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Pseudo–Arbitrage

A new Approach to Pricing and Hedging in Incomplete Markets

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Abstract

We develop a new approach to pricing and hedging contingent claims in incomplete markets. Mimicking as closely as possible in an incomplete markets framework the no–arbitrage arguments that have been developed in complete markets leads us to defining the concept of pseudo–arbitrage. Building on this concept we are able to extend the no–arbitrage idea to a world of incomplete markets in such a way that based on a concept of risk compatible with the axioms of Artzner et al. we can derive unique prices and corresponding optimal hedging strategies without invoking specific assumptions on preferences (other than monotonicity and risk aversion). Price processes of contingent claims are martingales under a unique martingale measure. A comparison to a version of the Hull and White stochastic volatility model shows that in contrast to their approach explicitly taking into account optimal hedging strategies leads to positive market prices of risk for volatility even if the latter is instantaneously uncorrelated with the stock price process. Our results are, however, in agreement with the findings of Lamoureux and Lastrapes.

JEL Classification: G12 G13

Keywords: arbitrage, pseudo–arbitrage, pricing of contingent claims, hedging, incomplete markets, stochastic volatility

Contents

1 Introduction

Since the pathbreaking papers by Black/Scholes [BS73] and Merton [Mer73] pricing derivatives in complete markets by no–arbitrage arguments has become standard. As is well known the normative pricing theory these authors have developed is so compelling for two main reasons. First, in order to derive their prices for derivatives it is not necessary to invoke specific preference relations of individual investors. This is so because, second, given the prices of the primitive securities there are dynamic trading strategies in the latter assets that perfectly duplicate the payoff of any derivative. Hence any deviation of the price of a derivative from the initial price of the duplicating portfolio strategy gives rise to trading strategies that imply riskless profit opportunities. These can, however, not prevail in equilibrium if investors have monotonic preferences, which can be taken for granted.

The preceding can be looked upon from a slightly different angle. Suppose in a world of complete markets somebody proposed an alternative normative pricing theory suggesting prices for derivatives that were different from arbitrage free prices. Then an investor following the theory of pricing by arbitrage and the trading strategies that go with it would have a riskless profitable strategy against any such pricing theory. Hence deviating from the alternative normative pricing theory would be individually rational for every risk averse investor with monotonic preferences. In complete markets the pricing theory for derivatives is thus an immediate consequence of the hedging theory.

Unfortunately actual markets are incomplete. There are many reasons for that. A number of prominent examples are stochastic volatility, jump risk, transaction costs, short sale restrictions or even credit risk . Although the present paper is not primarily on stochastic volatility the reason for incompleteness we have in mind in writing this paper is essentially volatility risk. Therefore, as examples of how the problem of market incompleteness has been addressed in the finance literature we shall mainly review in this introduction the literature on stochastic volatility.

The obvious difference between the pricing and hedging problem in complete markets as compared to incomplete markets is that in incomplete markets hedging arguments are not sufficient to derive unique prices for derivatives. Rather the hedging approach is used to derive partial differential equations for the prices of derivatives that unfortunately are specified only up to the market prices of the non–traded sources of risk. Hence in order to arrive at fully specified pricing equations assumptions need to be made about these market prices of risk. These assumptions are often based on more or less carefully developed general equilibrium arguments and include the use of specific utility functions often of the logarithmic or CRRA types. Another typical approach to solving the problem of specifying a market price of risk is to assume that the respective risk can be diversified away which implies a market price of zero for this source of risk. Based on these assumptions unique price processes for derivatives are calculated, which are subsequently used to derive fully specified hedging strategies.

In the literature on stochastic volatility there are a number of papers that take this approach. Important examples are Hull/White [HW87], Wiggins [Wig87], Scott [Sco87], Stein/Stein [SS91], Heston [Hes93], Ball/Roma [BR94], Duan [Dua95], and Renault and Touzi [RT96]. However, while these papers claim to propose normative pricing theories for derivatives under stochatic volatility the prices they propose rely essentially on the assumptions that are made on the market prices of volatility risk. In fact, taking as a typical example a European call option on a stock, as a consequence of the independent work of Frey and Sin [FS97] and Cvitanić, Pham and Touzi [CPT97] it turns out that in diffusion models for stochastic volatility where the support of the squared volatility is $(0, \infty)$ one can always specify the market price of risk of volatility in such a way that any price for the option between its inner value and the stock price itself can be supported. Given that neither the specific assumptions on preferences underlying the results in these papers nor the assumption of diversifiability of volatility risk (in fact it is a common experience among traders that volatilities for different stocks tend to move together) can safely be regarded as empirically warranted it is unclear in what sense the above papers do actually provide a normative pricing theory for derivatives under stochastic volatility. In any case these papers leave open the question why even if the market followed a different pricing theory than the one they propose, adhering to the respective pricing theories in these papers should be individually rational for a trader or investor who most likely does not have the preference relation assumed in the respective paper and who certainly has no knowledge about the preference relations of all the other market participants that he faces.

