^i \tilde{\eta}_t^{i,j}\right)^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{j=1}^d \left(\sum_{i=1}^m (\alpha^i (S^F)_t^i)^2 (\tilde{\eta}_t^{i,j})^2 + 2 \sum_{l<k\leq m} \alpha_k (S^F)_t^k \alpha_l (S^F)_t^l \tilde{\eta}_t^{k,j} \tilde{\eta}_t^{l,j}\right)\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^m (\alpha^i (S^F)_t^i)^2 \sum_{j=1}^d (\tilde{\eta}_t^{i,j})^2 + 2 \sum_{l<k\leq m} \alpha_k (S^F)_t^k \alpha_l (S^F)_t^l \sum_{j=1}^d \tilde{\eta}_t^{k,j} \tilde{\eta}_t^{l,j}\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^m (\alpha^i (S^F)_t^i)^2 \tilde{\sigma}_i^2 + 2 \sum_{l<k\leq m} \alpha_k (S^F)_t^k \alpha_l (S^F)_t^l \rho_{k,l} \tilde{\sigma}_k \tilde{\sigma}_l\right)^{\frac{1}{2}} \\
&\leq \sum_{i=1}^m \alpha^i (S^F)_t^i \tilde{\sigma}_i \leq \max_i \tilde{\sigma}_i \sum_{i=1}^m \alpha^i (S^F)_t^i
\end{aligned}
$$

Hence the claim follows from formula (9)

We take this reasoning as a theoretical justification for using – as is customary in practice – the model (8) as a proxy for the stochastic evolution of an index based on some arithmetic mean.

3.3 Hedging, the "LEGO-approach"

We now want to derive hedge portfolios for the options considered in the previous section. Again it is our aim to present a general procedure for determining them. Our approach is to derive first the general hedge portfolio for the options considered in Theorem 3.2 in terms of the simple derivatives X and Y and then to show how X and Y can themselves be duplicated by dynamic strategies in assets for which liquid markets exist.

3.3.1 General Lemma: Hedging with simple derivatives

Lemma 3.6 *The hedge portfolio $(P_t)_{0\le t\le T}$ for the option in Theorem 3.2 in terms of the simple derivatives X and Y is given by*

$$\begin{array}{lll} \delta_t^X(X_t,Y_t) := \mathcal{N}_1(d_1) & \textit{units of } X & \textit{and} \\ \delta_t^Y(X_t,Y_t) := -\mathcal{N}_1(d_2) & \textit{units of } Y & \end{array}$$

where d_1 and d_2 are as in Theorem 3.2.

Proof: We need to show that the above portfolio strategy is self-financing and duplicates the option. To this end we proceed in three steps.

(i) Since it is not a priori clear that P_t is actually self-financing we have to allow for a position in γ_t in domestic cash to meet this requirement. Denote the enlarged portfolio by $\tilde{P}_t$ and its value process by $\tilde{V}_t$. Let $[Z_t]^M$ denote the martingale part of a continuous semimartingale Z. We now show that the dynamics of the martingale parts of the option price and of the value process $\tilde{V}_t$ are the same. By Theorem 3.2 C_t is a smooth function of t, X_t and Y_t. Therefore we get from Itô's Lemma

$$[dC_t]^M = \frac{\partial C_t}{\partial X_t}[dX_t]^M + \frac{\partial C_t}{\partial Y_t}[dY_t]^M \; .$$

Now following [EKMV92] we may compute the derivatives of the option:

$$\begin{aligned} \frac{\partial C_t}{\partial X_t} &= \frac{\partial}{\partial X_t} E_t^P\left[\left(\beta_{t,T}^d\right)^{-1}[X_T - Y_T]^+\right] \\ &= E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} 1_{[X_T\ge Y_T]}\frac{\partial X_T}{\partial X_t}\right] \\ &= \frac{1}{X_t}E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} 1_{[X_T\ge Y_T]}X_T\right] \end{aligned}$$

As shown in the proof of Theorem 3.2 this expression equals $\mathcal{N}_1(d_t^1)$. Similarly we have

$$\frac{\partial C_t}{\partial Y_t} = -\mathcal{N}_1(d_t^2)\,.$$

On the other hand we get from the self–financing condition and $\left[\beta_{0,t}^d\right]^M = 0$ for the portfolio $\tilde{P}_t$

$$\left[d\tilde{V}_t\right]^M = \mathcal{N}_1(d_t^1)[dX_t]^M - \mathcal{N}_1(d_t^2)[dY_t]^M \text{ and hence } [dV_t]^M = [dC_t]^M$$

(ii) By (i) the self-financing portfolio which consists in buying the call and shorting the portfolio $\tilde{P}$ is locally riskless. By absence of arbitrage it must hence grow at the riskless rate, i.e.

$$d\left(C_t - \tilde{V}_t\right) = r_t\left(C_t - \tilde{V}_t\right)dt$$

Hence if γ_0 is chosen such that $\tilde{V}_0 = C_0$ we have $\tilde{V}_t = C_t\ \forall t$.

(iii) By Euler's Theorem we get from the linear homogeneity of C_t in X_t and Y_t

$$C_t = \frac{\partial C_t}{\partial X_t} X_t + \frac{\partial C_t}{\partial Y_t} Y_t = \mathcal{N}(d_t^1) X_t - \mathcal{N}(d_t^2) Y_t \,.$$

Now from (ii) $0 = C_t - \tilde{V}_t = -\gamma_t \beta_{0,t}^d$. Therefore $\gamma_t \equiv 0$ which proves the Lemma.

Remark: Consider the problem of finding a hedging-strategy for a contingent claim Z with price process $(Z_t)_{0 \le t \le T}$ using the risky domestic assets H_i^Z, $1 \le i \le L^Z$ and the domestic savings account. By a similar argument as in the proof of Lemma 3.6 it is easily shown that the following two step procedure always leads to the correct answer, at least in a complete market.

