ROBUSTNESS OF GAUSSIAN HEDGES AND THE HEDGING OF FIXED INCOME DERIVATIVES

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Abstract. The effect of model and parameter misspecification on the effectiveness of Gaussian hedging strategies for derivative financial instruments is analyzed, showing that Gaussian hedges in the "natural" hedging instruments are particularly robust. This is true for all models that imply Black/Scholes–type formulas for option prices and hedging strategies. In this paper we focus on the hedging of fixed income derivatives and show how to apply these results both within the framework of Gaussian term structure models as well as the increasingly popular market models where the prices for caplets and swaptions are given by the corresponding Black formulas. By explicitly considering the behaviour of the hedging strategy under misspecification we also derive the El Karoui, Jeanblanc-Picqué and Shreve (1995, 1998) and Avellaneda, Levy and Paras (1995) result that a superhedge is obtained in the Black/Scholes model if the misspecified volatility dominates the true volatility. Furthermore, we show that the robustness and superhedging result do not hold if the natural hedging instruments are unavailable. In this case, we study criteria for the optimal choice from the instruments that are available.

1. Introduction

Models for pricing by arbitrage are widely applied in practice despite, or perhaps because of, the fact that they imperfectly represent reality. The Black/Scholes formula for option pricing owes its popularity to two important features: For one, it can be derived in a totally preference–free modelling framework, without recourse to such unobservables as agents' utility functions. Secondly, the model is analytically very tractable and therefore does not require time–consuming numerical calculations. Models in other areas of derivative asset analysis are often measured by this standard. Thus for example Gaussian term structure models remain popular even though they imply negative interest rates with positive probability.

The question therefore arises whether the unrealistic and often empirically invalidated assumptions that are made on the stochastic processes driving the underlying prices, as well as the requirement that the process parameters are known exactly, are legitimate abstractions for the sake of model tractability. In practice, these two sources of misspecification are taken into account by repeatedly recalibrating the model to the market, an approach which contradicts the model assumptions. In particular term structure models are refitted time and time again to whichever yield curve is observed in the market.

The foundation for the pricing of derivative securities by arbitrage is given by the concept of a self-financing, duplicating strategy, *hedging strategy* for short. We analyze how

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effective such strategies are under model misspecification and recalibration. The hedging strategies are assumed to be carried out according to a model which differs from the true dynamics of market prices either in the process parameters or, more generally, in the way the stochastic differential equations driving market prices are specified. In particular, we relax the usual assumption that strategies are self-financing and only require that they duplicate the payoff to be hedged at maturity. This is due to the fact that misspecification necessarily introduces a non-vanishing cost process.

As strategies are no longer self-financing and duplicating at the same time, some authors [1] consider model misspecification to be a case of market incompleteness. However, although markets may be incomplete in our framework, incompleteness is not the issue central to this paper. Even if both the underlying "true" model and the model assumed for the purpose of calculating hedging strategies are complete in the Harrison and Pliska (1981, 1983) sense; contingent claims that could *theoretically* be duplicated by self-financing strategies are not, because of misspecification.

We study the case of a European option to exchange two assets. This is a suitably general payoff; at the same time an explicit hedging strategy can be calculated under the assumption that the relevant dynamics are lognormal. The strategy we consider is continuously recalculated under the assumed model, given the market prices generated by the true dynamics. Therefore we have inflows and/or outflows of funds from our hedging portfolio, defining a cost process along the lines of Föllmer and Sondermann (1986). We analyze the behaviour of this cost process under misspecification. In order to remain independent of agents' risk preferences, we focus on strategies which are superhedges over their entire lifetime, i.e. whose cost processes are almost surely monotonically decreasing. Such processes must necessarily be of finite variation. The main result is that the cost process is indeed always of finite variation for a Black/Scholes–type hedge in the natural instruments. Identifying the natural hedge instruments for fixed income derivatives, we can apply the exchange option framework to the hedging of bond options as well as interest rate derivatives such as caps, floors and swaptions, using Gaussian term structure models of the Vasicek (1977) type, or lognormal interest rate "market" models such as Miltersen, Sandmann and Sondermann (1997), Brace, Gatarek and Musiela (1997) or Jamshidian (1997).

By explicitly studying the hedging strategy, we arrive at a very straightforward proof of the El Karoui, Jeanblanc-Picqué and Shreve (1995) and Avellaneda, Levy and Paras (1995) result that a superhedge is obtained in the Black/Scholes model if the misspecified volatility dominates the true volatility. Clearly, if the option price is increasing in the volatility of the underlyings, under absence of arbitrage a hedging strategy based on an overestimated volatility must be a superhedge "on average", i.e. in expectation under the risk neutral measure. However, the robustness result is stronger than this; a portfolio strategy in the natural instruments constructed assuming lognormal dynamics with volatility dominating the true volatility is an almost sure superhedge, irrespective of the true asset dynamics.

This result has several implications. For one, if the misspecified volatility is the smallest upper bound on the true volatility, running a Black/Scholes (or "Gaussian") hedge is arguably the cheapest of (many possible) superhedges. Secondly, the market practice of simultaneously applying Black/Scholes–like formulas to bond options, swaptions, caps and floors leads to inconsistencies and theoretical arbitrage opportunities, but in view of our result attempting to force consistency by employing theoretically compatible non–Gaussian

[1] E.g. El Karoui, Jeanblanc-Picqué and Shreve (1995, 1998) or Ahn, Muni and Swindle (1997).

hedges for some products may be counterproductive given uncertainty about what is the "true" model.
The El Karoui, Jeanblanc-Picqué and Shreve (1995) result very much depends on the fact that the hedging strategy is carried out in the "natural" instruments, as we show in section 8. If a natural hedging instrument is unavailable, a two-step hedging strategy based on volatility overestimation no longer results in a superhedge and the problem arises how to select hedging instruments from traded assets in a manner which is by some criterion optimal.
The paper is organised as follows. The next section introduces the probabilistic setup. We then proceed to formalize the pragmatic approach of assuming the dynamics of a suitable process to be lognormal for hedging purposes while frequently recalibrating to market prices. Section 4 derives the finite variation of the cost process as well as the superhedging result; section 5 discusses the application of this result to fixed income derivatives and the corresponding models. In section 6, we add a few remarks on the cost of setting up a superhedge and section 7 commences the discussion of the case where "natural" hedging instruments are unavailable. This discussion is continued in sections 8 through 10, where we study the problem of duplicating a zero coupon bond with other bonds and state some criteria for the optimal choice of maturities. The last section concludes.

2. Probabilistic Framework

Due to the effects of misspecification, the trading strategies we consider will not be self-financing. Therefore, we adopt a more general definition of a trading strategy which does not include the self-financing requirement. Associated with each such strategy is a cost process, whose introduction is the main purpose of this section. We collect definitions and fix some terminology along the way.

All the stochastic processes we consider are defined on an underlying stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^*]}, P)$, which satisfies the usual hypotheses. Trading terminates at time $T^* > 0$. We assume that the price processes of underlying assets are described by strictly positive, continuous semimartingales. By a contingent claim X with maturity $T \in [0, T^*]$, we simply mean a random payoff received at time T, which is described by the $\mathcal{F}_T$-measurable random variable X.

Definition 2.1 (Trading Strategy, Duplication). Let $S^{(1)}, \dots, S^{(N)}$ denote the price processes of underlying assets. A trading strategy ϕ in these assets is given by an $\mathbb{R}^N-$ valued, predictable process which is integrable with respect to S. The value process $V(\phi)$ associated with ϕ is defined by

$$V(\phi) = \sum_{i=1}^{N} \phi^{(i)} S^{(i)}$$

If X is a contingent claim with maturity T, then ϕ duplicates X iff

$$V_T(\phi) = X \quad P-\text{a.s.}$$

Definition 2.2 (Cost Process). If ϕ is a trading strategy in the assets $S^{(1)}, \dots, S^{(N)}$, the cost process $L(\phi)$ associated with ϕ is defined as follows:

$$L_t(\phi) := V_t(\phi) - V_0(\phi) - \sum_{i=1}^{N} \int_0^t \phi_u^{(i)} \, dS_u^{(i)}$$

The infinitesimal increment $dL_t(\phi)$ is the incremental cost incurred at time t by rebalancing the portfolio as prescribed by the strategy ϕ. The portfolio strategies we consider in actual calculations will be continuous semimartingales themselves. If ϕ is a continuous semimartingale, the same is true of the value process $V(\phi)$ and the cost process $L(\phi)$. Itô's lemma then implies that we can calculate the increment of the cost process as follows

$$dL(\phi) = \sum_{i=1}^{N} \left(S^{(i)} \, d\phi^{(i)} + \, d\langle S^{(i)}, \phi^{(i)} \rangle \right) \tag{1}$$

The strategy ϕ is self-financing iff the cost process $L(\phi)$ is identically zero. In this case, the value process $V(\phi)$ is always a continuous semimartingale because it can be represented as a stochastic integral.