The essential question to ask would be whether based on these theories it would be possible to exploit market prices for derivatives that deviate from the prices given in these papers. This issue has been addressed in Chesney and Scott [CS89] who claim that option–trading based on their stochastic volatility model allows small gains to be made. However, in their study they leave out the risk dimension. Based on their stochastic volatility model they set up delta–neutral portfolios including a second option to hedge volatility risk. However, their hedging portfolios will only be delta–neutral if the market actually follows their pricing theory for the option in their hedging portfolio. This problem is recognised in Bakshi et al. [BCC96] who in a study of hedging effectiveness consider minimal variance hedges in the underlying and the bond besides delta–neutral hedges that also include a second option. Conceptually both these papers point in the direction that the present paper takes, namely developing a normative pricing theory for derivatives in incomplete markets that similar to the theory in complete markets draws its justification from a theory of hedging but due to market incompleteness necessarily takes risk explicitly into account.

In that the present paper tries to develop a normative pricing theory it also distinguishes itself clearly from a recent strand in the literature that focus entirely on the question of hedging. The best example of this approach is probably the literature on super hedging such as it has been developed in the papers by Avellaneda, Levy and Paras [ALP95] or El Karoui, Jeanblanc–Picquet and Shreve [KJPS98]. While this is an ingenious theory it is often impractical because the initial amount of money one has to invest in order to set up a super hedging strategy for a certain contingent claim is much higher than the price one can achieve for this contingent claim in the market. As examples we may again point to the previously mentioned results of Frey and Sin [FS97] and Cvitanić, Pham and Touzi [CPT97] on the range of option prices when the support of the volatility is $(0, \infty)$. Hence the papers dealing with super hedging typically assume that volatility is bounded and under this assumption determine super hedging strategies. If the actual volatility leaves the assumed boundaries nothing is said about the quality of the hedge that these strategies provide. This problem is avoided in the concept of quantile hedging proposed by Föllmer and Leukert [FL98]. For a given maximal probability of making a loss in the future these authors determine the cheapest hedging strategy that will achieve the desired loss probability.

Neither super nor quantile hedging can, however, give rise to a normative pricing theory. Rather they either take as an input the prices that the market is ready to pay for a contingent claim and determine the least loss probability or the widest interval of volatility that can be attained with an optimal hedging strategy whose costs do not exceed the given market price of the contingent claim considered. Or they start with a given loss probability or a given interval for the volatility and determine the cheapest super or quantile hedging strategy that is compatible with these assumptions.

The situation seems to be slightly different for the concept of (local) risk minimisation suggested by Föllmer and Sondermann [FS86], Schweizer [Sch91], Föllmer and Schweizer [FS90], and Hofmann, Platen and Schweizer [HPS92]. Given the stochastic processes of primitive securities, in their incomplete market models these authors are able to determine unique (locally) risk minimising hedging strategies for contingent claims based on the primitive securities. One might be tempted to regard the value processes of these strategies as suitable candidates for the price processes of the contingent claims under consideration. If this could actually be done their theory would indeed combine in incomplete markets a hedging theory with a normative pricing theory.

However, there are a number of problems with this approach. The least seems to be that there exists an inconsistency between discrete–time models and continuous–time models. While in continuous–time models the value processes of locally risk minimising hedging strategies can be determined by taking conditional expectations of the payoffs of the contingent claims considered under the so called minimal martingale measure no such martingale measure need exist in discrete–time models. This problem has been addressed by Elliott and Madan [EM98] who in discrete time have introduced a different concept of risk minimisation than the above authors. With this modification even in discrete–time models they can represent the value processes of their hedging strategies as martingales under a specific probability measure. The two connected problems that neither of the two approaches addresses are first that the objective function i.e. the concept of risk used in this literature is unintuitive and economically inappropriate for the problem at hand and second that if the actual price process of the contingent claim is different from the value process of the locally risk minimising strategy then simply no locally risk minimising strategy exists and the theory fails to give any answer as to how to hedge optimally. In particular this literature uses quadratic risk functions. This implies that any deviation — even positive — of the value of the hedging portfolio from the payoff of the contingent claim is considered a manifestation of risk. Obviously this is in sharp contrast to the ideas underlying the literature on super hedging which only regards as risk the danger of a shortfall of the value of the hedging portfolio behind the payoff of the contingent claim to be hedged. Clearly only the latter makes economic sense since in hedging no one is interested in hitting precisely the payoff of a contingent claim. What counts is not to lose money.