(i) Compute the portfolio holdings $\delta_i^Z(t)$, $1 \le i \le L^Z$ in the risky assets by equating the martingale parts of Z_t and of the portfolio's value process V_t. The latter is given by the self-financing condition:

$$[dV_t]^M = \sum_{i=1}^{L^Z} \delta_i^Z(t) d \left[H_i^Z(t) \right]^M \,,$$

while the former may often be computed using Itô's Lemma.

(ii) The holdings in the domestic savings account, denoted by γ_t, are determined by $V_0 = Z_0$ and the self-financing condition. Nonetheless they may be computed from the relation $V_t = Z_t$ yielding

$$\gamma_t = Z_t - \sum_{i=1}^{L^Z} \delta_i^Z(t) H_i^Z(t) \,.$$

3.3.2 Intermediate Portfolios

Whenever the simple derivatives X_t and Y_t are assets for which liquid markets exist we are done with Lemma 3.6. If this is not the case we must duplicate X_t and Y_t by a dynamic hedging strategy, too. The existence of such a strategy is guaranteed by our complete market assumption. We start with the following observation. Let the hedge portfolios for X_t and Y_t in terms of assets H_i^X and H_i^Y for which we assume the existence of liquid markets be given by

$$P_t^X = \sum_{i=1}^{L^X} \delta_i^X(t) H_i^X \quad \text{and} \quad P_t^Y = \sum_{i=1}^{L^Y} \delta_i^Y(t) H_i^Y \,.$$

Then the hedge portfolio for the simple option in terms of H_i^X and H_i^Y is given by

$$P_t = \mathcal{N}(d_t^1) \sum_{i=1}^{L^X} \delta_i^X(t) H_i^X - \mathcal{N}(d_t^2) \sum_{i=1}^{L^Y} \delta_i^Y(t) H_i^Y \,,$$

which explains the name "LEGO-aproach" for our derivation of hedging strategies. We may compute the hedge portfolios for those simple derivatives for which there are no liquid markets by going through the procedure described in the remark following Lemma 3.6. As to the above examples there are a few cases where we actually need to duplicate some of the simple derivatives involved. This is the case for the

- Option on the Product of Two Assets: Here we need to duplicate X_t. We get

$$\delta^X_{e^1 \cdot S^1}(t) = \frac{X_t}{e^1_t \cdot S^1_t}, \qquad \delta^X_{e^2 \cdot S^2}(t) = \frac{X_t}{e^2_t \cdot S^2_t}, \qquad \delta^X_{B^d}(t) = -\frac{X_t}{B^d(t,T)}$$

- Guaranteed Exchange Rate Option: Again we need to duplicate X_t. We get

$$\delta^X_{e \cdot S^f}(t) = \frac{X_t}{e_t \cdot S^f_t}, \qquad \delta^X_{e \cdot B^f}(t) = -\frac{X_t}{e_t \cdot B^f(t,T)}, \qquad \delta^X_{B^d}(t) = \frac{X_t}{B^d(t,T)}$$

- Options on Futures: Duplicating X_t in this case by a portfolio in the futures $\tilde{X}^f_t$ and in $B^d(t,T)$ gives

$$\delta^X_{\text{futures}}(t) = \frac{X_t}{\tilde{X}^f_t}, \qquad \delta^X_{B^d}(t) = \frac{X_t}{B^d(t,T)}$$

3.3.3 Hedging Zero Coupon Bonds with Futures

Since in reality liquid markets for zero coupon bonds of longer maturities usually do not exist we now consider duplicating them by positions in cash and futures on coupon bonds. For simplicity we restrict ourselves to domestic bonds. The fact that zero coupon bonds can indeed be duplicated by only taking positions in cash and futures is due to the assumption that the stochastic movements of the term structure in one country are given by a one-factor model. This is formalized as follows.

Assumption 3.7 *There is a one-dimensional Brownian Motion $\tilde{W}$ such that every domestic zero coupon bond solves the SDE*

$$dB(t,T) = B(t,T)r^d_t dt + B(t,T)\sigma(t,T)d\tilde{W}_t$$

for a deterministic function $\sigma(\cdot,T) : [0,T] \to I\!R^+$ with $\sigma(T,T) = 0$

Proposition 3.8 *Denote by V^f_t the futures-price of the futures contract maturing at $\overline{T}$ on a coupon bond V with deterministic coupons k_i, $1 \leq i \leq n$ at coupon dates $\overline{T} \leq T_1 < \ldots < T_n$. If Assumption 3.7 holds, every domestic zero coupon bond with maturity $T \leq \overline{T}$ can be duplicated by a self-financing portfolio strategy in futures and cash. The futures position at time t of this portfolio is given by*

$$\delta_{V^f}(t) = \frac{B(t,T)\sigma(t,T)}{\sum_{i=1}^n k_i \exp\left\{\int_t^{\overline{T}} \sigma^2(s,\overline{T}) - \sigma(s,T^i)\sigma(s,\overline{T})ds\right\} \frac{B(t,T^i)}{B(t,\overline{T})} \left(\sigma(t,T_i) - \sigma(t,\overline{T})\right)}$$

The amount to be held in the savings account is equal to $B(t,T)$.

