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Stock evolution under Stochastic Volatility:

A discrete approach

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Abstract

Stochastic volatility models model asset dynamics by a bivariate diffusion process. For practical calculation of prices of financial derivatives lattice models are necessary. In this paper we present a procedure to construct discrete process approximations converging to such models.

Keywords

binomial model, option valuation, lattice–approach, stochastic volatility

1 Introduction

In a highly simplified model Black and Scholes explained how the intrinsic risk in a call option could be completely removed by continuous trading in the underlying stock and a Bond. This has been the starting point for a theory of pricing and hedging contingent claims. In this paper we study an extension where the asset prices are modeled by bivariate diffusions (see Hull and White (1987), Chesney and Scott (1989) and Heston (1993)). These models are the continuous time limits of models used in the time–series literature, so called GARCH models. Recently Ritchken and Trevor (1997) and Duan and Simonato (1997) presented two different approaches to construct a GARCH approximation. Unfortunately the implementation is tricky and complicates the analysis of the algorithm. Although in actual calculations prices converge to the correct continuous state solution, both papers are lacking a proper convergence proof. The purpose of this article is to present a method by which a discrete lattice can be constructed for a class of stochastic volatility models, including those currently used in the literature.

Section two present the framework. In section 3 we present conditions to ensure convergence of prices by ensuring weak convergence (in distribution) of a discrete process. The structure of our problem recommends constructing first a (one–dimensional) lattice describing only the dynamics of the volatility (section 4) which is then extended to the two–dimensional dynamics of the joint volatility and stock dynamic (section 5).

2 The Diffusion setup

We suppose that we observe a constant interest rate r and that the stock–price can be described by the stochastic process $(S_t)_t$ which is solution to the following system of stochastic differential equations:

$$dv_t = \kappa(v_t - \bar{v})dt + \varphi(v_t)dW_t^1 \tag{1}$$
$$dS_t = \nu(v_t)S_t dt + \psi(v_t)S_t dW_t^2 \tag{2}$$

where (W^1, W^2) is a two–dimensional Wiener–process on a suitable probability space $(\Omega, \mathcal{F}, P)$ with constant correlation ρ and κ a suitable constant.

Following the equivalent martingale measure technique, developed by Harrison and Pliska (1981), Föllmer and Sondermann (1986) the value of financial derivatives will be calculated as discounted expectations under a suitable martingale measure. Here we choose $\nu(v) = r$, i.e. the so called minimal martingale measure introduced by Föllmer and Schweizer (1991) (see also Hofmann, Platen and Schweizer (1992)). In economic terms this states that there is no premium for volatility risk.

Different specifications of the function φ and ψ allow us to treat the models present in the literature in a unified way (see table 1). We will suppose the functional forms

$$\begin{aligned} & \varphi(v) = p_1 + p_2 v + p_3\sqrt{v} \\ \text{and} \qquad & \psi(v) = k_1 v + k_2\sqrt{v} + k_3 \exp(v) \end{aligned}$$

for suitable $k_1, k_2, k_3, p_1, p_2, p_3 \geq 0$, which ensure the existence of a solution to the system (1),(2). Here we excluded a time–dependency in the volatility. This will result in a time–homogeneous lattice, which is much easier to handle. However our technique could easily be extended to incorporate this feature.

We denote by

$$Z := \begin{pmatrix} v \\ X \end{pmatrix}$$

the joint volatility and (logarithmic) stock dynamics. Applying the Itô–formula

Table 1

Different model specifications

model	$\varphi(v)$	$\psi(v)$
Hull and White (1987)	v	v
Heston (1993)	$\sqrt{v}$	$\sqrt{v}$
Stein and Stein (1991)	$const.$	v
Chesney and Scott (1989)	$const.$	$\exp\{v\}$

we see that the Z dynamics can be described by the instantaneous drift

$$\mu(v) = \begin{pmatrix} \kappa(v - \bar{v}) \\ \nu(v) - \frac{\psi(v)^2}{2} \end{pmatrix}$$

and the instantaneous variance/covariance matrix

$$\sigma(v) = \begin{pmatrix} \varphi(v)^2 & \frac{\rho}{2}\varphi(v)\psi(v) \\ \frac{\rho}{2}\varphi(v)\psi(v) & \psi(v)^2 \end{pmatrix}$$

