Evolutionary Stability in Asymmetric Population Games

Dieter Balkenborg Karl H. Schlag

University of Bonn*

September 28, 1995

Abstract

We extend the notions of evolutionary stability and, for the first time, that of neutral stability to asymmetric games played between two populations. Stability with respect to simultaneous entry of a small proportion of mutants into both populations is considered. Allocations where neither mutant can ever spread are called neutrally stable. For bimatrix games, neutral stability in the asymmetric population game is found to be a weaker concept than neutral stability in the asymmetric contest. Moreover existence is guaranteed for 2 × 2 bimatrix games. Sets of neutrally stable strategy pairs such that for any pair of mutants not in the set at least one mutant is driven out are called evolutionarily stable. Evolutionarily stable sets are shown to be equivalent to strict equilibrium sets.

Additionally, uniformity considerations are investigated.

Keywords: evolutionarily stable set, evolutionarily stable strategy, neutral stability, strict equilibrium, mixed strategy model, asymmetric contest.

JEL *classification number*: C79.

1. Introduction

This paper introduces concepts of evolutionary stability for conflicts between two disjoint populations. The basic framework is set up and the definitions

---

[*] Address: Abt. Wirtschaftstheorie III, Department of Economics, University of Bonn, Adenauerallee 24, 53113 Bonn, Germany.
e-mail: dieter@glider.econ3.uni-bonn.de resp. schlag@glider.econ3.uni-bonn.de
Financial support by Deutsche Forschungsgemeinschaft through the Sonderforschungsbereich 303 at the University of Bonn is greatfully acknowledged.

are derived in the same way as done in Balkenborg and Schlag [2] for conflicts within a single population. We replicate a known result (Swinkels [13], Cressman [5]) in a more general context by showing that evolutionarily stable strategies are strict equilibria. Moreover, for the first time neutrally stable strategies and evolutionarily stable sets are defined and characterized in the two-population context. On the side, our results are contrasted to those obtained for asymmetric contests (see Balkenborg and Schlag [2]).

Evolutionary stability, introduced by Maynard Smith and Price [10], is a criterion for the robustness of an incumbent strategy against the entry of individuals or mutants using a different strategy. The framework considered is a conflict within a homogenous population. Originally only presented as an intuitive concept the following definition was given in Maynard Smith [8]: A strategy p is called *evolutionarily stable* if for every strategy $q \neq p$, in a population where a sufficiently large proportion of the individuals are playing p and the rest of the population plays a strategy q, the expected fitness of a p strategist is strictly higher than that of a q strategist. Consider any explicit dynamic model of the reproduction where strategies with higher payoffs reproduce at a higher rate. In a population consisting only of p and q strategists, the q strategists will die out in the long run provided that their initial proportion is sufficiently small.

Populations in which at most two different strategies are played are called *bimorphic*. The advantage of the bimorphic approach, i.e., of only considering dynamics in a bimorphic population, is the possibility to avoid explicit specification of the dynamic adjustment process and the possibility to obtain simple and tractable conditions.

Inexistence of evolutionarily stable strategies in many games lead to the weakening of the concept. Maynard Smith [9] introduced the concept of neutral stability where mutants are allowed to achieve at most the same expected fitness as individuals using the incumbent strategy. Thomas [15] introduced the notion of an evolutionarily stable set as a set-valued extension of the concept of an evolutionarily stable strategy. Both of these concepts were introduced without an explicit dynamic justification as done by Maynard Smith [9] for evolutionarily stable strategies (see above). This lead to the investigation of Balkenborg and Schlag [2] that aimed to take the dynamic approach seriously. They restrict the analysis to bimorphic populations while allowing for a wide class of reproduction processes. The common features the reproduction processes must satisfy are that they do not introduce new strategies and that strategies with a higher expected fitness reproduce at a higher rate. Balkenborg and Schlag [2] define what it means that a mutant strategy 'is driven out' or 'does not to spread' after entering a population playing a single "incumbent" strategy. These notions correspond to the concepts of a local attractor and a stable rest point in the theory of dynamical systems, when applied to bimorphic population states. More specifically, a mutant strategy entering a population of individuals playing a given strategy is *driven* out if its frequency tends to zero over time when its initial proportion is sufficiently small. Similarly, the mutant strategy *can not spread* if entry in a small proportion implies that this proportion stays small. In this terminology a strategy is called *evolutionarily stable* if any mutant strategy using a different strategy than the incumbent is driven out. A strategy is *neutrally stable* when no mutant can spread. Finally, the concept of an *evolutionarily stable set* G is defined as follows. No mutant can spread when the incumbent is using a strategy in G and mutants using a strategy not belonging to G are driven out. Moreover, uniformity requirements on the maximal mutation sizes lead to the concepts of uniformly neutrally stable strategies and uniformly evolutionarily stable sets.

Much of the work in Balkenborg and Schlag [2] was undergone in such detail to lay the grounds for this paper. The object of this paper is to extend the evolutionary stability concepts described above for the conflicts within a single population to the class of conflicts between two disjoint populations, which we will refer to as the *asymmetric population games.* These are conflicts where all members of the same population have the same strategy set and where the fitness of an individual depends only on his strategy and on the distribution of strategies in the opposite population.[1] The bimorphic approach of the one population model will be extended to the asymmetric population game as follows. For two populations we speak of a bimorphic population state if two strategies are played in each population and thus four strategies are played on total. A bimorphic population state is obtained if originally a single incumbent strategy is played in each population and small fractions in both populations switch to playing a mutant strategy. We define evolutionarily stable strategies and neutrally stable strategies just as in the one population model, except that now these concepts refer to *pairs* of strategies. Thus a pair of strategies is called evolutionarily stable in an asymmetric population game if for any pair of mutant strategies the mutant strategies are driven out in each population. Similarly, a pair of strategies is called neutrally stable if for any pair of mutant strategies no mutant strategy can spread. The notion of an evolutionarily stable set can not be adapted without some additional considerations. Since there are two populations the novel situation arises in which the mutant strategy is driven out of in one but not the other population. We use a weak necessary condition to define an ES set. Our definition requires in the two population setting for any pair of mutant strategies that no mutant strategy can spread and, if the pair of mutant strategies is not contained in the evolutionarily stable set, that at least one mutant strategy is driven out in one of the two populations.

Previous work on evolutionary stability using the bimorphic approach in the two population framework is as follows. Swinkels [13] follows a static approach. He suggests that an asymmetric ESS must have the property that at least one of

---

[1] In contrast an asymmetric contest (see Maynard Smith [8], Selten [12] and Balkenborg and Schlag [2]) is a contest between the members of a single population, where the players are randomly assigned a role before fitness is determined.

the two mutant strategies is strictly worse off when there are sufficiently many incumbents. Cressman [5] also gives the same static condition in a more general model that allows to combine the one and the two population framework. Moreover, Cressman [5] justifies the condition by showing that it is equivalent to the strategy pair being locally asymptotically stable in the replicator dynamics of Taylor [14] for bimorphic population states. Hence, concerning ESS we extend the approach of Cressman [5] and allow for a more general class of dynamics. Both Swinkels [13] and Cressman [5] show for their individual definition that evolutionarily stable strategy pairs are strict equilibria. An alternative approach for polymorphic population states was started by Cressman and Dash [6] in the more general model that combines the one and the two population framework. Balkenborg [1] studies a replicator dynamics for polymorphic populations playing mixed strategies in a normal form game. He introduces the concept of strict equilibrium sets as a generalization of strict equilibrium points and establishes a one-to-one correspondence between strict equilibrium sets and certain locally asymptotically stable sets of the dynamic.

We give a characterization of neutrally stable strategies that is easy to verify since it essentially reduces the conditions for neutral stability to the one dimensional dynamics that is obtained when a mutation occurs only in one of the two populations. Namely, in order to be neutrally stable, either the frequencies in both populations remain unchanged after the mutation or one of the two mutant strategies is driven out after those small mutations where it is the only mutant strategy. In the case of bilinear payoff functions (which are obtained as the expected payoff when randomly matched individuals from the two populations play a bimatrix game) we obtain static conditions for neutral stability. In particular we derive the following conclusions about neutrally stable strategy pairs in bimatrix games: i) each Nash equilibrium of a zero sum game is neutrally stable, ii) neutrally stable strategies always exist in 2×2 games, iii) existence is no longer guaranteed in larger games.