In the presence of misspecification, the duplication of a contingent claim by a self-financing strategy may not be possible. Superhedges are one concept designed to deal with this situation.

Definition 2.3 (Superhedge). Consider a contingent claim X maturing at time $T \in [0, T^*]$. A superhedge for X is a portfolio strategy ϕ which replicates X and for which the paths of the rebalancing cost process $L(\phi)$ are almost surely monotonically decreasing.

According to our definition, a strategy ϕ replicating a contingent claim X maturing at $T \in [0, T^*]$ is a superhedge iff at each time $t \in [0, T]$ the incremental cost $dL_t(\phi)$ of rebalancing the portfolio is non-positive, so that no funds need to be injected into the portfolio while still replicating the contingent claim at time T.

Note that in our definition of the cost process $L(\phi)$, for each $t \in [0, T]$ the increment $dL_t(\phi)$ is given in terms of money paid at time t, so that the increments are not given in terms of a single numeraire. In particular, future payments are not discounted. Suppose now that the asset $S^{(1)}$ is the numeraire. We denote discounted asset price and value processes by starring them, i.e. $V^* := \frac{1}{S^{(1)}} V$ and $S^{*(i)} := \frac{1}{S^{(1)}} S^{(i)}$. In contrast, the proper definition of the discounted cost process $L^*(\phi)$ is

Definition 2.4 (Discounted Cost Process). Let ϕ be a trading strategy in the assets $S^{(1)}, \dots, S^{(N)}$ and suppose that the asset $S^{(1)}$ is used as numeraire. Then, the discounted cost process $L^*(\phi)$ is given by

$$L_t^*(\phi) := V_t^*(\phi) - \sum_{i=1}^{N} \int_0^t \phi_u^{(i)} \, dS_u^{*(i)} \tag{2}$$

This is the definition of a cost process as it is introduced in Föllmer and Sondermann (1986). In that paper, it is shown how one arrives at this formula as the continuous time limit of the cost incurred when the portfolio is rebalanced at discrete points in time. The concept of a superhedge only refers to the local properties of the cost process. Therefore, as the following proposition shows, it makes no difference whether we define superhedging strategies using the cost process $L(\phi)$ or the discounted process $L^*(\phi)$.

PROPOSITION 2.5. *Let ϕ be a portfolio strategy. The two processes L and L^* are related as follows*

$$\begin{aligned} dL^* &= \frac{1}{S^{(1)}}\, dL + d\left\langle \frac{1}{S^{(1)}},\, L \right\rangle \\ dL &= S^{(1)}\, dL^* + d\left\langle S^{(1)},\, L^* \right\rangle \end{aligned}$$

In particular, the paths of L^* are almost surely locally of bounded variation iff the paths of L are. The paths of of L^* are almost surely monotonically decreasing iff the same is true of the paths of L.

PROOF: By definition of L^* we have

$$\begin{aligned} dL^* &= dV^* - \sum_{i=1}^{N} \phi^{(i)}\, dS^{*(i)} \\ &= \frac{1}{S^{(1)}} dV + V\, d\left(\frac{1}{S^{(1)}}\right) + d\left\langle \frac{1}{S^{(1)}}, V \right\rangle \\ &\quad - \sum_{i=1}^{N} \phi^{(i)} \left(\frac{1}{S^{(1)}}\, dS^{(i)} + S^{(i)}\, d\left(\frac{1}{S^{(1)}}\right) + d\left\langle S^{(i)}, \frac{1}{S^{(1)}} \right\rangle \right) \end{aligned}$$

Therefore

$$dL^* = \frac{1}{S^{(1)}}\, dL + d\left\langle \frac{1}{S^{(1)}}, L \right\rangle \tag{3}$$

On the other hand

$$\begin{aligned} dL &= dV - \sum_{i=1}^{N} = \phi^{(i)}\, dS^{(i)} \\ &= d\left(S^{(1)} V^*\right) - \sum_{i=1}^{N} \phi^{(i)}\, d(S^{(1)} S^{(i)^*}) \\ &= S^{(1)}\, dV^* + V^*\, dS^{(1)} + d\langle S^{(1)}, V^* \rangle - \sum_{i=1}^{N} \phi^{(i)} \left(S^{(1)}\, dS^{(i)^*} + S^{(i)^*}\, dS^{(1)} + d\langle S^{(1)}, S^{(i)^*} \rangle \right) \end{aligned}$$

so that

$$dL = S^{(1)}\, dL^* + d\langle S^{(1)}, L^* \rangle \tag{4}$$

We now suppose that the paths of L are locally of bounded variation. Equation (3) implies that for each $t \in \mathbb{R}_+$:

$$L_t^* = \int_0^t \frac{1}{S^{(1)}}\, dL_u$$

This shows that the paths of L^* are also locally of bounded variation. If the paths of L are monotonically decreasing, then because $S^{(1)}$ is strictly positive, the same is true of the paths of L^*. That the path properties of L^* imply those of L is proved in exactly the same way using equation (4).

□

3. Asset Dynamics Assumed by Gaussian Hedgers

Black/Scholes–like formulae for pricing derivatives follow from the assumption that the stochastic dynamics of the process relevant for hedging are driven by a geometric Brownian motion. In particular, this implies that the volatility is deterministic. Hedge ratios can then be expressed in terms of the cumulative distribution function of the standard normal distribution, therefore the term "Gaussian hedges". To formalize, along the lines of Frey and Sommer (1996) we state the following

Definition 3.1 (Lognormal Process). We call a stochastic process Z *lognormal* iff it can be written in the form

$$dZ_t = Z_t\left(\mu_t dt + \tilde{\sigma}_Z(t) dW_t\right) \tag{5}$$

with *deterministic* dispersion coefficients $\tilde{\sigma}_Z : [0,T[\rightarrow \mathbb{R}_+^d$.

This lognormality assumption allows the derivation of prices and hedges in the following way:

Theorem 3.2. *Let X, Y be the price processes of two assets. Consider an option to exchange X for Y at the maturity date T, i.e. a European option with payoff $[X_T - Y_T]^+$. In a model where the quotient process $Z := \frac{X}{Y}$ is lognormal, it holds that*

(a) *The price process $C = (C_t)_{0 \le t \le T}$ of the exchange option is given by*

$$C_t = C(t, X_t, Y_t) := X_t \mathcal{N}(h^{(1)}(t, Z_t)) - Y_t \mathcal{N}(h^{(2)}(t, Z_t))$$

where $\mathcal{N}$ denotes the one-dimensional standard normal distribution function, $\tilde{\sigma}_Z$ is the deterministic volatility of Z, and where the functions $h^{(1)}$ and $h^{(2)}$ are given by

$$h^{(1)}(t,z) = \frac{\ln(z) + \frac{1}{2}\int_t^T \|\tilde{\sigma}_Z(s)\|^2 ds}{\sqrt{\int_t^T \|\tilde{\sigma}_Z(s)\|^2 ds}} \tag{6}$$

$$h^{(2)}(t,z) = h^{(1)}(t,z) - \sqrt{\int_t^T \|\tilde{\sigma}_Z(s)\|^2 ds} \tag{7}$$

(b) *The hedge portfolio $\Phi = (\Phi)_{0 \le t \le T}$ for this option in terms of the assets X and Y is given by*

$$\begin{aligned} \phi_t^X &:= \mathcal{N}(h^{(1)}(t, Z_t)) \quad \text{units of } X \\ \text{and} \quad \phi_t^Y &:= -\mathcal{N}(h^{(2)}(t, Z_t)) \quad \text{units of } Y \end{aligned}$$

Proof: See Margrabe (1978) or Frey and Sommer (1996).

□

A few comments are in order concerning this theorem. The lognormality of X and Y is sufficient, but not necessary for its application. In a model where X and Y are lognormal processes with volatilities $\tilde{\sigma}_X$ and $\tilde{\sigma}_Y$, respectively, Z is lognormal as well with volatility $\tilde{\sigma}_X - \tilde{\sigma}_Y$. This remains valid in the degenerate cases where either X_T or Y_T is deterministic, so that theorem 3.2 can be applied to a standard put or call option on an asset with a lognormal price process.

The trading strategy Φ given in (b) duplicates the exchange option at maturity even in the case of model misspecification. However, it is only self-financing if the true dynamics of X and Y are such that Z is lognormal and the true volatility of Z equals the assumed volatility. It is remarkable that the hedging strategy can be specified exclusively in X and Y, regardless of the dimension of the driving Wiener process. In particular, the pricing and hedging in theorem 3.2 is the same as for a model driven by a one-dimensional Wiener process where

$$dZ_t^M = \sqrt{\int_t^T \|\tilde{\sigma}_Z(s)\|^2 ds} \; dW_t$$

with dZ^M denoting the martingale part of the Doob-Meyer decomposition of Z.

