Pricing the american put option: A detailed convergence analysis for binomial models

Dietmar P.J. Leisen*

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Dietmar P.J. Leisen: Department of Statistics
University of Bonn
Adenauerallee 24–42
D–53113 Bonn, Germany
Tel.: +49 228 739263
Fax.: +49 228 735050
e–mail: leisen@addi.finasto.uni-bonn.de

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Abstract

Viewing binomial models as a discrete approximation of the respective continuous models, the interest focusses on the notions of convergence and especially "fast" convergence of prices. Though many authors were proposing new models, none of them could successfully explain better performance for their models, since they all lacked a measure of convergence speed. In the case of the european call option Leisen and Reimer[96] examined convergence speed by the order of convergence in a rigorous framework. However the analysis could not be transformed to the case of american put options. For the model of Cox, Ross and Rubinstein[79] Lamberton[95] addressed the same problem. For american put options he derived that the error is bounded by suitable sequences with order of convergence 1/2 from above and by 2/3 from below. However the simulation results of Broadie and Detemple[96] suggest order one. One aim of this paper is to improve this result and extend it to different lattice approaches. We establish the result that the model of Cox, Ross and Rubinstein[79] converges with order one. From a general theorem follows for the models of Jarrow and Rudd[83] and Tian[93] that the error is bounded by order one from above and 1/2 from below. Thus none of these three models performs better in comparison to the other. In a further step an error representation is derived using the concept of order of convergence. This allows an error analysis of extrapolation. Moreover we study the Control Variate technique introduced by Hull and White[88]. Since the investigation reveals the need for smooth converging models in order to get smaller initial errors, such a model is constructed. The different approaches are then tested: simulations exhibit up to 100 times smaller initial errors.

Keywords

binomial model, option valuation, order of convergence, smoothing, extrapolation, Control Variate technique

1. Introduction

In their celebrated work Black and Scholes[73] introduced a new framework into the theory of option valuation using the notions of hedging and arbitrage–free pricing. Later Harrison and Kreps[79] and Harrison and Pliska[81] developed the concept of equivalent martingale measure. This concept gave an elegant technique to express and solve pricing problems. Bensoussan[84] and Karatzas[88] transformed this technique to the case of the american put option. In this context the price is determined by an optimal stopping problem; the price–process can be described as the smallest supermartingale majorant to the discounted payoff ("Snell enveloppe"). This problem was already studied by McKean[65] and transformed into a free boundary problem. Moreover he represented the stopping time in terms of the so called early–exercise boundary and the option price as a function of this boundary. Van Moerbeke[76] derived properties of the boundary. After McKean[65] many authors were dealing with representations of the price in terms of the boundary; a very intuitive in our eyes was given recently by Carr, Jarrow and Myneni[92]. For an overview of the state–of–the–art in continuous time we refer the reader to Myneni[92].

Though the american put option is of great interest in practice, up to now no closed–form or analytical solution to the price nor to the boundary is known, yet. Therefore there is an abundance of numerical work on this subject.

A straightforward approach is dealing with analytic approximations. The best known of these are quadratic approximations which were developed by MacMillan[86] and extended by Barone–Adesi and Whaley[87]. However such approximations cannot be made arbitrarily accurate.

Another approach starts from a discretization of the partial differential equation describing the free boundary problem. This method of finite differences was originally proposed by Brennan and Schwartz[77]. Using variational inequalities the algorithm was justified completely only recently by Jaillet, Lamberton and Lapeyre[90].

This paper sticks to the broad field of binomial models, of which the first was proposed by Cox, Ross and Rubinstein[79] (CRR). They are constructed in such a way that if the time between two trading dates shrinks to zero, convergence (weakly in distribution) to their continuous counterpart is achieved. In these models american put options can be priced very easily by the Bellman principle of dynamic optimization, which is justified very intuitively from arbitrage arguments.

Though in the case of european call and put options convergence of prices is ensured very easily from weak convergence of the processes, things are much more complicated in the case of the american put option, since in general convergence of maxima over expectations on functionals on the processes — which are the prices — cannot be derived from weak convergence only (see Aldous[81]). However a proof can be deducted from Kushner[77] in a sligthly different context and more recently in Lamberton and Pagès[90].

There are numerous binomial approaches and extensions. One mainstream is dealing with "better" stock price approximations in comparison to CRR. Jarrow and Rudd[83] (JR) adjusted their model to account for the local drift term. Tian[93] argued that matching discrete and continuous local moments should yield "better" convergence. Actually, though these works worry about better convergence, none of them resolved it fully for the lack of a proper definition.

These problems were addressed by Leisen and Reimer[96]. They defined the speed of convergence by the concept of order of convergence. It was shown a general theorem for determining this in the case of the european call option. Using this they concluded that in this sense the presented models of CRR, JR and Tian are equal; they all converge with order one. In a second step a model with higher order two was constructed.

In this paper we first give a short introduction to the (discrete and continuous) models and the basic notation (section 2). In a next step we will then extend the theorem derived by Leisen and Reimer[96] to the case of the american put option. This allows to determine order of convergence one for the models of CRR and one from above resp. 1/2 from below for the models of JR and Tian (section 3). The information about the type of convergence is then used for an error representation. This allows to analyze in detail two ad hoc improvements common in practice: Richardson extrapolation and the Control Variate technique introduced by Hull and White[88]. Since the analysis reveals the need for smoothing the convergence behaviour of price calculations, we construct a new model for calculating european put option prices (section 4). Though this model is very simple, it yields order two by extrapolation. In section 5 we present a numerical analysis of different binomial models for the american put option. It turns out that extrapolation yields initial errors that are up to 100 times smaller than previous binomial models.

2. The Framework

Throughout the following paper we suppose to be given a constant interest rate $r \geq 0$ and a constant volatility $\sigma > 0$. Continuous capital markets are modelled by a stock price process $(S_t)_{t\geq 0}$ following geometric Brownian motion, i.e.:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where $(W_t)_{t\geq 0}$ is a standard Wiener process on some probability space $(\Omega, \mathcal{F}, P)$. Please note, that here we immediatly introduced the risk–neutral probability measure P according to Harrison and Pliska[81].

In this theory the price $Put^e(t,S)$ of a european put with strike K when time–to–maturity equals $T-t$ and the stock–value equals S is the well known Black–Scholes formula:

$$\begin{aligned} Put^e(t,S) &= K \cdot e^{-r(T-t)} \mathcal{N}(-d_2) - S \cdot \mathcal{N}(-d_1) \\ d_{1,2} &= \frac{\ln(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

where $\mathcal{N}(\cdot)$ is the cumulative standard normal distribution function.
Things get complicated when dealing with american put options. Suppose we are given a fixed american put with strike K and maturity date T.
Denote the price function by $Put^a(t,S)$. From Van Moerbeke[76] it follows that there exists a critical stock price B_t, below which the option should always be exercised ($Put^a(t,S) = (K-S)^+$ for $S \leq B_t$) and above which it should never be exercised ($Put^a(t,S) > (K-S)^+$ for $S > B_t$). The function $t \mapsto B_t$ is a smooth, nondecreasing function of time t which terminates in the strike price ($B_T = K$). It is called the (early–exercise) **Boundary**.
The Boundary separates the domain $\mathcal{D} = [0,T] \times \mathbb{R}^+$ into the continuation region $\mathcal{C} := \{(t,S) \in \mathcal{D} | S > B_t\}$ and the stopping region $\mathcal{S} := \{(t,S) \in \mathcal{D} | S \leq B_t\}$.
Binomial models are a description of discrete asset price dynamics. They specify a number n of trading dates. Trading occurs only at the equidistant spots of time $t_i^n \in \mathcal{T}^n := \{0 = t_0^n, \cdots, t_n^n = T\}$ with $t_{i+1}^n - t_i^n := \Delta t_n := \frac{T}{n} \quad (i = 0, \ldots, n-1)$. In order to achieve a complete market model, the one–period returns $\overline{R}_{n,i} \quad (i = 1, \ldots, n)$ are modelled by two point iid binomial random variables

$$\overline{R}_{n,i} = \begin{cases} u_n & \text{with probability} & \overline{p}_n \\ d_n & \text{with complementary probability} & 1 - \overline{p}_n \equiv \overline{q}_n \end{cases}$$

on a suitable probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$. Therefore the discrete asset price dynamics is $(\overline{S}_{n,k})_k$ where the price $\overline{S}_{n,k}$ at time t_k is described by

$$\overline{S}_{n,k} = S_0 \cdot \prod_{i=1}^{k} \overline{R}_{n,i}$$

The specification of the one–period returns is a complete description of the discrete dynamics $\overline{S}_n$. We call such a finite sequence $\overline{R}_n = (\overline{R}_{n,i})_{i=1,\ldots,n}$ a **lattice (tree)**.
In the sequel we will suppose always that there is given a whole sequence of lattices. One should think of it as a triangular array

$$\begin{array}{llll} \overline{R}_{1,1} & & & \\ \overline{R}_{2,1} & \overline{R}_{2,2} & & \\ \overline{R}_{3,1} & \overline{R}_{3,2} & \overline{R}_{3,3} & \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

where each row represents a lattice.
A method which assigns to each refinement n a lattice is called a **lattice approach**.
In order to compare it with the continuous model denote for any $n \in \mathbb{N}$ for $i = 1, \ldots, n$ by $R_{n,i}$ the continuous return between times t_i^n and t_{i+1}^n. For n fixed they are iid random variables on $(\Omega, \mathcal{F}, P)$ such that $S_{t_k^n} = S_0 \prod_{i=1}^k R_{n,i} \quad \forall k = 0, \ldots, n$.