In contrast to symmetric population games the conditions for a strategy pair or a set of strategy pairs to be evolutionarily stable turn out to be quite restrictive. Evolutionarily stable strategy pairs are equivalent to strict equilibria. Similarly, evolutionarily stable sets are equivalent to strict equilibrium sets. Thus, in a certain sense we extend a result in Balkenborg [1] to a more general class of dynamics at the cost of restricting the number of types present in the population. It should be mentioned that the analysis of evolutionarily stable sets in the context of asymmetric contests also leads to the same equivalence of evolutionarily stable sets and strict equilibrium sets (see Balkenborg and Schlag [2]).

The definitions of the terms 'a mutant strategy is driven out' and 'a mutant strategy does not spread' refer to small initial sizes of a mutation. How small the initial proportions of the entering mutants must hereby be may depend on which mutant strategies are entering. We find the conditions of neutral and evolutionary stability not intuitive if their underlying conditions can only be upheld when the size of the mutation may also depend on which mutant strategy is entering. Considerations of this sort have been raised repeatedly for the symmetric population case (see, e.g., Vickers and Canning [16] and Bomze and Pötscher [3]).

Neutrally stable strategy pairs where the quantifiers in the condition 'do not spread' do not depend on the entering mutant strategies are called *uniformly neutrally stable.* In order to characterize uniformly neutrally stable strategy pairs we must restrict the class of dynamics that we consider. Concerning 'uniformness' questions, the relationship between the relative dynamic forces in the two population matters. We define a class of dynamics in which it can be argued that evolutionary forces apply in both populations in the same order of magnitude. For this class of dynamics, which includes the replicator dynamics (see Maynard Smith [9], Taylor [14]), we show for the prevalent area of applications, namely for bilinear fitness functions, that uniformness considerations are redundant: neutral stability and uniform neutral stability coincide. As a side result we obtain that uniform neutral stability in an asymmetric population game is a strictly weaker concept than that of uniform stability in a corresponding truly asymmetric contest. Balkenborg [1] observes that stability in the replicator dynamics of a truly asymmetric contest implies stability in the corresponding asymmetric population. Our counter example shows for neutral stability that the converse does not hold.

Uniformity considerations with respect to the condition 'mutant strategies are driven out' were first explicitly considered by Vickers and Canning [16] and led Bomze and Pötscher [3] to the definition of a uniform invasion barrier in the one population setting. Applied to the two population setting, an evolutionarily stable strategy pair has a *uniform invasion barrier* if, independent of the mutant strategies, mutants entering in a sufficiently small proportion are driven out. Evolutionarily stable sets that are, as a set, uniformly neutrally stable and have a uniform invasion barrier are called *uniformly evolutionarily stable sets.* This means that the maximal sizes of the initial mutations referred to in the definition can be chosen independently of both the strategy pair in the ES set and of the pair of mutant strategies. As evolutionarily stable strategies are strict equilibria it is not astonishing that we are able to show that singleton ES sets are also uniform ES sets. We replicate this result for general ES sets and bilinear fitness functions in the class of dynamics where evolutionary forces apply in both populations in the same order of magnitude.

The rest of the paper is organized as follows. Section one introduces asymmetric population games. Section two sets up the relationship between fitness of individuals and the evolutionary process. In Section three the condition that a mutant strategy 'does not spread' or is 'not driven out' are defined. Section four presents the definitions of neutrally stable strategies and of evolutionarily stable sets. In part one of Section five neutrally stable strategies are characterized. A static condition for neutral stability is derived for bilinear fitness functions, existence and relationship to Nash equilibria is analyzed. In part two it is shown that strict equilibrium sets and ES sets are equivalent concepts. In Section six uniformly neutrally stable strategies sets are defined and then analyzed in bimatrix games. In Section seven the relationship to evolutionary stability in asymmetric contests is demonstrated. Finally, Section eight considers uniformly evolutionarily stable sets.

2. Asymmetric Population Games

We consider conflicts between members of two different populations, indexed by one and two. All members of the same population i ($i \in \{1, 2\}$) have the same set of strategies Q_i available, $i = 1, 2$. A population state is defined as a *pair* of distributions of strategies for each population. Such a population state is called *monomorphic* if all members of the same population use the same strategy. It is called *bimorphic* if there are strategies $q_1, p_1 \in Q_1$ and $q_2, p_2 \in Q_2$ such that the members of population one use either q_1 or p_1 and the members of population two use either q_2 or p_2.

Bimorphic states are denoted by $((p_1, q_1, x), (p_2, q_2, y))$ where $0 \leq x \leq 1$ is the percentage of individuals in population one using q_1 and $0 \leq y \leq 1$ is the percentage of individuals in population two using q_2. The fitness which an individual in population one using strategy $r_1 \in \{p_1, q_1\}$ obtains from the conflict in the bimorphic population state $(p_1, q_1, x, p_2, q_2, y)$ is denoted by

$$E_1(r_1, (p_2, q_2, y)) .$$

Similarly an individual in population two using $r_2 \in \{p_2, q_2\}$ receives a fitness

$$E_2((p_1, q_1, x), r_2) .$$

Thus we assume that the fitness of the strategy an individual uses depends only on this strategy and the distribution of strategies in the *other* population.[2]

We call (Q_1, Q_2, E_1, E_2) an *asymmetric population game.* We restrict ourselves in the following to mixed strategy models. We call (Q_1, Q_2, E_1, E_2) an *asymmetric mixed strategy model* if the following three conditions are satisfied: a) The strategy spaces Q_i ($i \in \{1, 2\}$) are the sets of mixed strategies $\Delta(S_i)$ over finite sets of pure strategies S_i. b) *State substitutability* holds, i.e., we have

$$E_1(r_1, (p_2, q_2, y)) = E_1(r_1, (1 - y)p_2 + yq_2)$$

for population one and

$$E_2((p_1, q_1, x), r_2) = E_2((1 - x)p_1 + q_1, r_2).$$

for population two. c) $E_1(\cdot, \cdot) : \Delta(S_1) \times \Delta(S_2) \to \mathbb{R}$ is linear in the first and continuous in the second component while $E_2(\cdot, \cdot) : \Delta(S_1) \times \Delta(S_2) \to \mathbb{R}$ is continuous in the first and linear in the second component.

---

[2] A more general case is discussed in Cressman [5].

We will consider from now on only asymmetric mixed strategy models. We will nevertheless formulate the basic concepts such that they are meaningful in any asymmetric population game.

In the majority of the literature attention is restricted to asymmetric mixed strategy models with bilinear payoff functions. These correspond to asymmetric bimatrix games, i.e., to two-player normal form games.

3. Fitness and the Evolutionary Process

The change of the proportions of players in population one and population two using the various strategies is determined by a deterministic process in continuous time in which strategies not played remain extinct. In case of a bimorphic population $((p_1, q_1, x), (p_2, q_2, y))$ we therefore need only to consider the dynamic on $[0, 1] \times [0, 1]$ governing the change of the proportion x of q_1-players in population one and of the proportion y of q_2-players in population two. This can be written as

$$\begin{aligned} \dot{x} &= g_1(x, y)(1 - x)x \\ \dot{y} &= g_2(x, y)(1 - y)y \end{aligned} \tag{3.1}$$

where $g_1(x, y)(1 - x)$ and $g_2(x, y)(1 - y)$ are the growth rates of q_1 and q_2 respectively. We assume that $g_1(x, y)(1 - x)x$, and $g_2(x, y)(1 - y)y$ are Lipschitz-continuous on $[0, 1] \times [0, 1]$. We assume in addition that g_1 is continuous in x and g_2 is continuous in y, i.e., that (3.1) is a regular selection dynamic as defined in Samuelson and Zhang [11]. Following the lines of Hammerstein and Selten [7] we call

$$f_1(y, p_1, q_1, p_2, q_2) := E_1(p_1, (1 - y)p_2 + yq_2) - E_1(q_1, (1 - y)p_2 + yq_2)$$

with $0 \leq x \leq 1$ the *deviation fitness function* of p_1 against q_1 given (p_2, q_2), and

$$f_2(x, p_1, q_1, p_2, q_2) = E_2((1 - x)p_1 + xq_1, p_2) - E_2((1 - x)p_1 + xq_1, q_2)$$

with $0 \leq x \leq 1$ the *deviation fitness function* of p_2 against q_2 given (p_1, q_1). We will write $f_1(y)$ and $f_2(x)$ when it is clear which strategies are played in the bimorphic state.

