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Uniform Consistency of Modified Kernel Estimators in Nonparametric Multivariate VARCH-Models

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Abstract:
This paper shows the uniform convergence in probability of modified kernel estimators towards the Baire functions representing the conditional variances and contemporaneous conditional covariances provided the data generating process is given by a strictly stationary solution of a nonparametric multivariate VARCH(q,m)-model.

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Chapter 1

Introduction

In a recent paper (Cron(1995)) we showed the uniform consistency of a modified kernel estimator provided the data generating process is given by a strictly stationary solution of the univariate parametric ARCH(q)-model introduced in the econometric literature by Engle(1982). There the decisive property of the underlying data generating process to guarantee the consistency of the modified kernel estimator was its exponential ν-stability in L_1. Therefore it is natural to look for more general processes to fulfil this condition in order to exploit the full flexibility of kernel estimators. In this paper a class of multivariate nonparametric models is studied. Our main attention is directed to the solution theory of such models.

We consider a system of m difference equations

$$\begin{aligned}
\eta_1(t) &= \epsilon_1(t)h_1^{\frac{1}{2}}(\eta_1^2(t-1),\ldots,\eta_1^2(t-q),\eta_2^2(t-1),\ldots,\eta_m^2(t-q)) \quad (1.1)\\
\vdots \quad &\vdots \quad \vdots\\
\eta_m(t) &= \epsilon_m(t)h_m^{\frac{1}{2}}(\eta_1^2(t-1),\ldots,\eta_1^2(t-q),\eta_2^2(t-1),\ldots,\eta_m^2(t-q)),
\end{aligned}$$

$t \in Z$, where $\{\epsilon(t)\}_{t\in Z} = \{\epsilon_1(t),\ldots,\epsilon_m(t)\}_{t\in Z}$ denotes a sequence of i. i. d. random vectors with mean zero and finite variances and $h_i : I\!R^{qm} \to I\!R$ any measurable positive function. While the expectation of any vector solving the **nonparametric VARCH(q,m)-model** above conditioned on past values of the solution process is zero, its contemporaneous conditional covariances are given by

$$E\{\eta_i(t)\eta_j(t) \mid \eta_1(t-1),\ldots,\eta_m(t-q)\}$$

$= E\{\epsilon_i(t)\epsilon_j(t)\}h_i^{\frac{1}{2}}(\eta_1^2(t-1),\ldots,\eta_m^2(t-q))h_j^{\frac{1}{2}}(\eta_1^2(t-1),\ldots,\eta_m^2(t-q)) \quad a.\ s.\ ,$
$i,j = 1,\ldots,m$, which in the case of $i = j$ can be simplified to

$$E\{\eta_i^2(t) \mid \eta_1(t-1),\ldots,\eta_m(t-q)\} = h_i(\eta_1^2(t-1),\ldots,\eta_m^2(t-q)) \quad a.\ s.\ ,$$

if we assume the normalization $E\{\epsilon_i^2(t)\} = 1, i = 1,\ldots,m$, as will be done hereafter.

We can imagine that system 1.1 describes the price changes of m assets. While these changes are serially uncorrelated both conditional on past values and unconditional, their conditional variances and mutual conditional covariances depend on past price changes. Therefore we are interested in identifying the Baire functions

$$m_{i,j}(x) = E\{\eta_i(t)\eta_j(t) \mid \eta_1(t-1) = x_1,\ldots,\eta_m(t-q) = x_{qm}\},$$

$i,j = 1,\ldots,m$, which represent these conditional variances and covariances, e. g. ,in order to evaluate the risk of a portfolio consisting of these m assets based on observed past price changes of these assets [1]. For this purpose we make use of a modified kernel estimator.

Let $y_{i,j}(t) := \eta_i(t)\eta_j(t)$, $\hat{x}(t) := (\eta_1^2(t-1),\ldots,\eta_m^2(t-q))'$, and $x(t) := (\eta_1(t-1),\ldots,\eta_m(t-q))', t = 1,\ldots,T$ denote the data set and $\phi : \mathbb{R}^{qm} \to \mathbb{R}$ the truncation function

$$\phi(u) := \begin{cases} 1 & : \ \|u\| \leq C \\ \frac{C}{\|u\|} & : \ \|u\| > C \end{cases} \tag{1.2}$$

where C is a positive constant and $\|\cdot\|$ denotes the Euclidean norm. We define the modified kernel estimator by

$$\hat{m}_{i,j,T}(x) \ := \ \frac{\hat{g}_{i,j,T}(x)}{\hat{f}_T(x)}, \tag{1.3}$$

[1] From a theoretical point of view one could equally well consider the functions $E\{\eta_i(t)\eta_j(t) \mid \eta_1^2(t-1) = x_1,\ldots,\eta_m^2(t-q) = x_{qm}\}$ instead of the symmetric functions $m_{i,j}(x)$, provided the data generating process is indeed given by 1.1. In practice, however, one would give away information when replacing the conditioning σ-field $\sigma(\{\eta_i(t-k)\}_{i=1,\ldots,m,k\in\mathbb{N}_+})$ by its subset $\sigma(\{\eta_i^2(t-k)\}_{i=1,\ldots,m,k\in\mathbb{N}_+})$ when the (unknown) real data generating process is given e. g. by a solution of the asymmetric model we deal with in chapter 5.

$$
\begin{aligned}
\hat{g}_{i,j,T}(x) &:= \frac{1}{T}\sum_{t=1}^{T}\gamma_T^{-qm}\phi(\hat{x}(t))y_{i,j}(t)K\left(\frac{x-x(t)}{\gamma_T}\right), \\
\hat{f}_T(x) &:= \frac{1}{T}\sum_{t=1}^{T}\gamma_T^{-qm}K\left(\frac{x-x(t)}{\gamma_T}\right),
\end{aligned}
$$

with the usual properties of the kernel $K(u)$ and the bandwidth sequence $\{\gamma_T\}_{T\in\mathbb{N}}$.