3 How to discretize

The purpose is to construct a sequence of discrete trading models indexed by its refinement n and converging to the continuous time trading model of the previous section in a suitable sense. Such a discrete model corresponding to refinement n is specified by a discrete set $\mathcal{T}^n = \{0 = t_0^n < t_1^n \ldots < t_n^n = T\}$ of equidistant trading dates, i.e. for $i = 0, \ldots, n : t_{i+1}^n - t_i^n = \Delta t_n := \frac{T}{n}$.

We denote by

$$\overline{Z}^n := \begin{pmatrix} \overline{v} \\ \overline{X} \end{pmatrix}$$

the discrete dynamics. Since the original process is time–homogeneous, we restrict ourselves here to a time–homogeneous $\overline{Z}^n$, too. The appropriate convergence concept is in our case here to require weak–convergence in distribution:

$$\overline{Z}^n \stackrel{d}{\Longrightarrow} Z$$

$\overline{S}^n \stackrel{d}{\Longrightarrow} S$ is a straightforward consequence from the observation that the function exp is continuous.

We denote by P_x^n (E_x^n) the probability of P^n (the expectation) conditional on $\overline{Z} = x$ and

$$\begin{aligned}
\Delta^n(x) &:= \overline{Z}_{1+\Delta t_n} - x \\
\bar{\mu}^n(x) &:= \frac{E_x[\Delta^n(x)]}{\Delta t_n} \\
\bar{\sigma}^n_{i,j}(x) &:= \frac{\operatorname{cov}(\Delta^n_t(x)_i, \Delta^n_t(x)_j)}{\Delta t_n}
\end{aligned}$$

due to our assumption that $\overline{Z}^n$ is time–homogeneous these terms are time independent. $\Delta_t^n(x)$ is the instantaneous increment, $\bar{\mu}^n(x)$ the local drift and $\bar{\sigma}^n_{i,j}(x)$ the local variance/covariance matrix conditional on x. It is natural to require that the first two moments converge to their corresponding continuous–time counterparts. We will now state a theorem which is an immediate restriction of the martingale Central Limit Theorem (see e.g. Ethier and Kurtz (1986), p. 354). It claims that under the further restriction that the mesh becomes "sufficiently quick dense", that this is sufficient:

Theorem 1 *Suppose that for all $c > 0$:*

(1) the local drift converges uniformly on $\{|x| \leq c\}$

(2) the local variance/covariance matrix converges uniformly on $\{|x| \leq c\}$

(3) jump–sizes diminish

$$\forall q > 0 : \max_{x \leq q} |\Delta^n(x)| \stackrel{n}{\longrightarrow} 0 \quad a.s.$$

Then:

$$\overline{Z}^n \stackrel{d}{\Longrightarrow} Z$$

Please note that these are all local conditions. The conditions of this theorem will guide us in our construction.

4 Construction of the volatility–grid

We will proceed first by constructing a one–dimensional grid approximation of v and then extend this to a two–dimensional Z approximation.

In order to get rid of the dependency on $\varphi(v_t)$ in equation (1), Nelson and Ramaswamy (1990) suggest to define first the function

$$f(x) := \int^x \frac{1}{\varphi(v)} dv$$

Setting $Y := f(v)$, an application of the Itô–formula yields

Lemma 2

$$dY_t = \kappa_t^Y dt + dW_t^2$$

where κ^Y is a suitable adapted process.

Note that the model of Chesney and Scott (1989) is specified directly in terms of $Y_t = \ln v_t$ with $\kappa^Y = a - bY$ for some constants a, b.

A lattice–grid for Y is immediately constructed for each refinement n by taking the grid points $Y_j = j\sqrt{\Delta t_n}$ for $j \in \mathbb{Z}$.

It will turn out that it is not necessary to know the process κ^Y. It is only necessary to know how to transform back, i.e. we need we are looking for a function $R(z)$ with $f(R(z)) = z$. In table 2 we present f and R, for the models from the literature. It can be checked that they are true.