In view of this literature it seems fairly difficult to find a normative pricing theory in incomplete markets that draws its justification from a theory of hedging. Nevertheless, this is exactly what this paper tries to achieve. Just as has been done in the literature on super or quantile hedging we assume an asymmetric concept of risk namely the concept of an expected decrease in portfolio value. What still allows us to derive unique prices for contingent claims is that just as in arbitrage theory we do not only look at short positions in contingent claims but also at long positions. The question we ask is whether at given market prices for a contingent claim and for the primitive assets it is possible to make a strictly postive expected profit from entering the market for the contingent claim while running the least possible risk from doing so. Finding an answer to this question will sometimes require going long sometimes it will imply going short the contingent claim. With this approach we are able to modify the no–arbitrage concept so that it becomes extendable to the case of incomplete markets. The extended concept will be called the concept of pseudo–arbitrage.

Using this concept we obtain unique prices for contingent claims that do not give rise to pseudo–arbitrage opportunities. These prices are justified by the fact that whenever market prices deviate from pseudo–arbitrage free prices then similar to what is the case in complete markets we can offer trading strategies that yield strictly postive expected payoffs while being risk minimal in well defined sense. In fact, our concept of risk minimality in multi–period securities markets will be an adaptation of the concept of local risk minimsation introduced by Schweizer. Thus similar to what is the case in complete markets our theory offers an integrated approach to pricing, hedging and risk management. Moreover, in our framework pseudo–arbitrage free prices can be represented as expected payoffs of the contingent claims under a unique martingale measure even in the discrete–time framework, which we shall work in throughout the entire paper. Finally we shall apply our theory to a version of the Hull and White [HW87] stochastic volatility model. We shall show that unlike what was assumed by Hull and White our theory does not lead to a zero market price of risk for volatility even if the volatility process is instantaneously uncorrelated with the stock price process. Rather we obtain a positive market price of risk that diminishes with the size of the stock price volatility itself. This result is line with the interpretation that Lamoureux and Lastrapes [LL93] give to their empirical results on the predictive power of implied volatilities.

The present paper is organised as follows. In section two we shall explain our ideas in a simple two–period example. In section three we shall develop the general theory for the two–period case. This will be transferred to the multy–period framework in section four. Section five contains our analysis of the modified Hull/White stochastic volatility model. Section six finally concludes and gives some suggestions for further work.

2 An Example

The purpose of this section is to provide an intuitive understanding of the economic ideas that underly our approach to the pricing and hedging of contingent claims in incomplete markets.

Consider the two–period model represented in Figure 1. We assume that there are frictionless markets for two securities, a stock S and a riskless bond B. The prices for the stock and the bond are assumed given as are the transition probabilities p_{1j} from the origin to the different states in the second period. We want to find a unique price C_{00} in period 0 for the contingent claim which has payoffs $C_{1,\cdot}$ in period 1. With these payoffs the contingent claim can be interpreted as a call–option on the stock with a strike of 50.

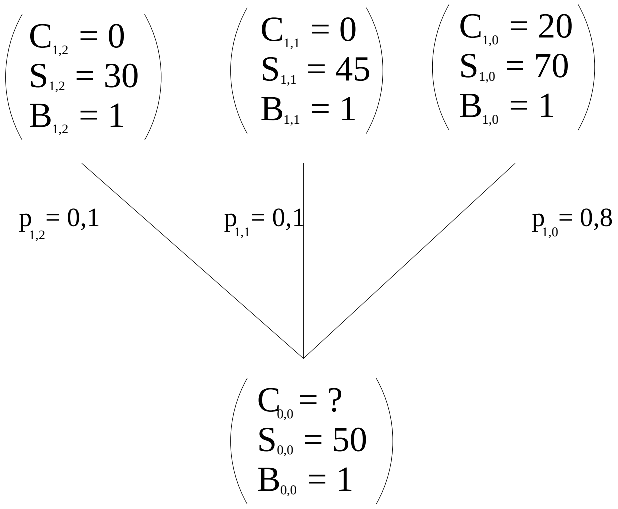

Figure 1: Market Structure

It is easy to see that by no–arbitrage arguments C_{00} must be between 4 and 10. Our aim is now to argue that there is an intuitive extension of the no–arbitrage concept that singles out a unique price C_{00} inside this interval.

To this end assume that C_{00} was slightly below 10. There would no longer be a riskless hedging strategy in stocks and bonds for the contingent claim. However, we can still ask how much risk an investor would inevitably have to accept if he went long or short the contingent claim and hedged this position in stocks and bonds in such a way that his risk of losing money from the entire transaction became minimal. Against this backgrond consider taking a short position in the contingent claim at a price of $C_{00} = 10 - \epsilon$. Assume you wanted to invest this amount of money in the stock and the bond so as to minimise the overall expected loss from this portfolio. This quantity is given by the following expression

$$0.1[\varphi(30-50)+(10-\epsilon)]^{-}+0.1[\varphi(45-50)+(10-\epsilon)]^{-}+0.8[-20+\varphi(70-50)+(10-\epsilon)]^{-}.$$

It is easy to see that its minimum is attained by choosing φ, the holdings in the stock, equal to $0.05(10+\epsilon)$. This implies that in states $(1,0)$ and $(1,1)$ in the second period the value of this portfolio will be nonnegative. In state $(1,2)$ it will have a value of -2ϵ. The expected loss from this portfolio will hence be 0.2ϵ. On the other hand the expected value of the portfolio is $0.025(30-13\epsilon)$, which is positive for $\epsilon < 30/13$ or $C_{00} > 100/13$.