For this reason, we call X and Y the *natural hedge instruments* for the contingent claim. The case where one of these instruments, say Y, is unavailable for trading leads to complications. If the driving Wiener process is one-dimensional and $\tilde{Y}$ is a lognormal asset not perfectly correlated with X, Y can be replicated by a self-financing strategy in X and $\tilde{Y}$. This is no longer the case in higher dimensions: An additional asset would be required for each additional dimension of the Wiener process.

4. The Cost Process for Gaussian Hedges

Let us now analyze the case where the trader hedges according to theorem 3.2. As inputs, this requires the actually observed prices of the underlyings X and Y as well as an assumption on the volatility of the quotient process Z in form of a deterministic function $\tilde{\sigma}_Z : [0,T[\to \mathbb{R}^d$. Again, note that this is an assumption for hedging purposes only. It is the discrepancy between reality and the hedgers assumptions that gives rise to the misspecification of hedging strategies which we seek to analyze; nothing is assumed about the true dynamics of the asset prices, other than that they are given by strictly positive, continuous semimartingales.
We set

$$\tilde{v}(t) := \sqrt{\int_t^T \|\tilde{\sigma}_Z(s)\|^2 ds}$$

and define stochastic processes $H^{(i)}$, $i = 1, 2$, on $[0, T[$ by $H_t^{(i)} := h^{(i)}(t, Z_t)$. We then have $\phi^X = \mathcal{N}(H^{(1)})$ and $\phi^Y = -\mathcal{N}(H^{(2)})$, and can state the following

Lemma 4.1. *Given that the hedger follows the hedging strategy in theorem 3.2, the cost process (1) can be written as*

$$\begin{aligned} dL_t &= \frac{1}{2}\tilde{v}(t) X_t \mathcal{N}'(H_t^{(1)}) d\langle H^{(2)}\rangle_t \\ &+ \tilde{v}'(t) X_t \mathcal{N}'(H_t^{(1)}) dt \end{aligned}$$

In particular, L is locally of bounded variation.

Proof: According to (1), the cost process is given by

$$dL_t = X_t d\phi_t^X + d\langle X, \phi^X\rangle_t + Y_t d\phi_t^Y + d\langle Y, \phi^Y\rangle_t \tag{8}$$

It follows from (7) that

$$dH_t^{(1)} = dH_t^{(2)} + \tilde{v}'(t) dt$$

In particular, we have $\langle H^{(1)}\rangle = \langle H^{(2)}\rangle$. Using these relationships, we calculate the right-hand side of (8) via Itô's formula to obtain:

$$
\begin{aligned}
dL_t &= \left\{X_t\mathcal{N}'(H_t^{(1)}) - Y_t\mathcal{N}'(H_t^{(2)})\right\} dH_t^{(2)} \\
&+ \frac{1}{2}\left\{X_t\mathcal{N}''(H_t^{(1)}) - Y_t\mathcal{N}''(H_t^{(2)})\right\} d\langle H^{(2)}\rangle_t \\
&+ \mathcal{N}'(H_t^{(1)})d\langle X, H^{(2)}\rangle_t - \mathcal{N}'(H_t^{(2)})d\langle Y, H^{(2)}\rangle_t \\
&+ X_t\mathcal{N}'(H_t^{(1)})\tilde{v}'(t)dt
\end{aligned}
\tag{9}
$$

It is easy to verify that the following equation is valid:

$$z\mathcal{N}'\left(h^{(1)}(t,z)\right) = \mathcal{N}'\left(h^{(2)}(t,z)\right) \tag{10}$$

We use this result in the form

$$X\mathcal{N}(H^{(1)}) = Y\mathcal{N}(H^{(2)}) \tag{11}$$

which immediately shows that the first expression on the right-hand side of (9) is in fact zero. Furthermore, (11) implies that

$$0 = \left\langle X\mathcal{N}(H^{(1)}) - Y\mathcal{N}(H^{(2)}), H^{(2)}\right\rangle \tag{12}$$

If we calculate the quadratic variation in (12) by the product rule and rearrange, we obtain

$$
\begin{aligned}
&\mathcal{N}'(H_t^{(1)})d\langle X, H^{(2)}\rangle_t - \mathcal{N}'(H_t^{(2)})d\langle Y, H^{(2)}\rangle_t \\
&\qquad = -\left(X_t\mathcal{N}''(H_t^{(1)}) - Y_t\mathcal{N}''(H_t^{(2)})\right) d\langle H^{(2)}\rangle_t
\end{aligned}
\tag{13}
$$

Using this result on the right side of (9) gives us:

$$
\begin{aligned}
dL_t &= -\frac{1}{2}\left\{X_t\mathcal{N}''(H_t^{(1)}) - Y_t\mathcal{N}''(H_t^{(2)})\right\} d\langle H^{(2)}\rangle_t \\
&+ \tilde{v}'(t)X_t\mathcal{N}'(H_t^{(1)})dt
\end{aligned}
$$

One easily verifies that $\mathcal{N}''(x) = -x\mathcal{N}'(x)$, so that

$$
\begin{aligned}
dL_t &= \frac{1}{2}\left\{X_tH_t^{(1)}\mathcal{N}'(H_t^{(1)}) - Y_tH_t^{(2)}\mathcal{N}'(H_t^{(2)})\right\} d\langle H^{(2)}\rangle_t \\
&+ \tilde{v}'(t)X_t\mathcal{N}'(H_t^{(1)})dt
\end{aligned}
$$

Applying (7) and (11) to the previous equation yields the desired result.

□

In order to analyze the cost process further, we must make some general assumption on how the "true" stochastic dynamics of the underlying claims may be represented. We place ourselves in a diffusion process setting by assuming that the probability space $(\Omega, \mathcal{F}, P)$ supports an n-dimensional Brownian motion W and that $\mathbb{F}$ is the augmented filtration generated by W. The martingale representation theorem implies that there is an n–dimensional process σ, integrable with respect to W, so that the martingale part dZ^M of the Doob-Meyer decomposition of Z can be written in the form

$$dZ^M = \sum_{i=1}^{n} Z\sigma^{(i)}dW^{(i)} \tag{14}$$

Other than integrability we are not restricting σ in any way. In particular, σ at time t may depend on t and Z_t, the entire path of Z up to t, or some other random variable. Thus the dynamics of X, Y, and Z need by no means be lognormal, volatility may be stochastic, and the "true" number of driving Brownian motions may differ from the number assumed by the hedger. This general representation permits the following

PROPOSITION 4.2. *Given assumption (14) and a hedging strategy according to theorem 3.2, we have the following equation on* $[0,T[$*:*

$$dL_t = \frac{1}{2} X_t \mathcal{N}'(H_t^{(1)}) \frac{1}{\tilde{v}(t)} \left(\|\sigma_t\|^2 - \|\tilde{\sigma}_Z(t)\|^2\right) dt \tag{15}$$

PROOF: Itô's formula tells us that

$$d\langle H^{(2)}\rangle_t = \frac{1}{\tilde{v}^2(t) Z_t^2} d\langle Z\rangle_t$$

By (14), we have

$$d\langle Z\rangle_t = Z_t^2 \|\sigma_t\|^2 dt$$

With this we obtain from lemma 4.1

$$dL_t = \frac{1}{2\tilde{v}(t)} X_t \mathcal{N}'(H_t^{(1)}) \left\{\|\sigma_t\|^2 + 2\tilde{v}'(t)\tilde{v}(t)\right\} dt \tag{16}$$

Since $\tilde{v}'(t)\tilde{v}(t) = -\frac{1}{2}\|\tilde{\sigma}_Z(t)\|^2$, this is the equation we claimed.

□

COROLLARY 4.3. *By equation (15), the strategy* Φ *is a superhedge for the exchange option iff for each* $t \in [0,T]$ *we have*

$$\|\sigma_t\| \leq \|\tilde{\sigma}_Z(t)\| \quad P\text{-}a.s. \tag{17}$$

5. Applications to Interest Rate Derivatives

In the previous sections we have discussed the robustness of Gaussian hedges in general terms. We go beyond the Black-Scholes framework usually considered in the literature on misspecification to show how these results can be applied to many interest rate derivatives.