In the following chapter we present a solution theory for system 1.1 under certain conditions on the functions h_i. Chapter 3 provides consistency results for the kernel estimator 1.3. In the final chapter we generalize model 1.1 replacing the squared lagged values which serve as arguments for the functions h_i by ordinary lagged values allowing in this way for asymmetric behaviour.

Chapter 2

The nonparametric VARCH(q,m)-model

2.1 Basic assumptions

The following assumptions specify the class of models 1.1 we deal with in this paper:

(A. 1) *The sequence of random vectors* $\{\epsilon_1(t), \ldots, \epsilon_m(t)\}_{t \in Z}$ *is i. i. d. with mean zero and variances* $E\{\epsilon_i^2\} = 1$ *for* $i = 1, \ldots, m$. *Moreover,*
$E\{\epsilon_i^4\} < +\infty$ *for* $i = 1, \ldots, m$.

(A. 2) *The distribution of* $\epsilon_i, i = 1, \ldots, m$, *is absolutely continuous with respect to Lebesgue measure. The corresponding density* $f_{\epsilon_i}(u)$ *is twice continuously differentiable and bounded on* $\mathbb{R}$.

(A. 3) *The nonnegative functions* $h_i : \mathbb{R}_+^{qm} \to \mathbb{R}, i = 1, \ldots, m$, *have unbounded support and are continuously differentiable on* $\mathbb{R}^{qm}$. *The derivatives are bounded, i. e.*

$$sup_{u \in \mathbb{R}^{qm}} \left| \frac{\partial h_i(u)}{\partial u_k} \right| =: \alpha(i, k) < +\infty$$

for $i = 1, \ldots, m$ *and* $k = 1, \ldots, qm$.

(A. 4) *Define the $q \times q$-matrix*

$$A(i,j) := \begin{pmatrix} \alpha(i,(j-1)q+1) & \dots & \dots & \alpha(i,jq) \\ & & \delta_{ij} I_{q-1} & 0 \\ & & & \end{pmatrix}$$

$\forall i,j = 1,\dots,q$ and the $qm \times qm$-matrix

$$A := \begin{pmatrix} A(1,1) & A(1,2) & \dots & A(1,m) \\ A(2,1) & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ A(m,1) & \dots & \dots & A(m,m) \end{pmatrix}.$$

The spectral radius $\rho := \rho(A)$ is less than unity.

2.2 Solution theory

Under the assumptions made above the following results hold:

Lemma 1
System 1.1 has a strictly stationary, square-integrable solution.

Proof:
Define

$$\Pi_i(t) := \begin{pmatrix} \epsilon_i^2(t) & 0 \\ 0 & I_{q-1} \end{pmatrix}, \Pi(t) := \begin{pmatrix} \Pi_1(t) & 0 & \dots & \ddots & 0 \\ 0 & \Pi_2(t) & 0 & \ddots & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \dots & \dots & \dots & \ddots & \Pi_m(t) \end{pmatrix},$$

$$Z_i(t) := \begin{pmatrix} \eta_i^2(t) \\ \vdots \\ \eta_i^2(t-q+1) \end{pmatrix}, X(t) := \begin{pmatrix} Z_1(t) \\ \vdots \\ Z_m(t) \end{pmatrix}$$

and functions

$$H_i : I\!R^{qm} \to I\!R^q,$$

$$H_i(u) = \begin{pmatrix} h_i(u_1, \ldots, u_{qm}) \\ u_{(i-1)q+1} \\ \vdots \\ u_{iq-1} \end{pmatrix}$$

and

$$H : I\!R^{qm} \to I\!R^{qm},$$
$$H(u) = \begin{pmatrix} H_1(u) \\ \vdots \\ H_m(u) \end{pmatrix}.$$

Then we obtain an equivalent representation of system 1.1 squared:

$$X(t) = \Pi(t) H(X(t-1)), \quad t \in Z. \tag{2.1}$$

We solve this system first: Let $I\!R^{qm \times qm}$ be the space of $qm \times qm$ real-valued matrices and $\otimes_\tau I\!R^{qm \times qm}$ its τ-fold product space. We define by recursion for any $\tau \in I\!N_+$

$$H^{(\tau)} : \otimes_\tau I\!R^{qm \times qm} \times I\!R^{qm} \to I\!R^{qm},$$

$$\begin{array}{rcl} H^{(1)}(P_1; X) & := & H(P_1 H(X)) \\ H^{(2)}(P_1, P_2; X) & := & H(P_1 H^{(1)}(P_2; X)) \\ \vdots & \vdots & \vdots \\ H^{(\tau)}(P_1, \ldots, P_\tau; X) & := & H(P_1 H^{(\tau-1)}(P_2, \ldots, P_\tau; X)). \end{array}$$

Note that

$$X(t) = \Pi(t) H^{(\tau)}(\Pi(t-1), \ldots, \Pi(t-\tau); X(t-\tau-1)). \tag{2.2}$$

For notational convenience we introduce the symbol

$$| u | := \begin{pmatrix} | u_1 | \\ \vdots \\ | u_n | \end{pmatrix}$$

for any vector $u \in I\!R^n$ and state some simple inequalities which hold componentwise:

(1) If A is a nonnegative $n \times n$ matrix, $\mid Au \mid \leq A \mid u \mid$.
(2) $\mid u - v \mid \leq \mid u - z \mid + \mid z - v \mid$.
(3) $\mid u \mid - \mid v \mid \leq \mid u - v \mid$.