Several different lattice–approaches have been proposed. The model of CRR takes

$$\begin{aligned} u_n &= \exp\left\{\sigma\sqrt{\Delta t_n}\right\} \\ d_n &= \exp\left\{-\sigma\sqrt{\Delta t_n}\right\} \end{aligned}$$

Moreover they take into account that the risk–neutrality argument of Harrison and Pliska[81] requires that the expected one–period return $\overline{E}[\overline{R}_{n,1}]$ must equal the one period return of the riskless bond $r_n = \exp\{r\Delta t_n\}$. This amounts to setting $\overline{p}_n = \frac{u_n - r_n}{u_n - d_n}$.
The risk–neutrality argument amounts to matching discrete and continuous first moment. Tian's parameter selection requires the second and third moments to be equal, too:

$$\begin{aligned} u_n &= \frac{r_n v_n}{2}\left(v_n + 1 + \sqrt{v_n^2 + 2v_n - 3}\right) \\ d_n &= \frac{r_n v_n}{2}\left(v_n + 1 - \sqrt{v_n^2 + 2v_n - 3}\right) \\ \text{where} \quad v_n &= \exp\{\Delta t_n\} \end{aligned}$$

JR argue in terms of the gross–return. Adding the local drift–term $\mu'\Delta t_n$ yields:

$$\begin{aligned} u_n &= \exp\left\{\sigma\sqrt{\Delta t_n} + \mu'\Delta t_n\right\} \\ d_n &= \exp\left\{-\sigma\sqrt{\Delta t_n} + \mu'\Delta t_n\right\} \\ \text{where } \mu' &= r - \frac{\sigma^2}{2} \end{aligned}$$

Moreover they have $\overline{p}_n = \frac{1}{2}$.

3. Characterization of errors

Now suppose we are given some fixed stock S_0 and a contingent claim. Denote its continuous time price by p_∞. Moreover suppose we study a lattice–approach yielding a sequence $(\overline{R}_n)_n$ of lattices. From this sequence we can calculate a sequence $(p_n)_n$ of discrete prices. We know from Kushner[77] and Lamberton and Pagès[90] that discrete american put prices converge to the continuous price p_∞. That means, if we denote by $e_n := |p_\infty - p_n|$ the error each lattice produces, we have $\lim_{n\to\infty} e_n = 0$.
A straightforward way to measure convergence speed is by comparing it with those of the sequences $\left(\frac{1}{n}\right)_n, \left(\frac{1}{n^2}\right)_n, \ldots$. That is, we use the mathematical concept of "order of convergence". Restated in our specific case here, we adopt the following

Definition 3.1:
Let $(\overline{R}_n)_n$ a sequence of lattices. A sequence of prices $(p_n)_n$ calculated from the lattices **converges with order** $\rho > 0$ *if there exists a constant $\kappa > 0$ such that*

$$\forall n \in \mathbb{N} : e_n \leq \frac{\kappa}{n^\rho}$$

In the sequel we will often write shortly $e_n = \mathcal{O}(\frac{1}{n^\rho})$ for this, too.

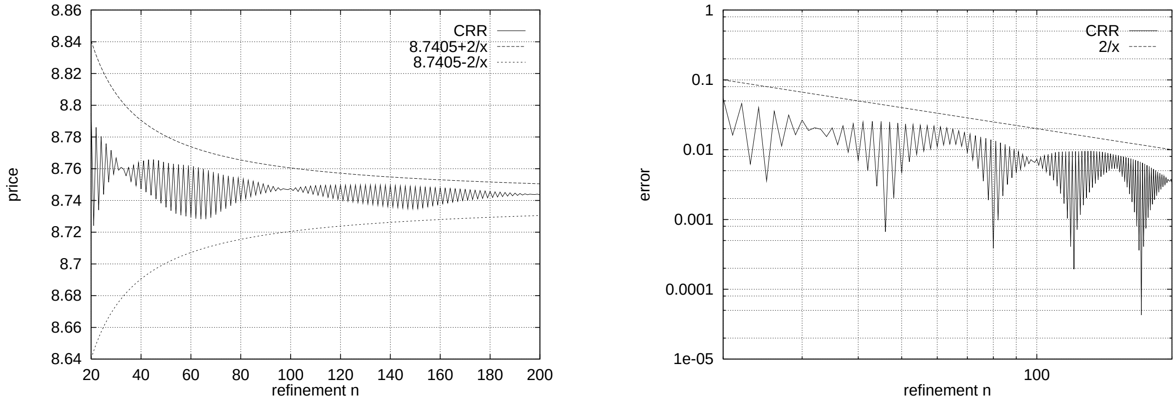

FIGURE 3.1. typical pattern with bounding error functions resulting from american put option price calculations with the CRR-Model and the following selection of parameters: $S = 100$, $K = 105$, $T = 1$, $r = 0.05$, $\sigma = 0.2$, $n = 10, \ldots, 200$

Please note that convergence of prices is implied by any order greater than 0. Moreover we remark that higher order means "quicker" convergence. Thus the theoretical concept of order of convergence is not unique: a lattice approach with order ρ has also order $\tilde{\rho} \leq \rho$. Though the concept of order of convergence may seem very theoretical, it is very easy to observe in actual simulations. Because of $\log \frac{\kappa}{n^\rho} = \log \kappa - \rho \log n$ the bounding function $\frac{\kappa}{n^\rho}$ becomes a straight line with slope equal $(-\rho)$ and shift κ on a log–log–scale. So when plotting e_n on a log–log–scale, determining the order of convergence consists in looking for the slope of a suitable bounding straight line (see figure 3.1).
Leisen and Reimer[96] were looking for factors of the lattice approach under consideration that determine the order of convergence for european call options. The following (pseudo–)moments turned out to fulfill this.

Definition 3.2:
For a sequence of lattices $\left(\overline{R}_n\right)_{n\in\mathbb{N}}$ we call for all $n \in \mathbb{N}$:

$$\begin{aligned} \mathfrak{m}_n^1 &:= \overline{E}\left[\overline{R}_{n,1} - 1\right] - E\left[R_{n,1} - 1\right] \\ \mathfrak{m}_n^2 &:= \overline{E}\left[\left(\overline{R}_{n,1} - 1\right)^2\right] - E\left[\left(R_{n,1} - 1\right)^2\right] \\ \mathfrak{m}_n^3 &:= \overline{E}\left[\left(\overline{R}_{n,1} - 1\right)^3\right] - E\left[\left(R_{n,1} - 1\right)^3\right] \end{aligned}$$

moments *and*

$$\mathfrak{p}_n := \overline{E}\left[\left(\ln \overline{R}_{n,1}\right)\left(\overline{R}_{n,1} - 1\right)^3\right]$$

pseudo–moment

These moments are mainly the difference between the ordinary moments of discrete and continuous approach. Therefore they represent a generalisation of the ordinary moments. The form of the pseudo–moment is of technical nature as it resulted from the proof of the following theorem 3.1. Please note that $\mathfrak{m}_n^1 = 0$ from the risk neutrality argument of Harrison and Pliska[81].

In the case here, where we have a discrete approximation of a continuous framework, it turns out that the order of convergence is determined by the difference of the ordinary moments, that is by that of our moments. This is exactly what theorem 3.1 stated and proven in Leisen and Reimer[96] says.

Theorem 3.1:

Let $(\overline{R}_n)_{n\in I\!N}$ be a sequence of lattices and $\mathfrak{m}_n^2, \mathfrak{m}_n^3, \mathfrak{p}_n$ its respective (pseudo–) moments. The order of convergence in calculating european call option prices is the smallest order contained in $\mathfrak{m}_n^2, \mathfrak{m}_n^3$ and $\mathfrak{p}_n$ reduced by 1, but not smaller than 1, i.e.:

$$\exists\ \kappa(S_0, K, r, \sigma, T) : e_n \leq \kappa \cdot \left\{ n \cdot \left(\mathfrak{m}_n^2 + \mathfrak{m}_n^3 + \mathfrak{p}_n\right) + \frac{1}{n} \right\}$$

Theorem 3.2:

Under the assumptions of theorem 3.1 the same results hold for european put options.

Proof. This is an immediate consequence of put–call parity. □

Proposition 3.1:

The lattice–approaches of CRR, JR and Tian fulfill:

$$\begin{aligned} \mathfrak{m}_n^2 &= \mathcal{O}\left(\frac{1}{n^2}\right) \\ \mathfrak{m}_n^3 &= \mathcal{O}\left(\frac{1}{n^2}\right) \\ \mathfrak{p}_n &= \mathcal{O}\left(\frac{1}{n^2}\right) \end{aligned}$$

Proof. see the Appendix in Leisen and Reimer[96] □

Theorem 3.2 and Proposition 3.1 immediatly yield the following result:

Corollary 3.1:

European put option prices calculated using the lattice–approaches of CRR, JR and Tian converge with order one.

The question may now arise, wheather it would be possible to strengthen the result of theorem 3.2 in order to prove higher order. We now state a theorem which says that this bound is actually best achievable. The idea and the proof are from David Heath.

Theorem 3.3:

Given a sequence $(\overline{R}_n)_{n\in I\!N}$ of lattices with $u_n/d_n = 1 + \mathcal{O}(\sqrt{\Delta t_n})$, there always exists a strike price K such that the prices calculated for this european put option have error $e_n \geq c^*/n$ for a suitable constant $c^* \in I\!R$.