We require that strategies with higher fitness spread faster, i.e., for $0 \leq x \leq 1$ and $0 \leq y \leq 1$,

$$f_1(y) \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases} \text{if and only if } g_1(x, y) \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases} \tag{3.2}$$

$$f_2(x) \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases} \text{if and only if } g_2(x, y) \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases} .$$

The dynamic (3.1) is monotonic in the sense of Samuelson and Zhang [11] if and only if (3.2) holds.[3]

Example 3.1. For the bilinear case with E_1,$E_2 > 0$ two dynamic processes (both referred to as the continuous replicator dynamics) satisfying 3.1 and 3.2 are frequent the literature:

i) $g_1(x, y) = -\ f_1(y)$ and $g_2(x, y) = -\ f_2(x)$ for $x, y\ \in [0, 1]$ (see Taylor [14]).

ii)

$$g_1(x, y) = \frac{-f_1(y)}{E_1((1-x)p_1 + xq_1, (1-y)p_2 + yq_2)}$$

and

$$g_2(x, y) = \frac{-f_2(x)}{E_2((1-x)p_1 + xq_1, (1-y)p_2 + yq_2)}$$

for $x, y \in\ [0, 1]$ (see Maynard Smith *[9]*).

4. Mutations of a Monomorphic Population

Consider mutations of a monomorphic population playing (p_1, p_2) of the following sort. At time $t = 0$ simultaneously a small frequency x of mutants playing q_1 enters population one and a small frequency y of mutants playing q_2 enters population two $(x, y > 0)$. Consequently we have a bimorphic state $((p_1, q_1, x(0)), (p_2, q_2, y(0)))$ at time zero. The *size* of a such a mutation will be defined as $\max\{x(0), y(0)\}$. The case of a pair of mutant strategies (p_1, q_2) can be interpreted as only population two being entered by a mutant.

Definition 4.1. *Given the monomorphic population (p_1, p_2) and the pair of mutant strategies (q_1, q_2) the mutant strategy q_1 is driven out if the fraction of individuals playing q_1 in population one vanishes in the long run for any sufficiently small mutation, provided no further mutants occur. Formally: there exists $0 < \varepsilon \leq 1$ such that $\lim_{t \to \infty} x(t) = 0$ if $\max\{x(0), y(0)\} < \varepsilon$. Similarly q_2 being driven out is defined. Especially, if $q_1 \neq p_1$ and $q_2 \neq p_2$ we obtain that $(0, 0)$ is a local attractor in our two dimensional dynamic (3.1) if and only if both q_1 and q_2 are driven out.*

Definition 4.2. *Given the monomorphic population (p_1, p_2) and the pair of mutant strategies (q_1, q_2) the mutant strategy q_1 spreads after any arbitrarily small mutation if there exist arbitrarily small (but strictly positive) initial entry proportions of q_1 and q_2 in population one and two respectively after which the frequency of the mutant strategy q_1 does not stay small. Formally: there exists $\varepsilon > 0$ such that for any $0 < \varepsilon' < \varepsilon$ there exists $x(0), y(0) < \varepsilon'$ and $t^* \in \mathbb{R}$ such that $x(t^*) > \varepsilon$. In other words, $(0, 0)$ is a stable rest point of the dynamic (3.1) if and only if neither q_1 nor q_2 can spread.*

---

[3] In fact it also aggregate monotone since there are at most two different types in each population.

5. Evolutionary Stability

Definition 5.1. *i) A strategy pair* $(p_1, p_2) \in \Delta(S_1) \times \Delta(S_2)$ *is (asymmetric) neutrally stable if there is no pair of mutant strategies* (q_1, q_2) *such that either* q_1 *or* q_2 *can spread.*

ii) A strategy pair $(p_1, p_2) \in \Delta(S_1) \times \Delta(S_2)$ *is (asymmetric) evolutionarily stable if for any pair of mutant strategies* (q_1, q_2), $q_1 \neq p_1$ *implies that* q_1 *is driven out and* $q_2 \neq p_2$ *implies that* q_2 *is driven out.*

In the following we will define the concept of an evolutionarily stable set. In the one population setting (see Balkenborg and Schlag [2]) this is a set neutrally stable strategies such that mutants that are not contained in the set are driven out. In the two population setting there are various ways in which mutations can be driven out. Either both mutants are driven out or only one is driven out. We adapt a seemingly weak condition and require that at least one of the two mutants is driven out when the pair of mutant strategies is not contained in the set.

Definition 5.2. *A non-empty set of strategy pairs* $G \subseteq \Delta(S_1) \times \Delta(S_2)$ *is called an (asymmetric) evolutionarily stable set (short,* ***ES set****) if for any given monomorphic population* $(p_1, p_2) \in G$ *and any pair of mutant strategies* (q_1, q_2) *neither mutant strategy spreads and if additionally* $(q_1, q_2) \notin G$ *then either* q_1 *or* q_2 *is driven out.*

Notice that an evolutionarily stable set is always a set of neutrally stable strategy pairs. It follows that $\{(p_1, p_2)\}$ is an ES set if and only if (p_1, p_2) is evolutionarily stable.

The above definition has an alternative interpretation. Consider the interpretation of an ES set in the one population setting as a set such that mutants that are not driven out are contained in the set. Adapting this condition to the two population case one might obtain the following condition. If neither mutant strategy is driven out then the pair of mutant strategies must be contained in the evolutionarily stable set. What if only one of the mutant strategies is driven out, e.g., q_1, but not q_2? In this case trajectories must converge to a population state where q_1 is not played. Hence the result of the entry of the pair of mutant strategies (q_1, q_2) looks as if a mutation had only occurred in population two where the mutant strategy was not driven out. Therefore in such a situation one might require (p_1, q_2) to be contained in the evolutionarily stable set. Formalizing these considerations leads to the above definition.

6. Characterizations

6.1. ... of Neutral Stability

Consider the pair of mutant strategies (q_1, p_2) entering the monomorphic population (p_1, p_2). Then q_1 can not spread if and only if $E_1(q_1, p_2) \le E_1(p_1, p_2)$. Applying the same argument to the pair of mutant strategies (p_1, q_2) we obtain that each neutrally stable strategy pair (p_1, p_2) is a *Nash equilibrium*, i.e., $E_1(q_1, p_2) \le E_1(p_1, p_2)$ and $E_2(p_1, q_2) \le E_2(p_1, p_2)$ for all $q_1 \in Q_1$, $q_2 \in Q_2$. Especially, if we had restricted mutants to enter in only one of the two populations then a neutrally stable strategy pair would have been equivalent to a Nash equilibrium. However as we allow for mutants to enter simultaneously in both populations we obtain a stronger concept characterized in the following theorem. Loosely speaking, it states that a strategy pair is neutrally stable if and only if either the proportions in both populations remain fixed after the mutation or one of the two mutant strategies is driven out when there is no selection pressure in the opposite population.

Theorem 6.1. *Let E_1 and E_2 be analytic. The strategy pair (p_1, p_2) is neutrally stable if and only if (p_1, p_2) is a Nash equilibrium and for each pair of mutant strategies (q_1, q_2) there exists $\varepsilon > 0$ such that for all $x, y \in (0, \varepsilon]$ one of the following three statements is true:*

i) $((1-x)p_1 + xq_1, (1-y)p_2 + yq_2)$ is a rest point, i.e., $f_1(y) = f_2(x) = 0$.
ii) $f_1(y) > 0$.
iii) $f_2(x) > 0$.

Notice that the requirement that (p_1, p_2) is a Nash equilibrium is redundant in the above theorem.

Remark 6.2. Let E_1 and E_2 be analytic. The 'if' statement in Theorem 6.1 holds if and only if for each pair of mutant strategies (q_1, q_2) there exists *no* $\varepsilon > 0$ such that for all $0 < x, y \le \varepsilon$: $f_1(y) \le 0$, $f_2(x) \le 0$ and $f_1(y) + f_2(x) < 0$.

Before getting to the proof we give some intuition. Notice that it might be the case that while q_1 is driven out, the proportion of q_2 starts to increase. The reason why (p_1, p_2) is never-the-less neutrally stable is that as q_1 being driven out, the nourishment that allows q_2 to spread locally vanishes, hindering the proportion of q_2 to grow substantially.

Proof. For $\varepsilon > 0$ we denote

$$U(\varepsilon) := \{(x, y) \mid 0 \le x \le \varepsilon,\ 0 \le y \le \varepsilon\}\ .$$

1) Assume that (p_1, p_2) is neutrally stable. As stated at the beginning of this section, it follows that (p_1, p_2) is a Nash equilibrium. Fix a pair of mutant strategies (q_1, q_2). Suppose there exists $\varepsilon > 0$ such that $f_1(y) \le 0$, $f_2(x) \le 0$ and

$f_1(y) + f_2(x) < 0$ holds for all $0 < x, y \leq \varepsilon$. Choose $\hat{\varepsilon} > 0$ such that trajectories starting in $U(\hat{\varepsilon})$ stay in $U(\varepsilon)$. If $f_1(y) < 0$ holds for all $0 < y \leq \varepsilon$ then y grows as long as $0 < y \leq \varepsilon$ which means that trajectories starting in $U(\hat{\varepsilon})$ with $y > 0$ leave $U(\varepsilon)$ in contradiction to the neutral stability of (p_1, p_2). If there exists an $y_0 < \hat{\varepsilon}$ with $f_1(y_0) = 0$, then $f_2(x) < 0$ for all $0 < x \leq \varepsilon$. Again this contradicts the neutral stability of (p_1, p_2).

