Dynamic Stability in Perturbed Games[*]

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Abstract

The effect that exogenous mistakes, made by players choosing their strategies, have on the dynamic stability for the replicator dynamic is analyzed for both asymmetric and symmetric normal form games. Through these perturbed games, the dynamic solution concept of limit asymptotic stability is motivated by insisting that such solutions be asymptotically stable for all sufficiently small perturbations (a robustness property). Limit asymptotic stability is then a refinement of the Nash equilibrium. For asymmetric normal form games, it is shown that a strategy pair is limit asymptotically stable if and only if it is a pure strategy pair that weakly dominates alternative best replies. For symmetric normal form games, all evolutionarily stable strategies (ESS's), whether pure or mixed, are limit asymptotically stable. Here, conditions are established for limit asymptotic stability of completely mixed (i.e. interior) strategies as well as strategies on the boundary. Consistency with solutions found by backwards and/or forwards induction is shown for elementary extensive form games. Limit asymptotically stable sets are introduced that generalize other set-valued solution concepts such as the "strict equilibrium set" and the "ES set" for asymmetric and symmetric normal form games respectively.

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1 Introduction

We consider the effect of mistakes on dynamic stability for the replicator dynamic. Players make mistakes that are exogenous, occur independent of the strategy chosen and have full support. Our setup falls in the framework of perturbed games introduced by Selten (1975).

Perturbed games were introduced for normal form games as a means to eliminate unreasonable Nash equilibria (NE) as solutions to these games. In particular, Selten (1975) defines the solution concept of normal form perfect equilibrium. The NE refinements (van Damme, 1991) that are selected in this fashion become limits of NE of the perturbed games. This is essentially a static procedure based on the intuitive static stability conditions of the NE.

Dynamic evolutionary game theory also selects particular NE such as an evolutionarily stable strategy (ESS) that is dynamically stable with respect to an underlying adjustment process whereby the frequencies of players using a particular strategy evolve according to their payoff relative to that of other strategies. In this paper, we consider exclusively the adjustment process known as the continuous replicator dynamic. This dynamic, originally based on the biological interpretation of evolutionary game theory (Taylor and Jonker 1978; Taylor 1979), has a firm basis from the economic perspective as well (Björnerstedt and Weibull, 1994; Börgers and Sarin, 1993; Cabrales, 1993; Gale et al., 1995; Schlag, 1994b). Here the dynamic stability notion is that of asymptotic stability of equilibria as in Definition 1 below which asserts that near such an equilibrium the frequencies of strategy-types evolve towards the equilibrium. It is well-known that an asymptotically stable equilibrium is a NE for two-person asymmetric normal form games (Samuelson and Zhang, 1992) and corresponds to a symmetric NE for symmetric normal form games (Bomze, 1986) but that the converses are not true. Thus, dynamic stability is also a refinement of NE that can be used to select among equilibria. However, there are many games that have no dynamically stable equilibria. In particular, this phenomenon occurs for two-person asymmetric normal form games when no strict NE exist and for the normal form of extensive form games when equilibrium outcomes do not reach all information sets with positive probability.

The main purpose of this paper is to combine the perturbed game approach with the dynamic approach of evolutionary game theory. The resulting solution concept is called limit asymptotic stability as in Definition 4 below. It differs from other limit solution concepts for perturbed games such as the limit ESS (Selten 1983, Samuelson, 1991) and the concepts of perfect and/or proper equilibria (van Damme, 1991) in two fundamental ways. The first is that limit asymptotic stability satisfies dynamic stability criteria as opposed to the static NE conditions. Secondly, limit asymptotic stability must hold for all sufficiently small perturbations (a robustness property) that is not required of other limit solution concepts where the limit property need only hold for a sequence of perturbations possibly chosen by the players. Both these fundamental differences emphasize our strong belief that, if dynamic evolutionary game theory (which is not based on rationality assumptions) is going to be used to select among possible NE, then to be consistent the solutions must have robust dynamic stability properties.

The paper introduces and analyzes limit asymptotic stability for both asymmetric normal form games (the bimatrix games of Section 2) and for symmetric normal form games (Section 3). Of particular interest is the comparison of results in these two categories, both to each other and to previous limit solution concepts. In the asymmetric case we consider two separate populations (one for each player in the game) matched against each other and each evolving according to their payoff relative to that of other strategies in their population. In the symmetric case there is only one population, with matching and selection occurring among the same individuals.

The main result (Theorem 1) for Section 2 is the complete static characterization of limit asymptotic stability for asymmetric games; namely, (p^*, q^*) is limit asymptotically stable if and only if it is a pure strategy pair that weakly dominates any alternative best replies. Through this characterization, examples are given that compare limit asymptotic stability to other limit solution concepts mentioned above and to solutions found by backwards and/or forwards induction in elementary extensive form games. Numerous analyses in the evolutionary game theory literature (Balkenborg and Schlag, 1995; Schlag, 1994a; Thomas, 1985; Weibull, 1995) have considered the possibility of defining certain sets of strategies as a solution to a game. Limit asymptotically stable sets are introduced in Subsection 2.3 and, although a complete static characterization is not known, partial results are given. In particular, each strict equilibrium set (Balkenborg, 1994) is limit asymptotically stable (Theorem 4 below).

A fundamental result (Theorem 5) of Section 3 shows that an ESS is limit asymptotically stable. An immediate corollary is that not all limit asymptotically stable solutions are pure strategies for symmetric normal form games. The remainder of Subsection 3.1 considers exclusively pure strategies and shows that the static characterization of the preceding paragraph is now equivalent to a stronger form of limit asymptotic stability, called strict limit asymptotic stability. This is again illustrated for various normal form and elementary symmetric extensive form games. Subsection 3.2 has no parallel for asymmetric games in that it deals with limit asymptotic stability of mixed strategies. Here, limit asymptotic stability criteria are proven for completely mixed strategies (Theorems 8 and 9) and for mixed strategies on the boundary (Theorem 10) by techniques that combine asymptotic stability in the unperturbed dynamic with the concept of weakly dominating alternative best replies. Finally, Subsection 3.3 briefly introduces limit asymptotically stable sets for symmetric normal form games and generalizes Theorem 5 to prove the limit asymptotic stability of evolutionarily stable sets (ES sets).