Assumption (A.3) together with Taylor's formula implies that

$$\mid H(u) - H(v) \mid \leq A \mid u - v \mid, \tag{2.3}$$

where A is defined in (A.4), and since $H(0)$ is nonnegative, we obtain in particular from (3)

$$\mid H(u) \mid \leq H(0) + A \mid u \mid . \tag{2.4}$$

Therefore,with

$$\begin{aligned} X^{(\tau)}(t) \quad &:= \Pi(t) H^{(\tau)}(\Pi(t-1), \ldots, \Pi(t-\tau); 0), \\ X^{(\tau+n)}(t) \quad &:= \Pi(t) H^{(\tau+n)}(\Pi(t-1), \ldots, \Pi(t-(\tau+n)); 0), \end{aligned}$$

$t \in Z, \tau \in I\!N_+$ and $\Gamma(t) := A\Pi(t)$ we have

$$\begin{aligned} & \left| X^{(\tau)}(t) - X^{(\tau+n)}(t) \right| \\ \leq \;& \Pi(t)\Gamma(t-1) \left| H^{(\tau-1)}(\Pi(t-2), \ldots, \Pi(t-\tau); 0) \right. \\ & \left. - \, H^{(\tau+n-1)}(\Pi(t-2), \ldots, \Pi(t-(\tau+n)); 0) \right|, \end{aligned}$$

since $\Pi(t)$ and $\Pi(t-1)$ are nonnegative. By recursion, we obtain

$$\begin{aligned} \left| X^{(\tau)}(t) - X^{(\tau+n)}(t) \right| \;\leq\; & \Pi(t) \prod_{d=1}^{\tau} \Gamma(t-d) \\ & \times \left\{ \sum_{s=\tau+1}^{\tau+n} \Gamma(t-(\tau+1)) \ldots \Gamma(t-s) H(0) \right\}. \end{aligned}$$

Markoff's inequality and the mutual independence of the $\Pi(t)$ imply that

$$\begin{aligned} & P\left(\sup_{n\geq 1} \left\| X^{(\tau)}(t) - X^{(\tau+n)}(t) \right\| > \epsilon \right) \\ \leq \;& P\left(\sup_{n\geq 1} \left\| \Pi(t) \prod_{d=1}^{\tau} \Gamma(t-d) \left\{ \sum_{s=\tau+1}^{\tau+n} \Gamma(t-(\tau+1)) \ldots \Gamma(t-s) H(0) \right\} \right\| > \epsilon \right) \end{aligned}$$

$$
\begin{aligned}
&\le P\left(\sup_{n\ge 1}\left\|\Pi(t)\prod_{d=1}^{\tau}\Gamma(t-d)\right\|\left\|\sum_{s=\tau+1}^{\tau+n}\Gamma(t-(\tau+1))\ldots\Gamma(t-s)H(0)\right\|>\epsilon\right)\\
&\le \frac{1}{\epsilon}E\left\{\left\|\Pi(t)\prod_{d=1}^{\tau}\Gamma(t-d)\right\|\right\}E\left\{\sup_{n\ge 1}\left\|\sum_{s=\tau+1}^{\tau+n}\Gamma(t-(\tau+1))\ldots\Gamma(t-s)H(0)\right\|\right\},
\end{aligned}
$$

where $\|\cdot\|$ denotes the infinity-norm. Moreover, we have

$$
\begin{aligned}
& E\left\{\left\|\Pi(t)\prod_{d=1}^{\tau}\Gamma(t-d)\right\|\right\}\\
\le\ & E\left\{\|\Pi(t)\|\right\}E\left\{\left\|\prod_{d=1}^{\tau}\Gamma(t-d)\right\|\right\}\\
\le\ & E\left\{1+\sum_{i=1}^{m}\epsilon_i^2(t)\right\}E\left\{\sum_{k,k^*}\left(\prod_{d=1}^{\tau}\Gamma(t-d)\right)_{(k,k^*)}\right\}\\
\le\ & (m+1)\sum_{k,k^*}(A^\tau)_{(k,k^*)}\\
=\ & O(\rho^\tau)
\end{aligned}
$$

and,if ρ is less than unity,Fatou's lemma together with the strict stationarity of $\{\Pi(t)\}_{t\in Z}$ yields

$$
\begin{aligned}
& E\left\{\sup_{n\ge 1}\left\|\sum_{s=\tau+1}^{\tau+n}\Gamma(t-(\tau+1))\ldots\Gamma(t-s)H(0)\right\|\right\}\\
\le\ & E\left\{\sum_{s=\tau+1}^{\infty}\|\Gamma(t-(\tau+1))\ldots\Gamma(t-s)H(0)\|\right\}\\
\le\ & \|H(0)\|\sum_{s=\tau+1}^{\infty}\left(E\left\|\prod_{d=\tau+1}^{s}\Gamma(t-d)\right\|\right)\\
=\ & \|H(0)\|\sum_{s=1}^{\infty}\left(E\left\|\prod_{d=1}^{s}\Gamma(t-d)\right\|\right)\\
<\ & +\infty.
\end{aligned}
$$