2) Assume conversely that (p_1, p_2) is a Nash equilibrium and that for each pair of mutant strategies (q_1, q_2) there exists $\varepsilon > 0$ such that one of the Conditions i)-iii) in Theorem 6.1 holds. Then $f_1(0), f_2(0) \leq 0$.

If i) is true then all points in $U(\varepsilon)$ are rest points, especially neither q_1 nor q_2 can spread.

Assume that ii) and iii) hold simultaneously. Then for each $0 < \varepsilon' < \varepsilon$ any trajectory starting in $U(\varepsilon')$ will stay in $U(\varepsilon')$. Again, neither q_1 nor q_2 can spread.

Assume next that ii) holds but iii) does not. Then necessarily $f_2(x) = 0$. We will show that we can find for each $\varepsilon > 0$ a neighborhood V of $(0, 0)$ contained in $U(\varepsilon)$ which is of the form

$$V = \{(x, y) \mid 0 \leq x < a,\ 0 \leq y < b \exp(-cx)\} \tag{6.1}$$

and for which trajectories starting in V remain in V and hence in $U(\varepsilon)$.

$g_2(x, y)(1 - y)y$ and hence $g_2(x, y)$ are Lipschitz continuous at $\left(x_0, \frac{\varepsilon}{2}\right)$. Moreover $g_2(0, y) = 0$ for all $0 \leq y \leq 1$. Therefore we can find constants $0 < \alpha < \frac{\varepsilon}{2}$ and $k > 0$ such that

$$|g_2(x, y)| \leq k \cdot x \text{ for all } x \in [0, \alpha] \text{ and } y \in [\frac{\varepsilon}{2} - \alpha, \frac{\varepsilon}{2} + \alpha]\,. \tag{6.2}$$

Because $f_1(y) > 0$ for all $y \in \left[\frac{\varepsilon}{2} - \alpha, \frac{\varepsilon}{2} + \alpha\right]$ and because g_1 is continuous we can assume that α is chosen so small that

$$\theta := \min\left\{g_1(x, y)\,(1 - y) \mid x \in [0, \alpha],\, y \in [\frac{\varepsilon}{2} - \alpha, \frac{\varepsilon}{2} + \alpha]\right\} > 0\,. \tag{6.3}$$

Since $\dot{x} < 0$ the only reason why (p_1, p_2) may not be neutrally stable is that $\dot{y}$ increases too strong. It will be enough if we can show stability when slowing the growth rate of x (increasing $\dot{x}$) and speeding up the growth of y. A trajectory starting in $(x(0), y(0))$ with $x(0) \leq \alpha$ and $y(0) \leq \frac{\varepsilon}{2}$ will either remain in $[0, \alpha] \times \left[0, \frac{\varepsilon}{2}\right]$ or will arrive in finite time at a point $\left(x, \frac{\varepsilon}{2}\right)$ with $x \leq x(0)$.

For a trajectory starting in $(x(0), y(0))$ with $x(0) \leq \alpha$ and $y(0) = \frac{\varepsilon}{2}$ (6.2) and (6.3) imply $\dot{x} \leq -\theta x$ and $\dot{y} \leq kxy$. Consider the extreme dynamic $\dot{x} = -\theta x$ and $\dot{y} = kxy$. Interested only in trajectories with $y = y(x)$ we obtain $\frac{\partial y}{\partial x} = \dot{y}/\dot{x} = -cy$ where $c := k/\theta$. It follows that $y(x) = \lambda e^{-cx}$ with $y(0) = \lambda \in R$. For $v > 0$ let $\lambda(v) > 0$ be such that the trajectory starting in $(v, \frac{\varepsilon}{2})$ converges to $(0, \lambda(v))$, i.e., $\lambda(v) = \frac{\varepsilon}{2} e^{cv}$. Since $\lambda(\cdot)$ is continuous and $\lambda(0) = \frac{\varepsilon}{2}$ we can choose $0 < a \leq \alpha$ such that $b := \lambda(a) < \frac{\varepsilon}{2} + \alpha$. It follows that for a trajectory starting at $\left(x(0), \frac{\varepsilon}{2}\right)$ with $x \leq a$, $x(t) \leq \frac{\varepsilon}{2} + \alpha$ will hold for all $t \geq 0$. Thus we have found values $a, b, c > 0$ such that the neighborhood V defined in (6.1) has the desired properties.

A symmetric argument if condition iii), but not condition ii) holds concludes the argument that (p_1, p_2) is neutrally stable. ■

Corollary 6.3. *Let E_1 and E_2 be bilinear and consider $(p_1, p_2) \in \Delta(S_1) \times \Delta(S_2)$. Then (p_1, p_2) is neutrally stable if and only if (p_1, p_2) is a Nash equilibrium and there do not exist $q_1 \in BR_1(p_2)$ and $q_2 \in BR_2(p_1)$ such that $E_1(p_1, q_2) \leq E_1(q_1, q_2)$ and $E_2(q_1, p_2) \leq E_2(q_1, q_2)$ with at least one inequality holding strict.*

Proof. "only if": Let $q_1 \in BR_1(p_2)$ and $q_2 \in BR_2(p_1)$ such that $E_1(p_1, q_2) < E_1(q_1, q_2)$ and $E_2(q_1, p_2) \leq E_2(q_1, q_2)$. Then $f_1(y) < 0$ and $f_2(x) \leq 0$ for all $0 < x, y \leq 1$. By Theorem 6.1 (p_1, p_2) cannot be neutrally stable.

"if": Assume $q_1 \notin BR_1(p_2)$. Then $f_1(y) > 0$ for small y and hence Condition ii) in Theorem 6.1 is satisfied. Assume $q_1 \in BR_1(p_2)$, $q_2 \in BR_2(p_1)$. If $E_1(p_1, q_2) < E_1(q_1, q_2)$ then we must have $E_2(q_1, p_2) > E_2(q_1, q_2)$. Consequently $f_2(x) > 0$ for all x, i.e., Condition iii) in Theorem 6.1 holds. If $E_1(p_1, q_2) = E_1(q_1, q_2)$ then $E_2(q_1, p_2) \geq E_2(q_1, q_2)$. Therefore $f_1(y) = 0$ for all y and $f_2(x) > 0$ or $f_2(x) = 0$ for all x which implies Condition i) or iii) in Theorem 6.1. ■

In the rest of this section we will give some applications of Corollary 6.3. At first we obtain that the concepts of Nash equilibrium and neutral stability are equivalent in constant sum games.

Definition 6.4. *A bimatrix game is called a constant sum game if there exists $c \in \mathbb{R}$ such that $E_1(w, z) + E_2(w, z) = c$ for all $w \in \Delta(S_1)$ and $z \in \Delta(S_2)$.*

Corollary 6.5. *Every Nash equilibrium of a constant sum game is neutrally stable.*

Proof. Let (p_1, p_2) be a Nash equilibrium. Let $q_1 \in BR_1(p_2)$ and $q_2 \in BR_2(p_1)$ such that $E_1(p_1, q_2) < E_1(q_1, q_2)$ and $E_2(q_1, p_2) \leq E_2(q_1, q_2)$. Then since the game is a constant sum game it follows that $E_1(q_1, q_2) \leq E_1(q_1, p_2)$. Moreover since (p_1, p_2) is a Nash equilibrium we obtain $E_1(q_1, p_2) \leq E_1(p_1, p_2) \leq E_1(p_1, q_2)$. Summarizing the above we have shown that $E_1(q_1, q_2) \leq E_1(q_1, p_2) \leq E_1(p_1, p_2) \leq E_1(p_1, q_2) < E_1(q_1, q_2)$ which gives a contradiction. ■

Example 6.6. *(Not every Nash equilibrium in mixed strategies of a coordination game is neutrally stable.)* Straightforward calculation shows that the mixed Nash equilibrium of a coordination game (a game as in Figure 6.1 where $a > b, a' > c', d > c, d' > b'$) is not neutrally stable.