Table 2

The transformations f, R for the models from the literature

model	$f(x)$	$R(z)$
Hull and White (1987)	$\ln x$	$e^{p_2 z}$
Heston (1993)	$2\sqrt{x}$	$\frac{z^2}{4}$; $z > 0$ 0 ; $otherwise$
Stein and Stein (1991)	$const.x$	$1/const.x$
Chesney and Scott (1989)	ln	exp

The transformation R on Y yields the grid for v, i.e.:

$$v_j = R\left(j\sqrt{\Delta t_n}\right)$$

As was pointed out by Nelson and Ramaswamy (1990), v may become so small, however the v–drift so great, that jumps into adjacent nodes are no longer sufficient. However multiple jumps in the grid become necessary, to ensure martingale properties are in $[0,1]$. That is the dynamics is given by

$$\begin{aligned} v_j^u &= \min\{v_{j+k} \mid v_{j+k} - v_j \geq \mu_2(v_j)\Delta t_n, k \text{ odd}\} \\ &\qquad \text{for an "up"–jump} \\ v_j^d &= \max\{v_{j+k} \mid v_{j-k} - v_j \leq \mu_2(v_j)\Delta t_n, k \text{ odd}\} \vee 0 \\ &\qquad \text{for a "down"–jump} \end{aligned}$$

i.e. in general v_j^u is just two grid nodes below v_j^u and the further jumps just ensure that the necessary drift lies between these two nodes.

We will denote the corresponding probabilities by $\bar{p}_j^u, \bar{p}_j^d$. The probability of a single jump will be completely specified in any node j by the drift–condition. Therefore we set

$$\bar{p}_j^u = \frac{\kappa(v_j - \bar{v})\Delta t_n + v_j - v_j^d}{v_j^u - v_j^d}$$

Since

$$\frac{\partial R}{\partial x} = \frac{1}{\frac{\partial f}{\partial x}} = \varphi$$

a Taylor series expansion of Y yields immediately that the local volatility is matched up to an error–term of order $\sqrt{\Delta t_n}$ for any specification of the probability $\bar{p}_j^u$.

Proposition 3 *Under the above assumptions we have*

(1) the drift converges properly

(2) the variance converges properly

(3) weak–convergence

PROOF. We check the assumptions of Nelson and Ramaswamy (1990). Their assumption 5 is vacuous in our case, since our model is time independent. Assumption 7 serves to establish the existence of the continuous solution, which we supposed from the very beginning. Moreover from the observation

$$\frac{\partial f(x)}{\partial x} = \varphi(f(x))$$

follows that assumption 8 and 10 hold. Now we need to distinguish two cases:

(1) $p_1 \neq 0$: Assumption 6 holds e.g. with $\Lambda_{T,R}$. Therefore we conclude from their theorem 2 on p. 405 that weak–convergence of our discrete approximation of the continuous volatility process v holds.

(2) $p_1 = 0$: Assumption 9 can be fulfilled with $\rho = \varphi$. Moreover it is easy to check that our definition of the jumps coincides with thoses of Nelson and Ramaswamy (1990). Therefore we conclude from their theorem 3 on p. 408 that weak–convergence of our discrete approximation of the continuous volatility process v holds.

5 Extension to the $\overline{Z}$–grid

Next we come to the construction of the grid in the X–direction at each node v_j. Constructing the grid points at node v_j in X direction is a very simple task:

$$X_{j,k} := \psi_j \sqrt{\Delta t_n} \cdot k \quad \text{for } k \in \mathbb{Z}$$

The jumps contained in the v–grid carry over and pose no further problem. Here at each time–point t_i we have now a two–dimensional grid, indexed by (j,k) where j continues to correspond to denote v dependence whereas k denotes X dependence.