We conclude that

$$
\lim_{\tau\to\infty}P\left(\sup_{n\ge 1}\|X^{(\tau)}(t)-X^{(\tau+n)}(t)\|>\epsilon\right)=0
$$

for all $\epsilon > 0$. Therefore $X^{(\tau)}(t)$ is a Cauchy-sequence a. s. (cf. GAENSSLER/STUTE(1977)) and hence

$$\begin{aligned} X(t) &:= \lim_{\tau \to \infty} X^{(\tau)}(t) \quad a.\ s. \\ &= \Pi(t) H^{(\infty)}(\Pi(t-1), \Pi(t-2), \ldots) \\ &= F(\Pi(t), \Pi(t-1), \ldots) \end{aligned}$$

exists a. s. . It is a nonnegative solution of 2.1, since

$$X^{(\tau)}(t) = \Pi(t) H(X^{(\tau-1)}(t-1)),$$

whence

$$\lim_{\tau \to \infty} X^{(\tau)}(t) = \Pi(t) H(\lim_{\tau \to \infty} X^{(\tau)}(t-1)).$$

Since $\{\Pi(t)\}_{t \in Z}$ is a strictly stationary process and F is a measurable and time-independent function, $\{X(t)\}_{t \in Z}$ is strictly stationary. Moreover, repeated application of 2.4 yields

$$X(t) \leq \Pi(t) \lim_{\tau \to \infty} \sum_{d=1}^{\tau} \Gamma(t-1) \ldots \Gamma(t-d) H(0) + \Pi(t) H(0) \quad a.\ s.\ .$$

Therefore the integrability of $X(t)$ follows from the monotone convergence theorem.
Finally, define

$$1(t) := \begin{pmatrix} 1_{\epsilon_1(t) \geq 0} & 0 & \ldots & 0 \\ 0 & 1_{\epsilon_2(t) \geq 0} & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 1_{\epsilon_m(t) \geq 0} \end{pmatrix}.$$

Then,with square roots taken positive,

$$\begin{pmatrix} \eta_1(t) \\ \vdots \\ \eta_m(t) \end{pmatrix} := 1(t) \begin{pmatrix} \sqrt{X_{(1)}(t)} \\ \sqrt{X_{(q+1)}(t)} \\ \vdots \\ \sqrt{X_{(m-1)q+1}(t)} \end{pmatrix} - (I - 1(t)) \begin{pmatrix} \sqrt{X_{(1)}(t)} \\ \sqrt{X_{(q+1)}(t)} \\ \vdots \\ \sqrt{X_{(m-1)q+1}(t)} \end{pmatrix},$$

where $X_{(i)}(t)$ denotes the i-th component of $X(t)$, is a strictly stationary,square-integrable solution of 1.1. ∎

Lemma 2
Let $\mathcal{F}_{t,\tau}$ denote the σ-field $\sigma(\epsilon(t), \epsilon(t-1), \ldots, \epsilon(t-\tau+1))$. Then for $0 < \kappa < \frac{1}{2}$ and for all $t \in Z$

$$\begin{array}{llcl} (i) & E\left|E\{\eta_i(t)\eta_j(t) \mid \mathcal{F}_{t,\tau}\} - \eta_i(t)\eta_j(t)\right| & = & o(e^{-\kappa\tau}) \\ (ii) & E\left|E\{\eta_i(t) \mid \mathcal{F}_{t,\tau}\} - \eta_i(t)\right| & = & o(e^{-\kappa\tau}) \end{array}$$

$\forall i, j = 1, \ldots, m$.

Proof:
We define the $\mathcal{F}_{t,\tau}$-measurable random vector

$$X(t,\tau) \;:=\; \Pi(t)H^{(\tau-1)}(\Pi(t-1), \ldots, \Pi(t-\tau+1); E\{X(t-\tau)\})$$

and note that

$$X(t) \;=\; \Pi(t)H^{(\tau-1)}(\Pi(t-1), \ldots, \Pi(t-\tau+1); X(t-\tau)).$$

From 2.3 we have

$$|X(t) - X(t,\tau)| \;\leq\; \Pi(t)\prod_{d=1}^{\tau-1}\Gamma(t-d)A\left|X(t-\tau) - E\{X(t-\tau)\}\right|.$$

Hence,

$$\begin{aligned} E\left|X(t) - X(t,\tau)\right| &\leq A^\tau E\left|X(t-\tau) - E\{X(t-\tau)\}\right| \\ &\leq 2A^\tau E\{X(0)\} \\ &= O(\rho^\tau)\iota. \end{aligned}$$

Denote $\eta_i^2(t,\tau) = E\{\eta_i^2 \mid \mathcal{F}_{t,\tau}\}$. Then we have for $i, j = 1, \ldots, m$,

$$\begin{aligned} & E\left|E\{\eta_i(t)\eta_j(t) \mid \mathcal{F}_{t,\tau}\} - \eta_i(t)\eta_j(t)\right| \\ = \;& E\Big|(1_{\epsilon_i(t)\geq 0} - 1_{\epsilon_i(t)<0})(1_{\epsilon_j(t)\geq 0} - 1_{\epsilon_j(t)<0})\sqrt{\eta_i^2(t)}\sqrt{\eta_j^2(t)} \\ & - (1_{\epsilon_i(t)\geq 0} - 1_{\epsilon_i(t)<0})(1_{\epsilon_j(t)\geq 0} - 1_{\epsilon_j(t)<0})E\{\sqrt{\eta_i^2(t)}\sqrt{\eta_j^2(t)} \mid \mathcal{F}_{t,\tau}\}\Big| \\ \leq \;& E\left|\sqrt{\eta_i^2(t)}\sqrt{\eta_j^2(t)} - E\{\sqrt{\eta_i^2(t)}\sqrt{\eta_j^2(t)} \mid \mathcal{F}_{t,\tau}\}\right| \\ \leq \;& 2E\left|\sqrt{\eta_i^2(t)}\sqrt{\eta_j^2(t)} - \sqrt{\eta_i^2(t,\tau)}\sqrt{\eta_j^2(t,\tau)}\right|, \end{aligned}$$