		R
T	a a'	c c'
B	b b'	d d'

Figure 6.1: Bimatrix games

Remark 6.7. *(Existence of neutrally stable strategies in* 2×2 *bimatrix games)* It is easily shown that every general 2×2 bimatrix game as in Figure 6.1 has a neutrally stable strategy pair.

		M	R
T	ε ε	-1 1	1 -1
C	1 -1	ε ε	-1 -1
B	-1 1	1 -1	ε ε

Figure 6.2: A rock-paper-scissor game

Example 6.8. *(Neutrally stable strategy pairs do not have to exist in bimatrix games.)* Consider the class of "Rock-Paper-Scissor"-games in Figure 6.2. It is straightforward to check that $((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$ is neutrally stable if and only if $\varepsilon \leq 0$. Additional calculations show that there is no neutrally stable strategy when $0 < \varepsilon < 1$.

6.2. ... of Evolutionary Stability

Definition 6.9. *(Balkenborg [1]): A non-empty set of strategy pairs G is called a strict equilibrium set (short,* **SE set***) if for any $(p_1, p_2) \in G$ and any pair of strategies $(q_1, q_2) \in \Delta(S_1) \times \Delta(S_2)$*

$$E_1(q_1, p_2) \leq E_1(p_1, p_2)$$

whereby equality implies $(q_1, p_2) \in G$ and

$$E_2(p_1, q_2) \leq E_2(p_1, p_2)$$

whereby equality implies $(p_1, q_2) \in G$.

We will denote the set of *best replies* of a player 1 against a strategy $p_2 \in \Delta(S_2)$ by $BR_1(p_2)$, i.e., $p_1 \in BR_1(p_2)$ if and only if $E_1(q_1, p_2) \leq E_1(p_1, p_2)$ for all $q_1 \in Q_1$. The set of best replies $BR_2(q_1)$ is defined symmetrically.

A strict equilibrium set G can thus be considered as a set of Nash equilibria which is *closed under best replies.*

Theorem 6.10. *$G \subseteq \Delta(S_1) \times \Delta(S_2)$ is an (asymmetric) evolutionarily stable set if and only if G is a strict equilibrium set. In particular an ES set is closed.*

Proof. 'only if': Let G be an (asymmetric) ES set and let $(p_1, p_2) \in G$. Let $q_1 \in BR_1(p_2)$ and consider the pair of mutant strategies (q_1, p_2). Since (p_1, p_2) is neutrally stable, $E_1(q_1, p_2) = E_1(p_1, p_2)$. Hence $\dot{x} = 0$ for all t. Since q_1 is not driven out it follows that $(q_1, p_2) \in G$. A symmetric argument for $q_2 \in BR_2(p_1)$ shows that G is a SE set. Similar arguments concerning the pair of mutant strategies (p_1, q_2) complete the proof. It was shown in Balkenborg [1] that SE sets are always closed.

'if': Let G be a SE set, let $(p_1, p_2) \in G$ and consider (q_1, q_2). Then $f_1(0) \geq 0$ and $f_2(0) \geq 0$. If $f_1(0) > 0$ and $f_2(0) > 0$ then continuity of the fitness functions implies that there exists $\varepsilon > 0$ such that both mutants are driven out if $x(0), y(0) < \varepsilon$. If $f_1(0) > 0$ and $f_2(0) = 0$ then continuity together with the proof of Theorem 6.1 implies that there exists $\varepsilon > 0$ such that neither q_1 nor q_2 can spread if $x(0), y(0) < \varepsilon$. Moreover, q_1 is driven out which leads to no contradiction since $f_2(0) = 0$ implies that $(p_1, q_2) \in G$. Finally assume that $f_1(0) = f_2(0) = 0$. Then $f_2(0) = 0$ implies $(p_1, (1-y)p_2 + yq_2) \in G$, therefore $p_1 \in BR_1((1-y)p_2 + yq_2)$ and hence $f_1(y) \geq 0$ for all $0 \leq y \leq 1$. Similarly $f_2(x) \geq 0$ for all $0 \leq x \leq 1$. It follows that neither q_1 nor q_2 can spread. Assume that q_1 is not driven out. Notice that $(q_1, p_2) \in G$ since $f_1(0) = 0$. Moreover $f_1(y') = 0$ for some $y' > 0$ since q_1 is not driven out. Since $(p_1, (1-y')p_2 + y'q_2) \in G$ we obtain $(q_1, (1-y')p_2 + y'q_2) \in G$ and hence $(q_1, q_2) \in G$. We see that in each case the conditions for an ES set are satisfied. ■

7. Uniform Stability

The following definition strengthens neutral stability by requiring that the condition that no mutant spreads is independent of the entering mutant strategies and depends only on the size of the mutation.

Definition 7.1. *A pair of strategies (p_1, p_2) is called uniformly neutrally stable if for every $\varepsilon > 0$ there exists $0 < \varepsilon' \leq \varepsilon$ such that for any pair of mutant strategies (q_1, q_2) and for any initial size of the mutation less than ε' the proportion of both mutant strategies will stay below ε, i.e., $\max\{x(0), y(0)\} < \varepsilon'$ implies $\max\{x(t), y(t)\} < \varepsilon$ for all $t > 0$.*

We will now show for asymmetric mixed strategy models with bilinear payoff functions – i.e., for the models defined by a bimatrix game – that a strategy pair is neutrally stable if and only if it is uniformly neutrally stable, provided the evolutionary process satisfies an additional restriction. We have to require that the dynamic in each population reacts in the same order of magnitude to the absolute value of the deviation fitness of each population. More precisely:

There exists $\varepsilon > 0$ and $M > 0$ such that for all $p_1, q_1 \in \Delta(S_1)$, $p_2, q_2 \in \Delta(S_2)$ and $(x,y) \in U(\varepsilon)$,

$$\begin{array}{l} \text{i) } |f_2(x)| \leq |f_1(y)| \text{ implies } \frac{|g_2(x,y)|}{|g_1(x,y)|} < M \\ \text{ii) } |f_1(y)| \leq |f_2(x)| \text{ implies } \frac{|g_1(x,y)|}{|g_2(x,y)|} < M. \end{array} \tag{7.1}$$

Example 7.2. *Let E_1 and E_2 be Lipschitz-continuous and let $g_1(x,y) = -h(f_1(y), x, y)$ and $g_2(x,y) = -h(f_2(x), y, x)$. Then a sufficient condition for (7.1) is that i) $h(\cdot, x, y)$ non decreasing, ii) $h(-z,x,y) = -h(z,x,y)$ and iii) $|h(z,x,y)| \leq k\,|z|$ for some $k > 0$ when z is small. Especially it follows that both versions of the replicator dynamics given in Example 3.1 satisfy (7.1).*

		M	R
T	1 1	1 1	0 0
C	0 0	0 1	0 0
B	1 1	0 0	1 0

Figure 7.1: A bimatrix game

Example 7.3. In the following we will demonstrate for the game in Figure 7.1 how restrictive (7.1) is. Let $g_1(x,y) = -\operatorname{sgn}(f_1(y))\,|f_1(y)|^{\alpha}$ and $g_2(x,y) = -\operatorname{sgn}(f_2(x))\,|f_2(x)|^{\beta}$ for given α, $\beta \geq 1$. We will first show how (7.1) influences α and β. Let $(p_1,p_2) = (T, \;)$ and $(q_1,q_2) = ((1-\lambda)T + \lambda C, M)$ for $0 \leq \lambda \leq 1$. Then $f_1(y) = \lambda$ and $f_2(x) = -\lambda x$. Therefore (7.1) implies $\lambda^{\beta-\alpha} \cdot x^{\beta} < M$ for all $\lambda \in [0,1]$ and hence $\beta \geq \alpha$. Similar arguments concerning $(q_1,q_2) = ((1-\lambda)T + \lambda B, R)$ imply $\beta \leq \alpha$ and hence (7.1) requires $\alpha = \beta$. Moreover it is easy to show that if $\alpha = \beta$ then (7.1) is satisfied.

However, it can be shown that $\alpha = \beta$ is necessary and sufficient for $(T, \;)$ to be uniformly neutrally stable in the game in Figure 7.1.

Theorem 7.4. *Let E_1 and E_2 be bilinear and consider $(p_1,p_2) \in \Delta(S_1) \times \Delta(S_2)$. If (7.1) is satisfied then (p_1,p_2) is neutrally stable if and only if (p_1,p_2) is uniformly neutrally stable.*

In order to prove the above theorem we will need the following lemma.