Now we explain what are the successors of a node $x_{j,k}$ and their transition probabilities. For a down–move in volatility we will allow the stock to move to one of four successors. We will refer to these as (u,u), (u,d), (d,u) and (d,d) in decreasing order. Each is the immediate grid successor in its layer (see figure 1). They are fixed one of these is chosen. Similarly to the preceding section,

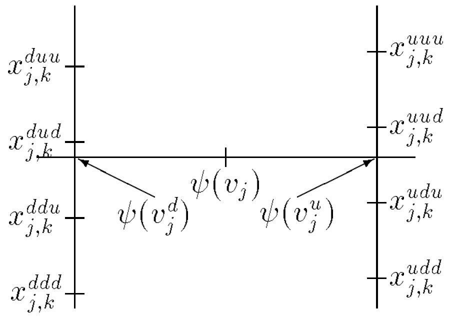

Fig. 1. part of the transition grid

in order to being able to match the drift, when the X–variance is too small and the X–drift too great, we need to allow for multiple jumps in X–direction. Here we choose the closest node above the drift:

$$x_{j,k}^{dud} = \min\{x \mid x = l \cdot v_j \cdot \sqrt{\Delta t_n}; l \in \mathbb{Z} \text{ and } x > x_{j,k} + \mu_2(v_j)\Delta t_n\}$$

Defining for $a, b, c \in \{u, d\}$

$$\begin{aligned} \alpha_{j,k} &:= \frac{x_{j,k}^{dud} - x_{j,k}}{\psi_j^d \sqrt{\Delta t_n}} \\ \text{and} \qquad \beta_{j,k} &:= \frac{x_{j,k}^{uud} - x_{j,k}}{\psi_j^u \sqrt{\Delta t_n}} \end{aligned}$$

we can easily index all successor–nodes of $x_{j,k}$ by $\alpha_{j,k}$ and $\beta_{j,k}$.

In the next step we fix the eight transition probabilities of the eight nodes. This defines a random variable $\overline{R}_{j,k}$. We denote by $\overline{R}_{1,j,k}$ ($\overline{R}_{2,j,k}$) its v (X) marginal. For convergence we require that we match the two drift terms, the two variances and the covariance. According to the observations of the previous section, drift and variance for the $(v_t)_t$ process are correct as long as $p_{j,k}^{udd} + p_{j,k}^{udu} + p_{j,k}^{uud} + p_{j,k}^{uuu} = \bar{p}_j^u$ holds independent of k.

So we have the following equation system:

$$\sum_{a,b,c \in \{u,d\}} p_{j,k}^{a,b,c} = 1 \tag{3}$$

$$\sum_{a,b \in \{u,d\}} p_{j,k}^{d,a,b} = \bar{p}_j^u \tag{4}$$

$$E\left[\overline{R}_{2,j,k}\right] = \mu_2(v_j)\Delta t_n \tag{5}$$

$$\mathrm{Var}\left(\overline{R}_{2,j,k}\right) = \psi(v_j)^2 \Delta t_n \tag{6}$$

$$E\left[\overline{R}_{1,j,k} \cdot \overline{R}_{2,j,k}\right] = 2\sigma_{1,2}(v_j)\Delta t_n \tag{7}$$

(4) corresponds to setting the v–drift, (5) to the X–drift, (6) to the X–variance and (7) to the covariance up to an error term of order $\mathcal{O}(\Delta t_n^2)$.

Instead of resolving this equation system we solved a slightly different one. First we simplify further by assuming that the difference between two nodes in X–direction is equal ψ_j regardless of an up– or a down–move. Moreover that $v_j^u - v_j = v_j - v_j^d$ in v–direction. It is straightforward to see that replacing all $x^{a,b,c}$ by $x^{a,b,c} - \mu(v_j)\Delta t_n$ and requiring $E\left[\overline{R}_{2,j,k}\right] = 0$ is equivalent to requiring condition (5), which we will do in the sequel. To simplify our further