Proof. The price Put^e of a european put is a strictly convex function in its strike price K. Therefore, for sufficiently high refinement n, we can always find some $C > 0, K_1 < K_2$ such that:

$$\frac{\partial^2 Put^e(K)}{\partial K^2} \geq C, \text{ for all } K \in [K_1, K_2]$$

Let $K^* = \frac{K_1+K_2}{2}$ and I_n denote the intervall between successive terminal stock prices which contain K^*.
Then:

$$|I_n| \geq \left(\frac{u_n}{d_n} - 1\right) K^*$$

Put^e_n is a linear function in the strike price K on the interval I_n.
Therefore:

$$\frac{\partial^2 Put^e(K)}{\partial K^2} = 0 \text{ on } I_n$$

Let $e_n(K) := Put^e(K) - Put^e_n(K)$ denote the error depending on the strike price K.
Then:

$$\frac{\partial^2 e_n}{\partial K^2} = \frac{\partial^2 Put^e(K)}{\partial K^2} \geq C \text{ on } I_n \subset [K_1, K_2]$$

Integrating twice yields

$$\sup_{K \in I_n} |e_n| \geq \frac{|I_n|^2}{16} \inf_{K \in I_n} \left|\frac{\partial^2 e_n}{\partial K^2}\right| \geq \frac{C \cdot |I_n|^2}{16} \geq \frac{C \cdot |K^*|^2}{16} \left(\frac{u_n}{d_n} - 1\right)^2 \geq \frac{C^*}{n}$$

for a suitable constant C^*. □

Obviously the lattice–approaches of CRR, JR and Tian fulfill the condition $u_n/d_n = 1 + \mathcal{O}(\sqrt{\Delta t_n})$ in theorem 3.3. The following two theorems will state a result similar to that of the european put option for the american put option.

Theorem 3.4:
Let $(\overline{R}_n)_{n \in \mathbb{N}}$ *be a sequence of lattices and* $\mathfrak{m}^2_n, \mathfrak{m}^3_n, \mathfrak{p}_n$ *its respective (pseudo–) moments. There exists a positive constant* $\kappa_u(S_0, K, r, \sigma, T)$ *such that:*

$$Put^a(0, S_0) - Put^a(0, S_0) \leq \kappa_u \cdot \left\{ n \cdot \left(\mathfrak{m}^2_n + \mathfrak{m}^3_n + \mathfrak{p}_n\right) + \frac{1}{n} \right\}$$

Proof. Denote by $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ the product space of $(\Omega, \mathcal{F}, P)$ and $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$. For all $n \in \mathbb{N}$ and $k = 0, \dots, n$ let $\mathcal{A}_{n,k} = \sigma(\overline{S}_{n,i} | i \leq k)$ denote the information structure.
From Carr, Jarrow and Myneni[92] we know that the price of the american put can be decomposed into the price of a european put and the early–exercise premium π, which takes the form:

$$\pi = rK \int_0^T e^{-rt'} \mathcal{N}(b_{2,0}(S_0, t'))dt'$$

where $b_{2,0}(x,t') = \frac{\ln B_t/x - (r - \frac{\sigma^2}{2})t'}{\sigma\sqrt{t'}}$.
Lemma 2 in the Appendix tells us that stopping the discrete process $(\overline{S}_{n,k})_{k=0,\dots,n}$ according to the rule rule $(B_{t_k^n})_{k=0,\dots,n}$ yields the premium:

$$\pi_n^B = \hat{E}\left[\sum_{k=0}^{n-1} e^{-rt_k^n} \cdot K(1-e^{-r\Delta t}) \cdot \hat{P}\left[\overline{S}_{n,k} < B_{t_k^n} \;\middle|\; \mathcal{A}_{n,k}\right]\right] + \mathcal{O}(\Delta t)$$

The optimal stopping policy, however, yields the higher premium π_n.
Therefore we have according to Lemma 5 in the Appendix:

$$\exists\ \kappa_u(S_0, K, r, \sigma, T) : \pi - \pi_n \leq \kappa_u \Delta t$$

Since

$$|Put^a(0,S_0) - Put_n^a(0,S_0)| \leq |Put^e(0,S_0) - Put_n^e(0,S_0)| + |\pi - \pi_n|$$

the theorem now follows immediatly from theorem 3.1. □

Theorem 3.5:
Let $(\overline{R}_n)_{n\in\mathbb{N}}$ be a sequence of lattices and $\mathfrak{m}_n^2, \mathfrak{m}_n^3, \mathfrak{p}_n$ its respective (pseudo–) moments. There exists a negative constant $\kappa_l(S_0, K, r, \sigma, T)$ such that:

$$Put^a(0,S_0) - Put^a(0,S_0) \geq \kappa_l \cdot \left\{ n \cdot \left(\mathfrak{m}_n^2 + \mathfrak{m}_n^3 + \mathfrak{p}_n\right) + \frac{1}{\sqrt{n}} \right\}$$

Moreover for CRR we have:

$$Put^a(0,S_0) - Put^a(0,S_0) \geq \kappa_l \cdot \left\{ n \cdot \left(\mathfrak{m}_n^2 + \mathfrak{m}_n^3 + \mathfrak{p}_n\right) + \frac{1}{n} \right\}$$

Proof. According to Lemma 6 in the Appendix:

$$\mathcal{O}(\Delta t) = \hat{E}\left[Put^a(t_k^n, \overline{S}_{n,k}) - Put_n^a(t_k^n, \overline{S}_{n,k})\right]$$

According to Lemma 1 in the Appendix this equals:

$$
\begin{aligned}
& \hat{E}\Big[Put^e(t_k^n,\overline{S}_{n,k}) + rK\int_{t_k^n}^{T} e^{-r(t-t_k^n)}\mathcal{N}(b_{2,t_k^n}(\overline{S}_{n,k},t))dt - Put_n^e(t_k^n,\overline{S}_{n,k}) \\
& \qquad -K\left(1-r_n^{-1}\right)\sum_{i=k}^{n} e^{-r(t_i^n-t_k^n)}\hat{P}\left[\overline{S}_{n,i}\le \overline{B}_{n,i}\right]\Big] + \mathcal{O}(\Delta t) \\
= \; & \hat{E}\left[Put^e(t_k^n,\overline{S}_{n,k}) - Put^e(t_k^n,S_{t_k^n})\right] \\
& + \underbrace{\hat{E}\left[Put^e(t_k^n,S_{t_k^n})\right]}_{=Put^e(0,S_0)} - \underbrace{\hat{E}\left[Put_n^e(t_k^n,\overline{S}_{n,k})\right]}_{=Put_n^e(0,S_0)} \\
& + rK\int_{t_k^n}^{T}\hat{E}\left[e^{-r(t-t_k^n)}\mathcal{N}(b_{2,t_k^n}(\overline{S}_{n,k},t))\right]dt - K\left(1-r_n^{-1}\right)\sum_{i=k}^{n}\hat{E}\left[e^{-r(t_i^n-t_k^n)}\overline{S}_{n,i}\le \overline{B}_{n,i}\right] \\
& + \mathcal{O}(\Delta t) \\
= \; & \hat{E}\left[Put^e(t_k^n,\overline{S}_{n,k}) - Put^e(t_k^n,S_{t_k^n})\right] \\
& + Put^a(0,S_0) - Put_n^a(0,S_0) \\
& - rK\int_0^{t_k^n} e^{-rt}\mathcal{N}(b_{2,0}(S_0,t))dt + K\left(1-r_n^{-1}\right)\sum_{i=0}^{k} e^{-rt_i^n}\hat{E}\left[\overline{S}_{n,i}\le \overline{B}_{n,i}\right] \\
& + \mathcal{O}(\Delta t)
\end{aligned}
$$

The proof of theorem 1 in Leisen and Reimer[96] contains as a special case the estimation of

$$
\hat{E}\left[Put^e(t_k^n,\overline{S}_{n,k}) - Put^e(t_k^n,S_{t_k^n})\right] = \mathcal{O}\left(n\cdot\left(\mathfrak{m}_n^2+\mathfrak{m}_n^3+\mathfrak{p}_n\right)+\frac{1}{n}\right)
$$

Now the assertion follows immediatly with Lemma 8 in the Appendix. □

Theorem 3.4 and 3.5 together with proposition 3.1 immediatly imply the following two corollaries:

Corollary 3.2:

American put option prices calculated using the lattice–approach of CRR converge with order one.

Corollary 3.3:

American put option prices calculated using the lattice–approaches of JR and Tian converge with order one from above and order 1/2 from below.

These results improve on that of Lamberton[95] who proved for the lower bound an order of 2/3 and for the upper bound of 1/2 for CRR. Moreover our results apply to general lattice–approaches.

Actually error pictures like figure 3.1 and simulations performed by Broadie and Detemple[96] suggest that the order of convergence is one for JR and Tian, too. We will subsequently assume that this holds for JR and Tian. Then the results in the previous section tell us that for a certain class of models, calculating either american or european put option prices, the error e_n is of the form $\frac{\kappa_1(n)}{n} + \textit{higher terms}$ for a suitable bounded function κ_1.

To take advantage of this information, let us suppose in a first approximation that $p_n = \frac{\kappa_1}{n} + p_\infty$. For any given refinement n this equation contains two unknowns: the constant κ_1 and the correct value p_∞. In order to resolve this, we need a pair of refinements (n_1, n_2) with $n_2 > n_1$ and corresponding prices (p_{n_1}, p_{n_2}). Denoting the approximation for p_∞ by $p_{(n_1,n_2)}$ we have the following system of equations:

$$\begin{aligned} \frac{\kappa_1}{n_1} + p_{(n_1,n_2)} &= p_{n_1} \\ \frac{\kappa_1}{n_2} + p_{(n_1,n_2)} &= p_{n_2} \end{aligned}$$

Resolving yields:

$$\begin{aligned} p_{(n_1,n_2)} &= p_{n_2} - \frac{(p_{n_1} - p_{n_2})n_1}{n_2 - n_1} \\ &= \frac{n_2 p_{n_2} - n_1 p_{n_1}}{n_2 - n_1} \end{aligned}$$

We will refer to this as the extrapolation rule.