Gaussian short-rate models. In one– and multifactor Vasicek (1977)–type term structure models[2], zero coupon bonds are lognormal assets. Thus, by assuming such a model the hedger can construct Gaussian hedges for options on zero coupon bonds. We denote the price at time t of a zero coupon bond with maturity T by $B(t,T)$. The payoff of a call option with maturity T and strike price K on a zero coupon bond with maturity $T+c$ can be written as

$$C_T = [B(T,T+c) - KB(T,T)]^+ \tag{18}$$

We see that it is an option to exchange the two lognormal assets $B(.,T+c)$ and $KB(.,T)$. The price and the hedging strategy for this option prescribed by the model are derived from theorem 3.2. Furthermore, the natural hedge instruments are the bond $B(.,T)$ whose maturity is equal to that of the option and the bond $B(.,T+c)$, which is the option's

[2]Such models have been extensively studied in the literature, e.g. see Jamshidian (1989), Hull and White (1990), Heath, Jarrow and Morton (1992), Brace and Musiela (1994), or Geman, El Karoui and Rochet (1995).

underlying. The quotient process Z, which determines the hedge strategy and the cost process is given by

$$Z_t = \frac{B(t, T+c)}{B(t,T)}$$

which is simply the forward price of the underlying bond for settlement at the option's maturity. We achieve a superhedge by dominating the volatility of this forward price process.

Money market derivatives such as caps and floors can also be treated in this model. We recall that forward LIBOR $f(t, T, \alpha)$ is defined via the equation

$$1 + \alpha f(t, T, \alpha) = \frac{B(t,T)}{B(t, T+\alpha)} \tag{19}$$

A caplet with reset date T and strike rate κ has a payoff of $\alpha(f(T,T,\alpha) - \kappa)^+$ at time $T + \alpha$. A straightforward calculation shows that this is equivalent to a payoff of

$$(1 + \alpha\kappa)\left[\frac{1}{1+\alpha\kappa} - B(T, T+\alpha)\right]^+$$

at time T. Thus, the caplet can be interpreted as a put option on the bond $B(., T+\alpha)$. Analogously, a floorlet corresponds to a call option. In these cases, the natural hedge instruments are the zero coupon bonds whose maturities correspond to the reset and settlement dates of the caplet (floorlet).

A more sophisticated derivative which can be considered in this model is a spread option on forward LIBOR.The associated payoff is given by

$$[f(t, T+c, \alpha) - f(t, T, \alpha)]^+ = \frac{1}{\alpha}\left[\frac{B(t, T+c)}{B(t, T+c+\alpha)} - \frac{B(t,T)}{B(t, T+\alpha)}\right]^+$$

As quotients of lognormal assets, the two processes appearing in the option's payoff are lognormal themselves, so that theorem 3.2 can be used to derive a model price and hedging strategy for this option. Unfortunately, in this case the "natural hedge instruments" are unavailable in the market: $B(t, T+c)/B(t, T+c+\alpha)$ may be interpreted as a forward on $B(t, T+c)$ with maturity $T + c + \alpha$, i.e. the forward would mature *after* its underlying.

LIBOR Market Models. As we have seen above, caps and floors may be decomposed into portfolios of puts and calls (*caplets* and *floorlets*) on zero coupon bonds and thus can be treated as options on lognormal claims in a Vasicek–type framework. However, practitioners see *rates* with actuarial compounding as the underlying of these contracts and therefore apply the Black (1976) formula to the rates. Miltersen, Sandmann and Sondermann (1997) (MSS) show this to be compatible with an arbitrage-free framework in which selected forward rates are modelled as lognormal diffusions. Hence the rate $f(t, T, \alpha)$ is a lognormal martingale under the $(T + \alpha)$–forward measure[3], i.e.

$$df(t, T, \alpha) = \tilde{\gamma}(t, T, \alpha) f(t, T, \alpha) dW_t$$

where W is a (possibly multidimensional) Brownian motion under the $(T + \alpha)$–forward measure and $\tilde{\gamma}(t, T, \alpha)$ is a deterministic function of its three arguments. This approach

[3]This modelling approach was pursued further by Brace, Gatarek and Musiela (1997) (BGM), Musiela and Rutkowski (1997) and Jamshidian (1997).

also has the advantage of precluding negative rates, which are assigned positive probability in models of the Vasicek type. The price of a caplet paying

$$\alpha\,[f(T,T,\alpha)-\kappa]^{+}$$

at time $T+\alpha$ is

$$B(t,T+\alpha)\left(\alpha f(t,T,\alpha)\mathcal{N}(h_t^{(1)})-\alpha\kappa\mathcal{N}(h_t^{(2)})\right) \tag{20}$$

with

$$\begin{aligned} h_t^{(1)} =& \frac{1}{\tilde{v}(t,T,\alpha)}\left(\ln\frac{f(t,T,\alpha)}{\kappa}+\frac{1}{2}\tilde{v}^2(t,T,\alpha)\right) \\ h_t^{(2)} =& h_t^{(1)}-\tilde{v}(t,T,\alpha) \end{aligned}$$

and

$$\tilde{v}^2(t,T,\alpha)=\int_t^T \|\tilde{\gamma}(u,T,\alpha)\|^2du$$

Inserting (19), we can write (20) as

$$(B(t,T)-B(t,T+\alpha))\mathcal{N}(h_t^{(1)})-\alpha\kappa B(t,T+\alpha)\mathcal{N}(h_t^{(2)})$$

giving us the natural hedge instruments $X_t := B(t,T)-B(t,T+\alpha)$ and $Y_t := \alpha\kappa B(t,T+\alpha)$. Thus, to construct the hedge, the zero coupon bond with the same maturity as the option and the "underlying" (either the zero coupon bond maturing in $T+\alpha$ or forward LIBOR) are required. Note that while neither X nor Y are assumed to be lognormal claims in this framework, the quotient process

$$\frac{X}{Y}=\frac{f(\cdot,T,\alpha)}{\kappa}$$

is lognormal under the appropriate probability measure by the hedger's assumption. We can apply the same arguments as in the previous section to arrive at the following results:

Assumption 5.1. *There is an n-dimensional Brownian motion W, and an n-dimensional process $\gamma(\cdot,T,\alpha)$, integrable with respect to W, so that the following is valid:*

$$df(t,T,\alpha)^M=\sum_{i=1}^n f(t,T,\alpha)\gamma^{(i)}(t,T,\alpha)dW_t^{(i)}$$

Again, the "true" volatility process $(\gamma(t,T,\alpha))_{0\le t\le T}$ may depend on parameters other than t, T, and α; in particular, γ may be stochastic.

Proposition 5.2. *Under assumption 5.1, the cost process L associated with the hedging strategy prescribed by the MSS model for the caplet satisfies the following equation on $[0,T[$:*

$$dL_t=\frac{1}{2}X_t\mathcal{N}'(H_t^{(1)})\frac{1}{\tilde{v}(t)}\left(\|\gamma(t,T,\alpha)\|^2-\|\tilde{\gamma}(t,T,\alpha)\|^2\right)dt \tag{21}$$

In particular, L is locally of bounded variation.

Corollary 5.3. *The MSS hedging strategy is a superhedge iff for each $t\in[0,T]$ we have:*

$$\|\gamma(t,T,\alpha)\|\le\|\tilde{\gamma}(t,T,\alpha)\| \quad P\text{-a.s.}$$

We see that in the market model, it is necessary to dominate the volatility of the forward rate underlying the caplet in order to obtain a superhedge. This is different from the Vasicek case where we had to dominate the forward price volatility. We see that different modelling strategies lead to different superhedges. The implications of this will be discussed in the next section.

Swap Market Models. The industry–standard approach to pricing European swaptions is again to apply a Black/Scholes–type formula, in this case to the underlying *forward swap rate*. Jamshidian (1997) makes this practice rigorous by developing an arbitrage–free model in which the forward swap rate is a lognormal martingale under the relevant pricing measure.

We consider a European payer swaption which, as Jamshidian (citing Neuberger (1990)) notes, is simply the option to exchange "fixed" cash flows for "floating" cash flows. We let the tenor structure of the swap be represented by dates T_i, $i \in \{0, \dots, n\}$, and define $\delta_i := T_i - T_{i-1}$ for $i \in \{1, \dots, n\}$. At each settlement date T_i, $i \in \{1, \dots, n\}$, a fixed payment of $\delta_i \kappa$ is exchanged in return for a floating payment of $\delta_i L(T_{i-1})$, where $L(T_{i-1})$ denotes the spot LIBOR prevailing at the reset date T_{i-1}, which is given by the usual equation

$$1 + \delta_i L(T_{i-1}) = \frac{1}{B(T_{i-1}, T_i)}$$

The value of the "fixed" side at option expiry is

$$\kappa Y_T := \kappa \sum_{i=1}^{n} \delta_i B(T, T_i)$$

and for "floating"

$$X_T := B(T, T_0) - B(T, T_n)$$

Thus the natural hedging instruments are defined and we can write the option payoff as $[X_T - \kappa Y_T]^+$. The *forward swap rate* at time t is the level κ_t^* such that $X_t - \kappa_t^* Y_t = 0$, defining a process $\kappa^* = X/Y$. Under the equivalent measure induced by choosing Y as numeraire (and under suitable regularity assumptions), κ^* is a martingale. Assuming κ^* to be a lognormal martingale under this measure leads to industry–standard swaption pricing by

$$C_t = (B(t, T_0) - B(t, T_n))\mathcal{N}(h_t^{(1)}) - \kappa \left(\sum_{i=1}^{n} \delta_i B(T, T_i) \right) \mathcal{N}(h_t^{(2)}) \tag{22}$$

with

$$h_t^{(1,2)} = \frac{\ln \kappa^*(t) \pm \frac{1}{2} \int_t^T \|\tilde{\gamma}_\kappa(s)\|^2 ds}{\sqrt{\int_t^T \|\tilde{\gamma}_\kappa(s)\|^2 ds}}$$

where $\tilde{\gamma}_\kappa(\cdot)$ denotes the deterministic volatility of the assumed lognormal dynamics of κ^*. Analogously to the previous results, running a hedge portfolio in X and Y according to (22) results in a cost process of finite variation, and in a superhedge if $\tilde{\gamma}_\kappa$ dominates the true forward swap rate volatility.

