CORRELATED EQUILIBRIA AND LOCAL INTERACTIONS[1]

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by George J. Mailath, Larry Samuelson, and Avner Shaked

Summary. This paper shows that Nash equilibria of a local-interaction game are equivalent to correlated equilibria of the underlying game.

1 Introduction

It is now common to interpret the Nash equilibria of a game as a description of equilibrium behavior of populations of agents who are randomly matched to play that game. In particular, if there is one population for each player in the underlying game and if the pattern of matching is uniform, then every agent in each population faces the same distribution of opponents' strategies. An equilibrium in which each agent chooses a best reply then corresponds to a Nash equilibrium of the underlying game (where the distribution of strategies in each population is interpreted as a mixed strategy).

However, the matching pattern may not be uniform. If not, then different agents in the same population may face different distributions of opponents' strategies. Moreover, these distributions may be correlated across the individuals. This note makes the obvious point that equilibria in such cases correspond to correlated equilibria (Aumann [1]) of the underying game.

We begin by describing a simple matching model in which the game is played by agents selected from a population. There is a fixed and "local" structure to the interactions, with each agent playing with a set of neighbors. We then observe that any Nash equilibrium, given the local interactions, corresponds to a correlated equilibrium of the underlying game. The different signals received by the players in a correlated equilibrium appear as different possibilities for meeting other agents that arise out of the local nature of the interactions. Conversely, for any correlated equilibrium of the underlying game, we can find a pattern of local interactions such that, fixing this pattern, there is a Nash equilibrium in strategy choices that gives the correlated equilibrium outcome.

Establishing these results requires nothing more than writing the definitions and noting that one set of equations is a straightforward manipulation of the other. The results suggest that when working with matching models, we should be interested in correlated, rather than simply Nash, equilibria. When intepreting the correlation device involved in a correlated equilibrium, we should thus add the possibility of local interactions to the assortment of existing interpretations: a referee, pre-play communication, and observations of correlated signals.

2 Correlated Equilibria and Local Interactions

Let $G = (S, \pi)$ denote a finite, n–player normal form game, where $S = \prod_i S_i$ is the joint strategy set and $\pi = (\pi_1, ..., \pi_n)$ is the payoff function. Suppose nature (or a referee) randomly determines a strategy profile $s \in S$ according to some distribution ξ, and then privately recommends the strategy s_k to player k. If it is a best reply for each player to follow the recommendation, then ξ is a correlated equilibrium:

Definition 1 *A* **correlated equilibrium** *is a probability distribution ξ on S , such that, for all i, if $\xi(s_i) > 0$, then*

$$\sum_{s_{-i} \in S_{-i}} \pi_i(s_i, s_{-i}) \xi(s_{-i}|s_i) \geq \sum_{s_{-i} \in S_{-i}} \pi_i(\hat{s}_i, s_{-i}) \xi(s_{-i}|s_i), \;\; \forall \hat{s}_i \in S_i, \qquad (1)$$

where $\xi(s_{-i}|s_i) = \xi(s_i, s_{-i}) / \xi(s_i)$.

A correlated equilibrium allows a player to contemplate changing his strategy, but does not allow the player to alter the information that is conveyed by the recommended strategy. Different recommendations may imply different payoffs, since they may correspond to different conditional distributions of opponents' recommendations and hence behavior.

We now describe the model of local interactions. For each i, there is a finite population N_i of agents who could fill the role of player i. We assume each population has at least as many members as pure strategies for that player, i.e., $|N_i| \geq |S_i|$. We denote by $\sigma_i : N_i \to S_i$ the function that associates to each member of population i a pure strategy, and write $\sigma \equiv (\sigma_1, ..., \sigma_n)$. A "meeting" is a selection of an agent k_i from each population, who then play the game. The vector $k \equiv (k_1, ..., k_n) \in \prod_i N_i$ is the **cast** of the meeting. The pattern of interactions is then described by the number of times that each cast of agents meets. The interactions are local, since agent 1 of population 1 may meet agent 1 of population 2, while never meeting agent 2 of population 2. [1] Without loss of generality, we normalize the total number of meetings to one. The proportion of meetings between agents k_i and k_j can then also be described as the probability that, given a meeting, it involves players k_i and k_j. The interactions between the populations are thus described by a probability distribution μ on the space $\prod_i N_i$, with $\mu(k)$ interpreted as either the proportion of meetings that involve the cast k, or as the probability that, given a meeting, it involves the cast k.

Definition 2 *An $(n+1)$-tuple $(\sigma;\mu)$, consisting of an assignment of strategies to agents σ and an interaction distribution μ, is an* **equilibrium with local interactions** *if, for all i, for all $k_i \in N_i$, and $\forall \hat{s}_i \in S_i$,*

$$\sum_{k_{-i}\in N_{-i}} \pi_i(\sigma_i(k_i),\sigma_{-i}(k_{-i}))\mu(k_i,k_{-i}) \geq \sum_{k_{-i}\in N_{-i}} \pi_i(\hat{s}_i,\sigma_{-i}(k_{-i}))\mu(k_i,k_{-i}). \tag{2}$$

Just as a correlated equilibrium does not allow a player to alter the information that is conveyed by a recommendation, an agent in a local-interaction model cannot affect the mix of opponents with whom he or she plays the game. And, just as for correlated equilibria, an agent may prefer a different mix of opponents.