Unless otherwise stated, we take the pair $(n, 2n)$. This is commonly referred to as Richardson extrapolation (see Kloeden and Platen[92]).

The above analysis needs to be refined for two reasons. The first stems from the fact that in general the constant will depend on the refinement, whereas above, we replaced the function $\kappa_1(n)$ by a constant κ_1. The second stems from the *higher order* terms, since these may distort extrapolation, such that our rule may no longer optimal. Therefore a detailed analysis of the error $e_{(n_1,n_2)} = p_{(n_1,n_2)} - p_\infty$ prevails:

Proposition 4.1:

Suppose $e_n = \frac{\kappa_1(n)}{n} + \frac{\kappa_2(n)}{n^2}$ *where* $\kappa_1, \kappa_2 : \mathbb{N} \to \mathbb{R}$ *are suitable functions.*
Then:

$$e_{(n_1,n_2)} = \frac{\kappa_1(n_2) - \kappa_1(n_1)}{n_2 - n_1} + \frac{n_1\kappa_2(n_2) - n_2\kappa_2(n_1)}{n_1 n_2 (n_2 - n_1)}$$

Proof. It is obvious that extrapolation yields the error:

$$e_{(n_1,n_2)} = \frac{n_2 e_2 - n_1 e_1}{n_2 - n_1} = \frac{n_2\left(\frac{\kappa_1(n_2)}{n_2} + \frac{\kappa_2(n_2)}{n_2^2}\right) - n_1\left(\frac{\kappa_1(n_1)}{n_1} + \frac{\kappa_2(n_1)}{n_1^2}\right)}{n_2 - n_1}$$

The statement of the proposition follows immediatly from this. □

From corollary 3.2 in the previous section is clear, that for the lattice–approaches of CRR $b_l^a := \liminf_{n\to\infty} \kappa_1(n)$ and $b_u^a := \limsup_{n\to\infty} \kappa_1(n)$ exist and are finite. For JR and Tian it follows from corollary 3.3 only that b_u^a is finite. However, according to the assumption at the beginning of this section, we assume that b_l^a is finite, too. The proposition tells us that in first order the absolute error resulting from extrapolation is bounded by $|e_{(n_1,n_2)}| \le \frac{b_u^a - b_l^a}{n_2 - n_1}$. For Richardson Extrapolation we get that the error $|e_{(n,2n)}| < \frac{b_u^a - b_l^a}{n}$. It means, that extrapolation replaces the constant $|b_u^a| \vee |b_l^a|$ by $|b_u^a - b_l^a|$. Therefore extrapolating makes sense only if $|b_u^a - b_l^a| < \max\{|b_u^a|, |b_l^a|\}$ and our aim in constructing new models should be to get models with very little $|b_u^a - b_l^a|$.
Please note that this observation explains the (obvious) fact that for the CRR model extrapolation does not make sense, since there we typically have $b_l^a < 0 < b_u^a$ yielding $|b_u^a - b_l^a| > b_u^a$. The same holds for the JR and Tian model.
Actually there is an optimal case, in which $b_u^a = b_l^a$. If κ_2 is bounded, an immediate consequence of Proposition 4.1 is that extrapolated prices converge with order of two. Whereas in general we need to select n_2 such that $n_2 - n_1 = \mathcal{O}(n_1)$ in order to get a series of extrapolated prices converging to the true price p_∞, in this special case it is possible to select n_2 such that $n_2 - n_1 = const.$ and still getting prices converge. Under the additional assumption that $\kappa_2 = const.$ we even get the scheme converging with order two. This is very interesting since the extra amount of computation time needed for extrapolation relatively becomes comparable to those needed for calculating the price for n_1. We shall therefore try to construct new models with $b_u^a = b_l^a$. For these models the error picture looks "smooth". We will therefore loosely speak of smoothing options when constructing better performing models.
Another major approach for improving results is the Control Variate technique (CV) proposed by Hull and White[88]. This technique uses the same lattice with refinement n to calculate the price approximations Put_n^a of the the american and Put^e of the european put. It is inspired by the observation that the order of convergence is the same for the european and american put. Then *assume* that errors to the true prices are approximately equal, i.e.:

$$\begin{aligned} Put^a(0,S_0) - Put_n^a(0,S_0) &\approx Put^e(0,S_0) - Put_n^e(0,S_0) \\ \Rightarrow \qquad Put_n^a(0,S_0) &\approx Put_n^a(0,S_0) + Put^e(0,S_0) - Put_n^e(0,S_0) \end{aligned}$$

However looking closely on the errors we immediatly get the following

Proposition 4.2:
Suppose $e_n^a = \frac{\kappa_1^a(n)}{n}$, $e_n^e = \frac{\kappa_1^e(n)}{n}$ *where* $\kappa_1^a, \kappa_1^e : I\!N \to I\!R$ *are suitable functions.*
Then:

$$e_n^{CV} = \frac{\kappa_1^a(n) - \kappa_1^e(n)}{n}$$

The price calculated using the CV–technique will be good only if good and bad price approximations follow at the same rhythm for european and american puts. However in general this will not hold. To perform a similar analysis as for extrapolation we deduce from theorem 1 of Leisen and Reimer[96] (see Theorems 3.1 and 3.2) that $b_l^e := \liminf_{n\to\infty} \kappa_1(n)$ and $b_u^e := \limsup_{n\to\infty} \kappa_1(n)$ exist and are finite. Then:

$$|e_n^{CV}| \leq \frac{(|b_u^a| \vee |b_u^e|) - (|b_l^a| \wedge |b_l^e|)}{n}$$

Therefore the CV–technique replaces the constant $|b_u^a| \vee |b_l^a|$ by $(|b_u^a| \vee |b_u^e|) - (|b_l^a| \wedge |b_l^e|)$ and all the conclusions drawn from proposition 4.1 for extrapolation carry over to the CV–technique. Especially we have the same task to get better performing models: smooth the option, i.e. to reduce price oscillations as much as possible. In the sequel we will stick to extrapolation only and show up a way how to smooth the option at least partially.
We have according to Carr, Jarrow and Myneni[92] and Lemma 1 in the Appendix:

$$\begin{aligned} Put^a(0, S_0) &= Put^e(0, S_0) + rK \int_0^T e^{-rt} P[S_t \leq B_t] \\ Put_n^a(0, S_0) &= Put_n^e(0, S_0) + rK \sum_{j=0}^n e^{-rt_k^n} \overline{P}\left[\overline{S}_{n,j} \leq \overline{B}_{n,j}\right] + \mathcal{O}(\Delta t) \end{aligned}$$

This means that errors result both from approximating the European Put component as well as the early exercise premium, whereas the errors in the early exercise premium component result from approximating the value of the cash–or–nothing options $P[S_t \leq B_{t_j^n}] - \overline{P}\left[\overline{S}_{n,j} \leq \overline{B}_{n,j}\right]$.
With barrier option valuation, Derman, Kani, Ergener and Bardhan[95] argue that price oscillations result from the fact, that a specific lattice under consideration implicitly determines the class of possible option contracts which can be priced, since exercise is only possible at nodes in the tree grid. They call this the "quantization error". More specifically, in the case of the european call option, Leisen and Reimer[96] determined as the origin of these errors the following: when taking a close look at terminal nodes especially at the nodes around the strike price K we see that with varying n, nodes shift upwards and downwards. Since they contain the whole probability mass, this causes the distortions.
Improving results for cash–or–nothing options is difficult, since we do not know the exercise Boundaries B, resp. $\overline{B}$. However we can profit from this observation in constructing a model which improves at least the european put component. This can be done by ensuring that the strike always lies fixed at a specific node, the center of the tree. In order to do this consistently we must assume that n is even, too. Therefore, suppose we are given a refinement n with n even and u_n, d_n according to CRR, that is $u_n = \exp\{\sigma\sqrt{\Delta t_n}\}$,

$d_n = 1/u_n$. Remember JR who adjusted the local drift–term to match the continuous drift–term. We are interested in fixing the strike at the center of the tree at maturity. Thus the new parameter selection u'_n, d'_n should fulfill:

$$\begin{aligned} u'_n &= u_n \cdot e^{c_n} \\ d'_n &= d_n \cdot e^{c_n} \\ S_0 \cdot (u'_n \cdot d'_n)^{n/2} &= K \end{aligned}$$

The third equation tells us $c_n = \frac{\ln K/S_0}{n}$. The equivalent martingale measure is obtained by setting $\overline{p}'_n = \frac{r_n - d'_n}{u'_n - d'_n}$.

In the sequel this model will be denoted by **SMO**.