Compare these results to those from going long the contingent claim and financing this with a portfolio in stocks and bonds. The expected loss is then given by

$$0.1[\varphi(30-50)-(10-\epsilon)]^- + 0.1[\varphi(45-50)-(10-\epsilon)]^- + 0.8[20+\varphi(70-50)-(10-\epsilon)]^- .$$

This expression is minimised by choosing $\varphi = -0.05(10+\epsilon)$ i.e. by selling $0.05(10+\epsilon)$ units of the stock. The remaining funds are invested in the bond. In this case the portfolio value in the second period will be nonnegative in states $(1,0)$ and $(1,2)$, whereas it will be negative in state $(1,1)$. The minimal expected loss turns out to be $1/8(6-\epsilon)$, which exceeds the minimal risk of the short position in the contingent claim for $\epsilon < 30/13$. Moreover, the expected value of this portfolio will be $13/40\epsilon - 3/4$, which is negative for $\epsilon < 30/13$ or $C_{00} > 100/13$.

These calculations reveal that as long as $C_{00} > 100/13$ there is actually a whole range of short positions in the contingent claim, which at zero cost offer a positive expected payoff at a lower risk (in terms of expected loss) than the minimal risk of any long position in the contingent claim that could be entered into at zero cost. Hence, even without specifying an exact tradeoff between risk and return we may conclude that for any $C_{00} > 100/13$ it is possible to find risk averse investors with monotonic preferences who would prefer taking a short position in the contingent claim to both taking a long position in this asset or staying out of this market completely.

Particular investors of this type would be arbitrageurs. In the classical sense these would be investors seeking riskless profit opportunities. Such riskless profit opportunities hardly ever exist in practice since actual markets are incomplete and small deviations of actual market prices for contingent claims from theoretical prices obtained in complete markets models need by no means indicate riskless profit opportunities. Nevertheless, we observe that so called "arbitrage desks" in banks are busily trading well inside theoretical arbitrage bounds for the prices of derivatives so that these bounds are actually never attained. Therefore, we would argue that it would be more realistic to describe the behaviour of arbitrageurs as that of investors seeking positive profit at as low a risk as possible. We suggest that when looking at a market for a particular contingent claim arbitrageurs would consider that zero–cost position in this asset that would imply the lowest possible risk. If this portfolio allowed them to achieve a payoff that would be sufficient to compensate them for taking the risk they would enter the market. Otherwise they would stay out.

This vision of arbitrageurs means that we do not regard them as infinitely risk averse investors. However, by the role assigned to them the set of positions or trading strategies arbitrageurs would be allowed to choose from would be restricted. In the market for the contingent claim they would only be allowed to take risk minimal zero–cost positions. Such restrictions do not seem implausible since banks do assign different trading policies to different trading books. That is they place restrictions on the types of positions that may be taken by the traders responsible for a particular book. "Arbitrage policies" comparable to the one described above are often popular with banks since they are viewed to yield more stable if sometimes smaller profits than taking outright positions.

If we adopted this — admittedly stylised — vision of arbitrageurs and the ensuing extended idea of "arbitrage", in our example it would not seem reasonable to assume that as the price of the contingent claim dropped slightly below 10 all the arbitrageurs would immediately leave the market. Rather we would expect many of them to remain active in the market preferring the short position to the long position in the contingent claim. However, arbitrageurs will differ in their perceptions of the tradeoff between risk and return. Hence, as the price for the contingent claim approached 100/13 arbitrage activity would fade. At $C_{00} = 100/13$ it would come to a complete end since at minimal risk no positive expected payoff could be made and hence, whatever the arbitrageurs' exact risk–return tradeoff, it would be preferabel for them to stay out of the market.

A similar argument could be made for $C_{00} < 100/13$ with arbitrageurs preferring the long to the short position. We would, therefore, argue that $C_{00} = 100/13$ is the unique price for the contingent claim at which in the market for this asset there would be no arbitrage activity of the type described above.

Figure 2 summarises the situation. The two schedules represent — for different values of C_{00} — the minimal risk that an investor would have to face when setting up at zero cost a short (decreasing schedule) or a long position (increasing schedule) respectively in the contingent claim. They intersect at $C_{00} = 100/13$. This can be seen as a point of indifference between long and short positions in the contingent claim since at this price risk and expected payoff from both positions in this asset are equal. This provides a further argument for why at this point all arbitrage activity should come to a halt since for an investor interested only in taking positions with minimal risk there would be no reason any more to prefer one position to the other.