PROOF: First notice that the futures price is given by

$$\begin{aligned} V_t^f &= E^P\left[\sum_{i=1}^n k_i B(\overline{T}, T_i) \mid \mathcal{F}_t\right] \\ &= \sum_{i=1}^n k_i \frac{B(t,T^i)}{B(t,\overline{T})} \underbrace{\exp\left\{\int_t^{\overline{T}} \sigma^2(s,\overline{T}) - \sigma(s,T^i)\sigma(s,\overline{T})ds\right\}}_{=:\,\exp^i(t)} \end{aligned}$$

Following [EKMV92] we compute the futures position $\delta_{V^f}(t)$ in such a way that the martingale parts of $B(t,T)$ and of the resettlement gain $\int_0^t \delta_{V^f}(t) dV_s^f$ are equal. By Itô's lemma we get for the latter

$$\left[\int_0^t \delta_{V^f}(t) dV_s^f\right]^M = \int_0^t \delta_{V^f}(s) \sum_{i=1}^n \frac{k_i \exp^i(s)}{B(t,\overline{T})} B(t,T_i)\left(\sigma(s,T^i) - \sigma(s,\overline{T})\right) d\tilde{W}_s$$

Since the martingale part of the bond is given by

$$[B(t,T)]^M = \int_0^t B(s,T)\sigma(s,T)d\tilde{W}_s,$$

choosing δ_{V^f} as in the proposition actually yields equal martingale parts. The cash position of the portfolio then follows immediately, because the value of the hedge portfolio has to be equal to $B^d(t,T)$.

4 Pricing and Hedging of Exchange Options on Simple Options

We shall now consider options to exchange two simple options — that is options of the form considered in Theorem 3.2 — for one another. The payoff of such options is given by

$$\left[\left[X_T^1 - Y_T^1\right]^+ - \left[X_T^2 - Y_T^2\right]^+\right]^+ \tag{10}$$

where X_T and Y_T are simple derivatives. This payoff is not only interesting in itself — for instance (10) gives the payoff of a spread-option[2] — but is also a building block for a number of other payoffs that can be obtained by linear combinations of this payoff, simple options and simple derivatives. The problem is, however, that in general there is no closed formula for the value of the above option. Yet, in our opinion there are two interesting simplifications which allow an analytical formula for this option to be derived. These are

a) that $Y_T^1 = Y_T^2$ or $X_T^1 = X_T^2$

[2]The payoff of a simple spread option equals $[X_T - Y_T - K]^+$, where K is a positive constant. If we define $X_T^1 := X_T$, $X_T^2 := Y_T$, $X_T^3 := K$ and $X_T^4 := 0$, we see that this payoff can be seen as a special case of an option to exchange two options for one another.

b) that $X_T^2 = Z_T X_T^1$ and $Y_T^2 = Z_T Y_T^1$, where Z_T is the time T price of a simple derivative.

Simplification a) covers the case of an option on the best of two simple derivatives, since

$$[\max[X_T^1, Y_T] - X_T^2]^+ = \left[[X_T^1 - Y_T]^+ - [X_T^2 - Y_T]^+\right]^+ + \left[Y_T - X_T^2\right]^+ .$$

As shown by Witt in [Wit93], a number of payoffs can be expressed as linear combinations of this payoff, simple options and simple derivatives so that the closed formula under assumption a) gives in fact rise to many more closed formulas for a number of other payoffs.

Assumption b) covers the case of currency optimized options which have been analysed by [KR94] in a model with deterministic interest rates. The payoff of such a contract equals

$$\max\left\{\overline{e} \cdot \left[X_T^f - Y_T^f\right]^+ ;\, e_T \cdot \left[X_T^f - Y_T^f\right]^+\right\} , \tag{11}$$

where X_T^f and Y_T^f are simple derivatives denominated in foreign currency , e_T denotes the exchange rate at the maturity date T and $\overline{e}$ the guaranteed exchange rate known at date 0. The simple derivatives could well be foreign bonds. If we set $X_T^1 := \overline{e} X_T^f$, $Y_T^1 := \overline{e} Y_T^f$ and $Z_T := e_T/\overline{e}$ we see that (11) equals

$$\begin{aligned} &\max\left\{[X_T^1 - Y_T^1]^+ ;\, Z_T[X_T^1 - Y_T^1]^+\right\} = \\ &= \left[[X_T^1 - Y_T^1]^+ - [Z_T X_T^1 - Z_T Y_T^1]^+\right]^+ + [Z_T X_T^1 - Z_T Y_T^1]^+ . \end{aligned}$$

In the following two sections we shall confine ourselves first to developing a pricing formula and a hedging strategy for the payoff in (10) under assumption a) and second to deriving a near explicit solution for the general pricing and hedging problem and briefly discussing numerical methods for the evaluation of the required expectations. While the derivation of a closed valuation formula under assumption b) is not exactly parallel to case a), given the techniques expounded in this paper, the solution to this valuation problem is rather straightforward. Hence we leave it to the reader.

4.1 Explicit Solutions

For future use define for arbitrary simple derivatives U^i

$$\begin{aligned} d^+(U^1, U^2, t, T) &:= \frac{\ln\left(\frac{U_t^1}{U_t^2}\right) + \frac{1}{2}\int_t^T |\eta_s^{U^1} - \eta_s^{U^2}|^2 ds}{\sqrt{\int_t^T |\eta_s^{U^1} - \eta_s^{U^2}|^2 ds}} \\ d^-(U^1, U^2, t, T) &:= d^+(U^1, U^2, t, T) - \sqrt{\int_t^T |\eta_s^{U^1} - \eta_s^{U^2}|^2 ds} \\ \rho((U^1, U^2); (U^3, U^4); t; T) &:= \frac{\int_t^T \left(\eta_s^{U^1} - \eta_s^{U^2}\right)\left(\eta_s^{U^3} - \eta_s^{U^4}\right) ds}{\sqrt{\int_t^T |\eta_s^{U^1} - \eta_s^{U^2}|^2 ds \int_t^T |\eta_s^{U^3} - \eta_s^{U^4}|^2 ds}} \end{aligned}$$

The explicit pricing formula for the payoff in (10) under the simplifying assumption a) is then given in the following