In figure 4.1 is represented the error for the CRR and SMO models in calculating european

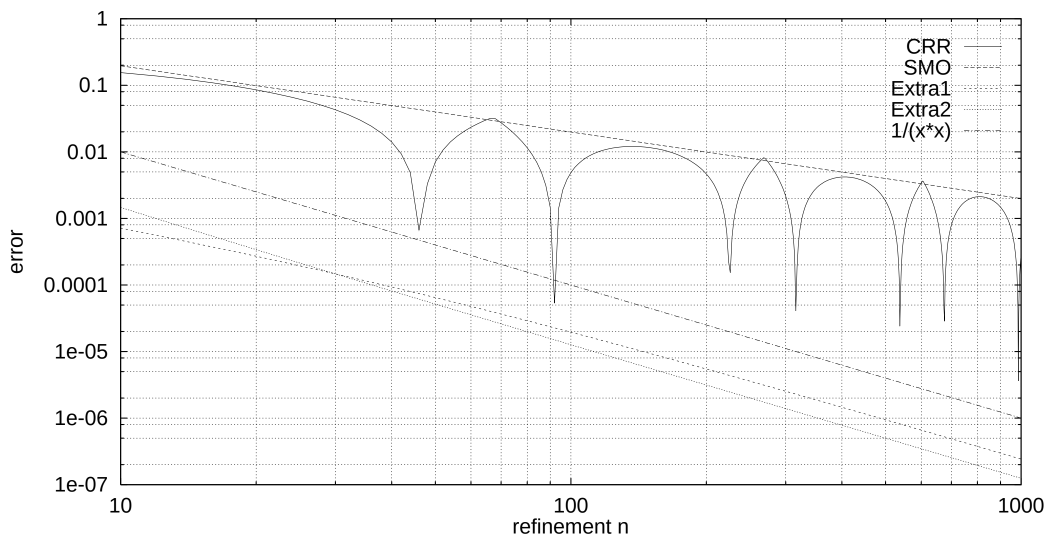

FIGURE 4.1. comparison of the CRR lattice approach with the SMO lattice approach and two extrapolations using the *SMO* lattice–approach for the case of the european put option: example with the following selection of parameters: $S = 100$, $K = 105$, $T = 1$, $r = 0.05$, $\sigma = 0.2$, $n = 10, \ldots, 1000$ even

put option prices where n is always taken even. Moreover they are compared to the following two extrapolations of SMO: *EXTRA1* uses the pair $(n, 2n)$, while *EXTRA2* uses the pair $(n, n+40)$. Actually we see that the SMO model performs bad, i.e. it always yields higher errors than CRR. This stems from the fact that we did not care for implementing the local–variance properly.

It would be possible to construct smooth models which behave much better. However we preferred this approach for its very simple and intuitive construction. Moreover here we see very drastically the effects of extrapolation, since the "slow" convergence speed of SMO is completely offset: When comparing it with the function $1/x^2$ we see that both extrapolations converge with order two in the long run. That is, we see completely

justified our remarks made after proposition 4.1, which told us that it is possible to win one order in the convergence and even with the simple EXTRA2.
For the american put option we may not expect to get smooth results, but at least smoother ones, replacing the constant by a much more little one. This means that we are starting with lower initial error, which means in turn, that a lower tree–refinement already yields the same precision level. A comparison of different lattice–approaches for the american put will be studied in more detail in the next section.

5. Numerical Evaluation

To evaluate properly the additional time needed for extrapolation, we stick to an analysis suggested by Broadie and Detemple[96]. There, within one analysis several methods using a large sample of randomly selected parameters are compared simultaneously over refinements with measurement of computation speed and approximation error. Computation speed is expressed by the number of option prices calculated per second. Since we stick to tree models with identical structure except for the tree parameters, we need not care on tuning our computer implementation of methods. The approximation error is measured by the relative root–mean–squared (RMS) error. RMS–error is defined by

$$\mathrm{RMS} = \sqrt{\frac{1}{m}\sum_{i=1}^{m} e_i^2}$$

where $e_i = (\hat{c}_i - c_i)/c_i$ is the relative error, $\hat{c}_i$ ist the estimated option value and c_i ist the true option value. The true option value is calculated using CRR with a refinement of 15000.
We chose the following distribution of parameters for the whole sample. Volatility is distributed uniformly between 0.1 and 0.6. Time to maturity is, with probability 0.75, uniform between 0.1 and 1.0 years and, with probability 0.25, uniform between 1.0 and 5.0 years. We fix the strike price at $K = 100$ and take the initial asset price $S \equiv S_0$ to be uniform between 70 and 130. Relative errors do not change if S and K are scaled by the same factor, i.e., only the ratio S/K is of interest. The riskless rate r is, with probability 0.8, uniform between 0.0 and 0.10 and, with probability 0.2, equal to 0.0. Each parameter is selected independently of the others. This selection of parameters matches the choice of Broadie and Detemple[96] except for dividends which we donot regard here.
To make relative error meaningful, that is to avoid senseless distortions because of very small option prices, options with $c_i \leq 0.50$ did not enter the sample.
We tested the CRR model with the SMO model and its extrapolation. Moreover we tested it in comparison to the PP model suggested by Leisen and Reimer[96] and its extrapolation, too. This model was constructed using the works of Pratt[68] and Peizer and Pratt[68] on inverted normal approximations such as to yield order of convergence two for the european put option.

To account for different behaviour with long/short maturities respectively in/out–of–the money options we splitted the whole sample into 4 subsamples.
The first two figures (5.1 and 5.2) deal with options with short time–to–maturities $T \leq 0.2$. More specifically the first figure deals with out–of–the–money options ($S \geq 100$). We see that SMO yields results that are 3 times worse than CRR, whereas PP yields 10 times better results than CRR. Surprisingly, however, extrapolating SMO and PP yields results that are again approximately 10 times better than PP, i.e. in total they have 100 times lower initial error than CRR. Moreover we see immediatly that extrapolation has a tremendous effect on the error since using it in a 200 step (together with a 400 step) tree exceeds already the precision level of a CRR tree with a refinement of 15000, such that we could have dropped higher extrapolations.
The second figure deals with in–the–money options ($S \leq 100$). Here the effects of extrapolation are still astonishing. Although extrapolating the PP and SMO models yields only 3 times better results than PP, this yields 10 times better results than CRR. Thus we are winning 30 in total in comparison to CRR.
The last two two figures (5.3 and 5.4) deal with options with long time–to–maturities $T \geq 0.2$. In the case of out–of–the–money options (figure 5.3) we see that PP performs 3 times better than CRR and that extrapolating PP and SMO improves this by a factor of 3 in comparison to PP, yet. Therefore it performs approximately 10 times better than CRR. In the case of in–the–money options (figure 5.4) extrapolation of PP and SMO improves by a factor of 2 in comparison to CRR, whereas PP shows only an improvement of 1.5 .
Generally spoken, out–of–the–money options converge much smoother and therefore yield much better convergence results with extrapolation. Moreover we want to remark, that extrapolation with $n = 24$ actually ensures in all cases that the error is less than 0.01. This means that it already yields a sufficient precision level, since in practice discrete and continuous models can no longer be distinguished.

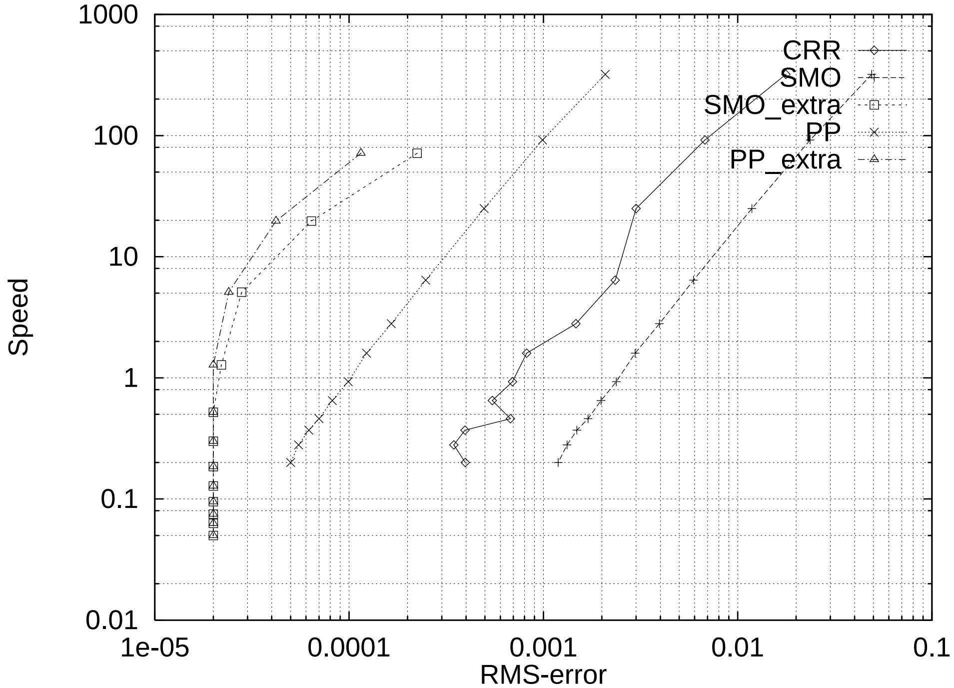

FIGURE 5.1. testing efficiency of binomial models for out–of–the–money American put options ($S \geq 100$) with $n_i = \{24, 50, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000\}$ and time–to–maturity $T \leq 0.2$ (subsample of 89 options)