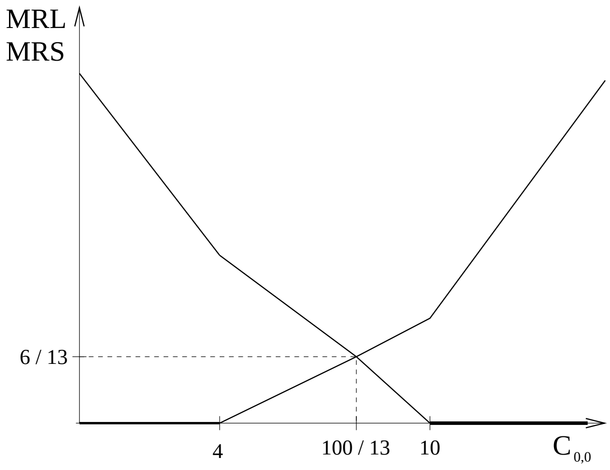

Figure 2: Minimum Risk Schedules

Summing up we argue that it makes sense to assume that there is some sort of "arbitrage activity" going on even within the arbitrage bounds for the prices of contingent claims. Given this we have identified a unique price $C_{00} = 100/13$ inside this interval at which all "arbitrage activity" as we have described it would cease since arbitrageurs would be indifferent between long and short positions in the contingent claim and, moreover, no positive expected payoff could be obtained from either a risk minimal long or short position. Put differently, at a price of 100/13 it is individually rational for every risk averse investor confined to taking only risk minimal positions in the contingent claim not to deviate from the pricing theory of the market. On the other hand for every price $C_{00} \neq 100/13$ there exist arbitrageurs for whom deviating from the pricing theory of the market is individually rational. The price of $C_{00} = 100/13$ obtains at the intersection point of the minimal risk schedules for the long and the short positions in the contingent claim.

The identification of this price depends on the specific measure of risk. However, this is easily seen to be the only measure of risk in our market model that would be consistent in the sense of the axioms of Artzner et al. [ADEH96]. Moreover, beyond assuming this specific risk measure the argument for why at every $C_{00} \neq 100/13$ there would be forces driving the price of the contingent claim to the intersection point between the minimal risk schedules does not depend on a full specification of the arbitrageurs' preferences. Notice, however, that using this measure of risk implies the assumption that investors do not default on their liabilities. This seems reasonable if the risk an individual investor takes is small relative to his endowment.

On the other hand, if we were to argue that $C_{00} = 100/13$ was actually the only price for the contingent claim that could prevail in equilibrium we would have to assume that arbitrage activity even in an arbitrarily small interval around C_{00} would be strong enough to swamp demand and supply from all the other market participants. This being a strong assumption we would still argue that the price we have idetified is of practical interest since by its properties described above it would give a good indication for the price around which a trader in this contingent claim should place bid and ask quotes in order to protect himself against being exploited by other market participants.

Finally, before passing to the general two–period model let us briefly note that the price of the contingent claim we have derived can be represented as the expected payoff of this asset under a martingale probability measure q with the q_{1j} given by

$$q_{1,0} = \frac{5}{13}, \quad q_{1,1} = q_{1,2} = \frac{4}{13}.$$

In fact if we were to find the price of an arbitrary contingent claim in our market model by determining the intersection of the respective minimal risk schedules we would find that this price could always be calculated as the expected payoff of this claim under the same martingale measure q. This means that even if we were to introduce more than one contingent claim in the above model at the prices determined following the above procedure this would not give rise to arbitrage opportunities.

3 The Two–Period Model

3.1 Basic Definitions and Results

In this and the following sections we shall consider a two–period securities market similar to the one represented in Figure 1. Before stating our assumptions about this market we shall introduce some useful notation. We shall immediately make it sufficiently general so that it will also fit the multi–period markets that we shall discuss in the sequel.

The time and information structure in our market models will be described by event trees. The vertices in these trees will be indexed by pairs $(ij) \in \mathbb{N}_0^2$ with i being the time index and j the state index in period i.

In our model there will be three securities, a contingent claim, a stock and a bond. $P'_{ij} := (C_{ij}; S_{ij}; B_{ij})$ represents the column–vector of the prices of the contingent claim, C_{ij}, the stock, S_{ij}, and the bond, B_{ij}, in vertex (ij). The entire price process will be denoted $P := (C; S; B)$ with C, S and B representig the price processes of the contingent claim, the stock and the bond respectively. By $\Delta X_{ij} := X_{ij} - X_{ij-}$ we shall denote the increment of the price of asset X (contingent claim, stock or bond) in vertex (ij), where $(ij-)$ indicates the vertex preceeding the vertex (ij).