2 Limit Asymptotic Stability in Asymmetric Normal Form Games

2.1 The Replicator Dynamic - Preliminaries

We start out by considering conflicts between two disjoint populations. The replicator dynamic (1) was introduced by Taylor (1979) to model the following behavioral evolution in the two populations, each of which is assumed to be very large. Individuals in either population exhibit one of a finite number of possible behaviors (these are called pure strategies and usually depend on which population the individual belongs to) and engage in random pairwise interactions with individuals in the opposite population. In population one, we assume that there are m possible behaviors labelled 1 to m, in population two n. Let A be the $m \times n$ payoff matrix whose entry A_{ij} gives the fitness of the i^{th} pure strategy of population one when interacting with the j^{th} pure strategy of population two. Similarly, let B be the $n \times m$ payoff matrix whose entry B_{kl} gives the fitness of the k^{th} pure strategy of population two when interacting with the l^{th} pure strategy of population one. In this context we call (A, B) an asymmetric game in normal form. The evolution of the frequencies of strategies played in the two populations, introduced by Taylor (1979), is governed by

$$\begin{aligned} \dot{p}_i &= p_i\,(e_i - p) \cdot Aq \\ \dot{q}_j &= q_j\,(f_j - q) \cdot Bp \end{aligned} \tag{1}$$

where p_i is the frequency of individuals in population one using the i^{th} pure strategy at time t; q_j is the frequency in population two using the j^{th} pure strategy; the derivatives are with respect to time t; $p \in \Delta^m = \{x \in \mathbb{R}^m \text{ s.t. } x_i \geq 0, \sum x_i = 1\}$ is the strategy frequency vector in population one; $q \in \Delta^n$ the corresponding vector in population two; $e_i = (0, .., 0, 1, 0, .., 0) \in \Delta^m$ is the i^{th} unit coordinator vector that represents the i^{th} pure strategy in population one; $f_j \in \Delta^n$ the corresponding unit vector in population two; and the standard inner product in $\mathbb{R}^k$ is given by $y \cdot z = \sum\limits_{i=1}^{k} y_i z_i$ (hence

$\cdot Aq = \sum_{i=1}^{m} \sum_{j=1}^{n} \; {}_i a_{ij} q_j$ and $q \cdot B \; = \sum_{i=1}^{m} \sum_{j=1}^{n} q_j b_{ji} \; {}_i)$. The dynamic on $\Delta^m \times \Delta^n$ is illustrated in the flow diagram, Figure 2, of the following example.

Example 1 *The Male Desertion Game (Selten, 1983; Samuelson, 1991) is the extensive form game in Figure 1.*

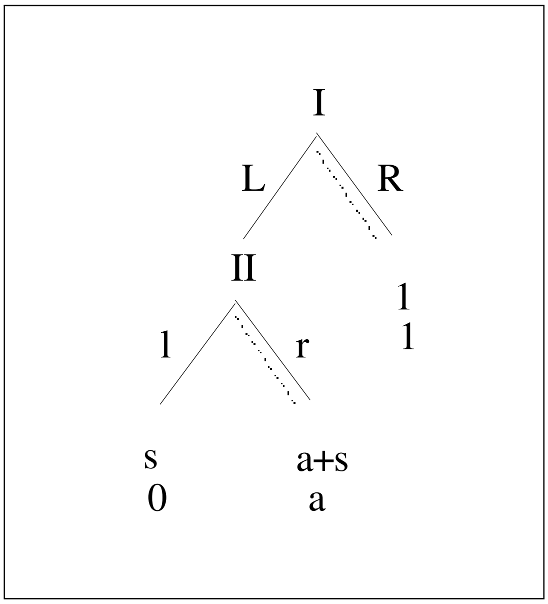

Figure 1: The Male Desertion Game in extensive form. Player one is male and two is female.

The game is a simple model of male/female competition over the care of their offspring. The male decides either to leave (L) his offspring to look for another mate or to remain (R) and help raise the offspring. If the male chooses L, the female may leave (l) and receive payoff 0 *or remain (r) to raise her offspring on her own in which case both receive a payoff of* a*. Suppose that both male and female receive a (normalized) payoff of* 1 *if the male chooses R and that a male who chooses L receives a further incremental payoff of* s *from the possibility of finding another mate.*

The associated normal form has payoff matrices

$$A = \begin{bmatrix} s & a+s \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ a & 1 \end{bmatrix},$$

where we assume $0 < a < a + s < 1$*. For notational convenience and to show the connection to Game Theory we will often summarize the two payoff* *matrices in one table* $(a_{ij}, b_{ji})_{\substack{i=1,..,m\\ j=1,..,n}}$ *as follows (here* $m = n = 2$*):*

$$\begin{array}{ccc} & l & r \\ L & s, 0 & a+s, a \\ R & 1, 1 & 1, 1 \end{array}.$$

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In order to characterize trajectories of the dynamic (1) we will use the following standard dynamic stability concepts (see Hirsch and Smale, 1974). Let $U_\varepsilon(\ \ , q)$ be the ε *neighborhood* of $(\ \ , q)$, i.e.,
$U_\varepsilon(\ \ , q) = \{(\ \ ', q') \in \Delta^m \times \Delta^n \text{ s.t. } dist((\ \ , q), (\ \ ', q')) < \varepsilon\}$.

Definition 1 *A pair of strategies* $(\ \ ^*, q^*)$ *is called* ***(Lyapunov) stable*** *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that trajectories starting in* $U_\delta(\ \ ^*, q^*)$ *stay in* $U_\varepsilon(\ \ ^*, q^*)$.