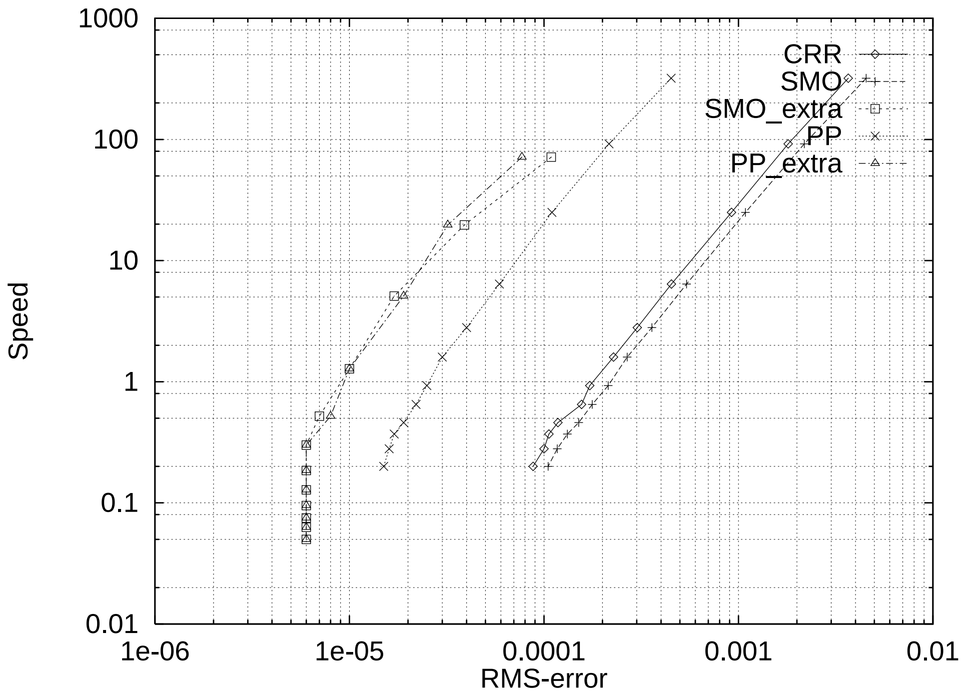

FIGURE 5.2. testing efficiency of binomial models for in–the–money American put options ($S \leq 100$) with $n_i = \{24, 50, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000\}$ and time–to–maturity $T \leq 0.2$ (subsample of 150 options)

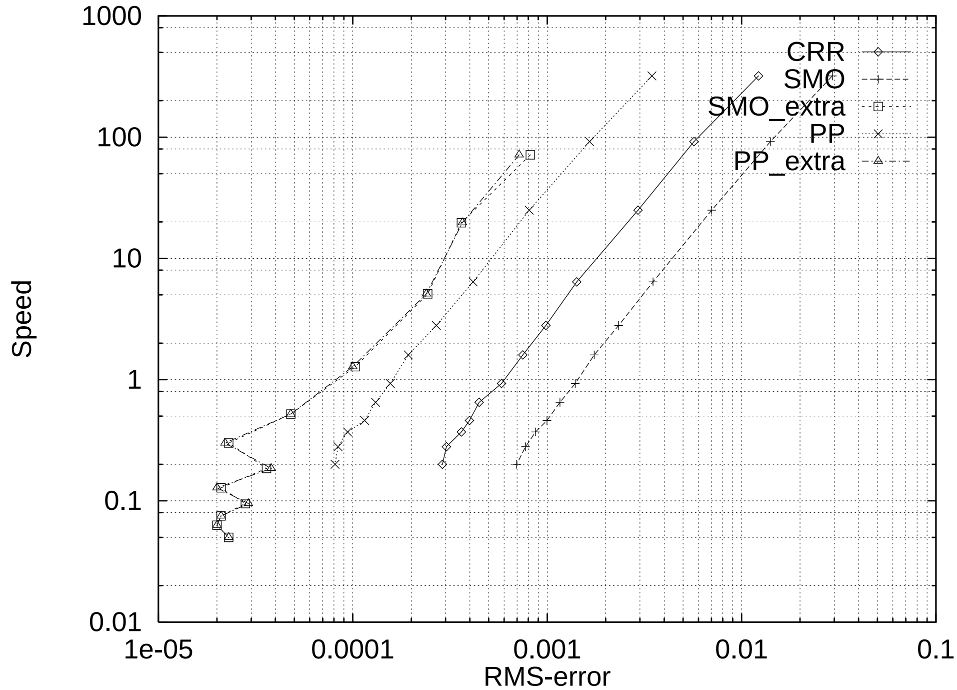

FIGURE 5.3. testing efficiency of binomial models for out–of–the–money American put options ($S \geq 100$) with $n_i = \{24, 50, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000\}$ and time–to–maturity $T \geq 0.2$ (subsample of 427 options)

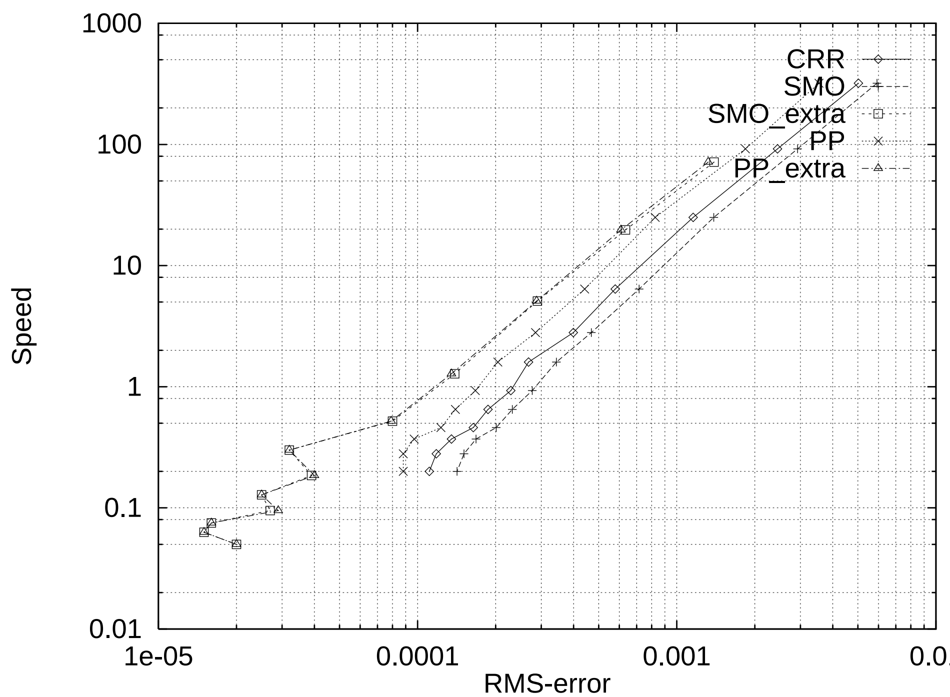

FIGURE 5.4. testing efficiency of binomial models for in–the–money American put options ($S \leq 100$) with $n_i = \{24, 50, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000\}$ and time–to–maturity $T \geq 0.2$ (subsample of 450 options)

6. Conclusion

In this paper we examined the convergence and more detailed the order of convergence for the american put option. The results of Leisen and Reimer[96] were extended to the american put option. It was thus shown that the models of CRR, JR and Tian are similar. In a next step we used this for an extrapolation rule and its error analysis. Here we saw the astonishing effects that a proper extrapolation may have. Actually although the approach we have taken here is rather simple it already yields up to 100 times better results. Better smoothing should be able to improve this further up to yielding one order as in the case of the european put option.

Let $\mathcal{C} := \left\{(n,k)\,|\,k = \min\{i|B_{t_i^n} < \overline{B}_{n,i}\}\text{ if this set is not empty }\right\}$, $\mathcal{D}_{n,k} := \{S_0 u_n^j d_n^{n-j}|0 \leq j \leq k\}$, and $\mathfrak{n}$ denote the density of the normal distribution function.

For $i = 0, \ldots, n$ we will call $I_{n,i} := \left\{t_k^n \in \mathcal{T}^n | S_0 u_n^i d_n^{n-i} \leq B_{t_k^n} < S_0 u_n^{i+1} d_n^{n-(i+1)}\right\}$ a domain. Domains are disjoint intervalls, i.e. $I_{n,i} = [l_{n,i}, r_{n,i}[$ for suitable $l_{n,i} < r_{n,i}$. Obviously we have $l_{n,0} < r_{n,0} = l_{n,1} < \ldots < r_{n,i}$

Lemma 1:

$$\pi_n = \hat{E}\left[\sum_{k=0}^{n-1} e^{-rt_k^n} \cdot K(1 - e^{-r\Delta t}) \cdot \hat{P}\left[\overline{S}_{n,k} \leq \overline{B}_{n,k} \;\middle|\; \mathcal{A}_{n,k}\right]\right] + \mathcal{O}(\Delta t)$$

Proof. For the cases $\overline{S}_{n,k} \leq \overline{B}_{n,k}$, $u_n \overline{S}_{n,k} > \overline{B}_{n,k+1}$ we have:

$$\begin{aligned}
& Put_n^{a,B}\left(t_{k+1}^n, \overline{R}_{n,k+1} \cdot \overline{S}_{n,k}\right) \\
= \;& f\left(\overline{R}_{n,k+1} \cdot \overline{S}_{n,k}\right) + \mathcal{O}\left(\sqrt{\Delta t}\right) \qquad \text{since } u_n - 1 = \mathcal{O}\left(\sqrt{\Delta t}\right) \\
\Rightarrow \;& \hat{E}\left[\left(K - \overline{S}_{n,k}\right)^+ - e^{-r\Delta t} Put_n^a\left(t_{k+1}^n, \overline{R}_{n,k+1} \cdot \overline{S}_{n,k}\right) \;|\; \mathcal{A}_{n,k}\right] \\
= \;& \mathcal{O}\left(\sqrt{\Delta t}\right) K\left(1 - r_n^{-1}\right) \\
= \;& \mathcal{O}\left(\sqrt{\Delta t}^3\right) \qquad \text{since } 1 - r_n^{-1} = \mathcal{O}(\Delta t)
\end{aligned}$$

If $\overline{S}_{n,k} \leq \overline{B}_{n,k}$, $u_n \overline{S}_{n,k} \leq \overline{B}_{n,k+1}$:

$$\begin{aligned}
& \hat{E}\left[(K - \overline{S}_{n,k})^+ - e^{-r\Delta t} Put_n^a(t_{k+1}^n, \overline{R}_{n,k+1} \cdot \overline{S}_{n,k}) \;\middle|\; \mathcal{A}_{n,k}\right] \\
= \;& \hat{E}[(K - \overline{S}_{n,k}) - e^{-r\Delta t}(K - \overline{R}_{n,k+1} \cdot \overline{S}_{n,k}) \mid \mathcal{A}_{n,k}] \\
= \;& (K - \overline{S}_{n,k}) - e^{-r\Delta t}(K - e^{r\Delta t}\overline{S}_{n,k}) \\
= \;& K\left(1 - e^{-r\Delta t}\right)
\end{aligned}$$