By $\varphi_{ij} := (\varphi_{ij}^C; \varphi_{ij}^S; \varphi_{ij}^B)$ we denote the portfolio that an investor holds when leaving vertex (ij). φ_{ij}^C, φ_{ij}^S and φ_{ij}^B represent the amounts of the contingent claim, the stock and the bond respectively in this portfolio. Negative numbers indicate short positions.

Next we introduce some concepts related to portfolios. Again we state them in a way general enough to use them in multi–period models as well.

Definition 3.1

a) *The value of a portfolio in vertex (ij) is given by $V(\varphi_{ij}, P_{ij}) := \varphi_{ij} \bullet P_{ij}$, "$\bullet$" representing the inner product.*

b) *A portfolio φ_{ij} is called admissible if it relies only on the path of the price process P up to vertex (ij) and if $V(\varphi_{ij}, P_{ij}) = 0$.*

c) *A short (long) position in the contingent claim in vertex (ij) is an admissible portfolio with $\varphi_{ij}^C = -1$ ($\varphi_{ij}^C = 1$). The set of all short (long) positions in the contingent claim in vertex (ij) at prevailing prices will be denoted $\Phi^s(P_{ij})$ ($\Phi^l(P_{ij})$).*

d) *A portfolio strategy, φ, is a stochastic process that assigns an admissible portfolio to every vertex (ij) in the event tree.*

e) *A short (long) strategy in the contingent claim is a portfolio strategy φ^s (φ^l) that assigns a short (long) position in the contingent claim to every vertex (ij) in the event tree before the maturity of the contingent claim.*

Notice that the zero–value condition in part b) of this assumption is nothing but an accounting rule. The following assumption now describes our two–period model. It is a generalisation of the model in Figure 1.

Assumption 3.2

a) The event tree consists of two periods, $i \in \{0,1\}$, with one state in period 0 and three states in period 1. The transition probabilities from state $(0,0)$ to the three different states in period 1 are given exogenously and are assumed known.

b) There are two assets, a stock and a bond, which trade frictionlessly in both periods. The price process B is assumed constant and equal to 1. S is exogenous, stochastic and adapted to the event tree. $\Delta S_{1,j} \neq 0\,\forall j$. Moreover, the $\Delta S_{1,j}$ will not all have the same sign (absence of arbitrage between stock and bond). Finally, if $\Delta S_{1,0} < 0$ then $\Delta S_{1,j} > 0$, $j \in \{1,2\}$ and $p_{1,0}|\Delta S_{1,0}| > p_{1,j}\Delta S_{1,j}$, $j \in \{1,2\}$. If $\Delta S_{1,0} > 0$ then $\Delta S_{1,j} < 0$, $j \in \{1,2\}$ and $p_{1,0}\Delta S_{1,0} > p_{1,j}|\Delta S_{1,j}|$, $j \in \{1,2\}$.

c) There is a contingent claim with payoffs $(c_{1,\cdot}) = (C_{1,\cdot})$ in period 1, which can be bought or sold in period 0 at a price of $C_{0,0}$.

The previous assumption requires two comments. First, what may seem a restrictive assumption on the relative sizes of the $\Delta S_{1,\cdot}$ is in fact a harmless technical assumption that rules out some artefacts that may otherwise occur in a discrete–time model. Notice in particular that this assumption does neither restrict in any way the expected value nor the variance of $\Delta S_{1,\cdot}$. Second, by assuming that S is given exogenously we imply that the price of the contingent claim unilateraly adapts to the price of the stock. While this is a problematic assumption from a general equilibrium point of view it makes sense from the point of view of a derivatives' trader if we assume that the volume of trades in the stock induced by the trading in contingent claims in period 0 is small relative to the general turnover in the stock market.

We now introduce the key–concept of risk that we shall use throughout the paper.

Definition 3.3
The risk of a portfolio in state (ij) is given by

$$R(\varphi_{ij}, P_{ij}) := E^p[\,[V(\varphi_{ij}, P_{i+1,\cdot})]^- \mid (ij)\,]\,,$$

where the expectation is taken with respect to the transition probabilities $p_{i+1,\cdot}$ over all vertices $(i+1,\cdot)$ in period $i+1$ that succeed vertex (ij).

As already stated in the previous section the choice of this measure of risk can be justified on grounds of the axioms given in Artzner et al. [ADEH96]. In fact, under Assumption 3.2 it is the only consistent measure of risk in the sense of these axioms. Notice also that unlike the quadratic measures of risk that underly the theory of (local) risk minimisation according to Föllmer, Schweizer and Sondermann this measure of risk does not punish overhedging. In this sense it is compatible with the ideas underlying the literature on super hedging. It is, moreover, compatible with our intuition that risk means the danger of losing money. Finally, this measure of risk has received considerable interest in the literature on portfolio theory (see Dembo [Dem91], Jaeger, Rudolf and Zimmermann [JRZ95], Reichling [Rei96], Kaduff and Spremann [KS96]) and is also meanwhile widely recognized as being superior to VaR.