Given an assignment of strategies to agents σ and an interaction distribution μ, the probability that the strategy profile $s^* \in S$ is played in a meeting is denoted $\xi_{\sigma,\mu}(s^*) = \sum_{\{k:\sigma(k)=s^*\}} \mu(k)$.

It is straightforward to support *some* outcomes that are correlated, but not Nash, equilibria as equilibria with local interactions. Suppose the game G is a battle of the sexes. Suppose each population has two agents (or two groups of agents), called α_i and β_i. Let the matching be such that the only casts that result are the $\alpha = (\alpha_1,\alpha_2)$ cast and the $\beta = (\beta_1,\beta_2)$ cast. One equilibrium is then for the α cast to play one of the pure strategy Nash equilibria of G and for the β cast to play the other. This equilibrium corresponds to a convex combination of Nash equilibria of the game G.

A few lines of algebra shows that, in fact, *any* correlated equilibrium can be supported as an equilibrium with local interactions, and the converse holds as well:

[1] This is to be contrasted with global (also sometimes called uniform, or symmetric) interactions, where the probability of cast k meeting is $|\prod_i N_i|^{-1}$ for all k. In this case, the interactions are symmetric: If there is a meeting involving agent 2 of population 1 and agent 2 of population 2, then there is the same chance of a meeting involving agent 1 of population 1 and agent 2 of population 2.

Proposition 1 *(1.1) If $(\sigma;\mu)$ is an equilibrium with local interactions, then $\xi_{\sigma,\mu}$ is a correlated equilibrium.*

(1.2) If ξ is a correlated equilibrium, then there exists an equilibrium with local interactions $(\sigma;\mu)$ such that $\xi_{\sigma,\mu}=\xi$.

Proof (1.1) Fix an equilibrium with local interactions $(\sigma;\mu)$, and an agent k_i of player i. We first observe that

$$\sum_{k_{-i}}\pi_i(\hat{s}_i,\sigma_{-i}(k_{-i}))\mu(k_i,k_{-i})=\sum_{s_{-i}}\pi_i(\hat{s}_i,s_{-i})\sum_{\{k_{-i}:\sigma_{-i}(k_{-i})=s_{-i}\}}\mu(k_i,k_{-i}),$$

where (as usual), if $\{k_{-i}:\sigma_{-i}(k_{-i})=s_{-i}\}=\emptyset$, then summing over that index set yields zero. Equation (2) can then be rewritten as

$$\sum_{s_{-i}}\pi_i(\sigma_i(k_i),s_{-i})\sum_{\{k_{-i}:\sigma_{-i}(k_{-i})=s_{-i}\}}\mu(k)\geq\sum_{s_{-i}}\pi_i(\hat{s}_i,s_{-i})\sum_{\{k_{-i}:\sigma_{-i}(k_{-i})=s_{-i}\}}\mu(k).$$

Fix a strategy s_i that is played by some agent of player i and sum over all agents playing this strategy. This yields

$$\sum_{s_{-i}}\pi_i(s_i,s_{-i})\sum_{\{k:\sigma(k)=s\}}\mu(k)\geq\sum_{s_{-i}}\pi_i(\hat{s}_i,s_{-i})\sum_{\{k:\sigma(k)=s\}}\mu(k).$$

Dividing both sides by $\sum_{\{k_i:\sigma_i(k_i)=s_i\}}\sum_{k_{-i}}\mu(k)$ (>0) yields equation (1), since

$$\xi_{\sigma,\mu}(s_{-i}\mid s_i)=\frac{\xi_{\sigma,\mu}(s_i,s_{-i})}{\xi_{\sigma,\mu}(s_i)}=\frac{\sum_{\{k:\sigma(k)=s\}}\mu(k)}{\sum_{\{k_i:\sigma_i(k_i)=s_i\}}\sum_{k_{-i}}\mu(k)}.$$

Hence, $\xi_{\sigma,\mu}$ is a correlated equilibrium.