Theorem 4.1 *Let $(X_t^i)_{0\leq t\leq T}$, $i=1,2$ and $(Y_t)_{0\leq t\leq T}$ be simple derivatives. Then the value C_t at time t of a European option with payoff*

$$\left[\left[X_T^1 - Y_T\right]^+ - \left[X_T^2 - Y_T\right]^+\right]^+$$

is given by

$$\begin{aligned} C_t &= X_t^1\left(\mathcal{N}_1\left(d_t^1\right) + \mathcal{N}_1\left(d_t^2\right) + \mathcal{N}_2\left(d_t^3, d_t^4, \rho_t^1\right) - 1\right) \\ &\quad - Y_t\mathcal{N}_2\left(d_t^5, d_t^6, \rho_t^2\right) - X_t^2\mathcal{N}_2\left(d_t^7, d_t^8, \rho_t^3\right) \end{aligned}$$

where

$$\begin{aligned} d_t^1 &= d^+(X^1, Y, t, T) & d_t^2 &= d^+(X^1, X^2, t, T) \\ d_t^3 &= d^-(Y, X^1, t, T) & d_t^4 &= d^-(X^2, X^1, t, T) \\ d_t^5 &= d^-(X^1, Y, t, T) & d_t^6 &= d^+(Y, X^2, t, T) \\ d_t^7 &= d^-(X^1, X^2, t, T) & d_t^8 &= d^+(X^2, Y, t, T) \\ \rho_t^1 &= \rho((X^1, Y); (X^1, X^2); t; T) & \rho_t^2 &= \rho((Y, X^1); (X^2, Y); t; T) \\ \rho_t^3 &= \rho((Y, X^1); (X^2, Y); t; T) \end{aligned}$$

PROOF:
As a first step we have to find a suitable decomposition of the payoff:

$$\begin{aligned} &\left[\left[X_T^1 - Y_T\right]^+ - \left[X_T^2 - Y_T\right]^+\right]^+ = \\ &\quad = (X_T^1 - Y_T)1_{\{X_T^1 > Y_T\}}1_{\{X_T^2 < Y_T\}} + (X_T^1 - X_T^2)1_{\{X_T^1 > X_T^2\}}1_{\{X_T^2 > Y_T\}} \\ &\quad = X_T^1\left(1_{\{X_T^1 > Y_T\}}1_{\{X_T^2 < Y_T\}} + 1_{\{X_T^1 > X_T^2\}}1_{\{X_T^2 > Y_T\}}\right) - Y_T 1_{\{X_T^1 > Y_T\}}1_{\{X_T^2 < Y_T\}} \\ &\quad\quad - X_T^2 1_{\{X_T^1 > X_T^2\}}1_{\{X_T^2 > Y_T\}} \end{aligned}$$

Now notice that the event $(X^1 > Y \wedge Y > X^2) \vee (X^1 > X^2 \wedge X^2 > Y)$ is simply the event: "X^1 is the largest of the asset prices". This is equivalent to the event: "There is no asset price which is larger than X^1". Hence we have

$$\begin{aligned} &\left[\left[X_T^1 - Y_T\right]^+ - \left[X_T^2 - Y_T\right]^+\right]^+ = \\ &= X_T^1\left(1 - 1_{\{X_T^1 < Y_T\}} - 1_{\{X_T^1 < X_T^2\}} + 1_{\{X_T^1 < Y_T\}}1_{\{X_T^1 < X_T^2\}}\right) - Y_T 1_{\{X_T^1 > Y_T\}}1_{\{X_T^2 < Y_T\}} \\ &\quad - X_T^2 1_{\{X_T^1 > X_T^2\}}1_{\{X_T^2 > Y_T\}} \end{aligned}$$

We can now calculate the relevant expectations.

$$\begin{aligned} E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} X_T^1 1_{[X_T^1 < Y_T]}\right] &= X_t^1\left(1 - Q_t^{X_T^1}\left(\ln\frac{Y_T}{X_T^1} \leq 0\right)\right) \\ &= X_t^1\left(1 - \mathcal{N}(d_t^1)\right) \end{aligned}$$

$$\begin{aligned} E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} X_T^1 1_{[X_T^1 < X_T^2]}\right] &= X_t^1\left(1 - \mathcal{N}(d_t^2)\right) \\ E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} X_T^1 1_{[X_T^1 < Y_T]}1_{[X_T^1 < X_T^2]}\right] &= X_t^1\mathcal{N}_2\left(d_t^3, d_t^4, \rho_t^1\right) \end{aligned}$$

where $\mathcal{N}_2$ is the bivariate normal distribution function, since $\ln(X_T^1/Y_T)$ and $\ln(X_T^1/X_T^2)$ are jointly normally distributed with correlation ρ_t^1. Moreover,

$$\begin{aligned}
E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} Y_T 1_{[X_T^1 \geq Y_T]} 1_{[X_T^2 < Y_T]}\right] &= Y_t \mathcal{N}_2\left(d_t^5, d_t^6, \rho_t^2\right) \\
E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} X_T^2 1_{[X_T^1 \geq X_T^2]} 1_{[X_T^2 > Y_T]}\right] &= X_t^2 \mathcal{N}_2\left(d_t^7, d_t^8, \rho_t^3\right)
\end{aligned}$$

This completes the proof of the theorem.

For this option there exists an explicit hedge portfolio. It is given in the following Lemma.

Lemma 4.2 *The hedge portfolio for the option in Theorem 4.1 in terms of the simple derivatives is given by*

$$\begin{aligned}
\delta_t^{X1} &= 1 + \mathcal{N}\left(d_t^1\right) + \mathcal{N}\left(d_t^2\right) + \mathcal{N}_2\left(d_t^3, d_t^4, \rho_t^1\right) \\
\delta_t^{Y} &= -\mathcal{N}_2\left(d_t^5, d_t^6, \rho_t^2\right) \\
\delta_t^{X2} &= -\mathcal{N}_2\left(d_t^7, d_t^8, \rho_t^3\right) ,
\end{aligned}$$

where the arguments of the normal distributions are as in Theorem 4.1.