As motivated in the previous section we are interested in determining the minimal risk from entering the market for the contingent claim by taking either a long or a short position in this asset. The following lemma asserts that for every price $C_{0,0}$ of the contingent claim it is indeed possible to find risk minimising long and short positions.

Lemma 3.4
Under Assumtion 3.2 $\forall\, C_{0,0} \in I\!R\ \exists\, \varphi^*_{0,0} \in \Phi^l(P_{0,0})$ *and* $\varphi^{**}_{0,0} \in \Phi^s(P_{0,0})$ *such that*

$$\begin{aligned} R(\varphi^*_{0,0}, P_{0,0}) &\leq R(\varphi_{0,0}, P_{0,0})\ \forall\, \varphi_{0,0} \in \Phi^l(P_{0,0}) \quad \textit{and} \\ R(\varphi^{**}_{0,0}, P_{0,0}) &\leq R(\varphi_{0,0}, P_{0,0})\ \forall\, \varphi_{0,0} \in \Phi^s(P_{0,0})\,. \end{aligned}$$

Proof
Consider $\Phi^l(P_{0,0})$. In this case $R(\varphi_{0,0})$ is of the following form

$$R(\varphi_{0,0}, P_{0,0}) \;=\; E^p[\,[\,C_{1,\cdot} \,+\, \varphi^S_{0,0}\Delta S_{1,\cdot} \,-\, C_{0,0}\,]^-\,]\,.$$

This is easily seen to be a convex function of $\varphi^S_{0,0}$. Moreover, since some of the $\Delta S_{1,\cdot}$ are positive while others are negative we have

$$\lim_{\substack{\varphi^S_{0,0}\to\infty \\ \varphi_{0,0}\,\in\,\Phi^l(P_{0,0})}} R(\varphi_{0,0}, P_{0,0}) \;\to\; \infty \;\text{ and }\; \lim_{\substack{\varphi^S_{0,0}\to-\infty \\ \varphi_{0,0}\,\in\,\Phi^l(P_{0,0})}} R(\varphi_{0,0}, P_{0,0}) \;\to\; \infty\,,$$

which together with the convexity of R in $\varphi^S_{0,0}$ ensures the existence of a risk minimal long position in the contingent claim. The argument for the short position being similar this proves the lemma. □

Clearly, from our market structure the contingent claim will in general not be a redundant asset. Therefore, a riskfree hedging strategy in the stock and the bond will — special cases apart — not be available. Our aim is nonetheless to formulate a single pricing rule that is based on trading strategies, gives unique prices for redundant and non–redundant contingent claims and can be interpreted as a generalisation of the no–arbitrage principle. In keeping with our intuition explained in the previous section that arbitrageurs search for profit opportunities while trying to minimise risk we propose the following concept of a “pseudo–arbitrage opportunity” in the two–period model.

Definition 3.5
In the two–period market model of Assumption 3.2 at prices $P_{0,0}$ *there exists a pseudo–arbitrage opportunity for the contingent claim iff there is a portfolio* $\varphi^*_{0,0}$ *with the following properties:*

a) $\varphi^*_{0,0} \in \Phi^l(P_{0,0}) \cup \Phi^s(P_{0,0})$,

b) $R(\varphi^*_{0,0}, P_{0,0}) \leq R(\varphi_{0,0}, P_{0,0}) \;\forall\; \varphi_{0,0} \in \Phi^l(P_{0,0}) \cup \Phi^s(P_{0,0})$,

c) $E^p[\,V_{1,\cdot}(\varphi^*_{0,0})\,] > 0$.

The definition makes sense in that in view of Lemma 3.4 it will always be possible to find risk minimising long and short positions in the contingent claim. Moreover, if there exists a pseudo–arbitrage opportunity there exist risk averse investors with monotonic preferences who would be ready to exploit it i.e. invest in the portfolio $\varphi^*_{0,0}$. Furthermore, as required the concept of pseudo–arbitrage is an extension of the no–arbitrage concept in the following sense:

Proposition 3.6
If at prices $P_{0,0}$ there is an arbitrage opportunity for the contingent claim there is also a pseudo–arbitrage opportunity for this claim at these prices.