$(\ \ ^*, q^*)$ *is called* ***attracting*** *if there exists* $\sigma > 0$ *such that all trajectories starting in* $U_\sigma(\ \ ^*, q^*)$ *converge to* $(\ \ ^*, q^*)$.

$(\ \ ^*, q^*)$ *is called* ***(locally) asymptotically stable*** *if it is stable and attracting.*

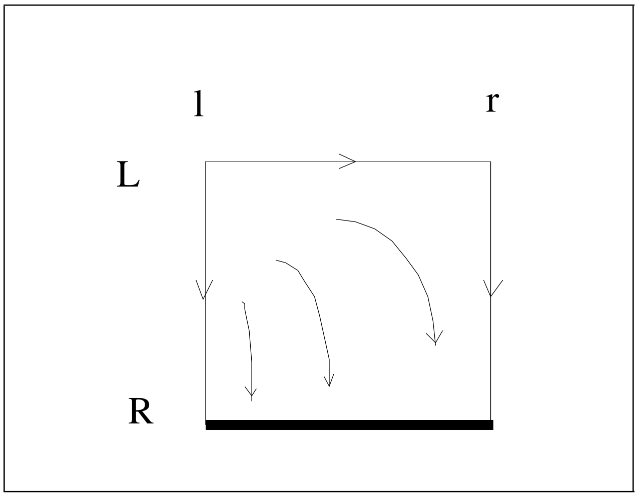

Figure 2: The Flow Diagram for the Male Desertion Game.

Figure 2 shows the flows of (1) in the Male Desertion Game of Example 1. Notice that there are no asymptotically stable outcomes although R is a strictly dominant strategy for player one and r is a weakly dominant strategy for player two. The reason for this is that the Nash equilibria are all strategies of the form $(R, (1-\lambda)\,l + \lambda r)$ where $0 \leq \lambda \leq 1$. These are automatically rest points of (1) (i.e., $\dot{}_i = \dot{q}_j = 0$ for all i and j). However, in a setup where player one (male) sometimes makes a mistake and plays L, then r is the unique best response for player two. In the resulting game, (R, r) becomes the unique Nash equilibrium which is in fact a strict equilibrium and hence asymptotically stable following Remark 1 below.

Definition 2 *$(\ {}^*, q^*)$ is called a **strict Nash equilibrium** of the asymmetric game (A, B) if $e_i \cdot Aq^* = \ {}^* \cdot Aq^*$ implies $e_i = \ {}^*$ and $f_j \cdot B\ {}^* = q^* \cdot B\ {}^*$ implies $f_j = q^*$.*

Remark 1 *The following facts concerning the dynamics stability of (1) for asymmetric games are well-known and so cited without proof.*

i) Any stable $(\ {}^*, q^*)$ is a Nash equilibrium pair (Samuelson and Zhang, 1992).

ii) Any attracting $(\ {}^*, q^*)$ is a Nash equilibrium pair (Samuelson and Zhang, 1992).

iii) Strict NE are equivalent to asymptotically stable strategy pairs (Cressman, 1992b).

Thus all three concepts in Definition 1 are refinements of the Nash equilibrium solution concept.

2.2 Perturbed Asymmetric Normal Form Games and Limit Asymptotically Stable Strategy Pairs

In this paper we will be interested in the asymptotically stable outcomes (i.e., strategy pairs) when players make mistakes. In each population it will be assumed that when a player wants to play a pure strategy, then sometimes he makes a mistake that is independent of the strategy that he uses and independent of the mistakes or play of others. We will also assume that, when a player makes a mistake, then each strategy will be played with a strictly positive probability. The resulting game is called a *perturbed game* following Selten (1975). The perturbation is characterized by vectors $\mu \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^n$ where μ_i is the probability of player one playing e_i conditional on the event of him making a mistake when playing e_k with $k \neq i$ and η_j is the probability of player two playing f_j conditional on the event of making a mistake when playing f_k with $k \neq j$.

Definition 3 *Let (A, B) be an asymmetric normal form game with m pure strategies for player one and n pure strategies for player two. Let $\mu \in \mathbb{R}^m$*

and $\eta \in \mathbb{R}^n$ *be such that* $\sum \mu_i < 1$, $\sum \eta_j < 1$ *and all* $\mu_i, \eta_j > 0$. *Let* $\left(\tilde{A}, \tilde{B}\right)$ *be the asymmetric normal form game with the same number of pure strategies for each player given by the* $m \times n$ *payoff matrix for population one with entries* $\tilde{A}_{ij} = S_i \cdot AT_j$ *and the* $n \times m$ *payoff matrix for population two with entries* $\tilde{B}_{ij} = T_i \cdot BS_j$ *where* $S_i = \left(\mu_1, .., \mu_{i-1}, 1 - \sum\limits_{k \neq i} \mu_k, \mu_{i+1}, .., \mu_m\right)$, $T_j = \left(\eta_1, .., \eta_{j-1}, 1 - \sum\limits_{k \neq j} \eta_k, \eta_{j+1}, .., \eta_n\right)$, $1 \leq i \leq m, 1 \leq j \leq n$. *Then* $\left(\tilde{A}, \tilde{B}\right)$ *is called the* ***perturbed asymmetric normal form game*** *generated by* (A, B, μ, η). *The* ***size*** *of the perturbation is the maximum of* $\sum \mu_i$ *and* $\sum \eta_j$.

For a given strategy frequency vector x we will denote by $\tilde{x}$ the realized strategy frequency vector (i.e., including mistakes) in the perturbed game, i.e., $\tilde{\ } = \sum \ _i S_i$ for strategy frequency $\ \in \Delta^m$ in population one and $\tilde{q} = \sum q_j T_j$ for strategy frequency $q \in \Delta^n$ in population two. In particular, $\tilde{e}_i = S_i$ and $\tilde{f}_j = T_j$. Notice that the above definition is equivalent to the standard definition of a perturbed asymmetric normal form game (Selten, 1975; see also Definition 2.2.1 in van Damme, 1991) where payoffs are generated by the original matrices A and B. In these references minimum probabilities μ_i, $1 \leq i \leq m$, and η_j, $1 \leq j \leq n$, on the pure strategies of player one and player two respectively are specified when choosing a strategy in the game.