The proof of Lemma 4.2 is similar to those of Lemma 3.6 and Theorem 4.3 part b) and hence left to the reader.

4.2 Near Explicit Solutions

4.2.1 The Theoretical Result

We now return to the general payoff in equation (10). As already mentioned there is no explicit solution in this case. However, we have a near-closed-form solution for the pricing and hedging of this option, which is given in

Theorem 4.3 *Let $(X_t^i)_{0\leq t\leq T}$, $(Y_t^i)_{0\leq t\leq T}$, $i = 1, 2$ be simple derivatives. Let C_t be the value at time t of a European option with payoff*

$$\left[\left[X_T^1 - Y_T^1\right]^+ - \left[X_T^2 - Y_T^2\right]^+\right]^+$$

1. We have the following near explicit pricing formula

$$\begin{aligned}
C_t &= X_t^1 \mathcal{N}_2\left(d_t^1, d_t^2, \rho_t\right) - Y_t^1 \mathcal{N}_2\left(d_t^3, d_t^4, \rho_t\right) \\
&\quad + X_t^1 Q_t^{X^1}[A] + Y_t^2 Q_t^{Y^2}[A] - \left(X_t^2 Q_t^{X^2}[A] + Y_t^1 Q_t^{Y^1}[A]\right)
\end{aligned}$$

where the arguments of the normal distributions are as defined in equations (12), (13) and where the set A is given by

$$A = \left\{\omega \in \Omega \mid X_T^2 \geq Y_T^2 \wedge X_T^1 - Y_T^1 \geq X_T^2 - Y_T^2\right\} .$$

The probability measures $Q_t^{X^i}$, $i=1,2$ and $Q_t^{Y^i}$, $i=1,2$ are as defined in Lemma 3.3. The exercice probabilities $Q_t^{X^i}[A]$ and $Q_t^{Y^i}[A]$ are given by the measure of a domain $\tilde{A} \subset I\!R^4$ under — possibly degenerate — four-dimensional normal distributions whose parameters are smooth functions of at most t and X_t^i, $i=1,2$ and Y_t^i, $i=1,2$.

2. The hedge portfolio (P_t) in terms of the simple derivatives is given by

$$\begin{aligned}
\delta_t^{X^1} &= \mathcal{N}_2\left(d_t^1, d_t^2, \rho\right) + Q_t^{X_T^1}[A]\,, & \delta_t^{X^2} &= -Q_t^{X_T^2}[A] \\
\delta_t^{Y^1} &= -\left(\mathcal{N}_2\left(d_t^3, d_t^4, \rho\right) + Q_t^{Y_T^1}[A]\right)\,, & \delta_t^{Y^2} &= Q_t^{Y_T^2}[A]\,.
\end{aligned}$$

Remark: It is worth noting that we are still able to give near explicit solutions for the hedge portfolios without an explicit pricing formula.

Proof: We consider first the pricing problem. The decomposition of the payoff is:

$$\begin{aligned}
&\left[\left[X_T^1 - Y_T^1\right]^+ - \left[X_T^2 - Y_T^2\right]^+\right]^+ \\
&= \left(X_T^1 - Y_T^1\right) 1_{\{X_T^1 \geq Y_T^1\}} 1_{\{X_T^2 \leq Y_T^2\}} + \left(X_T^1 - Y_T^1 - X_T^2 + Y_T^2\right) 1_{\{X_T^2 \geq Y_T^2\}} 1_{\{X_T^1 - Y_T^1 \geq X_T^2 - Y_T^2\}}
\end{aligned}$$

The first two expectations can be calculated explicitely.

$$\begin{aligned}
E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} X_T^1 1_{[X_T^1 \geq Y_T^1]} 1_{[X_T^2 \leq Y_T^2]}\right] &= X_t^1 \mathcal{N}_2\left(d_t^1, d_t^2, \rho_t\right) \\
E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} Y_T^1 1_{[X_T^1 \geq Y_T^1]} 1_{[X_T^2 \leq Y_T^2]}\right] &= Y_t^1 \mathcal{N}_2\left(d_t^3, d_t^4, \rho_t\right)
\end{aligned}$$

The arguments of the normal distributions are given by

$$\rho_t = \frac{\int_t^T \left(\eta_s^{Y^1} - \eta_s^{X^1}\right)\left(\eta_s^{X^2} - \eta_s^{Y^2}\right)}{\sqrt{\int_t^T |\eta_s^{Y^1} - \eta_s^{X^1}|^2 ds \int_t^T |\eta_s^{X^2} - \eta_s^{Y^2}|^2 ds}} \tag{12}$$

$$\begin{aligned}
d_t^1 &= \frac{\ln\left(\frac{X_t^1}{Y_t^1}\right) + \frac{1}{2}\int_t^T |\eta_s^{X^1} - \eta_s^{Y^1}|^2 ds}{\sqrt{\int_t^T |\eta_s^{X^1} - \eta_s^{Y^1}|^2 ds}} \\
d_t^2 &= \frac{\ln\left(\frac{Y_t^2}{X_t^2}\right) + \frac{1}{2}\int_t^T |\eta_s^{X^2}|^2 - 2\left(\eta_s^{X^2} - \eta_s^{Y^2}\right)\eta_s^{X^1} - |\eta_s^{Y^2}|^2 ds}{\sqrt{\int_t^T |\eta_s^{X^2} - \eta_s^{Y^2}|^2 ds}} \\
d_t^3 &= \frac{\ln\left(\frac{X_t^1}{Y_t^1}\right) - \frac{1}{2}\int_t^T |\eta_s^{X^1} - \eta_s^{Y^1}|^2 ds}{\sqrt{\int_t^T |\eta_s^{X^1} - \eta_s^{Y^1}|^2 ds}} \\
d_t^4 &= \frac{\ln\left(\frac{Y_t^2}{X_t^2}\right) + \frac{1}{2}\int_t^T |\eta_s^{X^2}|^2 - 2\left(\eta_s^{X^2} - \eta_s^{Y^2}\right)\eta_s^{Y^1} - |\eta_s^{Y^2}|^2 ds}{\sqrt{\int_t^T |\eta_s^{X^2} - \eta_s^{Y^2}|^2 ds}}
\end{aligned} \tag{13}$$