Table 3

Nodes and probabilities

node		probability
$\bar{x}^{duu}_{j,k}$	$= (\bar{\alpha}_{j,k}+1)\psi_j\sqrt{\Delta t_n}$	$p^{duu}_{j,k}$
$\bar{x}^{dud}_{j,k}$	$= \bar{\alpha}_{j,k}\psi_j\sqrt{\Delta t_n}$	$p^{dud}_{j,k}$
$\bar{x}^{ddu}_{j,k}$	$= (\bar{\alpha}_{j,k}-1)\psi_j\sqrt{\Delta t_n}$	$p^{ddu}_{j,k}$
$\bar{x}^{ddd}_{j,k}$	$= (\bar{\alpha}_{j,k}-2)\psi_j\sqrt{\Delta t_n}$	$p^{ddd}_{j,k}$
$\bar{x}^{uuu}_{j,k}$	$= (\bar{\beta}_{j,k}+1)\psi_j\sqrt{\Delta t_n}$	$p^{udd}_{j,k}$
$\bar{x}^{uud}_{j,k}$	$= \bar{\beta}_{j,k}\psi_j\sqrt{\Delta t_n}$	$p^{udu}_{j,k}$
$\bar{x}^{udu}_{j,k}$	$= (\bar{\beta}_{j,k}-1)\psi_j\sqrt{\Delta t_n}$	$p^{uud}_{j,k}$
$\bar{x}^{udd}_{j,k}$	$= (\bar{\beta}_{j,k}-2)\psi_j\sqrt{\Delta t_n}$	$p^{uuu}_{j,k}$

calculations, we *suppose* that X adopts the values $\bar{x}^{abc}_{j,k}$ as in table 3 where we define:

$$\begin{aligned}
\bar{x}^{dud} &= x^{dud} - \mu(v_j)\Delta t_n \\
\bar{x}^{uud} &= x^{uud} - \mu(v_j)\Delta t_n \\
\bar{\alpha}_{j,k} &:= \frac{\bar{x}^{dud}_{j,k} - x_{j,k}}{\psi^d_j\sqrt{\Delta t_n}} \\
\text{and} \qquad \bar{\beta}_{j,k} &:= \frac{\bar{x}^{uud}_{j,k} - x_{j,k}}{\psi^u_j\sqrt{\Delta t_n}}
\end{aligned}$$

Moreover we replace condition (6) by

$$E\left[\overline{R}^2_{2,j,k}\right] = \psi^2_j\Delta t_n \tag{8}$$

and condition (7) by

$$E\left[\overline{R}_{1,j,k}\cdot\overline{R}_{2,j,k}\right] = \rho\cdot\psi_j\cdot\Delta_j\cdot\sqrt{\Delta t_n} \tag{9}$$

To exclude ambiguity about the solution, for $a, b \in \{u, d\}$ we added the fol-

Table 4

Definition of functions

$\delta_1(\gamma) = \dfrac{1-\gamma^2}{\gamma^2+2\gamma}$	$\delta_2(\gamma) = \dfrac{2\gamma-\gamma^2}{\gamma^2-4\gamma+3}$
$\kappa_1(\gamma) = -2\gamma+3$	$\lambda_1(\gamma) = 2\gamma+1$
$\kappa_2(\gamma) = -2\gamma^2+6\gamma-6$	$\lambda_2(\gamma) = 2\gamma^2+2\gamma+2$
$\kappa_3(\gamma) = \dfrac{\gamma^2+2\gamma}{-\delta_2(\gamma)}$	$\lambda_3(\gamma) = \dfrac{\gamma^2-4\gamma+3}{\delta_2(\gamma)}$

lowing four equations to our system:

$$\begin{aligned}&\left(\bar{x}^{abu}_{j,k}\right)^2 \cdot p^{abu}_{j,k} + \left(\bar{x}^{abd}_{j,k}\right)^2 \cdot p^{abd}_{i,j,k}\\ &= \psi_j^2 \cdot \Delta t_n \cdot \left(p^{abu}_{j,k} + p^{abd}_{j,k}\right)\end{aligned}$$

which expresses $p^{abu}_{j,k}$ in terms of $p^{abu}_{j,k}$ and $\bar{\alpha}_{j,k}$ resp. $\bar{\beta}_{j,k}$. It is easy to see that under these four conditions, equation (8) is automatically fulfilled.