We are interested in the dynamic stability properties that can be sustained whenever mistakes are very small, strictly positive but otherwise arbitrary. The emphasis here on stable dynamic properties that hold for any specification of mistakes as long as they are small makes our approach different from the use of perturbed games in the refinement literature (see van Damme, 1991) where mistakes are typically chosen in a specific way to create a stable situation. For instance, Selten (1983) (see also Samuelson, 1991) defines a limit ESS as the limit of a specific sequence of ESS's for perturbed games whose size of perturbation tends to zero.

Perturbations in our dynamic framework are due to exogenous influences. We feel strongly that selecting between outcomes in a game based on a specific structure of the perturbations does not make sense if we do not know where the perturbations come from. Hence we consider outcomes that can be sustained for arbitrary perturbations provided that they are small.

Definition 4 *i) A pair of strategies* $(\ ^*, q^*)$ *is called* ***limit stable*** *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *and* $\pi > 0$ *such that, for any perturbations* (μ, η) *of size at most* π, *trajectories starting in* $U_\delta(\ ^*, q^*)$ *stay in* $U_\varepsilon(\ ^*, q^*)$.

ii) $(\ ^*, q^*)$ *is called* ***limit attracting*** *if there exists an open neighborhood* U *of* $(\ ^*, q^*)$ *such that, for every* $\varepsilon > 0$, *there exists* $\pi > 0$ *such that, for any* *perturbation* (μ, η) *of size at most* π*, all* ω *limits of trajectories starting in* U *are contained* $U_\varepsilon(\ ^*, q^*)$.

iii) $(\ ^*, q^*)$ *is called* ***limit asymptotically stable*** *if it is both limit stable and limit attracting.*

Remark 2 *In general, the above definition does not require that the realized strategy pair* $(\tilde{\ }^*, \tilde{q}^*)$ *corresponding to a limit asymptotically stable outcome is asymptotically stable in the perturbed game. It does require that the set of* ω *limits in the perturbed game converges to* $(\ ^*, q^*)$ *as the size of the perturbation tends to zero (however, see Corollary 1 below). Conversely, the fact that an outcome is asymptotically stable in any perturbed game does not necessarily imply that this outcome is limit asymptotically stable. Limit asymptotic stability requires the stability and attracting properties (i.e., the neighborhoods given by* δ *and* σ *in Definition 4) to be independent of the perturbation provided that its size is sufficiently small.*

In the Male desertion game in Example 1, we see that the realized strategy pair corresponding to (R, r) is a strict equilibrium for any specification of the perturbation and hence an asymptotically stable equilibrium of the perturbed game regardless of the size of the perturbation. In fact, (R, r) is limit asymptotically stable by the proof of Theorem 1 since "remain" is a strictly dominant strategy for either player in any perturbed game.

In the following we aim to characterize limit asymptotically stable outcomes for general games.

Definition 5 *Let* (A, B) *be an asymmetric normal form game (i.e., bimatrix game). i) The two strategies* e_i *and* e_j *are called* ***role equivalent (for player one)*** *if* $e_i \cdot Af_k = e_j \cdot Af_k$ *for all* k,

ii) ***weakly dominates*** x *(for player one) if* $\ \cdot Af_j \geq x \cdot Af_j\ \forall j$ *with a strict inequality for at least one* j.

iii) ***strictly dominates*** x *(for player one) if* $\ \cdot Af_j > x \cdot Af_j$ *for all* $j = 1, .., n$.

Analogous definitions apply for player two.

Lemma 1 *Consider the perturbed game* (A, B, μ, η)*. If* e_i *weakly dominates* e_k *for player one in the bimatrix game* (A, B) *then* S_i *strictly dominates* S_k *in the perturbed game* $\left(\tilde{A}, \tilde{B}\right) = (A, B, \mu, \eta)$*. If* e_i *and* e_k *are role equivalent then* S_i *and* S_k *are role equivalent in the perturbed game.*

Proof. Suppose e_i weakly dominates e_k. Then

$$\tilde{A}_{ij} = S_i \cdot AT_j = \left(1 - \sum \mu_r\right) e_i \cdot AT_j + \mu_i e_i \cdot AT_j + \mu_k e_k \cdot AT_j + \sum_{r \neq i,k} \mu_r e_r \cdot AT_j$$
$$> S_k \cdot AT_j = \tilde{A}_{kj}$$

since $e_i \cdot AT_j > e_k \cdot AT_j$ for all j. The proof for the case where e_i and e_k are role equivalent follows similarly. ■

Definition 6 *Given a Nash equilibrium* $({}^*, q^*)$ *of a bimatrix game,* e_i *is called an* ***alternate pure best reply*** *to* q^* *if* $e_i \neq {}^*$ *and* $e_i \cdot Aq^* = {}^* \cdot Aq^*$. *Similarly,* f_j *is called an* ***alternate pure best reply*** *to* * *if* $f_j \neq q^*$ *and* $f_j \cdot B {}^* = q^* \cdot B {}^*$.

Theorem 1 *The strategy pair* $({}^*, q^*)$ *is limit asymptotically stable if and only if* $({}^*, q^*)$ *is a pair of pure strategies where* * *weakly dominates any alternate pure best reply to* q^* *and* q^* *weakly dominates any alternate pure best reply to* *.