For the remaining four expectations for $i = 1, 2$ we immediately get using the appropriate probability measures

$$E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} X_T^i 1_{[A]}\right] = X_t^i Q_t^{X^i}[A] \quad \text{and} \quad E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} Y_T^i 1_{[A]}\right] = Y_t^i Q_t^{Y^i}[A].$$

Now observe that under each of the four probability measures the four random variables $\ln X_1$, $\ln Y_1$, $\ln X_2$, $\ln Y_2$, obey a four-dimensional normal distribution[3]. The mean μ of this distribution — which depends on the particular probability measure — is a smooth function of time and the initial values X_t^i, $i = 1, 2$ and Y_t^i, $i = 1, 2$, whereas Σ, the covariance matrix[4] of this normal distribution, depends only on time and the instantaneous volatilities and correlations of the four assets under consideration. Defining $\tilde{A} \subset \mathbb{R}^4$ by

$$\tilde{A} = \left\{(x_1, \ldots, x_4) \in \mathbb{R}^4 \mid \exp(x_3) > \exp(x_4) \wedge \exp(x_1) - \exp(x_2) > \exp(x_3) - \exp(x_4)\right\}$$

we see that $Q_t^{X^i}[A]$ and $Q_t^{Y^i}[A]$ are given by the probability of $\tilde{A}$ under the four-dimensional normal distribution with the mean corresponding to the respective measure and with covariance matrix Σ, and are therefore smooth functions of time and initial values.

Next we deal with the hedging problem. By Itô's Lemma we have for the martingale part of C_t

$$[dC_t]^M = \sum_{i=1}^{2} \frac{\partial C_t}{\partial X_t^i}\left[dX_t^i\right]^M + \sum_{i=1}^{2} \frac{\partial C_t}{\partial Y_t^i}\left[dY_t^i\right]^M$$

Now for the partial derivatives we get

$$\begin{aligned}
\frac{\partial C_t}{\partial X_t^1} &= \frac{\partial}{\partial X_t^1} E_t^P\left[\left(\beta_{t,T}^d\right)^{-1}\left[\left[X_T^1 - Y_T^1\right]^+ - \left[X_T^2 - Y_T^2\right]^+\right]^+\right] \\
&= \frac{\partial}{\partial X_t^1} E_t^P\left[\left(\beta_{t,T}^d\right)^{-1}\left[X_T^1 - Y_T^1\right]^+ 1_{\{X_T^2 \le Y_T^2\}}\right] \\
&\quad + \frac{\partial}{\partial X_t^1} E_t^P\left[\left(\beta_{t,T}^d\right)^{-1}\left[X_T^1 - Y_T^1 - \left(X_T^2 - Y_T^2\right)\right]^+ 1_{\{X_T^2 \ge Y_T^2\}}\right]
\end{aligned}$$

Now by arguments similar as in the proof of Lemma 3.6 this equals

$$E_t^P\left[\left(\beta_{t,T}^d\right)^{-1}\frac{X_T^1}{X_t^1} 1_{\{X_T^1 \ge Y_T^1\}} 1_{\{X_T^2 \le Y_T^2\}}\right] + E_t^P\left[\left(\beta_{t,T}^d\right)^{-1}\frac{X_T^1}{X_t^1} 1_{\{X_T^1 - Y_T^1 > X_T^2 - Y_T^2\}} \cdot 1_{\{X_T^2 > Y_T^2\}}\right].$$

By the same argument as in the the first part of the proof this equals $\mathcal{N}_2\left(d_t^1, d_t^2, \rho\right) + Q_t^{X^1}[A]$. Similarly it is easy to show that

$$\frac{\partial C_t}{\partial Y_t^1} = -\left(\mathcal{N}_2\left(d_t^3, d_t^4, \rho\right) + Q_t^{Y_T^1}[A]\right). \quad \frac{\partial C_t}{\partial X_t^2} = -Q_t^{X_T^2}[A] \text{ and } \frac{\partial C_t}{\partial Y_t^2} = Q_t^{Y_T^2}[A]$$

Notice that $C_t\left(X_t^i, Y_t^i, t, T\right)$ is linear homogeneous in its first four arguments, since the payoff is linear homogeneous in X_T^i and Y_T^i, and since X_T^i and Y_T^i are linear functions of the initial values X_t^i and Y_t^i. The remainder of the argument is, therefore, as in the proof of Lemma 3.6.

[3] This distribution might be degenerate, i.e. the covariance matrix might not have full rank

[4] Σ is the same for all probability measures

4.2.2 Numerical Results

We now discuss a specific example to illustrate how the exercice probabilities from Theorem 4.3 can be evaluated numerically. Assume that we want to price the option

$$\left[\left[S_T^d - K^d\right]^+ - e_T\left[S_T^f - K^f\right]^+\right]^+$$

i.e. $X_t^1 = S_t^d$, $X_t^2 = e_t S_t^f$, $Y_t^1 = K^d B^d(t,T)$, $Y_t^2 = e_t K^f B^f(t,T)$. To proceede further we have to specify our general model in more detail. We assume that the volatilities of the stock and the exchange rates are constant and that the bond volatilities are given by

$$\sigma^d(t,T) = \sigma^d \frac{1}{\lambda^d}\left(1 - \exp\left(-\lambda^d(T-t)\right)\right) \qquad \sigma^f(t,T) = \sigma^f \frac{1}{\lambda^f}\left(1 - \exp\left(-\lambda^f(T-t)\right)\right)$$

where σ^d and λ^d are given positive constants. This means that bond prices are given by a generalized one-factor Vasicek-model. Moreover in our simulations we have assumed constant instantaneous correlations between the assets[5].