(1.2) Fix a correlated equilibrium ξ, and, for each i, select $|S_i|$ agents from N_i. We refer to these as the "active" agents. Next, construct the functions σ_i by assigning each of the pure strategies in S_i to one of the $|S_i|$ agents we have selected from N_i. Then, set $\mu(k)$ (the probability that the cast k meets) to $\xi(\sigma(k))$ if every agent in k is active, and to zero otherwise. It is straightforward to verify that the inequalities contained in (1) coincide with those of (2), and the fact that ξ is a correlated equilibrium implies that we have constructed an equilibrium with local interactions. $\square$

It may be helpful to expand on the argument in part (1.1). Beginning with an equilibrium with local interactions, we construct a correlated equilibrium by recommending the pure strategy combination s with the same probability that the equilibrium with local interactions produces a match in which s is played. Now, suppose player i receives a recommendation to play strategy s_i^*. The distribution over S_{-i}, describing the opponents' strategy profile conditional on receiving a recommendation to play s_i^*, is a weighted average of the distributions over opponents' strategy profiles faced by the player-i agents in the equilibrium-with-local-interactions who play s_i^*. But if s_i^* is a best reply to the distribution over opponents' strategies facing each of these agents, then it is a best reply to the weighted average of these distributions. Hence, it is a best reply to play s_i^* when it is recommended, ensuring that we have a correlated equilibrium.

The construction of the local interaction equilibrium in part (1.2) potentially leaves large numbers of agents with no possibility of meeting other agents and so of playing the game. As the example of the next section shows, it is straightforward to bring these players into the game by replacing the individual players in our construction with groups of players.

Suppose G is symmetric.[2] The model of local interactions described above assumes that there is a distinct population of agents for each player. In the parlance of evolutionary game theory, there is role identification. An alternative assumption is that of no role identification. In this case, there is only one population of agents and a meeting involves a drawing of k agents from this single population. Each agent then chooses a strategy, not knowing which player's role they are filling. It is straightforward to show, along the same lines as the proof of the proposition, that every equilibrium with local interactions (where all agents are drawn from the same population) induces a symmetric correlated equilibrium, and conversely, that every symmetric correlated equilibrium can be represented as an equilibrium with local interactions (where all agents are drawn from the same population).

Finally, it is also possible for a general game G to model the local interactions using a single population, as long as any meeting involves not only a specification of the cast, but also the assignment of roles. We could also have defined a correlated equilibrium for the game G where *a priori* the players do not know which role they will play. [3] There are then two candidates for the definition of a correlated equilibrium. In the first, as before, only a strategy is recommended. In the second, a strategy is recommended and the player is informed of his or her role. The same style of argument as in

[2]A game G is symmetric if $S_i = S_j$, and if, for all permutations θ of $\{1, ..., n\}$ and for all s and t such that $s_j = t_{\theta(j)}$, we have $\pi_i(s_i, s_{-i}) = \pi_{\theta(i)}(t_{\theta(i)}, t_{-\theta(i)})$.

[3]If the player strategy sets are pairwise disjoint, then the strategy recommended obviously unambiguously reveals the role.

part (1.1) shows that the sets of equilibrium outcomes are identical. If the same strategy is optimal in different roles, then it is optimal for any beliefs over those roles.

3 An Example

Consider the following version of the game "chicken":

	L	R
T	$4,4$	$1,5$
B	$5,1$	$0,0$

The Nash equilibria of this game are the two pure strategy equilibria (B, L) and (T, R), as well as the mixed strategy $(.5T + .5B, .5L + .5R)$, with payoffs of $(5,1)$, $(1,5)$, and $(5/2, 5/2)$, respectively. There is also a continuum of correlated equilibria that are not Nash equilibria. The symmetric efficient one places probability $1/3$ on each of the outcomes (B, L), (T, L), and (T, R).

We describe two simple models of local interactions with equilibria that coincide with the symmetric efficient correlated equilibrium. In the first, each population consists of two agents (or select two agents from each population). In population 1, one of these agents plays T and one plays B. In population 2, one of these agents plays L and one plays R. The selected agents in population 1 we name "agent 1_a" and "agent 1_b " and those in population 2 we name " agent 2_a" and "agent 2_b". The interaction distribution μ is given by the following, where the number in each cell is the probability that the corresponding row and column agent meet and where each agent's strategy is shown in parentheses:

	agent 2_a (L)	*agent* 2_b (R)
agent 1_a (T)	$\frac{1}{3}$	$\frac{1}{3}$
agent 1_b (B)	$\frac{1}{3}$	0

This yields an equilibrium with local interactions that coincides with the symmetric efficient correlated equilibrium.

As remarked above, since each population may contain more than two agents, this construction may leave most agents with no opportunity for playing the game. We can avoid this by dividing the matching probabilities over multiple agents. For example, we can specify three active agents in each population with strategies and interactions be given by:

	$agent\ 2_a\ (L)$	$agent\ 2_b\ (L)$	$agent\ 2_c\ (R)$
$agent\ 1_a\ (T)$	$\frac{1}{6}$	0	$\frac{1}{6}$
$agent\ 1_b\ (T)$	0	$\frac{1}{6}$	$\frac{1}{6}$
$agent\ 1_c\ (B)$	$\frac{1}{6}$	$\frac{1}{6}$	0

This again yields an equilibrium with local interactions that corresponds to the symmetric efficient correlated equilibrium. Similar constructions allow all agents to be active.

References

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