REMARK 5.4. The natural hedge instruments X and Y can be constructed by taking appropriate positions in the swap market. For $i = 1, 2$, we denote by $FS^{(i)}$ the value process of a swap with a fixed rate κ_i and a tenor structure identical to the one above. Then

$$FS_t^{(i)} = X_t - \kappa_i Y_t$$

If $\kappa_1 \neq \kappa_2$, we have

$$\begin{aligned} X_t &= \frac{1}{\kappa_2 - \kappa_1}\left(\kappa_2 FS_t^{(1)} - \kappa_1 FS_t^{(2)}\right) \\ \kappa Y_t &= \frac{\kappa}{\kappa_2 - \kappa_1}\left(FS_t^{(1)} - \kappa_1 FS^{(2)}\right) \end{aligned}$$

6. Finding the Cheapest Possible Superhedge

In the previous sections, we showed that Gaussian hedging strategies can be used to achieve a superhedge if there is an upper bound on the volatility, no matter what the true dynamics of the underlying process are. Similar results were obtained by El Karoui, Jeanblanc-Picqué and Shreve (1995) and Frey (1998) by considering self-financing strategies which superreplicate the payoff at maturity. It is well known that (in the context of the Black/Scholes model) the upper volatility bound is a necessary condition for finding a nontrivial superhedging strategy for every contingent claim with a convex value function, i.e. under stochastic volatility without the upper volatility bound the cheapest almost sure superhedge for a call option is holding the underlying.[4]

As our hedging strategies replicate the option payoff at maturity by construction, we need not be explicitly concerned with the convexity of option prices under all possible volatility scenarios. This is one advantage of not restricting strategies to those which are self-financing. By showing that the cost process implied by Gaussian hedging strategies is of finite variation and can be chosen to be monotonically decreasing by setting the assumed volatility to the upper bound, we have met the sufficient conditions for a superhedge. In addition, if no further information about the dynamics of the underlying is available, then the Gaussian hedge at the upper volatility bound is the cheapest superhedge by a simple argument: If the true dynamics are such that volatility is constant at the upper bound, then the hedging strategy replicates the option perfectly at all times, i.e. it is self-financing. Thus the existence of any cheaper hedge would represent an arbitrage opportunity. Since we are requiring that the hedging strategy superreplicate the option, this argument remains valid if the probability that $\|\sigma_t\|^2 > \sigma_t^u - \varepsilon \ \forall \ t \in [0,T]$, where σ_t^u represents the upper volatility bound, is positive for all $\varepsilon > 0$. Thus it does not depend on the choice of equivalent probability measure.

The second advantage of considering superhedging strategies which are not required to be self-financing is that funds in the portfolio are freed as soon as they are no longer required: The value of the portfolio is always equal to the option price under the assumption that the realized volatility will be equal to the upper bound during the *remaining* life of the option; funds which were necessary to construct the superhedge for the time past but not drawn on because the “worst case” did not come about are “paid out” continuously by the monotonically decreasing cost process. Thus the portfolio represents the cheapest superhedge given the upper bound on the volatility and the realized dynamics of the underlying up to any time including option maturity; at option maturity the required payoff is replicated exactly.

In the case of caps, floors and swaptions we have the choice of constructing Gaussian hedges according to a Vasicek–type model or using the “market” model for each product, e.g. MSS

[4] see Frey and Sin (1997).

or BGM for caps and floors or Jamshidian (1997) for swaptions. Which model leads to a cheaper superhedge depends on how volatility information is specified. Since practitioners routinely quote Black volatilities for forward LIBOR and swap rates, the market models will usually be preferable. Of course, volatility bounds on forward rates can be translated into volatility bounds on zero coupon bond prices and vice versa by simple application of Itô's lemma. However, the translated bounds depend on forward rate resp. bond price realizations and then moving to a deterministic upper bound loses information, making the resulting superhedge more expensive.

7. Lack Of "Natural" Hedge Instruments

The main result of the above sections is that "Gaussian" hedges, if carried out in the "natural" hedging instruments, are robust in the sense that they imply a cost process of finite variation irrespective of the true dynamics of the underlying assets. If an upper bound for the volatility of the underlying is known, the Gaussian hedging strategy obtained for the maximum volatility superreplicates the option. This is arguably the cheapest superhedge if no further information on the volatility process is available. At the same time we have seen that a non-trivial superhedge is only possible given a minimum amount of information on the hedge instruments.

However, in many applications, in particular fixed income derivatives, the "natural" instruments leading to a cost process of finite variation are not always traded. As an example, consider even the plain vanilla European call option. The natural hedging instruments in this case are the underlying and the zero coupon bond maturing at option expiry. Typically, such a zero coupon bond will not be liquidly traded, and therefore not available for hedging purposes.

This means that we also need information on the correlation of the natural instruments with those available in the market. Given this information in a complete market, the natural hedging instruments can at least theoretically be synthesized by a dynamic hedging strategy. Thus we have to analyse the dependence of the cost process on the choice of hedge instruments.

Suppose that we are given a hedging strategy $\phi = (\phi^X, \phi^Y)$ for a contingent claim C_T which involves positions in the underlying assets X and Y. Also we assume that there are additional assets $Y^{(1)}, \dots, Y^{(n)}$, where $Y^{(1)} = Y$. If we want to hedge C_T without using asset X, a natural way to proceed is to find a strategy $\tilde{\phi} = (\tilde{\phi}^1, \dots, \tilde{\phi}^n)$ involving positions in $Y^{(1)}, \dots, Y^{(n)}$, which identically replicates X, so that

$$\forall t \in [0,T] : X_t = \sum_{i=1}^{n} \tilde{\phi}_t^i Y_t^{(i)} \tag{23}$$

This immediately gives a hedging strategy $\psi = (\psi^1, \dots, \psi^n)$ for C_T in $Y^{(1)}, \dots, Y^{(n)}$ by

$$\psi^1 = \phi^Y + \phi^X \tilde{\phi}^1 \quad , \quad \psi^i = \phi^X \tilde{\phi}^i \quad \forall i \geq 2$$

Lemma 7.1. *The cost processes of ϕ and ψ are related as follows*

$$dL(\psi) = dL(\phi) + \phi^X dL(\tilde{\phi}) \tag{24}$$

Proof: The cost process of ψ is given by

$$dL(\psi) = dV(\psi) - \sum_{i=1}^{n} \psi^i dY^{(i)}$$

By construction, $V(\psi) = V(\phi)$, so that

$$\begin{aligned} dL(\psi) &= dV(\phi) - \sum_{i=1}^{n} \psi^i dY^{(i)} \\ &= dL(\phi) + \phi^X dX + \phi^Y dY - \sum_{i=1}^{n} \psi^i dY^{(i)} \\ &= dL(\phi) + \phi^X \left\{ dX - \sum_{i=1}^{n} \tilde{\phi}^i dY^{(i)} \right\} \\ &= dL(\phi) + \phi^X dL(\tilde{\phi}) \end{aligned}$$

□

Suppose now that $\phi = (\phi^X, \phi^Y)$ is a superreplicating strategy for C_T, i.e. it holds that

$$dL(\phi) \leq 0$$

The condition for ψ to be superreplicating is

$$dL(\psi) = dL(\phi) + \phi^X dL(\tilde{\phi}) \leq 0$$

Without any information on the interdependence of the two cost processes $L(\phi)$ and $L(\tilde{\phi})$, the most obvious superhedging strategy would be to demand that each term in the equation above is non-positive. From the hedging formula presented in theorem 3.2, one can assume that ϕ^X does not change its sign. Therefore, $L(\tilde{\phi})$ must be monotonic, i.e. a sub- or superhedge for X. If ϕ^X is positive, this effectively means that the superhedge is constructed in two steps. First the missing asset X is superhedged with the available assets. This superhedge is then used as an input for the original superhedging strategy. However, in the next section we show that, since $\tilde{\phi}$ is a strategy identically replicating the asset X, the cost process $L(\tilde{\phi})$ has a non-vanishing martingale part. This implies that a superhedge is not possible using such a two-step strategy.