Lemma 7.5. *Let E_1 and E_2 be bilinear. If (p_1,p_2) is neutrally stable then there exists $\alpha > 0$ and $k \geq 1$ such that for all (q_1, q_2) one of the following statements is true:*

i) $f_1(y) = f_2(x) = 0$ for $0 \leq x, y < \alpha$.

ii) $f_1(y) > 0$ and $f_2(x) \geq -f_1(y)$ for $kx \leq y \leq \alpha$.

iii) $f_2(x) > 0$ and $f_1(y) \geq -f_2(x)$ for $ky \leq x \leq \alpha$.

Proof. Let

$$\begin{aligned} &-k_{12} \\ = \quad &\min\left\{ \frac{E_2(q_1,p_2)-E_2(q_1,q_2)}{E_1(p_1,q_2)-E_1(q_1,q_2)} \,\middle|\, \begin{array}{c} (q_1,q_2)\in\{BR_1(p_2)\cap S_1\}\times\{BR_2(p_1)\cap S_2\}, \\ E_1(p_1,q_2)>E_1(q_1,q_2) \end{array}\right\}. \end{aligned}$$

We will now show that

$$-k_{12} = \inf\left\{ \frac{E_2(q_1,p_2)-E_2(q_1,q_2)}{E_1(p_1,q_2)-E_1(q_1,q_2)} \,\middle|\, \begin{array}{c} (q_1,q_2)\in BR_1(p_2)\times BR_2(p_1), \\ E_1(p_1,q_2)>E_1(q_1,q_2) \end{array}\right\}. \tag{7.2}$$

Fix $z_1, q_1 \in BR_1(p_2)$, $q_2 \in BR_2(p_1)$ and let $q_1(\lambda) = (1-\lambda)\,q_1 + \lambda z_1$ for $\lambda \in [0,1]$. We will write in the following $f_1(y,\lambda) := f_1(y, p_1, q_1(\lambda), p_2, q_2)$ and $f_2(x,\lambda) := f_2(x, p_1, q_1(\lambda), p_2, q_2)$. Assume that $f_1(y,\lambda') > 0$ for some $\lambda' \in [0,1]$. Since $E_1(\cdot,\cdot)$ is bilinear we may assume by symmetry that $f_1(y,0) > 0$. We will investigate the quotient

$$f(\lambda) := \frac{f_2(x,\lambda)}{f_1(y,\lambda)}$$

for all $\lambda \in [0,1]$ such that $f_1(y,\lambda) > 0$. If $f_1(y,1) > 0$ then $f(\lambda)$ is defined for all $\lambda \in [0,1]$. Since f is as a quotient of two linear functions a monotonic function it follows that $f(\lambda) \geq \min\{f(0), f(1)\}$. Assume that $f_1(y,1) \leq 0$. Let $\lambda^* \in (0,1]$ be such that $f_1(y,\lambda^*) = 0$. Then since (p_1,p_2) is neutrally stable and $q_1(\lambda) \in BR_1(p_2)$, the first part of theorem 6.3 implies that $f_2(x,\lambda^*) \geq 0$. With $f(\lambda) = \frac{a-\lambda b}{c-\lambda d}$ we obtain $c > 0$, $c - \lambda^* d = 0$, $d > 0$, $a - \lambda^* b \geq 0$ and therefore $ad - bc \geq 0$. It follows that $f(\lambda) \geq f(0)$ for all $\lambda \in [0,\lambda^*)$ and $f_1(y,\lambda) < 0$ for $\lambda > \lambda^*$. By induction it follows that $\inf\{\ldots, f_1(y,q_1,q_2) > 0\} = \inf\{\ldots, f_1(y,q_1,q_2) > 0, (q_1,q_2) \in \{BR_1(p_2)\cap S_1\} \times BR_2(p_1)\}$. Repeating now the above argument with fixed q_1 and varying q_2 completes the proof of the statement in 7.2. Reversing the roles of 1 and 2 we similarly obtain k_{21}.

In the following we will show that the above lemma holds for $k = \max\{k_{12}, k_{21}, 1\}$ and $\alpha = \theta/(4Mk^2)$. Fix (q_1,q_2). Let $M > 0$ be such that $|E_i(a,b) - E_i(c,d)| < M$ for all $a, c \in \Delta(S_1)$, $b, d \in \Delta(S_2)$, $i = 1,2$. Let $\theta > 0$ be such that $\theta < \min\{E_1(p_1,p_2) - E_1(e_1,p_2), E_2(p_1,p_2) - E_2(p_1,e_2)\}$ for any $e_1 \in S_1 \backslash BR_1(p_2)$ and $e_2 \in S_2 \backslash BR_2(p_1)$. Let $\lambda, \mu \in [0,1]$, $u_1 \in BR_1(p_2)$,

$v_1 \in \Delta(S_1 \backslash BR_1(p_2))$, $u_2 \in BR_2(p_1)$, $v_2 \in \Delta(S_2 \backslash BR_2(p_1))$ be such that $q_1 = (1-\lambda)\, u_1 + \lambda v_1$ and $q_2 = (1-\mu)\, u_2 + \mu v_2$. From the first part of theorem 6.3 it follows that either $d := E_1(p_1, u_2) - E_1(u_1, u_2) \geq 0$ or $E_2(u_1, p_2) - E_2(u_1, u_2) \geq 0$. W.l.o.g. assume that $d \geq 0$. Let $\mu^* := \frac{\lambda M + (1-\lambda) dk}{\lambda M + (1-\lambda) dk + 2kM}$. It follows that μ^* is bounded above by $1/3$ since $d \leq M$.

Consider now the case where $\mu > \mu^*$. With $E_2(u_1, p_2) - E_2(u_1, u_2) \geq -dk$ we obtain

$$f_2(x, q_1, q_2) \geq \mu\theta - (1-\mu)\, \lambda M x - (1-\mu)\,(1-\lambda)\, x dk$$

Therefore, $f_2(x, q_1, q_2) > 0$ since

$$\mu^* > \frac{x\,[\lambda M + (1-\lambda)\, dk]}{x\,[\lambda M + (1-\lambda)\, dk] + \theta}$$

Moreover,

$$\begin{aligned} f_1(y, q_1, q_2) + f_2(x, q_1, q_2) &\geq \lambda\,[\theta - (1-\mu)\, Mx] + \mu\,[\theta - (1-\lambda)\, My] \\ &\quad - (1-\mu)\,(1-\lambda)\, dkx \\ &\geq \mu^*\,(\theta - My) - (1-\mu^*)\, dkx \\ &= \frac{(\theta - My)\,(\lambda M + (1-\lambda)\, dk) - 2k^2 M dx}{\lambda M + (1-\lambda)\, dk + 2kM} \end{aligned}$$

Since $x, y < \alpha$ implies $x < (\theta - My)/4k^2 M$ it follows that $f_1(y, q_1, q_2) + f_2(x, q_1, q_2) \geq 0$. Especially this is true for $x \geq ky$.

Assume now that $\mu \leq \mu^*$. Then

$$f_1(y, q_1, q_2) + f_2(x, q_1, q_2) \geq (\lambda + \mu)\,\theta - (\mu y + \lambda x)\, M + (1-\mu)\,(1-\lambda)\, d\,(y - kx) \geq 0$$

when $\alpha \geq y \geq kx$ or $d = 0$. Moreover,

$$\begin{aligned} f_1(y, q_1, q_2) &\geq \lambda\theta - \mu\,(1-\lambda)\, My + (1-\mu)\,(1-\lambda)\, yd && (7.3) \\ &\geq \lambda\theta - \mu^*\,(1-\lambda)\, My + \frac{2}{3}\,(1-\lambda)\, yd && (7.4) \end{aligned}$$

Multiplying the above by $\lambda M + (1-\lambda)\, dk + 2kM$ the right hand side turns

$$\lambda^2 \theta M\,[1 + 2k] + \lambda\,(1-\lambda)\left[-M^2 y + \frac{2}{3} y d M\,(1 + 2k) + \theta\,(dk + 2kM)\right]$$

into

$$+ (1-\lambda)^2\, yd\left[-Mk + \frac{2}{3}\,(dk + 2kM)\right] > 0\,.$$

If $d > 0$ then $f_1(y, q_1, q_2) > 0$ since each term in the square brackets is strictly positive. Together with (7.3) it also follows that $f_1(y, q_1, q_2) > 0$ if $d = 0$ and either $\lambda > 0$ or $\mu > 0$. If on the other hand $\lambda = \mu = d = 0$ then $f_1(y, q_1, q_2) = 0$ and $f_2(x, q_1, q_2) = x\,[E_2(u_1, p_2) - E_2(u_1, u_2)] \geq 0$. ■

Given the above lemma we are now able to prove Theorem 7.4.