where the last inequality follows from the fact that for any integrable $\mathcal{F}$-measurable φ

$$\begin{aligned} E\left|\xi - E\{\xi \mid \mathcal{F}\}\right| &\leq E\left|\xi-\varphi\right| + E\left|\varphi - E\{\xi \mid \mathcal{F}\}\right| \\ &= E\left|\xi-\varphi\right| + E\left|E\{\varphi-\xi \mid \mathcal{F}\}\right| \\ &\leq 2E\left|\xi-\varphi\right|. \end{aligned}$$

Finally, the Cauchy-Schwarz inequality yields

$$\begin{aligned} & E\left|\sqrt{\eta_i^2(t)\eta_j^2(t)} - \sqrt{\eta_i^2(t,\tau)\eta_j^2(t,\tau)}\right| \\ \leq & \sqrt{E\left(\sqrt{\eta_i^2(t)} - \sqrt{\eta_i^2(t,\tau)}\right)^2 E\eta_j^2(t)} + \sqrt{E\left(\sqrt{\eta_j^2(t)} - \sqrt{\eta_j^2(t,\tau)}\right)^2 E\eta_i^2(t,\tau)} \\ = & O\left(\max\left\{\sqrt{E\left(\sqrt{\eta_i^2(t)} - \sqrt{\eta_i^2(t,\tau)}\right)^2}, \sqrt{E\left(\sqrt{\eta_j^2(t)} - \sqrt{\eta_j^2(t,\tau)}\right)^2}\right\}\right) \\ = & O(\rho^{\frac{1}{2}\tau}), \end{aligned}$$

since

$$\begin{aligned} E\left(\sqrt{\eta_i^2(t)} - \sqrt{\eta_i^2(t,\tau)}\right)^2 &\leq E\left|\eta_i^2(t) - \eta_i^2(t,\tau)\right| \\ &\leq \left\|E\left|X(t) - X(t,\tau)\right|\right\| \end{aligned}$$

for all $i = 1, \ldots, m$. We obtain for $0 < \kappa < \frac{1}{2}$

$$E\left|E\{\eta_i(t)\eta_j(t) \mid \mathcal{F}_{t,\tau}\} - \eta_i(t)\eta_j(t)\right| = o(e^{-\kappa\tau}).$$

If we replace $\eta_j(t)$ by 1 in the preceding argument, we obtain the second statement. ■

Lemma 3
The distribution of $(\eta_1(t-1), \ldots, \eta_m(t-q))$ is absolutely continuous w. r. to Lebesgue measure with twice continuously differentiable density $f(u_1, \ldots, u_{qm})$ on any compact subset $E \subset I\!R^{qm}$.

Proof:
It can be seen by algebraic induction and the rule for density transformation that for any $B \in \mathcal{B}^*$ and $i = 1, \ldots, m$,

$$\int_B h_i^{-\frac{1}{2}}(\eta_1^2(t-1), \ldots, \eta_m^2(t-q)) f_{\epsilon_i}(u h_i^{-\frac{1}{2}}(\eta_1^2(t-1), \ldots, \eta_m^2(t-q))) du$$

is a version of the conditional probability $P\{\eta_i(t) \in B \mid \eta_1(t-1), \ldots, \eta_m(t-q)\}$. Therefore we can write

$$
\begin{aligned}
& P\{\eta_i(t) \in B\} \\
= \; & E\{P\{\eta_i(t) \in B \mid \eta_1(t-1), \ldots, \eta_m(t-q)\}\} \\
= \; & E\{\int_B h_i^{-\frac{1}{2}}(\eta_1^2(t-1), \ldots, \eta_m^2(t-q)) f_{\epsilon_i}(u h_i^{-\frac{1}{2}}(\eta_1^2(t-1), \ldots, \eta_m^2(t-q))) du\} \\
= \; & \int_B E\{h_i^{-\frac{1}{2}}(\eta_1^2(t-1), \ldots, \eta_m^2(t-q)) f_{\epsilon_i}(u h_i^{-\frac{1}{2}}(\eta_1^2(t-1), \ldots, \eta_m^2(t-q)))\} du \\
= \; & \int_B f_{\eta_i}(u) du,
\end{aligned}
$$

say. This shows the existence of a marginal density $f_{\eta_i}(u)$ and implies immediately the existence of a qm-dimensional density $f(u_1, \ldots, u_{qm})$ of the vector $(\eta_1(t-1), \ldots, \eta_m(t-q))$. The further properties follow from well-known results from calculus. (see e. g. APOSTOL(1977),10.39). ■

2.3 Examples

The preceding section reveals that assumptions (A.3) and (A.4) are crucial for the existence of strictly stationary and ν-stable solutions of 1.1. These assumptions restrict the effect of changes in squared lagged variables to changes in mean of squared current variables. Therefore a closer look on conditions that ensure the validity of (A.3) and (A.4) is of some interest. We give some examples:

Example 1:
The parametric ARCH(q)-model

$$\eta(t) = \epsilon(t) \left\{ \alpha_0 + \sum_{i=1}^{q} \alpha_i \eta^2(t-i) \right\}^{\frac{1}{2}}$$

with

$$\alpha_i \geq 0, \quad \sum_{i=1}^{q} \alpha_i < 1, \quad \alpha_0 > 0,$$

is included.