A solution is given by

$$\begin{aligned}
p^{dud}_{j,k} &= \left(-\kappa_1(\bar{\alpha}_{j,k})\rho\sqrt{\Delta t_n} - (1-\bar{p}^u_j)\kappa_2(\bar{\alpha}_{j,k})\right)\\
&\quad \cdot\kappa_3(\bar{\alpha}_{j,k})\\
p^{ddu}_{j,k} &= \left(-\lambda_1(\bar{\alpha}_{j,k})\rho\sqrt{\Delta t_n} - (1-\bar{p}^u_j)\lambda_2(\bar{\alpha}_{j,k})\right)\\
&\quad \cdot\lambda_3(\bar{\alpha}_{j,k})\\
p^{uud}_{j,k} &= \left(\kappa_1(\bar{\beta}_{j,k})\rho\sqrt{\Delta t_n} - \bar{p}^u_j\kappa_2(\bar{\beta}_{j,k})\right)\cdot\kappa_3(\bar{\beta}_{j,k})\\
p^{udu}_{j,k} &= \left(\lambda_1(\bar{\beta}_{j,k})\rho\sqrt{\Delta t_n} - \bar{p}^u_j\lambda_2(\bar{\beta}_{j,k})\right)\cdot\lambda_3(\bar{\beta}_{j,k})
\end{aligned}$$

together with

$$\begin{aligned}
p^{duu}_{j,k} &= \delta_1(\bar{\alpha}_{j,k})p^{dud}_{j,k}\\
p^{ddd}_{j,k} &= \delta_2(\bar{\alpha}_{j,k})p^{ddu}_{j,k}
\end{aligned}$$

where we used the definitions of table 4.

Lemma 4 *All transition probabilities $p^{a,b,c}_{j,k}$ are between 0 and 1.*

PROOF. We check that all probabilities are positive. Since they sum to one this will ensure that they are less than one. For $\rho \in [-1,1]$ and $\gamma \in]0,1]$ we have

$$\begin{aligned}
0 &< \kappa_1(\gamma)\rho\Delta t_n - (1-\bar{p}_j^u)\kappa_2(\gamma) \\
0 &< \kappa_1(\gamma)\rho\Delta t_n - \bar{p}_j^u\kappa_2(\gamma) \\
0 &> \lambda_1(\gamma)\rho\Delta t_n - (1-\bar{p}_j^u)\lambda_2(\gamma) \\
0 &> \lambda_1(\gamma)\rho\Delta t_n - \bar{p}_j^u\lambda_2(\gamma)
\end{aligned}$$

Moreover it follows easily that $\kappa_3(\gamma) \geq 0$, since

$$\begin{aligned}
0 &\leq \gamma^2 + 2\gamma \\
0 &< -8\gamma + 6\gamma + 12
\end{aligned}$$

and $\lambda_3(\gamma) \leq 0$, since

$$\begin{aligned}
0 &\leq \gamma^2 - 4\gamma + 3 \\
0 &> 8\gamma^2 - 8\gamma - 12
\end{aligned}$$

This implies immediately the assertion.

Please note that the discrete process is pathindependent and recombining. The following proposition states that in the limit, with our choice of the probabilities the first two moments converge properly and that weak convergence to the continuous solution holds (please note that with the notations on page 4, here the local increment is $\Delta^n(x_{j,k}) = (\overline{R}_{1,j,k}, \overline{R}_{2,j,k})$). Thus we do now check the conditions of theorem 1:

Lemma 5

$$\forall c > 0 : \sup_{x_{j,k} \leq c} \frac{E\left[\overline{R}_{2,i,j}\right]}{\Delta t_n} \xrightarrow{n} \mu(v_j)$$

PROOF.

$$
\begin{aligned}
& E\left[\overline{R}_{2,j,k}\right] \\
= & \sum_{a,b,c\in\{u,d\}} p_{j,k}^{a,b,c}(x_{j,k}^{a,b,c} - x_{j,k}) \\
= & \sum_{a,b,c\in\{u,d\}} p_{j,k}^{a,b,c}(\bar{x}_{j,k}^{a,b,c} - x_{j,k}) \\
& \quad +\mu_2(v_j)\Delta t_n \\
& \quad +\sqrt{\Delta t_n} \underbrace{\sum_{a,b,c\in\{u,d\}} p_{j,k}^{a,b,c}(\psi(v_j^a) - \psi_j)}_{=E[\overline{R}_{1,j,k}]=\mathcal{O}(\Delta t_n)} \\
= & \, \mu_2(v_j)\Delta t_n + \mathcal{O}(\sqrt{\Delta t_n}^3)
\end{aligned}
$$

uniformly

Lemma 6

$$
\forall c > 0 : \sup_{x_{j,k}\leq c} \frac{Var\left(\overline{R}_{2,i,j}\right)}{\Delta t_n} - \psi_j^2 \xrightarrow{n} 0
$$