Knowledge of instantaneous correlations and volatilities is all that is required for computing the parameters μ and Σ of the joint normal distribution of $\ln X_T^1$, $\ln X_T^2$, $\ln Y_T^1$ and $\ln Y_T^2$, cf. section 2.1. In our example Σ is degenerate since $Y_T^2 = KB^d(T,T)$ is deterministic such that it suffices to consider a three-dimensional normal distribution with covariance matrix $\tilde{\Sigma}$.

For each of the numerical techniques discussed here the following observation is usefull: Suppose that A is a 3×3 matrix with[6] $\tilde{\Sigma} = A \cdot A^t$ and that (z_1, z_2, z_3) is a vector of independent standard normal variates. Then for every $\mu \in \mathbb{R}^3$ the random variable $y := A \cdot z + \mu$ is $\mathcal{N}_3(\mu, \tilde{\Sigma})$ distributed.

We have used two techniques for the evaluation of the exercice probabilities:

Monte Carlo Simulation: Similar to Boyle [Boy77] we simulated a large number of independent 3-dimensional standard normal variates z_n and transformed them as above to obtain independent Bernoulli variables $s^i(z_n)$, $1 \leq i \leq 4$ with $P[s^i(z_n) = 1] = Q^i[A]$. The estimator for the exercice probabilities is then given by the arithmetic mean $S_k^i := k^{-1} \sum_{n=1}^k s^i(z_n)$ of the sample.

In order to assess the quality of the simulation results we use results from the theory of large deviations (see e.g. Durrett [Dur91]). In the case of ordinary Monte Carlo simulation as described above we can immediately apply the following result to give a rather sharp upper bound for the error probability $P\left[|S_k^i - Q^i[A]| > \delta\right]$. Suppose that $(y_n)_{n=1,2,\ldots}$ is a sequence of i.i.d. random variables on some probability space $(\Omega, \mathcal{F}, P)$. Let S_k denote the arithmetic mean of $y_1, \ldots, y_k$. Then

$$P\left[S_k > a\right] \leq \exp\left(-k\gamma(a)\right). \tag{14}$$

[5] The precise parameter values can be found in the appendix.

[6] Such a decomposition always exists, if $\tilde{\Sigma}$ is non-degenerate.

Here $\gamma(a)$ is given by $\gamma(a) = \max_{\theta>0}\{a\theta - \ln\varphi(\theta)\}$, where $\varphi(\theta, p) := E^P[\exp(\theta y_1)]$ is the moment generating function of y_1. It can be shown that $\gamma(a)$ is always positive such that we get convergence at an exponential rate in (14). Moreover the estimate (14) is asymptotically sharp since

$$\lim_{k\to\infty} \frac{1}{k} \ln\left(P\left[S_k > a\right]\right) = -\gamma(a)\,.$$

To compute the error probabilities for the ordinary Monte Carlo simulation method set $y_n := s^i(z_n)$. Since the $s^i(z_n)$ are 0-1 variables their moment generating function is given by $\varphi(\theta, Q^i[A])$ where for $p \in [0,1]$ $\quad\varphi(\theta,p) := 1 - p + p\exp(\theta)$. Define for $\delta > 0$ and $p \in [0,1]$ the function $\tilde{\gamma}$ by

$$\tilde{\gamma}(\delta, p) := \max_{\theta>0}\{(p+\delta)\theta - \ln(\varphi(\theta,p))\}$$

If we now apply (14) to both, $s^i(z_n)$ and $1 - s^i(z_n)$ we get

$$\begin{aligned} P\left[|S_k^i - Q^i[A]| > \delta\right] &\leq \exp\left(-k\tilde{\gamma}\left(\delta, Q^i[A]\right)\right) + \exp\left(-k\tilde{\gamma}\left(\delta, 1 - Q^i[A]\right)\right) & (15)\\ &\leq 2\cdot\exp\left(-\tilde{\gamma}_{\min}(\delta)\right) & (16)\end{aligned}$$

with $\tilde{\gamma}_{\min}(\delta) = \min_{p\in[0,1]}\tilde{\gamma}(\delta,p)$. This upper bound is independent of the a priori unknown parameter $Q^i[A]$. such that estimates four the error probabilitiy using (16) hold simultaneously for all payoffs considered in Theorem 4.3. Moreover the numerical examples we ran showed that the minimum is attained for $p \approx \frac{1}{2}$ such that estimate (16) is sharp for certain parameter values.

Besides the ordinary Monte Carlo simulation we also considered the antithetic variable technique introduced in [Boy77]. Here the estimator $\bar{S}_k^i$ for the exercice probabilities is given by

$$\bar{S}_k^i = \frac{1}{k}\sum_{n=1}^{k}\frac{1}{2}\left(s^i(z_n) + s^i(-z_n)\right) \tag{17}$$

When using this technique one hopes that the negative correlation between z_n and $-z_n$ partially carries over to $s^i(z_n)$ and $s^i(-z_n)$, leading to a reduced variance of $\bar{S}_k^i$. Since (17) involves only k simulations but $2k$ evaluations of the functions s^i, the antithetic variable technique is an improvement over ordinary Monte Carlo simulation, if the error probabilities corresponding to a sample size of k are not greater than those corresponding to $2k$ ordinary simulations. We have again applied the estimate (14) with $y_n := \frac{1}{2}(s^i(z_n) + s^i(-z_n))$. Here we used ordinary Monte Carlo simulations to get an approximation for the moment generating function. The results are reported in table 1 for $\delta = 0.01$ and in table 2 for $\delta = 0.005$. The first two lines give our estimates for the error probabilities for different sample sizes k where we used once the estimate (16) and once estimate (15), both for the ordinary Monte Carlo simulation; the third line contains the results for the antithetic variable technique with sample size $k/2$.