Assume that the following examples meet the requirements of (A.3).

Example 2:
There is no interdependence, i. e. $A(i,j) = 0$ if $i \neq j$. In this case a sufficient condition for (A.4) is

$$\sum_{k=(i-1)q+1}^{iq} \alpha(i,k) < 1$$

$\forall i = 1, \ldots, m$.

Example 3:
The model is free from feedback, i. e. $A(i,j) = 0$ if $i < j$. This reduces the eigenvalues of A to those of $A(i,i), i = 1, \ldots, m$, and therefore the condition given above is again sufficient.

Finally, we give a numerical (though insignificant in economic theory)
Example 4:

$$\begin{aligned}
\eta_1(t) &= \epsilon_1(t)\sqrt{1 + 0.4\eta_1^2(t-1) + \cos(0.5\eta_2^2(t-1))} \\
\eta_2(t) &= \epsilon_2(t)\sqrt{1 + 0.5\eta_1^2(t-1) + \sin(0.4\eta_2^2(t-1))}.
\end{aligned}$$

In this case we have

$$A = \begin{pmatrix} 0.4 & 0.5 \\ 0.5 & 0.4 \end{pmatrix},$$

and therefore the spectral radius of A is 0.9, as a simple calculation shows.

Chapter 3

Consistency results

3.1 Assumptions on the kernel

The subsequent results hold under the following assumptions concerning the filter 1.3:

(K. 1) *$K(u)$ is a nonnegative, bounded, continuous and symmetric real-valued function on $\mathbb{R}^{qm}$ that integrates to one.*

(K. 2) $\|u\|^{qm} K(u) \to 0 \quad for \quad \|u\| \to +\infty.$

(K. 3) *$K(u)$ has an absolutely integrable characteristic function $\beta(v) := \int_{\mathbb{R}^{qm}} \exp(iv'u)K(u)du$, for which the additional integrability condition*

$$\int_{\mathbb{R}^{qm}} \|v\| \mid \beta(v) \mid dv < +\infty$$

holds.

3.2 Uniform Consistency

Lemma 4
Define $M(C,\epsilon) := \{u \in I\!R^{qm} : \|u\| \leq C - \epsilon\}$ *and* $g_{i,j}(x) := m_{i,j}(x)f(x)$. *Suppose that* $\{\gamma_T\}_{T\in I\!N}$ *is a sequence of positive numbers satisfying*

$$\lim_{T\to\infty} \gamma_T = 0.$$

Then we have for all $i,j = 1,\ldots,m$,

$$\sup_{x\in M(\sqrt{C},0)} \mid E\{\hat{g}_{i,j,T}(x)\} - g_{i,j}(x) \mid = o(1). \tag{3.1}$$

Proof:
See Cron(1995),Lemma 5. ■

Lemma 5
Suppose that $\{\tau(T)\}_{T\in I\!N}$ *is any sequence of natural numbers. Then, for any sequence* $\{\gamma_T\}_{T\in I\!N}$ *of positive numbers and* $i,j = 1,\ldots,m$,

$$\gamma_T^{qm} E \sup_{x\in I\!R^{qm}} \left| E\{\hat{g}_{i,j,T}(x) \mid \mathcal{F}_{t,\tau(T)}\} - E\{\hat{g}_{i,j,T}(x)\}\right| = O\left(\sqrt{\frac{\tau(T)}{T}}\right). \tag{3.2}$$

Proof:
In order to be able to apply Bierens(1983), Lemma 1 and Cron(1995), Lemma 6 [1], respectively, only the square-integrability of $\phi(\hat{x}(t)y_{i,j}(t)$ is left to be verified. But under (A.3),

$$\begin{aligned}
& \mid \phi(\hat{x}(t))y_{i,j}(t) \mid \\
\leq\ & \phi(\hat{x}(t)) \mid \epsilon_i(t)\epsilon_j(t) \mid \\
& \times \{h_i(0)h_j(0) + h_i(0)a_j\hat{x}(t) + h_j(0)a_i\hat{x}(t) + (a_i\hat{x}(t))(a_j\hat{x}(t))\}^{\frac{1}{2}} \\
\leq\ & C_0 \mid \epsilon_i(t)\epsilon_j(t) \mid,
\end{aligned}$$

[1] Lemmata 6 and 7 of Cron(1995) are notationally incorrect: Replace $m(x), \hat{m}_T(x)$ by $g(x)$ and $\hat{g}_T(x)$, respectively.

say,where $a_i := (\alpha(i,1),\ldots,\alpha(i,qm))$. Therefore,

$$E\{\phi^2(\hat{x}(t))y_{i,j}^2(t)\} \leq C_0^2 E\{\epsilon_i^2(t)\epsilon_j^2(t)\} \leq C_0^2\sqrt{E\{\epsilon_i^4(t)\}}\sqrt{E\{\epsilon_j^4(t)\}}$$

from the Cauchy-Schwarz inequality. ■

Lemma 6

Suppose that $\{\tau(T)\}_{T\in\mathbb{N}}$ *is any sequence of natural numbers. Then, for any sequence* $\{\gamma_T\}_{T\in\mathbb{N}}$ *with* $\lim_{T\to\infty}\gamma_T = 0$,$i,j = 1,\ldots,m$ *and some constant* $\kappa > 0$,

$$E\sup_{x\in\mathbb{R}^{qm}}\left|E\{\hat{g}_{i,j,T}(x) \mid \mathcal{F}_{t,\tau(T)}\} - \hat{g}_{i,j,T}(x)\right| = o(e^{-\kappa\tau(T)}\gamma_T^{-(qm+1)}). \tag{3.3}$$