PROOF. This follows from the general fact that the observation $\mathrm{Var}(\overline{R}_{2,j,k} - \mu(v_j)) = \mathrm{Var}(\overline{R}_{2,j,k})$ together with the fact that equation (8) is fulfilled and $\mathrm{Var}(\overline{R}_{2,j,k}) = E[\overline{R}_{2,j,k}^2] - E[\overline{R}_{2,j,k}]^2 = E[\overline{R}_{2,j,k}^2] - \mathcal{O}(\Delta t_n^2)$ by Lemma 5.

Lemma 7

$$
\rho\psi_j\Delta v_j\sqrt{\Delta t_n} = 2\sigma_{1,2}(v_j)\Delta t_n + \mathcal{O}\left(\sqrt{\Delta t_n}^3\right)
$$

i.e. requiring (7) is equivalent to (9).

PROOF. We have:

$$
\begin{aligned}
\Delta v_j &= v_j^u - v_j \\
&= \frac{\partial R}{\partial x}(v_j)\sqrt{\Delta t_n} + \mathcal{O}(\Delta t_n) \\
&= \varphi(v_j)\sqrt{\Delta t_n} + \mathcal{O}(\Delta t_n)
\end{aligned}
$$

Therefore:

$$\begin{aligned}\rho\psi_j\Delta v_j\sqrt{\Delta t_n} &= \rho\psi_j\varphi(v_j)\Delta t_n + \mathcal{O}\left(\sqrt{\Delta t_n}^3\right)\\ &= 2\sigma_{1,2}(v_j)\Delta t_n + \mathcal{O}\left(\sqrt{\Delta t_n}^3\right)\end{aligned}$$

Lemma 8

$$\forall c > 0 : \sup_{x_{j,k}\le c} \frac{E\left[\overline{R}_{1,i,j}\cdot\overline{R}_{2,i,j}\right]}{\Delta t_n} \xrightarrow{n} \rho$$

PROOF. With Lemma 7 similarly to Lemma 5.

Lemma 9 *Denote:*

$$\begin{aligned}&\textit{for } q > 0 : \ \mathcal{G}_q \ := \{(j,k)\,|\,|x_{j,k}|, v_j \le q\}\\ &\textit{for } (j,k) : m_{j,k} := \max_{a,b,c\in\{u,d\}} |x_{j,k}^{a,b,c} - x_{j,k}|\end{aligned}$$

Jump–sizes diminish, i.e.:

$$\forall q > 0 : \max_{(j,k)\in\mathcal{G}_q} \max\{m_{j,k}, v_j^+ - v_j, v_j^- - v_j\} \xrightarrow{n} 0$$

PROOF. Since R is increasing, $x_{j,k} \le R(q)\sqrt{\Delta t_n}$ and $v_j^+ - v_j$ as well as $v_j^- - v_j$ converge with order $\mathcal{O}(\sqrt{\Delta t_n})$ to 0.

Theorem 10 *A sequence of discrete processes whith the above properties converges weakly in distribution to the two–dimensional process (v,S) which is solution to 1,2.*

PROOF. With the notations as on page 4 we have for all $c > 0$:

$$\begin{aligned}&\max_{\{(j,k)|\bar{x}_{j,k}\le c\}} \max_{a,b\in\{1,2\}} |\bar{\mu}_{a,b}^n| \xrightarrow{n} 0\\ &\max_{\{(j,k)|\bar{x}_{j,k}\le c\}} \max_{a,b\in\{1,2\}} |\bar{\sigma}_{a,b}^n| \xrightarrow{n} 0\end{aligned}$$

The proof concludes by applying Lemma 5 to 9.

This Theorem allows us immediately to deduce convergence of prices for the European put option and thus through put–call–parity for European call options as well.

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