8. The Cost Process When a Natural Hedging Instrument is Unavailable

We now study the case where the hedge instrument X is not available in the market and a potential hedger must use other assets $Y^{(1)}, ..., Y^{(n)}$ to synthesize X. The model used by the hedger must allow the self-financing replication of X. This is only true irrespective of concrete choice of X if the model used for hedging is dynamically complete. It is helpful to clarify the minimum asset structure necessary for the dynamic completeness of the market described by the hedgers model. This is especially so because we wish to focus on trading only a limited number of available assets. In particular, it is unrealistic to assume that one of these is a continuously compounded savings account, i.e. instantaneously risk-free.

Model Completeness. Once more, we place ourselves in a diffusion process setting. Consider a model where asset prices $Y^{(1)}, ..., Y^{(n)}$ are defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$ supporting a $\tilde{d}$-dimensional P-Brownian motion $\tilde{W}$, and $\mathbb{F}$ is the augmented filtration generated by $\tilde{W}$. We assume that the market determined by $Y^{(1)}, ..., Y^{(n)}$ is arbitrage-free, implying the existence of a martingale measure for each choice of numeraire. By the martingale representation theorem, we can write the martingale part of $Y^{(i)}$ as

$$dY^{(i)^M} = Y^{(i)} \tilde{\sigma}_{Y^{(i)}} \, d\tilde{W}$$

where $\tilde{\sigma}_{Y^{(i)}}$ is a suitably integrable, $\mathbb{R}^{\tilde{d}}$-valued predictable process. The characterization of completeness in this case is a slight generalization of the classical Black-Scholes framework, which has been well-studied in the literature (cf. chapter 6 of Duffie (1996)).

DEFINITION 8.1 (COMPLETENESS). The market determined by the assets $Y^{(1)}, ..., Y^{(n)}$ is complete iff any (suitably integrable) contingent T-claim is attainable; that is, if for any such claim C there exists a self-financing portfolio strategy ϕ in $Y^{(1)}, ..., Y^{(n)}$ such that $C = V_T(\phi)$. In the opposite case, the model is said to be *incomplete.*

PROPOSITION 8.2. *The market given by the assets $Y^{(1)}, \ldots, Y^{(n)}$ is complete iff, for $\lambda^1 \otimes P$-almost all $(t, \omega) \in [0, T^*] \times \Omega$, the affine subspace generated by $\tilde{\sigma}_{Y^{(1)}}(t, \omega), \ldots, \tilde{\sigma}_{Y^{(n)}}(t, \omega)$ has dimension $\tilde{d}$.*

PROOF: 1. We choose $Y^{(n)}$ as numeraire and denote the corresponding martingale measure by Q. We will start by showing that the rank condition is necessary. Let C be an attainable contingent claim settling at time T. Its price process X is given by

$$X_t := Y_t^{(n)} E^Q \left[\frac{C}{Y_T^{(n)}} \middle| \mathcal{F}_t \right] \tag{25}$$

The dynamics of the discounted price process $X^* = \dfrac{X}{Y^{(n)}}$ can be written as

$$dX^* = X^* (\tilde{\sigma}_X - \tilde{\sigma}_{Y^{(n)}}) \, dW^*$$

where W^* is a $\tilde{d}$-dimensional Q-Brownian motion and $\tilde{\sigma}_X$ is a $\tilde{d}$-dimensional predictable process. Since C is attainable, there is a self-financing portfolio ψ such that $X = \sum_{i=1}^n \psi^{(i)} Y^{(i)}$. The self-financing condition can be written as

$$dX^* = \sum_{i=1}^{n-1} \psi^{(i)} dY^{(i)*}$$

This is equivalent to

$$X^* \left(\tilde{\sigma}_X - \tilde{\sigma}_{Y^{(n)}} \right) \, dW^* = \sum_{i=1}^{n-1} \psi^{(i)} Y^{(i)*} \left(\tilde{\sigma}_{Y^{(i)}} - \tilde{\sigma}_{Y^{(n)}} \right) \, dW^*$$

Therefore, it must be true that $\lambda^1 \otimes P$-almost surely

$$\tilde{\sigma}_X - \tilde{\sigma}_{Y^{(n)}} = \sum_{i=1}^{n-1} \psi^{(i)} \frac{Y^{(i)}}{X} \left(\tilde{\sigma}_{Y^{(i)}} - \tilde{\sigma}_{Y^{(n)}} \right)$$

In particular, $\tilde{\sigma}_X(t, \omega)$ must lie in the affine subspace generated by $\tilde{\sigma}_{Y^{(i)}}(t, \omega)$, $i = 1, \ldots, n$, for $\lambda^1 \otimes P$-almost all $(t, \omega) \in [0, T] \times \Omega$.

If the rank condition is not fulfilled, we can construct a process $\hat{\sigma}$ such that there is a set $F \subset [0, T] \times \Omega$ with $\lambda^1 \otimes P[F] > 0$ so that $\hat{\sigma}_t(\omega)$ does not lie in the affine subspace generated by $\tilde{\sigma}_{Y^{(i)}}(t, \omega)$, $i = 1, \ldots, n$, for $(t, \omega) \in F$. We let the process $\hat{X}^*$ be a solution of the SDE

$$d\hat{X}^* = \hat{X}^* (\hat{\sigma} - \tilde{\sigma}_{Y^{(n)}}) dW^*$$

If we define the claim $\hat{C}$ by $\hat{C} := Y_T^{(n)} \hat{X}_T^*$, then it is clear from the arguments above that $\hat{C}$ is not attainable.

2. We now show that the rank condition is sufficient. Again, let C be a (suitably integrable) claim settling at T and define X by (25). Because the rank condition is fulfilled, we can find predictable processes $\lambda^{(i)}$ such that

$$\tilde{\sigma}_X - \tilde{\sigma}_{Y^{(n)}} = \sum_{i=1}^{n-1} \lambda^{(i)} (\tilde{\sigma}_{Y^{(i)}} - \tilde{\sigma}_{Y^{(n)}})$$

The process $\lambda^{(n)}$ is defined by

$$\lambda^{(n)} := 1 - \sum_{i=1}^{n-1} \lambda^{(i)}$$

We define $\psi^{(i)}$ by

$$\psi^{(i)} := \frac{\lambda^{(i)} X}{Y^{(i)}}$$

and show that ψ is a self-financing portfolio which replicates X. It is immediately clear that $\psi Y = X$.
Furthermore

$$\begin{aligned} dX^* &= X^* \left(\tilde{\sigma}_X - \tilde{\sigma}_{Y^{(n)}}\right) dW^* \\ &= \sum_{i=1}^{n-1} X^* \lambda^{(i)} \left(\tilde{\sigma}_{Y^{(i)}} - \tilde{\sigma}_{Y^{(n)}}\right) dW^* \\ &= \sum_{i=1}^{n-1} \psi^{(i)} Y^{(i)*} \left(\tilde{\sigma}_{Y^{(i)}} - \tilde{\sigma}_{Y^{(n)}}\right) dW^* \qquad (26) \\ &= \sum_{i=1}^{n-1} \psi^{(i)} \, dY^{(i)*} \end{aligned}$$

Since the self-financing property is invariant under a change of numeraire and a change of measure, (26) shows that the portfolio ψ is indeed self-financing.

□

REMARK 8.3. 1. The previous result is a straight-forward generalization of the rank condition in the classical Black-Scholes setting. However, in this case it is the affine structure of the volatility that matters. This is obscured in the case where one asset is the continuously compounded savings account, because one only needs to consider the volatility structure of the remaining "risky" assets.

2. Suppose that $n = \tilde{d}+1$ and that the rank condition is satisfied. Then, since $\tilde{\sigma}_{Y^{(1)}}, \ldots, \tilde{\sigma}_{Y^{(n)}}$ are affine independent, the weights $\lambda^{(i)}$ are uniquely determined by the fact that $\sum_{i=1}^{d+1} \lambda^{(i)} \tilde{\sigma}_{Y^{(i)}}$ is an affine combination of $\tilde{\sigma}_{Y^{(1)}}, \ldots, \tilde{\sigma}_{Y^{(d+1)}}$ which is equal to $\tilde{\sigma}_X$. Therefore, they are independent of the choice of $Y^{(d+1)}$ as numeraire. In particular, the replicating portfolio is unique.