Proof. Assume that $({}^*, q^*)$ satisfies the 'if' condition. Let $G = \{r \in \Delta^n$ s.t. $r_i > 0$ implies $e_i \cdot Aq^* < {}^* \cdot Aq^*\}$. By continuity there exists $\pi > 0$ and a neighborhood $U \subseteq \Delta^n$ of q^* such that $\tilde{}^* \cdot A\tilde{y} > \tilde{r} \cdot A\tilde{y}$ for all $r \in G$ and any perturbation with size smaller than π. Let $x = (1 - \alpha - \beta) {}^* + \alpha \hat{} + \beta r$ where $\hat{} \neq {}^*$, $\hat{} \cdot Aq^* = {}^* \cdot Aq^*$and $r \in G$. Then ${}^* \cdot Ay - x \cdot Ay = \alpha ({}^* - \hat{}) \cdot Ay + \beta ({}^* - r) \cdot Ay$. By assumption and Lemma 1, $({}^* - \hat{}) \cdot Ay > 0$. Hence, $y \in U$ and a perturbation with size less than π, implies $\tilde{}^* \cdot A\tilde{y} - \tilde{x} \cdot A\tilde{y} > 0$. Symmetric arguments for player two imply that there exists $\varepsilon > 0$ and $\pi > 0$ such that $\tilde{}^* \cdot A\tilde{y} - \tilde{x} \cdot A\tilde{y} > 0$ and $\tilde{q}^* \cdot Bx - y \cdot Bx > 0$ for all $(x, y) \in U_\varepsilon ({}^*, q^*)$ and perturbations in size less than π. This means that, for such perturbations, the frequency of $\tilde{}^*$ in population one and the frequency of $\tilde{q}^*$ in population two are strictly increasing for all initial states in $U_\varepsilon (\tilde{}^*, \tilde{q}^*) \setminus \{(\tilde{}^*, \tilde{q}^*)\}$. These frequencies are then (local) Lyapunov functions (Hofbauer and Sigmund, 1988) for the dynamic (1) in the perturbed game. Thus $(\tilde{}^*, \tilde{q}^*)$ is asymptotically stable in such a way that the neighborhoods in Definition 1 are independent of the perturbation. Therefore, $({}^*, q^*)$ is limit asymptotically stable.

Assume that $({}^*, q^*)$ is limit asymptotically stable. Since the replicator dynamic is volume conserving (Eshel and Akin, 1983) in the interior of $\Delta^m \times \Delta^n$ (and on any of its invariant faces of dimension at least one), an attracting set must contain a pure strategy profile (S_i, T_j) (Schlag, 1994a). Since this must be true for arbitrarily small perturbations, it follows that $({}^*, q^*)$ is a pair of pure strategies.

If * does not weakly dominate an alternate pure best reply e_i to q^*, then either * is role equivalent to e_i or $e_i \cdot Af_j > {}^* \cdot Af_j$ for some j.

If * is role equivalent to e_i, then S_i is role equivalent to $\tilde{\ }^*$ and hence $((1-\lambda)\,S_i + \lambda\tilde{\ }^*, \tilde{q}^*)$ is a rest point of (1) for all $0 \le \lambda \le 1$. This contradicts limit asymptotic stability of $(\ {}^*, q^*)$ since this point is then not limit attracting.

If $e_i \cdot Af_j > {}^* \cdot Af_j$ for some j, then choosing perturbations with most of the weight on f_j yields a perturbed game in which $S_i \cdot A\tilde{q}^* > \tilde{\ }^* \cdot A\tilde{q}^*$. Hence, trajectories starting in $((1-\lambda)\,\tilde{\ }^* + \lambda S_i, \tilde{q}^*)$ for $\lambda > 0$ converge to $(S_i, \tilde{q}^*)$ which contradicts the fact that $(\ {}^*, q^*)$ is limit asymptotically stable. ■

Corollary 1 *Let* $(\ {}^*, q^*)$ *be limit asymptotically stable. Then*

i) $(\tilde{\ }^*, \tilde{q}^*)$ *is asymptotically stable in any perturbed game* $\left(\tilde{A}, \tilde{B}\right)$ *of sufficiently small size and*

ii) $(\ {}^*, q^*)$ *is a strictly perfect equilibrium (but not conversely).*[1]

Proof. Part i) follows from the proof of Theorem 1. Part ii) then follows from part i) since $(\tilde{\ }^*, \tilde{q}^*)$ is asymptotically stable and hence a Nash equilibrium by Remark 1. Concerning the converse, consider a game in which $x \cdot Ay = y \cdot Bx = 0$ for all $x \in \Delta^m$ and $y \in \Delta^n$. Then each pair (x, y) is strictly perfect but there is no limit asymptotically stable outcome. ■

Remark 3 *The solution concept of limit asymptotic stability is weaker than that of asymptotic stability and stronger than that of stability. Why? Asymptotically stable strategies strategy pairs under (1) are equivalent to strict equilibria of the asymmetric game (Remark 1). Following Theorem 1 we hence obtain that asymptotically stable outcomes are limit asymptotically stable. On the other hand, uniform continuity of the trajectories of (1) (Weibull, 1995) implies that a limit (asymptotically) stable strategy pair must also be stable in the unperturbed game.*

From Theorem 1, it is easy to construct examples with no limit asymptotically stable strategy such as the following example. Example 2 also illustrates how our insistence that dynamic stability properties holds for arbitrarily small, strictly positive perturbations is a stronger concept than either that of perfect equilibrium or of limit ESS (Selten, 1983) - both of which are defined through static considerations.