Proof:

To obtain a result equivalent to Lemma 7 and its underlying Lemma 4 of Cron(1995), we start showing the Lipschitz-continuity of $\phi(u)h_i^{\frac{1}{2}}(u)h_j^{\frac{1}{2}}(u)$: On $\{\|u\| \leq C\}$ the Lipschitz-continuity with constant L_α follows from the continuous differentiability of $h_i^{\frac{1}{2}}(u)h_j^{\frac{1}{2}}(u)$. Moreover it is easy to check that under (A.3) for all $i = 1,\ldots,m$,

$$\max_{\|u\|\geq C-\epsilon}\left\|\frac{\partial\left(\|u\|^{-1}h_i^{\frac{1}{2}}(u)h_j^{\frac{1}{2}}(u)\right)}{\partial u_k}\right\| =: L_{i,j,k} < +\infty \tag{3.4}$$

holds for ϵ small, where $\|\cdot\|$ denotes the Euclidean norm and u_k the k-th component of $u \in \mathbb{R}^{qm}$. Thus $\phi(u)h_i^{\frac{1}{2}}(u)h_j^{\frac{1}{2}}(u)$ is Lipschitz-continuous with constant L_β on $\{\|u\| \geq C\}$,too. Finally, if $\|u\| \leq C, \|v\| \geq C$, there is a c^* with norm C that lies on the line segment connecting u and v , such that

$$\begin{aligned}
& \mid \phi(u)h_i^{\frac{1}{2}}(u)h_j^{\frac{1}{2}}(u) - \phi(v)h_i^{\frac{1}{2}}(v)h_j^{\frac{1}{2}}(v) \mid \\
\leq\ & \mid \phi(u)h_i^{\frac{1}{2}}(u)h_j^{\frac{1}{2}}(u) - \phi(c^*)h_i^{\frac{1}{2}}(c^*)h_j^{\frac{1}{2}}(c^*) \mid \\
+\ & \mid \phi(c^*)h_i^{\frac{1}{2}}(c^*)h_j^{\frac{1}{2}}(c^*) - \phi(v)h_i^{\frac{1}{2}}(v)h_j^{\frac{1}{2}}(v) \mid \\
\leq\ & L_\alpha\|u - c^*\| + L_\beta\|c^* - v\| \\
\leq\ & \max\{L_\alpha, L_\beta\}\|u - v\|
\end{aligned}$$

holds.
Similar to Lemma 4 ,Step 4, of CRON(1995) we then obtain from Lemma 2

$$E\left|E\{\phi(\hat{x}(t))y_{i,j}(t) \mid \mathcal{F}_{t,\tau}\} - \phi(\hat{x}(t))y_{i,j}(t)\right| = o(e^{-\kappa\tau}).$$

Lemma 7 of CRON(1995) follows directly. ■

Theorem 1 (Uniform Consistency)
For any sequence $\{\gamma_T\}_{T \in I\!N}$ *satisfying*

$$\gamma_T = MT^{-\mu}, \quad \mu \in \left(0, \frac{1}{2qm}\right), M > 0$$

and any pair

$$\left\{ \delta \in \left(0, \sup_{x \in I\!R^{qm}} f(x)\right], C > 0 \right\}$$

we have

$$p\lim \sup_{x \in \{f(u) \geq \delta\} \cap M(\sqrt{C}, 0)} \mid \hat{m}_{i,j,T}(x) - m_{i,j}(x) \mid = 0.$$

Proof:
As in Cron(1995),Theorem 1, Lemmata 4 to 6 can be combined to prove the assertion. ∎

3.3 The Rate of Convergence

Under the additional assumptions

(A. 5) *The functions* $h_i : I\!R_+^{qm} \to I\!R, i = 1, \ldots, m$, *are twice continuously differentiable on* $I\!R^{qm}$.

(K. 4)

$$\left\| \int_{I\!R^{qm}} uu' K(u) du \right\| < +\infty.$$

(K. 5) $\|u\|^{qm+2} K(u) \to 0 \quad for \quad \|u\| \to +\infty.$

we have the following result:

Theorem 2
For any sequence $\{\gamma_T\}_{T \in I\!N}$ *satisfying*

$$\gamma_T = MT^{-\mu}, \quad \mu \in \left(0, \frac{1}{2qm}\right), M > 0$$

and any triple

$$\left\{\delta \in \left(0, \sup_{x \in \mathbb{R}^{qm}} f(x)\right], \epsilon > 0, C > 0\right\}$$

with $\sqrt{C} > \epsilon$ *we have*

$$p\lim \zeta_T \sup_{x \in \{f(u) \geq \delta\} \cap M(\sqrt{C}, \epsilon)} | \hat{m}_{i,j,T}(x) - m_{i,j}(x) | = 0$$

where $\zeta_T = o(min(\gamma_T^{-2}, \xi_T)), \xi_T = o(\gamma_T^{qm}\sqrt{T})$.