Volatility Misspecification. Now we want to analyze the cost process resulting from volatility misspecification. The asset X is to be synthesized using assets $Y^{(1)}, \ldots, Y^{(n)}$. The hedger assumes a model which is driven by a $\tilde{d}$-dimensional Brownian motion and where the volatilities of X and $Y^{(i)}$ are given by $\tilde{d}$-dimensional stochastic processes $\tilde{\sigma}_X$ and $\tilde{\sigma}_{Y^{(i)}}$, respectively. Furthermore, $\tilde{\sigma}_{Y^{(1)}}, \ldots, \tilde{\sigma}_{Y^{(n)}}$ fulfill the rank condition of proposition (8.2) and $n = \tilde{d} + 1$. This means that the model used by the hedger is complete and dynamically spanned by the assets $Y^{(1)}, \ldots, Y^{(n)}$, none of which is redundant. In particular, the hedger is using the minimum number of assets to synthesize X that he thinks is necessary. We let $\lambda^{(1)}, \ldots, \lambda^{(d+1)}$ denote the weights of the uniquely determined affine combination such that

$$\tilde{\sigma}_X = \sum_{i=1}^{d+1} \lambda^{(i)} \tilde{\sigma}_{Y^{(i)}}$$

We assume that $\lambda^{(i)}$ can be written in the form $\lambda_t^{(i)} = \lambda^{(i)}(t, X_t, Y_t^{(1)}, \ldots, Y_t^{(n)})$, i.e. as a deterministic function of time and the price levels.

To analyze the cost process, in this case we assume that the true dynamics are driven by a d-dimensional Brownian motion, i.e. all asset prices are defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$ supporting a d-dimensional Brownian motion W and the filtration $\mathbb{F}$ is the augmented filtration generated by W. The martingale parts of the price processes can be written as

$$\begin{aligned} dY^{(i)M} &= Y^{(i)} \sigma_{Y^{(i)}} dW \quad , \; i = 1, ..., n \\ dX^M &= X \sigma_X dW \end{aligned}$$

where $\sigma_X, \sigma_{Y^{(1)}}, \ldots, \sigma_{Y^{(n)}}$ are d-dimensional predictable processes. The replication strategy ψ chosen by the hedger is given by

$$\psi^{(i)} = \frac{\lambda^{(i)} X}{Y^{(i)}}$$

and we have the following representation for the cost process.

PROPOSITION 8.4. *Let $L(\psi)$ denote the cost process associated with ψ, then*

$$dL(\psi)^M = X \left(\sigma_X - \sum_{i=1}^{n} \lambda^{(i)} \sigma_{Y^{(i)}} \right) dW$$

Suppose one of the assets $Y^{(i)}$ is used as numeraire, then the discounted cost process $L^*(\psi)$ satisfies

$$dL^*(\psi)^M = X^* \left(\sigma_X - \sum_{i=1}^{n} \lambda^{(i)} \sigma_{Y^{(i)}} \right) dW$$

PROOF: 1. For the undiscounted cost process we have

$$dL(\psi) = dV(\psi) - \sum_{i=1}^{n} \psi^{(i)} dY^{(i)}$$

Since ψ duplicates X, this is equivalent to

$$dL(\psi) = dX - \sum_{i=1}^{n} \psi^{(i)} dY^{(i)}$$

It follows

$$\begin{aligned} dL(\psi)^M &= dX^M - \sum_{i=1}^{n} \psi^{(i)} dY^{(i)M} \\ &= \left(X\sigma_X - \sum_{i=1}^{n} \psi^{(i)} Y^{(i)} \sigma_{Y^{(i)}} \right) dW \\ &= X \left(\sigma_X - \sum_{i=1}^{n} \lambda^{(i)} \sigma_{Y^{(i)}} \right) dW \end{aligned}$$

2. Now we consider the discounted cost process. Without loss of generality we can assume that asset $Y^{(n)}$ is used as numeraire. Once again, by equation (2)

$$dL^*(\psi) = dX^* - \sum_{i=1}^{n} \psi^{(i)} dY^{(i)*}$$

This gives

$$\begin{aligned} dL^*(\psi)^M &= \left(X^*(\sigma_X - \sigma_{Y^{(n)}}) - \sum_{i=1}^{n} \psi^{(i)} Y^{(i)} (\sigma_{Y^{(i)}} - \sigma_{Y^{(n)}}) \right) dW \\ &= X^* \left(\sigma_X - \sum_{i=1}^{n-1} \lambda^{(i)} \sigma_{Y^{(i)}} - \left(1 - \sum_{i=1}^{n-1} \lambda^{(i)} \right) \sigma_{Y^{(n)}} \right) dW \\ &= X^* \left(\sigma_X - \sum_{i=1}^{n} \lambda^{(i)} \sigma_{Y^{(i)}} \right) dW \end{aligned} \tag{27}$$

□

Notice that σ_X and $\sigma_{Y^{(i)}}$ denote the "true" volatilities. Thus we have the result that the Black/Scholes hedge of the missing asset yields a cost process of bounded variation if and only if

$$\sigma_X = \sum_{i=1}^{n} \lambda^{(i)} \sigma_{Y^{(i)}} \tag{28}$$

Generically, if the true volatilities are not known, the $\lambda^{(i)}$ chosen by the hedger will not satisfy (28). Note that, in particular, overestimating volatilities is not a sufficient condition for achieving a superhedge, i.e. if a natural hedging instrument is unavailable, we cannot construct a superhedge using a Black/Scholes–type strategy.

From equation (24) we see that the choice of hedge instruments used to synthesize the unavailable asset plays a role in determining the cost of the hedging strategy and proposition 8.4 gives us the martingale component of the cost process for a given choice of hedging instruments $Y^{(i)}$. In the next section, we analyze the problem of choosing the optimal instruments so as to minimize this martingale component. Since selecting the best hedging instruments requires some knowledge about relationships between asset volatilities, we focus on fixed income securities, where some "stylized facts" about the term structure of volatility are available.

9. Duplication of Bonds

The term structure of volatility is particularly transparent for zero coupon bonds. This leads us to consider the problem of duplicating a zero-coupon bond with bonds of different maturities. For simplicity we restrict ourselves to the case of duplicating a zero coupon bond using only two other bonds. To be consistent, this implies that the hedger assumes a one-factor term structure model. For each maturity $T \in \mathbb{R}_+$ we denote the assumed lognormal bond price dynamics by

$$dB(t,T)^M = B(t,T)\tilde{\sigma}(t,T)dW_t \tag{29}$$

where $\tilde{\sigma}$ is a deterministic function which is monotonic in T and W is a one-dimensional Brownian motion. For maturities T_1 and T_2, Itô's lemma tells us that

$$d\left(\frac{B(t,T_2)}{B(t,T_1)}\right)^M = \frac{B(t,T_2)}{B(t,T_1)}\Big\{\tilde{\sigma}(t,T_2) - \tilde{\sigma}(t,T_1)\Big\}dW_t$$

With respect to the assumed model, the volatility of the forward price process of the bond T_2 with respect to T_1 is given by $\tilde{\sigma}(t,T_2,T_1) := \tilde{\sigma}(t,T_2) - \tilde{\sigma}(t,T_1)$. As we only consider bond price volatilities and volatilities of forward prices, we will refer to $\tilde{\sigma}(t,T_2,T_1)$ as the forward volatility and also use the shorter notation $\tilde{\sigma}^{T_2,T_1}(t)$.

Lemma 9.1. *Let maturities $T, T_1, T_2 \in \mathbb{R}$ be given. In a one-factor term structure model as described above, there exists a unique self-financing strategy $\tilde{\phi} = \left(\tilde{\phi}^1, \tilde{\phi}^2\right)$ in the bonds T_1 and T_2 which identically replicates the bond T, i.e.*

$$B(.\,,T) = \tilde{\phi}^1 B(.\,,T_1) + \tilde{\phi}^2 B(.\,,T_2)$$

The trading strategy is given by

$$\begin{aligned}
\tilde{\phi}^1_t &= \frac{B(t,T)}{B(t,T_1)}\frac{\tilde{\sigma}(t,T_2,T)}{\tilde{\sigma}(t,T_2,T_1)} \\
\tilde{\phi}^2_t &= \frac{B(t,T)}{B(t,T_2)}\frac{\tilde{\sigma}(t,T,T_1)}{\tilde{\sigma}(t,T_2,T_1)}
\end{aligned}$$

Proof: Since the volatilities $\tilde{\sigma}(.,T_1)$, $\tilde{\sigma}(.,T_2)$ are not equal, they are affine independent and proposition 8.2 tells us that the market determined by the two bonds is complete. Therefore, there exists a self-financing strategy $\tilde{\phi}$ which identically replicates the bond T. In particular, for $i = 1, 2$

$$\tilde{\phi}^i_t = \frac{\lambda^i(t)B(t,T)}{B(t,T_i)}$$

where λ^1, λ^2 are determined by the equations

$$\begin{aligned}
\tilde{\sigma}(t,T) &= \lambda^1(t)\tilde{\sigma}(t,T_1) + \lambda^2(t)\tilde{\sigma}(t,T_2) \\
1 &= \lambda^1(t) + \lambda^2(t)
\end{aligned}$$

This immediately gives

$$\lambda^1(t) = \frac{\tilde{\sigma}(t,T_2,T)}{\tilde{\sigma}(t,T_2,T_1)} \tag{30}$$

$$\lambda^2(t) = \frac{\tilde{\sigma}(t,T,T_1)}{\tilde{\sigma}(t,T_2,T_1)} \tag{31}$$

resulting in the claimed strategy.