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[1] One might also note that each weak ESS (as defined by Hofbauer and Sigmund, 1988 on p. 288) is limit asymptotically stable (but not conversely).

xample 2 *The asymmetric normal form game in (2) has no limit asymptotically stable outcome.*

$$\begin{array}{ccc} & D & A \\ D & 1,0 & 1,0 \\ AD & 0,2 & 4,0 \\ AA & 0,2 & 3,3 \end{array} \tag{2}$$

Strategy pairs of the form $(D, (1-\lambda) D + \lambda A)$ *where* $0 \leq \lambda \leq \frac{1}{4}$ *are the only Nash equilibria. However, none of them are limit asymptotically stable since the only pure strategy NE is* (D, D)*, However, when player one plays* $\tilde{D}$*,* A *is strictly better than* D *for player two when* $\mu_{AA} > 2\mu_{AD}$*. Nevertheless,* (D, D) *is a perfect equilibrium, enforced by choosing a sequence of perturbations with size tending to zero and* $\mu_{AA} < 2\mu_{AD}$*. In the same fashion it follows that* (D, D) *is a limit ESS (Selten, 1983; Samuelson, 1991). Since our concepts must hold for all sufficiently small perturbations, a limit asymptotically stable outcome does not exist in this game.*

In the Male desertion game (Example 1) we saw how stable equilibria $((R, (1-\lambda) l + \lambda r)$ for $\lambda < 1)$ of the unperturbed game can fail to be stable in the perturbed game. However, one of the stable equilibria $((R, r))$ in the same component became stable in the perturbed game. The following example shows a situation where an entire component of stable outcomes becomes unstable in the perturbed game.

xample 3 *Consider the Ultimatum Mini Game as described by Gale et al. (1995). It is given in extensive form in Figure 3.*

Table (3) shows the corresponding normal form,

$$\begin{array}{ccccc} & LL & LR & RL & RR \\ U & 2,2 & 2,2 & 0,0 & 0,0 \\ D & 3,1 & 0,0 & 3,1 & 0,0 \end{array} \tag{3}$$

where AB *(with* A *and* B *in* $\{L, R\}$*) describes the strategy of player two of playing* A *in response to player one choosing* U *and* B *in response to player one choosing* D*.*

Any strategy pair $(U, (1-\lambda) LL + \lambda LR)$ *for* $\lambda > \frac{1}{3}$ *is stable, although none is asymptotically stable, in the unperturbed game. Only* (D, LL) *is asymptotically stable. Gale et al. (1995) show how the addition of a specific noise term to the replicator dynamic (*1*) causes an asymptotically stable equilibrium close to* $\left(U, \frac{1}{2}LL + \frac{1}{2}LR\right)$ *to emerge. The noise results from player two making mistakes (or experimenting) at a substantially higher rate than*

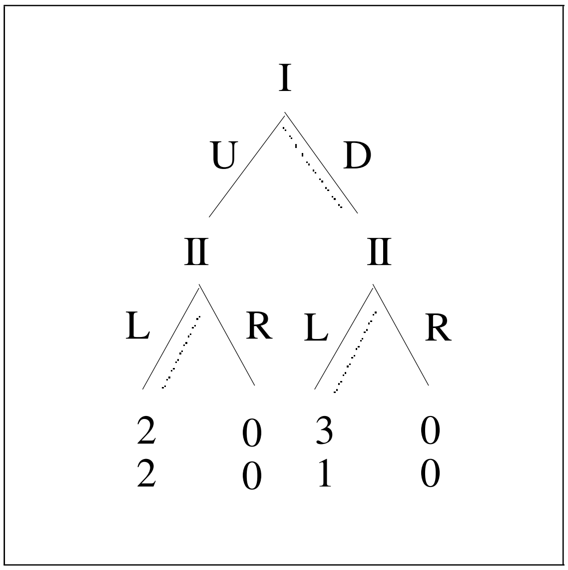

Figure 3: The Ultimatum Mini Game in extensive form.

player one. It should be noted here that, if the rate of mistakes of player one is of the same order as that of player two, then this result no longer holds.

In our framework, mistakes occur at the level of strategy choice. In this case, only the outcome (D, LL) is limit asymptotically stable. This is because LL weakly dominates all other strategies of player two.

In the Male Desertion Game and in the Ultimatum Mini Game, the equilibrium induced by the backwards induction algorithm (i.e., the subgame perfect equilibrium) is the unique limit asymptotically stable outcome. This is generally true in extensive form games where each player never moves twice and there is a unique subgame perfect equilibrium.

Although we do not attempt a complete analysis of dynamic stability of perturbed extensive form games in this paper, the following two theorems and examples give more insight into the relationship between backwards induction and limit asymptotically stable outcomes.

Theorem 2 *In an asymmetric normal form game associated to an extensive form game, limit asymptotically stable outcomes are subgame perfect.*

Proof. If (p^*, q^*) is limit asymptotically stable then (p^*, q^*) is a perfect equilibrium for all sufficiently small perturbations (Corollary 1), especially for perturbations at information sets in the extensive form (see van Damme Def. 6.4.1). Hence (p^*, q^*) is extensive form perfect and consequently a subgame perfect equilibrium (a combination of Theorems 6.4.3 and 6.3.2 in van Damme, 1991). ■

Theorem 3 *Generically, in an extensive form game of perfect information in which no player moves twice on any path, the unique subgame perfect equilibrium is the unique limit asymptotically stable outcome.*

Proof. Assume that the subgame perfect equilibrium of the extensive form game with perfect information is unique, which is generically true. Assume that player one is the player who moves first. Then it follows easily that the strategy of player two induced by the backwards induction algorithm weakly dominates any other strategy of player two. Moreover, the best reply of player one to this strategy of player two is unique. Hence, by Theorem 1 the subgame perfect equilibrium is limit asymptotically stable. Moreover there are no other limit asymptotically stable outcomes because the subgame perfect equilibrium is the unique pure strategy Nash equilibrium of the game. ■

The following example illustrates the statement of the above Theorem.