Proof. We will prove the claim based on the conclusion of Lemma 7.5 without otherwise using the bilinearity of E_1 and E_2. Let (p_1, p_2) be neutrally stable and let α and k be given from Lemma 7.5. Let $M > 0$ be given from (7.1). Let $\varepsilon \leq \alpha$. Let $\varepsilon' = \varepsilon'(\varepsilon) > 0$ be such that $\varepsilon > k\varepsilon' e^{\frac{M\varepsilon'}{(1-\varepsilon)}}$. Notice that $\varepsilon' < \varepsilon$. We will show that for any (q_1, q_2), trajectories starting in $U(\varepsilon')$ stay in $U(\varepsilon)$.

If part i) in Lemma 7.5 is true then all elements of $U(\varepsilon')$ are rest points and the claim is true.

Assume that part ii) is true. Then $\dot{x} < 0$. Similar to the proof of Theorem 6.1 we will assume a worst case and let y increase maximally. When $y < kx \leq \varepsilon$ assume that $\dot{x} = 0$ and $\dot{y} > 0$. When $kx \leq y \leq \varepsilon$, $f_2(x, q_1, q_2) \geq -f_1(y, q_1, q_2)$ and hence following (7.1) we will assume that $\frac{\dot{y}}{\dot{x}} = -\frac{My}{(1-\varepsilon)x}$. With $m = \frac{M}{1-\varepsilon}$ the trajectories of the solutions satisfy $\frac{\partial y}{\partial x} = -\frac{my}{x}$. Hence $y(x) = de^{-mx}$ with $d \in \mathbb{R}$. Starting at $(x, y) = (x, kx)$ with $x \leq \varepsilon'$ the trajectory converges to $(0, d(x))$ where $d(x) = kxe^{mx} \leq \varepsilon$ by definition of ε'. Let $U = \{(x, y) \mid x < \varepsilon', y < \varepsilon e^{-mx}\}$. Then $\varepsilon' < \varepsilon$, $U(\varepsilon') \subseteq U \subseteq U(\varepsilon)$ and trajectories starting in $U(\varepsilon')$ stay in U. Using symmetry completes the proof. ■

8. Relationship to the Asymmetric Contest

There are two possible ways of modelling evolutionary stability in asymmetric mixed strategy models. Modelling it as a conflict between two disjoint populations that each evolve separately is the objective of this paper. The alternative is to consider a conflict among homogenous individuals where roles in the asymmetric game are randomly assigned anew in each round. This translates the asymmetric population game into a symmetric population game where fitness equals expected fitness a priori to being assigned a role. This resulting game is called the truly asymmetric contest (see [12], [2]). Clearly there is no correct approach, the relevant model depends on the underlying situation considered. In the following we compare the results of these two approaches.

Following Balkenborg [1] and Balkenborg and Schlag [2] evolutionary stable sets of the truly asymmetric contest are equivalent to strict equilibrium sets, just as we have shown for the two population model in this paper (see Theorem 6.10). In the following we will show how neutrally stable strategies in the truly asymmetric contest relate to neutrally stable strategy pairs in the correponding asymmetric population game.

Throughout this section let $(\Delta(S_1), \Delta(S_2), E_1, E_2)$ be an asymmetric mixed strategy model. Consider the mixed strategy model as a conflict between two different types of individuals or players. In the following, individuals in a homogenous population will be allocated to a type in this asymmetric conflict. Each individual in the population will have a mixed strategy to use in the role of player one and a mixed strategy to use in the role of player two. A strategy in the asymmetric conflict, called a *behavioral strategy,* is hence a *pair* of strategies. When two individuals are matched, each playing a behavioral strategy, nature determines by a random draw with equal probabilities which of the two individuals will be in the role of the player one and which will be in the role of player two. The payoffs in this asymmetric conflict is then defined as the ex ante expected payoffs. The resulting conflict is called the *truly asymmetric contest*, denoted by $((\Delta(S_1), \Delta(S_2)), (E_1, E_2))$. For a formal definition see [2]. In this context we will say that the symmetric population game (S^c, E^c) results from the truly asymmetric contest $(\Delta(S_1), \Delta(S_2), E_1, E_2)$ if $S^c = \Delta(S_1) \times \Delta(S_2)$ and the payoff function $E^c : S^c \times S^c \to \mathbb{R}$ is defined by

$$E^c((p_1, p_2), (z_1, z_2)) = \frac{1}{2} E_1(p_1, z_2) + \frac{1}{2} E_2(z_1, p_2)$$

for (p_1, q_1), $(z_1, z_2) \in S^c$.

Neutral stability in the truly asymmetric contest is called *direct* neutral stability (see Selten [12]). Balkenborg and Schlag [2] develop the concept of direct uniformly neutral stability for the truly asymmetric contest. Among others they show for bilinear payoff functions that the concepts of direct neutral stability and direct uniformly neutral stability are equivalent. In the following we refer to the companion paper (Balkenborg and Schlag [2]) for definitions and motivation and only give here the following characterization.

Lemma 8.1. *Let E_1 and E_2 be analytic. The strategy pair $(p_1, p_2) \in S^c$ is direct uniformly neutrally stable if and only if there exists $\varepsilon > 0$ such that for all x, $y < \varepsilon$ and $(q_1, q_2) \in S^c$,*

$$x f_1(y) + y f_2(x) \geq 0 \,. \tag{8.1}$$

Theorem 8.2. *Let E_1 and E_2 be analytic and let (7.1) hold. If (p_1, p_2) is direct uniformly neutrally stable in the truly asymmetric contest then (p_1, p_2) is asymmetric uniformly neutrally stable in the asymmetric mixed strategy model, the converse is not true.*

Proof. Let (p_1, p_2) be direct uniformly neutrally stable with $\varepsilon > 0$ given by Lemma 8.1. We will first show that one of the following statements is true:

$$\begin{array}{l} \text{i) } f_1(y) = f_2(x) = 0 \text{ for all } x, y \leq \varepsilon \\ \text{ii) } f_1(y) > 0 \text{ and } f_2(x) \geq -f_1(y) \text{ for all } x \leq y \leq \varepsilon \\ \text{iii) } f_2(x) > 0 \text{ and } f_1(y) \geq -f_2(x) \text{ for all } y \leq x \leq \varepsilon \end{array} \tag{8.2}$$

Assume that $f_1(y') < 0$ for some $y' < \varepsilon$. Then $f_2(x) > 0$ for all $0 < x < \varepsilon$. Moreover setting $x = y$ in (8.1) establishes case iii) of (8.2). If $f_1(y') = 0$ for some $y' < \varepsilon$ then $f_2(x) \geq 0$ for all $0 < x < \varepsilon$. If case iii) does not hold then $f_2(x') = 0$ for some $x' < \varepsilon$. Hence $f_1(y) \geq 0$ for all $0 < y < \varepsilon$ and case i) is established. Finally, if $f_1(y) > 0$ for all $y < \varepsilon$ the same argument as above establishes case ii) and hence (8.2) is proven.

ow let $\varepsilon'(\cdot)$ be defined as in the proof of Theorem 6.3 given $k = 1$. In the proof of Theorem 6.3 it was shown that trajectories starting in $U(\varepsilon'(\beta'))$ stay in $U(\beta')$ for all $\beta' \leq \varepsilon$ and hence (p_1, p_2) is asymmetric uniformly neutrally stable. ■

The following example establishes that the converse is not true.

Example 8.3. *(Neutral stability in an asymmetric mixed strategy model does not necessarily imply neutral stability in the corresponding truly asymmetric contest.) Consider the* 2×2 *bimatrix game in Figure 8.1. Using Theorem 9.6 it is easy to verify that* (T, L) *is an (asymmetric) uniformly neutrally stable strategy pair of the asymmetric mixed strategy model defined by this game. Let* $(p_1, p_2) = (T, L)$ *and* $(q_1, q_2) = (B, R)$*. Then* $f_1(y) = y$*,* $f_2(x) = -2x$ *and hence* $xf_1(y) + yf_2(x) < 0$ *if* $xy \neq 0$*. Therefore* (T, L) *is not a direct neutrally stable strategy of the corresponding truly asymmetric contest.*

	L	*R*
T	*1 1*	*1 1*
B	*1 0*	*0 2*

Figure 8.1: A bimatrix game

The above example will play an important role in the relationship between the replicator dynamics in the truly asymmetric contest and the two population setting. Balkenborg [1] shows that stability in the truly asymmetric contest implies stability in the two population setting. This example shows that the converse is not true.