We clearly see that the antithetic variable technique is an improvement. Moreover, the results confirm the usual observation that the number of simulations must be multiplied by four in order to double their precision.

Table 1: Error Probabilities if $\delta = 0.01$

	number of simulations		
simulation technique	10000	25000	50000
ordinary MC, estimate (16)	0.27066	0.01347	0.00009
ordinary MC, estimate (15)	0.15600	0.00341	$5.8 10^{-6}$
antithetic technique	0.10572	0.00132	$9.5 10^{-7}$

Table 2: Error Probabilities if $\delta = 0.005$

	number of simulations		
simulation technique	40000	100000	200000
ordinary MC, estimate (16)	0.27066	0.01347	0.00009
ordinary MC, estimate (15)	0.15603	0.00341	$5.8 10^{-6}$
antithetic technique	0.05864	0.00083	$6.9 10^{-7}$

Grid Search: Starting from the observation that the density of the normal distribution is approximately zero outside the intervall $I := [-3, 3]$ we partitioned I into n sub-intervalls. We considered two types of partitioning: Type P_1 was such that the n sub-intervals had equal length while P_2 was chosen such that the n sub-intervals had equal mass under the standard normal distribution. Let $M_{P_l}(n) = \{m_{P_l}(1), \ldots, m_{P_l}(n)\}$ denote the set of the midpoints of the n sub-intervals of I under partitioning P_l. Then a three-dimensional grid is defined by $(M_{P_l}(n))^3 := M_{P_l}(n) \times M_{P_l}(n) \times M_{P_l}(n)$. Further denote by $\Xi_{P_l}(n) = \{\xi_{P_l}(1), \ldots, \xi_{P_l}(n)\}$ the set of increments of the standard normal distribution function over the n sub-intervals generated by the partition P_l. Then we associate with every element $m = (m_{P_l}(j_1), m_{P_l}(j_2), m_{P_l}(j_3))$ belonging to $(M_{P_l}(n))^3$ the weight $\pi(m) := \xi_{P_l}(j_1) \cdot \xi_{P_l}(j_2) \cdot \xi_{P_l}(j_3)$. Similar to the Monte-Carlo case we can transform the elements $m \in (M_{P_l}(n))^3$ by

$$s^i : (M_{P_l}(n))^3 \to \{0, 1\}, \quad m \mapsto s^i(m)$$

The approximation for $Q^i[A]$ is then given by

$$Q^i[A] = \sum_{m \in (M_{P_l}(n))^3} s^i(m)\pi(m).$$

Finally we considered constructing a three dimensional grid not from the mid-points but from the boundaries of the intervals generated by P_1. In this case we evaluated s^i on the corners of every cube, added up the results, divided by eight and then summed up the results thus obtained weighing them by $\pi(m)$ to compute $Q^i[A]$.

In table 3 we report the results for $Q^{X^1}[A]$ as the number of sub-intervals, n, increases from 12 to 120 for the three different methods. We see that for methods 1 and 3 which involve P_1 the value of $Q^{X^1}[A]$ is reasonably stable for 60 sub-intervals which corresponds to the evaluation of 216,000 points for method 1 and 226,981 points for method 3. Convergence for method 2 is somewhat smoother but also slower than for the other two methods.

Table 3: $Q^{X^1}[A]$ computed by Grid Search

	$Q^{X^1}[A]$		
number of sub-intervals	Method 1	Method 2	Method 3
12	28.273	25.204	28.554
24	26.442	26.250	26.391
48	26.665	26.548	26.569
60	26.647	26.575	26.616
120	26.655	26.653	26.641

As compared to the Monte-Carlo approach we see that in our specific example $Q^{X^1}[A]$ can be computed more exactly by a grid search. For, the error probabilities in the Monte-Carlo case with antithetic variable technique show that comparable precision also requires the simulation of $\approx 100,000$ three dimensional, i.e. 300,000, independent normal variates and the evaluation of $\approx 200,000$ points. However, had the Σ-matrix been non degenerate the Monte-Carlo approach would have been more favourable since, broadly speaking, in order to maintain the same degree of precision the number of simulations required grows linearly with the rank of Σ. In contrast to that the number of points to be evaluated in a grid search grows exponentially with the rank of Σ. Hence, for rank $(\Sigma) \leq 3$ a grid search approach seems to be advantageous, while for rank$(\Sigma) > 3$ we prefer the Monte-Carlo method.

5 Conclusion

The paper treats the valuation and hedging of non–pathdependent European options on several underlyings with interest rate risk. Using martingale techniques in many cases we are able to provide general closed form solutions together with a procedure for applying these general solutions to some specific pay–off at hand. In cases where explicit solutions do not exist we give near explicit solutions for both pricing and hedging.

The main restriction of our approach is the assumption of deterministic volatilities which is particularly bothersome in the case of bonds. However, it is known from the interest rate literature that in general this hypothesis cannot be dispensed with if one is interested in explicit solutions.

A Appendix

A.1 Parameters of the Simulations

asset prices, exchange rate, and mean reversion:

$$
\begin{aligned}
S_0^d &= 100 & S_0^f &= 100 \\
B^d(0,T) &= 0.9 & B^f(0,T) &= 0.9 \\
K^d &= 100 & K^f &= 100 \\
e_0 &= 1 & T &= 1 \\
\lambda^d &= 0.1 & \lambda^f &= 0.1
\end{aligned}
$$

variance-covariance-matrix:

	S^d	S^f	B^d	B^f	e
S^d	0.2^2	0	0.0015	0	0
S^f	0	0.2^2	0	0.0015	0
B^d	0.0015	0	0.015^2	0	0.00045
B^f	0	0.0015	0	0.015^2	-0.00045
e	0		0.00045	-0.00045	0.15^2

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