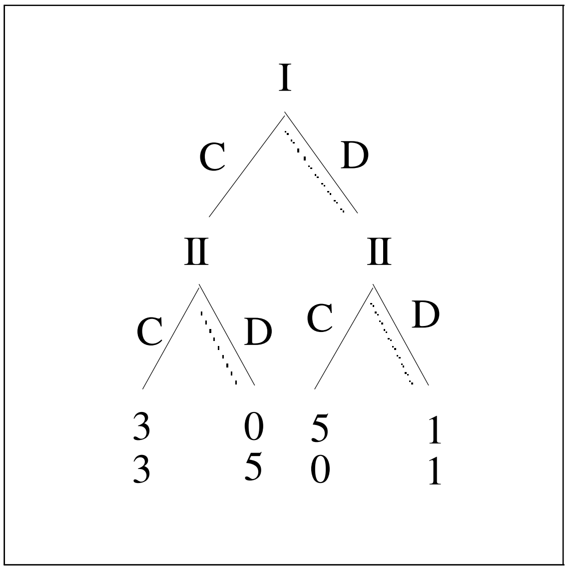

Figure 4: A Sequential Prisoners' Dilemma Game in Extensive Form.

xample 4 *The Sequential Prisoners' Dilemma Game of Figure 4.*
Let AB be the strategy of player two to play A (B) in response to player one playing C (D). Then (D, DD) is the unique subgame perfect equilibrium that is also limit asymptotically stable. Figure 5 shows a partial flow diagram for the dynamic (1) in the unperturbed game. Notice that, in contrast to the Examples 1 and 3, the unperturbed dynamic has no asymptotically stable outcome or set of stable outcomes that are attracting.

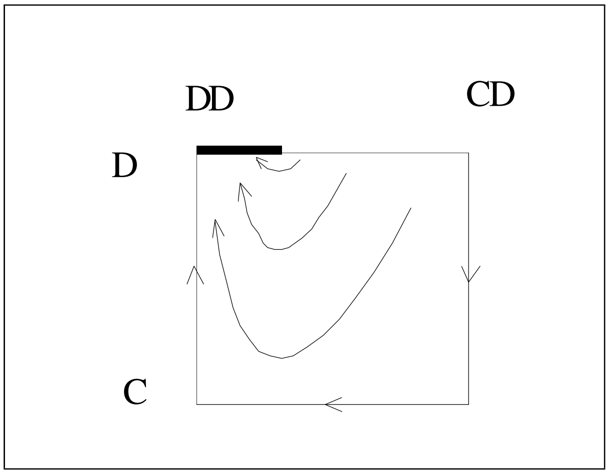

Figure 5: The Flow Diagram for Example 4 on the Face $\{D, C\} \times \{DD, CD\}$.

In simple extensive form games such as those of Theorem 3 and Example 4, limit asymptotic stability selects the backwards induction outcome. The following example shows that this is no longer true in games where a player possibly moves twice.

xample 5 *The Centipede Game of Figure 6.*
Let XY denote the strategy of player one to play X in his first move and Y in the case he is given the opportunity to move again, $X, Y \in \{A, D\}$. Then, regardless of the specific payoffs, if (DD, D) is a subgame perfect equilibrium then it cannot be limit asymptotically stable outcome. This is because the strategies DD and DA of player one are role equivalent which is not allowed by Theorem 1. *One might argue that DD and DA are the same strategy and hence should be summarized under the name D since they cannot be distinguished through the payoffs they generate for either player in this game. Writing up this reduced normal form we obtain the payoffs in (2), a game that has no limit asymptotically stable outcome (see Example 2).*

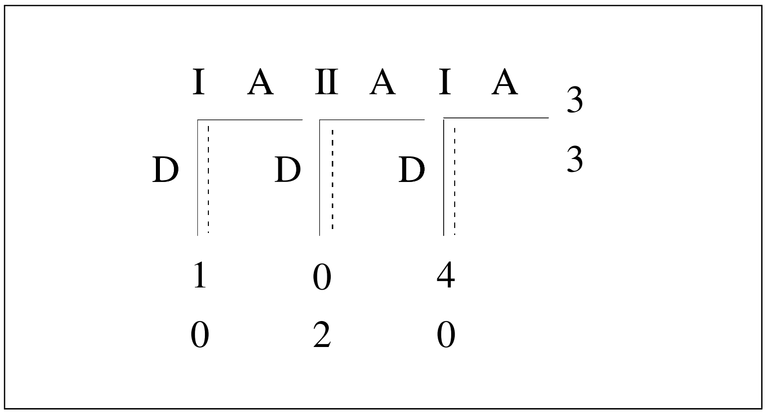

Figure 6: The Centipede Game in Extensive Form.

To conclude our discussion of limit asymptotically stable strategy pairs, we present an extensive form game with two subgame perfect equilibria in which only the equilibrium consistent with forwards induction is limit asymptotically stable.

xample 6 *Consider the extensive form game in Figure 7 for parameters* $a > c$, $d > b$ *and* $a < x < d$. *The corresponding normal form is*

$$\begin{array}{ccc} & L & R \\ A & x, y & x, y \\ T & d, d & c, b \\ B & b, c & a, a \end{array}$$

It is easy to verify that the subgame perfect equilibrium (T, L) *which is consistent with forwards induction is also limit asymptotically stable. On the other hand, the subgame perfect equilibrium* (A, R) *is not limit asymptotically stable since* L *is a pure best reply for player two that is not weakly dominated (*L *is strictly better than* R *against* T*).*

2.3 Limit Asymptotically Stable Sets

Consider the extensive form game in Figure 7 for parameters $a > c$, $d > b$ and $x > d$. We will see that this game has no limit asymptotically stable outcome. Player one chooses his strictly dominant strategy A. However, since both L and R are pure best replies of player two to player one choosing A, neither dominating the other the claim follows. Although the individuals