From the arguments of Harrison and Pliska[81] follows:

$$\begin{aligned}
\pi_n \;&=\; Put_n^a(0, S_0) - e^{-rT}\hat{E}[(K - \overline{S}_{n,n})^+] \\
&=\; Put_n^a(0, S_0) - e^{-rT}\hat{E}\Big[\underbrace{Put_n^e(T, \overline{S}_{n,n})}_{= Put_n^a(T, \overline{S}_{n,n})}\Big] \\
&=\; \hat{E}\left[\sum_{k=0}^{n-1} e^{-rt_k^n} \hat{E}\left[Put_n^a(t_k^n, \overline{S}_{n,k}) - e^{-r\Delta t} Put_n^a(t_{k+1}^n, \overline{S}_{n,k+1}) \;\middle|\; \mathcal{A}_{n,k}\right]\right]
\end{aligned}$$

This implies according to the above case study:

$$
\begin{aligned}
& \hat{E}\left[Put_n^a(t_k^n,\overline{S}_{n,k}) - e^{-r\Delta t} Put_n^a(t_{k+1}^n,\overline{S}_{n,k+1}) \;\middle|\; \mathcal{A}_{n,k}\right] \\
= \; & \hat{E}\left[1_{\overline{S}_{n,k}\le \overline{B}_{n,k}} \cdot \left((K-\overline{S}_{n,k})^+ - e^{-r\Delta t} Put_n^a(t_{k+1}^n,\overline{S}_{n,k+1})\right) \;\middle|\; \mathcal{A}_{n,k}\right] \\
= \; & \hat{E}\left[1_{\overline{S}_{n,k}\le \overline{B}_{n,k}} \cdot K\left(1-e^{-r\Delta t}\right) \;\middle|\; \mathcal{A}_{n,k}\right] + \mathcal{O}\left(\sqrt{\Delta t}^3\right) \\
= \; & \hat{P}\left[\overline{S}_{n,k} \le \overline{B}_{n,k} \;\middle|\; \mathcal{A}_{n,k}\right] \cdot K\left(1-e^{-r\Delta t}\right) + \mathcal{O}\left(\sqrt{\Delta t}^3\right)
\end{aligned}
$$

Since:

$$
\begin{aligned}
\hat{P}\left[\,\overline{S}_{n,k} \le \overline{B}_{n,k}, u_n\overline{S}_{n,k} > \overline{B}_{n,k+1}\right] &= \mathcal{O}\left(\frac{1}{\sqrt{k}}\right) \\
\Rightarrow \qquad \sum_{k=1}^{n} \hat{P}\left[\,\overline{S}_{n,k} \le \overline{B}_{n,k}, u_n\overline{S}_{n,k} > \overline{B}_{n,k+1}\right] &= \mathcal{O}\left(\frac{1}{\sqrt{\Delta t}}\right)
\end{aligned}
$$

the assertion follows. □

Lemma 2:

Stopping the discrete process $(\overline{S}_{n,k})_{k=0,\ldots,n}$ according to the rule rule $(B_{t_k^n})_{k=0,\ldots,n}$ yields the premium:

$$
\pi_n^B = \hat{E}\left[\sum_{k=0}^{n-1} e^{-rt_k^n} \cdot K(1-e^{-r\Delta t}) \cdot \hat{P}\left[\overline{S}_{n,k} < B_{t_k^n} \;\middle|\; \mathcal{A}_{n,k}\right]\right] + \mathcal{O}(\Delta t)
$$

Proof. Follows exactly as those of Lemma 1. □

Lemma 3:

$$
\begin{aligned}
\mathcal{N}(b_{2,0}(S_0,t_k^n)) - \hat{P}[\overline{S}_{n,k} \le \overline{B}_{n,k}] &= \mathcal{O}(\Delta t) + \mathcal{N}(b_{2,0}(S_0,t_k^n)) - \mathcal{N}(z_{2,0}(S_0,t_k^n)) \\
\textit{where } z_{2,0}(S_0,t_k^n) &= \frac{\ln \overline{B}_{n,k}/S_0 - n\ln d_n}{(\ln u_n - \ln d_n)\sigma_k} - \frac{n\overline{p}_n}{\sigma_k} + \frac{1}{\sigma_k} \\
\textit{and } \sigma_k &= \sqrt{k\overline{p}_n(1-\overline{p}_n)}
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\hat{P}\left[\overline{S}_{n,k} \le \overline{B}_{n,k}\right] &= \hat{P}\left[\prod_{j=1}^{k} \overline{R}_{n,j} \le \frac{\overline{B}_{n,k}}{S_0}\right] \\
&= \hat{P}\left[\sum_{j=1}^{k} \ln \overline{R}_{n,j} \le \ln \frac{\overline{B}_{n,k}}{S_0}\right]
\end{aligned}
$$

Moreover:

$$
\begin{aligned}
\overline{B}_{n,k} &= S_0 u_n^{j_k} d_n^{n-j_k} \\
\Rightarrow \quad \frac{\overline{B}_{n,k}}{S_0} &= u_n^{j_k} d_n^{n-j_k} \\
\Rightarrow \quad \ln \frac{\overline{B}_{n,k}}{S_0} &= j_k \ln u_n + (n - j_k) \ln d_n \\
&= j_k(\ln u_n - \ln d_n) + n \ln d_n \\
\Rightarrow \quad j_k &= \frac{\ln \frac{\overline{B}_{n,k}}{S_0} - n \ln d_n}{\ln u_n - \ln d_n}
\end{aligned}
$$

Obviously σ_k is the variance of $\sum_{j=1}^{k} \ln \overline{R}_{n,j}$ and $z_{2,0}(S_0, t_k^n) = \frac{j_k - n\overline{p}_n}{\sigma_k} + \frac{1}{2\sigma_k}$.
Let $z_{1,0}(S_0, t_k^n) = -\frac{n\overline{p}_n}{\sigma_k} - \frac{1}{2\sigma_k}$.
Then, according to Prohorov and Rozanov[69] we have:

$$
\begin{aligned}
\mathcal{N}\left(b_{2,0}(S_0, t_k^n)\right) - \hat{P}\left[\overline{S}_{n,k} \leq \overline{B}_{n,k}\right] &= \underbrace{\mathcal{N}(z_{2,0}(S_0, t_k^n)) - \mathcal{N}(z_{1,0}(S_0, t_k^n)) - \hat{P}\left[\overline{S}_{n,k} \leq \overline{B}_{n,k}\right]}_{=\mathcal{O}(\Delta t)} \\
&\quad - \mathcal{N}(z_{1,0}(S_0, t_k^n)) + \mathcal{N}\left(b_{2,0}(S_0, t_k^n)\right) + \underbrace{\mathcal{N}(z_{2,0}(S_0, t_k^n))}_{=\mathcal{O}(\Delta t^2)}
\end{aligned}
$$

□

Lemma 4:

$$
\exists \kappa \in I\!R : \mathcal{N}(b_{2,0}(S_0, t_k^n)) - \hat{P}[\overline{S}_{n,k} \leq B_{t_k^n}] \leq \kappa \Delta t
$$

Proof. Denote $z'_{2,0}(S_0, t_k^n) = b_{2,0}(S_0, t_k^n) - \frac{n\overline{p}_n}{\sigma_k} + \frac{1}{\sigma_k}$
Similarly to Lemma 3 we have:

$$
\mathcal{N}(b_{2,0}(S_0, t_k^n)) - \hat{P}[\overline{S}_{n,k} \leq B_{t_k^n}] = \mathcal{O}(\Delta t) + \mathcal{N}(b_{2,0}(S_0, t_k^n)) - \mathcal{N}(z'_{2,0}(S_0, t_k^n))
$$

The assertion follows immediatly from the observation $b_{2,0}(S_0, t_k^n) \leq z'_{2,0}(S_0, t_k^n)$ □

Lemma 5:

$$
\exists \kappa \in I\!R : \pi - \hat{E}\left[\sum_{k=0}^{n-1} e^{-rt_k^n} \cdot K(1 - e^{-r\Delta t}) \cdot \hat{P}\left[\overline{S}_{n,k} < B_{t_k^n} \;\middle|\; \mathcal{A}_{n,k}\right]\right] \leq \kappa \Delta t
$$

Proof. From a series expansion of the exponential function we get $K(1 - e^{-r\Delta t}) = Kr\Delta t + \mathcal{O}(\Delta t^2)$. Since $\mathcal{N}\left(b_{2,0}(S_0, t_k^n)\right) - \mathcal{N}\left(z_{2,n}(S_0, t_k^n)\right) \leq 0$ it follows from Lemma 3 immediately that this is lower than:

$$
\mathcal{O}\left(\Delta t\right) + rK \sum_{k=0}^{n-1} e^{-rt_k} \mathcal{N}\left(b_{2,0}(S_0, t_{k+1}^n)\right) \Delta t
$$

where $b_{2,0}(x, t) = \frac{\ln(B_t/x) - (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$.
The summation–term can be viewed as an approximation to the respective integral. From

the trapezoidal formula of numerical integration we get immediately that it equals for a suitable $\xi \in [0, T]$:

$$rK \int_0^T e^{-rt} \mathcal{N}\left(b_{2,0}(S_0, t_{k+1})\right) dt + \Delta t^2 \cdot \mathcal{N}(b_{2,0}(S_0, \xi)) + \mathcal{O}\left(\Delta t\right)$$