The above example also displays why the concept of unbeatable (see Bomze and Weibull [4]) is inappropriate in the two population setting. Although (T, L) is uniformly neutrally stable, the proportion of R increases strictly over time given the pair of mutant strategies (B, R).

9. Uniformly Evolutionarily Stable Sets

In this section we consider evolutionarily stable sets where the condition for how small the mutation must be such that the "not driven out" and the "does not spread" conditions hold uniformly on the entire set. How small the mutation must be such that the "not driven out" condition of an ES set holds will be characterized by the concept of an invasion barrier.

Definition 9.1. *Let* G *be an evolutionarily stable set, let* (p_1,p_2) *be contained in* G *and let* (q_1,q_2) *be a pair of mutant strategies. We define the invasion barrier*

$\varepsilon(p_1, p_2, q_1, q_2)$ of (p_1, p_2) against (q_1, q_2) by the supremum of all $0 \leq \varepsilon \leq 1$ such that if $\max\{x(0), y(0)\} < \varepsilon$ then either q_1 or q_2 is driven out.

otice that the invasion barrier $\varepsilon(p_1, p_2, q_1, q_2)$ is strictly positive because (p_1, p_2) is contained in an ES set.

Definition 9.2. *An ES set G has a uniform invasion barrier if there exists $\varepsilon > 0$ such that $\varepsilon(p_1, p_2, q_1, q_2) \geq \varepsilon$ for all $(p_1, p_2) \in G$ and (q_1, q_2). In this case we define the uniform invasion barrier $\varepsilon(G)$ by*

$$\varepsilon(G) = \inf\{\varepsilon(p_1, p_2, q_1, q_2) \mid (p_1, p_2) \in G, (q_1, q_2)\} .$$

The fact that the condition of 'can spread' in the definition of an ES set is independent of the incumbent and of the mutant strategies is formalized in the following definition.

Definition 9.3. *An ES set is called uniformly stable if for every $\varepsilon > 0$ there exists $0 < \varepsilon' \leq \varepsilon$ such that for any pair of incumbant strategies $(p_1, p_2) \in G$ and any pair of mutant strategies (q_1, q_2), if $\max\{x(0), y(0)\} < \varepsilon'$ then $\max\{x(t), y(t)\} < \varepsilon$ for all $t > 0$.*

For the case of singleton sets the above definition is equivalent to the definition of uniform neutral stability.

We find that both the most desirable and the most plausible condition for claiming the evolutionary stability of a given set is for this set to be an ES set that is uniformly stable with a uniform invasion barrier. This leads to the following definition.

Definition 9.4. *The set G is called a uniformly evolutionarily stable set if G is an ES set that is uniformly stable and that has a uniform invasion barrier.*

In the following we will give a strengthening of Theorem 6.10 for asymmetric evolutionarily stable strategy pairs.

Proposition 9.5. *$\{(p_1, p_2)\}$ is a uniformly evolutionarily stable set if and only if (p_1, p_2) is a strict equilibrium.*

Proof. Due to theorem 6.10 all that is needed to prove is "if" statement. Since (p_1, p_2) is strict there exists $\varepsilon > 0$ such that $f_2(x) > 0$ and $f_1(y) > 0$ for all $x, y \in (0, \varepsilon)$. Therefore (p_1, p_2) is uniformly neutrally stable and there is a uniform invasion barrier. ■

ext we strengthen Theorem 6.10, showing that in bimatrix games ES sets are uniform ES sets. This result is analogous to a result in the symmetric setting where the conflict occurs within the same population (see Balkenborg and Schlag [2]).

Theorem 9.6. *Let E_1 and E_2 be bilinear, let $G \subseteq \Delta(S_1) \times \Delta(S_2)$ and let (7.1) hold. Then the following statements are equivalent:*

i) G is a strict equilibrium set.

ii) G is an (asymmetric) evolutionarily stable set.

iii) G is a uniformly evolutionarily stable set.

Proof. Due to Theorem 6.10 i) is equivalent to ii). Moreover, by definition iii) implies ii). In the following we will show that i) implies ii). Let G be a SE set. Theorem 3.7 in Balkenborg and Schlag [2] states that each element of G is direct uniformly neutrally stable in the asymmetric contest and that there exists $\alpha > 0$ such $\varepsilon = \alpha$ satisfies the conditions of Definition 8.1 for each $(p_1, p_2) \in G$.

From the proof of Theorem 8.2 it follows with $\varepsilon = \alpha$ that for all $(p_1, p_2) \in G$ and (q_1, q_2) (at least) one of the statements in 8.2 is true. Let $\varepsilon'(\cdot)$ be defined as in the proof of Theorem 6.3 given $k = 1$. Then $\varepsilon'(\cdot)$ is independent of $(p_1, p_2) \in G$ and (q_1, q_2). Moreover in the proof of Theorem 6.3 it was shown that trajectories starting in $U(\varepsilon'(\beta'))$ stay in $U(\beta')$ for all $\beta' \leq \alpha$. Therefore G is uniformly stable. We will now show for given $(p_1, p_2) \in G$ and (q_1, q_2) that the invasion barrier of (p_1, p_2) against the entrant (q_1, q_2) is at least $\varepsilon'(\beta)$. Consider $x(0), y(0) \in (0, \varepsilon'(\alpha))$.

If part i) of 8.2 is true then no mutant is driven out and as in the proof of Theorem 6.10 it follows that $(q_1, q_2) \in G$.

Assume that part ii) is true. Then $\dot{x} < 0$ and q_1 is driven out. Moreover $\alpha = \beta$ and $k = 1$ satisfy 8.2 and hence trajectories starting in $U(\varepsilon'(\beta\ ))$ stay in $U(\beta)$. Especially $y(t) < \beta\ < 1$ for all t. Assume that $f_2(0) > 0$. Then there exists $\sigma > 0$ such that $f_2(x) > 0$ for all $x < \sigma$. Since $\dot{x} < 0$, eventually $x(t) < \sigma$ (and $y(t) < \beta < 1$) and hence q_2 is driven out. On the other hand, if $f_2(0) = 0$ then $(p_1, q_2) \in G$. This completes the claim given that part ii) holds. Using symmetry the same arguments hold when part iii) is true and the proof is complete. ■

References

[1] Dieter Balkenborg. Strictness and evolutionary stability. Disc. Paper 52, The Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, July 1994.

[2] Dieter Balkenborg and Karl H. Schlag. On the interpretation of evolutionarily stable sets. SFB Disc. Paper B-313, University of Bonn, April 1995.

[3] Immanuel M. Bomze and Benedikt M. Pöetscher. *Game theoretical foundations of evolutionary stability.* Springer Verlag, Berlin, 1989.

[4] Immanuel M. Bomze and Jörgen W. Weibull. Does neutral stability imply Lyapunov stability? Technical Report 93-13R, University of Vienna, ov 1994.

[5] Ross Cressman. *The stability concept of evolutionary game theory.* Springer Verlag, Berlin, 1992.

[6] Ross Cressman and A. T. Dash. Strong stability and evolutionarily stable strategies with two types of players. *Journal of Mathematical Biology*, 30:89 – 99, 1991.

[7] Peter Hammerstein and Reinhard Selten. Game theory and evolutionary biology. In Robert J. Aumann and Sergiu Hart, editors, *Handbook of Game Theory*, volume II. orth Holland, Amsterdam, 1994.

[8] John Maynard Smith. The theory of games and the evolution of animal conflicts. *Journal of Theoretical Biology*, 47:209 – 221, 1974.

[9] John Maynard Smith. *Evolution and the theory of games.* Cambridge University Press, Cambridge, 1982.

[10] John Maynard Smith and G.R. Price. The logic of animal conflict. *Nature*, 246:15 – 18, 1973.

[11] Larry Samuelson and Jianbo Zhang. Evolutionary stability in asymmetric games. *J. Econ. Theory*, 57:363 – 391, 1992.

[12] Reinhard Selten. A note on evolutionarily stable strategies in asymmetric animal conflicts. *J. Theor. Biol.*, 84:93 – 101, 1980.

[13] Jeroen M. Swinkels. Evolutionary stability with equilibrium entrants. *J. Econ. Theory*, 57:306 – 332, 1992.

[14] Peter. D. Taylor. Evolutionary stable strategies with two types of players. *J. Appl. Prob.*, 16:76 – 83, 1979.

[15] B. Thomas. On evolutionarily stable sets. *J. Math. Biology*, 22:105 – 115, 1985.

[16] G. T. Vickers and C. Cannings. On the definition of an evolutionarily stable strategy. *J. theor. Biol.*, 129:349 – 353, 1987.