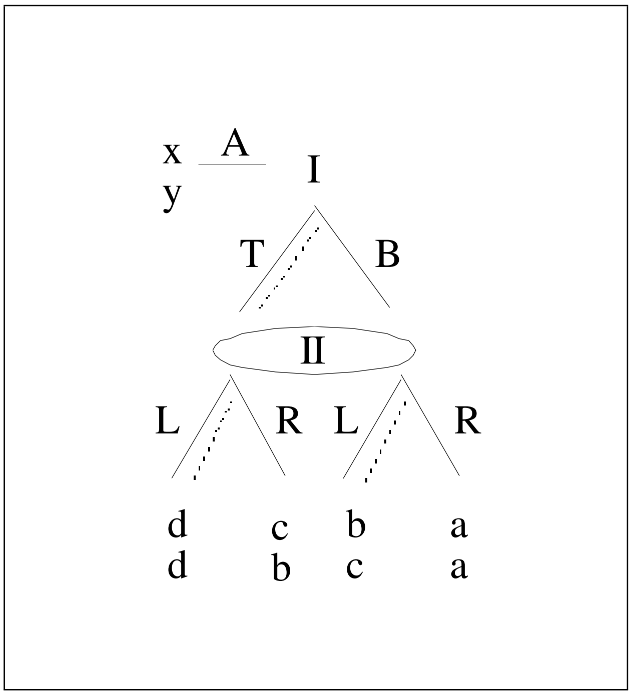

Figure 7: Forwards Induction.

in population two will not agree on how to play independent of the way mistakes occur, the overall play in the population remains most of the time constant, player one choosing A and player two making no decision. This motivates an extension of the concept of limit asymptotic stability to sets.

Another motivation for extending our concept to sets is the fact that spurious duplication of a strategy causes, as is the case for ESS (see Schlag, 1992), limit asymptotically stable pairs to fail to exist. Spurious duplication means a strategy is added that cannot be distinguished in its play from a present strategy except for its name (e.g., DA and DD in the Centipede game of Figure 6). It can be argued (see Schlag, 1992) that spurious duplication should not affect the existence property of an evolutionary concept because this means that the solution depends on how the infinite number of players are collected into behavioral classes. When spurious duplication does occur, it is reasonable to consider the limit asymptotic stability of the face containing a strategy and its duplicates.

Definition 7 *For* $G \subseteq \Delta^m \times \Delta^n$, *let*
$U_\varepsilon(G) = \{(p,q) \text{ s.t. } dist((p',q'),(p,q)) < \varepsilon \text{ for some } (p',q') \in G\}$ *be the* ε

neighborhood *of G. Then the set of strategy frequency pairs $G \subseteq \Delta^m \times \Delta^n$ is called* ***limit stable*** *if each $(p,q) \in G$ is a rest point of the unperturbed dynamic (1) and for every $\varepsilon > 0$ there exists $\delta > 0$ and $\pi > 0$ such that for any perturbations (μ, η) of size at most π, trajectories starting in $U_\delta(G)$ stay in $U_\varepsilon(G)$.*

$G \subseteq \Delta^m \times \Delta^n$ is called ***limit attracting*** *if there exists an open neighborhood U of G such that for every $\varepsilon > 0$ there exists $\pi > 0$ such that for any perturbations (μ, η) of size at most π, all ω limits of trajectories starting in U are contained $U_\varepsilon(G)$.*

G is called ***limit asymptotically stable*** *if it is both limit stable and limit attracting.*

Remark 4 *In the above definition, we required a limit stable set to be a set of rest points. Without this requirement, $\Delta^m \times \Delta^n$ would be limit asymptotically stable. Definition 7 rules out such trivial examples of limit asymptotically stable sets which we do not regard this as a "stable" situation since generally population frequencies in these sets change dramatically over time. In fact, we require that adjustment is slow near G, i.e., that frequencies change arbitrarily slow in a neighborhood of G provided that the perturbations are sufficiently small. A necessary and sufficient condition for this is that G is a set of rest points for the unperturbed dynamic.*

An alternative stronger condition in Definition 7 would be to require each element of G to be limit stable (compare to the definition of set-wise asymptotic stability in Schlag, 1994a). However, this condition is too strong as seen by Remarks 5 and 6 below. With the stronger condition, strict equilibrium sets, the corresponding definition of an evolutionarily stable set (Definition 11) in the two population setting (Balkenborg and Schlag, 1995) need not be limit asymptotically stable.

Definition 8 *A set of strategies $G \subseteq \Delta^m \times \Delta^n$ is called a* ***strict equilibrium set*** *(Balkenborg, 1994) if $(p,q) \in G$ implies*

- *(p,q) is a Nash equilibrium,*
- *$p' \cdot Aq = p \cdot Aq$ implies $(p',q) \in G$ and*
- *$q' \cdot Bp = q \cdot Bp$ implies $(p,q') \in G$.*

Theorem 4 *Any strict equilibrium set is limit asymptotically stable.*

We will prove the above theorem with the help of the one population setting of the next section and hence will postpone it until after Theorem 11.

Remark 5 *In the extensive form game of Figure 7 for parameters* $x > \max\{a, b, c, d\}$*,* $G = \{(A, (1-\alpha)\, L + \alpha R) \text{ s.t. } \alpha \in [0,1]\}$ *is a strict equilibrium set and hence limit asymptotically stable. Notice that none of the elements of* G *are limit stable.*

Remark 6 *Considering dynamic stability for arbitrary perturbations may select outcomes among strict equilibrium sets, as to be seem in Male Desertion Game (Example 1). Here,* $G = \{(R, (1-\lambda)\, l + \lambda r) \text{ s.t. } 0 \leq \lambda \leq 1\}$ *is a strict equilibrium set and hence limit asymptotically stable. Moreover,* $(R, r) \in G$ *is limit asymptotically stable. Notice that in this example,* $(R, l) \in G$ *is not stable in any perturbed game. This does not contradict the definition of limit asymptotically stable sets since in this definition we required set-wise stability and not point-wise stability.*

3 Limit Asymptotic Stability in Symmetric Normal Form Games

3.1 Perturbed Matrix Games and Limit Asymptotically Stable Pure Strategies

The standard model of evolutionary game theory for one population (Taylor and Jonker, 1978) is based on a symmetric normal form game. Each indi-