□

There is no reason to assume a one-factor model perfectly reflects reality. So, in contrast to the hedger's assumptions, we let the real bond price dynamics be driven by an n–dimensional Brownian motion W. For each maturity $T \in [0, T^*]$, there is an $\mathbb{R}^n$-valued stochastic volatility process $\sigma^T := \sigma(., T)$ such that

$$dB(t,T)^M = B(t,T)\sigma(t,T)dW_t$$

Once again, σ^{T_1,T_2} denotes the forward volatility process $\sigma^{T_1,T_2} := \sigma^{T_1} - \sigma^{T_2}$.

LEMMA 9.2. *Let maturities T, T_1, T_2 be given and let $\tilde{\phi}$ be the replication strategy given in the previous lemma. Then we have the following two representations for the martingale part of the cost process*

$$dL(\tilde{\phi})_t^M = B(t,T)\left\{\sigma_t^T - \frac{\tilde{\sigma}^{T_2,T}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_1} - \frac{\tilde{\sigma}^{T,T_1}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_2}\right\} dW_t \tag{32}$$

$$dL(\tilde{\phi})_t^M = B(t,T)\,\tilde{\sigma}^{T_2,T}(t)\left\{\frac{1}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_2,T_1} - \frac{1}{\tilde{\sigma}^{T_2,T}(t)}\,\sigma_t^{T_2,T}\right\} dW_t \tag{33}$$

PROOF: Equation (32) can be obtained by inserting the values for λ^1, λ^2 above into the first equation of proposition 8.4. By definition, $\tilde{\sigma}^{T,T_1}(t) = \tilde{\sigma}^{T,T_2}(t) + \tilde{\sigma}^{T_2,T_1}(t)$.
Replacing $\tilde{\sigma}^{T,T_1}$ results in

$$dL(\tilde{\phi})_t^M = B(t,T)\left\{\sigma_t^{T,T_2} + \frac{\tilde{\sigma}^{T_2,T}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_2,T_1}\right\} dW_t$$

Simple regrouping now yields (33).

□

10. Optimal Selection of Bond Maturities

In this section, we give some optimality criteria for the choice of bond maturities when duplicating a zero-coupon bond. We use the instantaneous variance of the cost process to determine optimality. From equation (32) we see that

$$d\langle L(\tilde{\phi})\rangle_t = B(t,T)^2\left\|\sigma_t^T - \frac{\tilde{\sigma}^{T_2,T}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_1} - \frac{\tilde{\sigma}^{T,T_1}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_2}\right\|^2 dt$$

We introduce the following notation for the volatility mismatch.

$$K_t^1 := \left\|\sigma_t^T - \frac{\tilde{\sigma}^{T_2,T}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_1} - \frac{\tilde{\sigma}^{T,T_1}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_2}\right\| \tag{34}$$

A bond price model is determined by the bond volatilities. In particular, we can only find a self-financing replicating strategy for the bond with maturity T if it is possible to write its volatility as a linear combination of the hedge instruments' volatilities, (cf. proposition 8.4). Due to misspecification, it is no longer possible to match the bond volatility exactly. The choice of hedge instruments is determined by the attempt to make this mismatching as small as possible. Intuitively the best hedge instruments are those whose volatility structure is as close to that of the bond being hedged as possible, i.e. those bonds with the closest maturity dates. The next question which presents itself is whether to use longer or shorter bonds. Typically, bond volatility is increasing in the time to maturity. Therefore one might be tempted to prefer bonds with shorter maturities as hedge instruments due to their lower volatility. However, the effect of misspecification is determined by the relationship between true and assumed forward volatilies and not by the absolute value of the volatilities.

In practical applications one deals with an exogenously given finite set of bond maturities, so that the following definition can be used to determine the optimal pair of maturities for hedging.

Definition 10.1 (Robustness of hedge instruments). Let (T_1, T_2) and (T_1', T_2') be two pairs of bond maturities available for hedging the unavailable bond with maturity T. The pair (T_1, T_2) is *preferable* to (T_1', T_2') *iff*

$$\left\| \sigma_t^T - \frac{\tilde{\sigma}^{T_2,T}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\, \sigma_t^{T_1} - \frac{\tilde{\sigma}^{T,T_1}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\, \sigma_t^{T_2} \right\| \leq \left\| \sigma_t^T - \frac{\tilde{\sigma}^{T_2',T}(t)}{\tilde{\sigma}^{T_2',T_1'}(t)}\, \sigma_t^{T_1'} - \frac{\tilde{\sigma}^{T,T_1'}(t)}{\tilde{\sigma}^{T_2',T_1'}(t)}\, \sigma_t^{T_2'} \right\|$$

In this case it might be possible to find an optimal pair of maturities, given additional assumptions on the relationship between the true and the assumed volatility structure. A different approach which is possible is to look at the behaviour of the cost process along the whole yield curve.

As before we want to duplicate a bond with maturity T. We fix the maturity of one bond and are interested in the effect of varying the maturity of the other bond. Since the problem is symmetric in T_1 and T_2 we choose to fix $T_2 \neq T$. From equation (33) we see that in this case we must study the process K^2 defined by:

$$K_t^2 := \left\| \frac{1}{\tilde{\sigma}^{T_2,T_1}(t)}\, \sigma_t^{T_2,T_1} - \frac{1}{\tilde{\sigma}^{T_2,T}(t)}\, \sigma_t^{T_2,T} \right\|$$

In the same manner as above, we compare different maturities as follows:

Definition 10.2. Let T and T_2 be fixed maturities of zero coupon bonds, T denotes the maturity of the bond to be duplicated and T_2 the maturity of one bond which we fix as a hedge instrument. We call the bond maturing at T_1 *preferable* to the one maturing at T_1' *iff* the instantaneous variance of the cost process is smaller,i.e.

$$\forall t \in [0,T] \quad \left\| \frac{1}{\tilde{\sigma}^{T_2,T_1}(t)}\, \sigma_t^{T_2,T_1} - \frac{1}{\tilde{\sigma}^{T_2,T}(t)}\, \sigma_t^{T_2,T} \right\| < \left\| \frac{1}{\tilde{\sigma}^{T_2,T_1'}(t)}\, \sigma_t^{T_2,T_1'} - \frac{1}{\tilde{\sigma}^{T_2,T}(t)}\, \sigma_t^{T_2,T} \right\|$$

A general solution of the problems stated above is not possible without further assumptions both on the set of available bond maturities as well as the true volatility structure. Therefore, further analysis requires a more specialized framework, which goes beyond the scope of this paper.

11. Conclusion

The results of this paper represent a strong argument for the use of Black/Scholes-type strategies to hedge derivatives, not only in equity and foreign exchange markets, but also in the case of fixed income security instruments. The finite variation of the resulting cost process under arbitrary model misspecification succintly captures the robustness of Gaussian hedges, and this carries over to the "market model" setting of pricing caps and floors or swaptions by Black-type formulae. Thus employing (theoretically incompatible) lognormal models of forward LIBOR and swap rates may be justified by uncertainty in the specification of the "true" model. Furthermore, when volatility can be bounded from above and a superhedge obtained, a Gaussian hedging strategy arguably gives the cheapest superhedge, with funds being freed as soon as they are no longer needed to hedge against the worst-case (volatility) scenario.

On the other hand, the robustness result must be qualified by the availability of the "natural" hedging instruments. If they are unavailable and must in turn be synthesized by a dynamic trading strategy, finite variation of the cost process under model misspecification is lost and consequently a non-trivial superhedge cannot be obtained even if volatility is bounded. Therefore, it makes sense to define the optimal selection of hedge instruments by the criterion of minimizing the local variance of the cost process, allowing us to define criteria for the optimal choice of hedge instruments. Given that we are able to characterize market completeness in a multi-dimensional Black/Scholes–type framework by a criterion on (finite–dimensional) volatility vectors in a manner quite similar to the discrete time, discrete state space case, research continues with the goal to analogously describe optimally robust hedging strategies when "natural" instruments are unavailable.

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