Since the normal–function is bounded, we have proven the Lemma. □

Lemma 6:

$$\exists \kappa \, \forall (n,k) \in \mathcal{C} : E[Put^a(t_k^n, \overline{S}_{n,k}) - Put_n^a(t_k^n, \overline{S}_{n,k})] \geq \kappa \Delta t$$

Proof. Let $\kappa_0 := \mathfrak{n}(B_0)/S_0$.
For $\overline{S} \in \mathcal{D}_{n,k}$ $(n \in \mathbb{N}, 0 \leq k \leq n)$ denote

$$\begin{aligned} \Delta^e_{n,k}(\overline{S}) &:= Put^e(t_k^n, \overline{S}) - Put_n^e(t_k^n, \overline{S}) \\ \Delta^p_{n,k}(\overline{S}) &:= \pi(t_k^n, \overline{S}) - \pi_n(t_k^n, \overline{S}) \end{aligned}$$

According to Leisen and Reimer[96]:

$$\exists \kappa_1 \in \mathbb{R} \, \forall n \in \mathbb{N} \, \forall 0 \leq k \leq n \, \forall \overline{S} \in \mathcal{D}_{n,k} : \kappa_1 \cdot \left(n \cdot \left(\mathfrak{m}_n^2 + \mathfrak{m}_n^3 + \mathfrak{p}_n \right) + \frac{1}{n} \right) \geq |\Delta^e_{n,k}(\overline{S})|$$

Now take $(n,k) \in \mathcal{C}$.
Since $B_{t_k^n} < \overline{B}_{n,k}$ we have:

$$\Delta^e_{n,k}(\overline{B}_{n,k}) + \Delta^p_{n,k}(\overline{B}_{n,k}) \geq 0$$

This implies:

$$\begin{aligned} \kappa_1 \cdot \left(n \cdot (\mathfrak{m}_n^2 + \mathfrak{m}_n^3 + \mathfrak{p}_n) + \tfrac{1}{n} \right) &\geq \Delta^e_{n,k}(\overline{B}_{n,k}) \\ &\geq -\kappa_0 \Delta^p_{n,k}(\overline{B}_{n,k}) \\ \Rightarrow \quad -\tfrac{\kappa_1}{\kappa_0} \cdot \left(n \cdot (\mathfrak{m}_n^2 + \mathfrak{m}_n^3 + \mathfrak{p}_n) + \tfrac{1}{n} \right) &\leq \Delta^p_{n,k}(\overline{B}_{n,k}) \\ \Rightarrow \quad \Delta^e_{n,k}(\overline{B}_{n,k}) + \Delta^e_{n,k}(\overline{B}_{n,k}) &\geq \left(\kappa_1 - \tfrac{\kappa_1}{\kappa_0} \right) \cdot \left(n \cdot (\mathfrak{m}_n^2 + \mathfrak{m}_n^3 + \mathfrak{p}_n) + \tfrac{1}{n} \right) \end{aligned}$$

□

Lemma 7:
There exists $\kappa \in \mathbb{R}$ such that for $(n,k) \in \mathcal{C}$ and $i \in \mathbb{N}$ with $I_{n,i} \subset [0, t_k^n]$ we have:

$$\sum_{i=0}^{k} \mathcal{N}(z_{2,0}(S_0, t_k^n)) - \mathcal{N}(b_{2,0}(S_0, t_k^n)) \geq \kappa$$

Proof. We denote:

- by $B^l_{n,i} := B_{l_{n,i}}, B^r_{n,i} := B_{r_{n,i}}$.

- by $g_{n,i} : I_{n,i} \to [0,\infty]$ the function

$$\begin{aligned} g_{n,i}(t) &:= \ln B^l_{n,i} + \mu_{n,i} \cdot (t - l_{n,i})) \\ \text{where} \qquad \mu_{n,i} &:= \frac{\ln B^r_{n,i} - \ln B^l_{n,i}}{r_{n,i} - l_{n,i}} \end{aligned}$$

- by $\nu_{n,i} := \frac{r_{n,i}-l_{n,i}}{\Delta t}$ the number of discrete time–points in $I_{n,i}$.
- by $\overline{\overline{B}}_{n,k} := \max\left\{\overline{S} \in \mathcal{D}_{n,k} | \overline{S} \leq B_{t^n_k}\right\}$ the highest node below $B_{t^n_k}$
- by $\rho_n(k)$ the number of up–steps necessary to reach $\overline{\overline{B}}_{n,k}$, i.e. $\overline{\overline{B}}_{n,k} = S_0 u_n^{\rho_n(k)} d_n^{k-\rho_n(k)}$.

Take $(n,k) \in \mathcal{C}$ and $i \in \mathbb{N}$ and $I_{n,i} \subset [0, t^n_k]$. Let us assume in the sequel that $B^r_{n,i} = S_0 u_n^{i+1} d_n^{n-(i+1)}$ and $B^l_{n,i} = S_0 u_n^i d_n^{n-i}$. This yields an error of order $\mathcal{O}(\Delta t)$.
On $I_{n,i}$ $\overline{\overline{B}}$ is alternately equal to $S_0 u_n^i d_n^{n-i}$ and $S_0 u_n^{i-1} d_n^{n-(i-1)}$. Therefore for $l_{n,i} + j\Delta t \in I_{n,i} \quad (\Leftrightarrow 0 \leq j \leq \nu_{n,i})$ we have:

$$\frac{\ln g_{n,i}(l_{n,i} + j\Delta t)}{\overline{\overline{B}}} = \begin{cases} (\ln u_n)\frac{j}{\nu_{n,i}} & j \ even \\ (\ln u_n)\frac{j}{\nu_{n,i}} + \ln\frac{1}{d_n} & j \ odd \end{cases}$$

This implies:

$$\begin{aligned} \sum_{t^n_j \in I_{n,i}} \frac{\ln g_{n,i}(l_{n,i} + j\Delta t)/\overline{\overline{B}}_{n,j}}{\sigma\sqrt{\Delta t}} - 1 &= \sum_{j=0}^{\nu_{n,i}} \frac{j}{\nu_{n,i}} + \frac{\nu_{n,i}}{2} - \nu_{n,i} \\ &= \frac{\nu_{n,i}(\nu_{n,i}+1)}{2\nu_{n,i}} - \frac{\nu_{n,i}}{2} \\ &= \frac{1}{2} \end{aligned}$$

Since $u_n = \exp\{+\sigma\sqrt{\Delta t}\}$ we have $|I_{n,i}| = \mathcal{O}\left(\sqrt{\Delta t}\right)$.
Moreover we have according to the mean value theorem for each $t \in I_{n,i}$ a suitable $\xi^t_{n,i} \in I_{n,i}$ such that:

$$\begin{aligned} B_t &= B_{r_{n,i}} + B'(\xi^t_{n,i})(t - r_{n,i}) \\ &= B_{r_{n,i}} + B'(r_{n,i})(t - r_{n,i}) + (B'(\xi^t_{n,i}) - B'(r_{n,i}))(t - r_{n,i}) \end{aligned}$$

Since B is continuously differentiable (see Myneni[92], McKean[65], Van Moerbeke[76]), B' is uniformly bounded on $[0,T]$ by a suitable $\kappa_1 \in \mathbb{R}$. Thus:

$$\begin{aligned} \sum_{t^n_j \in I_{n,i}} \frac{\ln B_{t^n_j}/\overline{B}_{n,j}}{\sigma\sqrt{\Delta t}} - 1 &\geq \sum_{t^n_j \in I_{n,i}} \frac{\ln B_{t^n_j}/\overline{\overline{B}}_{n,j}}{\sigma\sqrt{\Delta t}} - 1 \\ &= \sum_{t^n_j \in I_{n,i}} \frac{\ln g_{n,i}(l_{n,i} + j\Delta t)/\overline{\overline{B}}_{n,j}}{\sigma\sqrt{\Delta t}} - 1 + 2\kappa_1 \\ &= \frac{1}{2} + 2\kappa_1 \end{aligned}$$

Since $z_{2,0}(S_0, t_k^n) - b_{2,0}(S_0, t_k^n) = \mathcal{O}\left(\sqrt{\Delta t}\right)$ we have:

$$\begin{aligned}\mathcal{N}(z_{2,0}(S_0, t_k^n)) - \mathcal{N}(b_{2,0}(S_0, t_k^n)) &= \left(\mathfrak{n}(b_{2,0}(S_0, r_{n,i})) + \mathcal{O}\left(\sqrt{\Delta t}\right)\right) \cdot (z_{2,0}(S_0, t_k^n) - b_{2,0}(S_0, t_k^n)) \\ &= \mathfrak{n}(b_{2,0}(S_0, r_{n,i})) \cdot (z_{2,0}(S_0, t_k^n) - b_{2,0}(S_0, t_k^n)) + \mathcal{O}\left(\sqrt{\Delta t}\right)\end{aligned}$$

The assertion follows now from the fact that $\sqrt{\Delta t}\, \sum_{i=0}^{n} \sqrt{r_{n,i}}^{-1}$ is uniformly bounded. □

Lemma 8:

$$rK \int_0^{t_k^n} e^{-rt} \mathcal{N}(b_{2,0}(S_0, t))dt - K\left(1 - r_n^{-1}\right) \sum_{i=0}^{k} e^{-rt_i^n} \hat{E}\left[\overline{S}_{n,i} \leq \overline{B}_{n,i}\right] = \mathcal{O}(\Delta t)$$

Proof. Lemmata 3 and 7 and $1 - r_n^{-1} = \mathcal{O}(\Delta t)$ □

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