Proof:
See CRON(1995),Theorem 1. ■

Chapter 4

Asymmetric Models

In this final chapter we generalize model 1.1 replacing the squared lagged values which serve as arguments for the functions h_i by ordinary lagged values. In this way we allow for asymmetric reactions on lagged deviations of different sign. Equations 1.1 become

$$\begin{aligned}
\eta_1(t) &= \epsilon_1(t)h_1^{\frac{1}{2}}(\eta_1(t-1),\ldots,\eta_1(t-q),\eta_2(t-1),\ldots,\eta_m(t-q)) \quad (4.1)\\
\vdots \quad &\vdots \quad \vdots \\
\eta_m(t) &= \epsilon_m(t)h_m^{\frac{1}{2}}(\eta_1(t-1),\ldots,\eta_1(t-q),\eta_2(t-1),\ldots,\eta_m(t-q)),
\end{aligned}$$

$t \in Z$. We replace assumption (A.3) by

(A. 3*) *The nonnegative functions $h_i : I\!R^{qm} \to I\!R, \quad i = 1,\ldots,m$, have unbounded support and are partially differentiable with Lipschitz-continuous derivatives on $I\!R^{qm}$. Moreover there exist nonnegative constants $\alpha(i,k), i = 1,\ldots,m, k = 1,\ldots,qm$, such that*

$$|h_i(u) - h_i(v)| \le \sum_{k=1}^{qm} \alpha(i,k) \left| sgn(u_k)u_k^2 - sgn(v_k)v_k^2 \right|$$

$\forall u, v \in I\!R^{qm}$ and $i = 1,\ldots,m$.

With

$$V_i(t) := diag\left(sgn(\epsilon_i(t)), sgn(\epsilon_i(t-1)),\ldots,sgn(\epsilon_i(t-q+1))\right),$$

$$V(t) := \begin{pmatrix} V_1(t) & 0 & \dots & \ddots & \dots \\ 0 & V_2(t) & 0 & \ddots & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \dots & \dots & \dots & \ddots & V_m(t) \end{pmatrix}$$

and

$$H_i : I\!R^{qm} \to I\!R^q,$$

$$H_i(u) = \begin{pmatrix} h_i(sgn(u_1)\sqrt{\mid u_1 \mid}, \dots, sgn(u_{qm})\sqrt{\mid u_{qm} \mid}) \\ \mid u_{(i-1)q+1} \mid \\ \vdots \\ \mid u_{iq-1} \mid \end{pmatrix}$$

the inflated state representation of 4.1 squared becomes

$$X(t) = \Pi(t)H(V(t-1)X(t-1)), \quad t \in Z. \tag{4.2}$$

With the arguments used to prove Lemma 1 one easily shows that 4.1 has a strictly stationary, square-integrable solution. Moreover, if we redefine for $\tau^* := \tau - q, \tau > q$ the $\mathcal{F}_{t,\tau}$-measurable random vector

$$\begin{aligned} & X(t,\tau) \\ := \; & \Pi(t)H^{(\tau^*-1)}(V(t-1)\Pi(t-1), \dots, V(t-\tau^*+1)\Pi(t-\tau^*+1); E\{X(t-\tau^*)\}), \end{aligned}$$

Lemma 2 carries over.

With the slightly modified estimator

$$\hat{g}_{i,j,T}(x) := \frac{1}{T}\sum_{t=1}^{T} \gamma_T^{-qm}\,\phi(x(t))y_{i,j}(t)K\left(\frac{x - x(t)}{\gamma_T}\right),$$

where

$$\phi(u) := \begin{cases} 1 & : \; \|u\| \le C \\ \frac{C^2}{\|u\|^2} & : \; \|u\| > C \end{cases}$$

all results of section 3. 2 remain valid.

Finally, we illustrate the model by a simple

Example 5:
The asymmetric ARCH(1)-model

$$\eta(t) = \epsilon(t)\left\{\alpha_0 + 1_{\eta(t-1)\geq 0}\alpha_1\eta^2(t-1)\right\}^{\frac{1}{2}}$$

with $\alpha_0 > 0, \quad 0 \leq \alpha_1 < 1$, is included.

Bibliography

[1] Apostol,T.M.(1974)
Mathematical Analysis
Addison-Wesley Publishing Company.

[2] Bierens,H.J.(1983)
Uniform Consistency of Kernel Estimators of a Regression Function Under Generalized Conditions
Journal of the American Statistical Association,**78**,699-707.

[3] Bougerol,P. and Picard,N.(1992)
Stationarity of GARCH processes and of some nonnegative time series
Journal of Econometrics,**52**,115-127.

[4] Cron,A.(1995)
Uniform Consistency of Modified Kernel Estimators in Parametric ARCH-Models
SFB Discussion Paper No. 316,University of Bonn.

[5] Engle,R.F.(1982)
Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of U.K. Inflation
Econometrica,**50**,987-1008.

[6] Haerdle,W.(1989)
Applied nonparametric regression
Cambridge University Press.

[7] Jennrich,R.I.(1969)
Asymptotic Properties of Nonlinear Least Squares Estimators
The Annals of Mathematical Statistics,**40**,633-643.

[8] Kesten,H. and Spitzer,F.(1984)
Convergence in Distribution of Products of Random Matrices
Zeitschrift f. Wahrscheinlichkeitstheorie und verwandte Gebiete,**67**,363-386.

[9] Singh,R.S. and Ullah,A.(1985)
Nonparametric time series estimation of joint DGP,conditional DGP and vector autoregression
Econometric Theory, **1**.