Proof
If at prices $P_{0,0}$ there is an arbitrage opportunity for the contingent claim there obviously exists $\varphi^*_{0,0} \in \Phi^l(P_{0,0}) \cup \Phi^s(P_{0,0})$ such that $R(\varphi^*_{0,0}, P_{0,0}) = 0$ and $E^p[\,V_{1,\cdot}(\varphi^*_{0,0})\,] > 0$, which by definition is a pseudo–arbitrage opportunity for this contingent claim. □

Beyond these results we should like to convince ourselves that similar to requiring absence of arbitrage in complete markets absence of pseudo–arbitrage in the setting of Assumption 3.2 implies a unique price for any arbitrary contingent claim. Moreover, in the previous section we have introduced arbitrageurs as risk averse investors with monotonic preferences who are — if they enter the market for the contingent claim — confined to taking risk minimal zero–cost positions in this asset. We should like to see our intuition confirmed that if there was a pseudo–arbitrage opportunity one could find arbitrageurs who would enter the market for the derivative so as to drive the price towards the — hopefully unique — price prescribed by the absence of pseudo–arbitrage opportunities. On the other hand, if there was no pseudo–arbitrage opportunity we should like arbitrageurs not to be active in the market for this contingent claim. The following section will be devoted to the discussion of these issues.

3.2 Pseudo–arbitrage free Prices

As indicated by item b) of Definition 3.5 an essential step in determining whether a price system gives rise to a pseudo–arbitrage opportunity is to find the risk minimal long and short positions in the contingent claim at going prices and to determine the expected payoffs they imply. This motivates

Definition 3.7
The functions $MRL_{0,0}$ and $MRS_{0,0}$ will be called the minimum risk functions for a long or a short position respectively in the contingent claim. They are defined as follows

$$\begin{array}{rcccl} MRL_{0,0} & : & \mathbb{R} & \to & \mathbb{R}_+ \\ & & C_{0,0} & \mapsto & \min_{\varphi_{0,0}\in\Phi^l(P_{0,0})} R(\varphi_{0,0}, P_{0,0}) \\ MRS_{0,0} & : & \mathbb{R} & \to & \mathbb{R}_+ \\ & & C_{0,0} & \mapsto & \min_{\varphi_{0,0}\in\Phi^s(P_{0,0})} R(\varphi_{0,0}, P_{0,0})\,. \end{array}$$

Under Assumption 3.2 these functions are well defined by Lemma 3.4. They have an intuitive economic interpretation. For every $C_{0,0}$ $\mathrm{MRS}_{0,0}$ gives the minimum risk that an investor has to accept up to the next trading date from selling the contingent claim at going prices and investing the proceeds optimally in a hedging portfolio of stocks and bonds. Likewise for every $C_{0,0}$ $\mathrm{MRL}_{0,0}$ gives the minimum risk that an investor would have to incur from buying this claim and financing it optimally with a portfolio of stocks and bonds. Examples of graphs of these functions are depicted in Figure 2. The following lemma states that these graphs will in fact always be of a similar shape as those in Figure 2.

Lemma 3.8
$MRL_{0,0}$ and $MRS_{0,0}$ are both convex and continuous in $C_{0,0}$. $MRL_{0,0}$ is either strictly monotonically increasing in $C_{0,0}$ or constant and equal to zero. $MRS_{0,0}$ is either strictly monotonically decreasing in $C_{0,0}$ or constant and equal to zero.

Proof
see Appendix A □

With the help of the MRS and MRL functions it is now easy to identify a candidate for a pseudo–arbitrage free price for the contingent claim in our two–period securities market. Proceeding as in the example of Section 2 we shall be interested in the intersection point of the MRS and MRL schedules.

Proposition 3.9
Under Assumption 3.2 there is a unique $C^*_{0,0}$ such that $MRS_{0,0}(P^*_{0,0}) = MRS_{0,0}(P^*_{0,0})$, where $P^*_{0,0} := (C^*_{0,0}, S_{0,0}, 1)$.

Proof
By the monotonicity properties of the MRL and MRS functions we have

$$\lim_{C_{0,0}\to\infty} (MRS_{0,0}(P_{0,0}) - MRL_{0,0}(P_{0,0})) \leq 0$$

and

$$\lim_{C_{0,0}\to-\infty} (MRS_{0,0}(P_{0,0}) - MRL_{0,0}(P_{0,0})) \geq 0\,.$$

Hence, by continuity there exists at least one intersection between MRL and MRS. If this intersection is in the region where MRL and MRS are both positive then by the strict monotonicity of the two functions it is unique. It therefore remains to show that it is also unique if the intersection is in the region where both functions are equal to zero.

Assume to the contrary that there were to values $\overline{C}_{0,0} > \underline{C}_{0,0}$ giving rise to otherwise identical price vectors $\overline{P}_{0,0}$ and $\underline{P}_{0,0}$ such that $\mathrm{MRL}(\overline{P}_{0,0}) = \mathrm{MRL}(\underline{P}_{0,0}) = \mathrm{MRS}(\overline{P}_{0,0}) = \mathrm{MRS}(\underline{P}_{0,0})$. Hence there are zero–risk portfolios $\underline{\varphi}_{0,0} \in \Phi^s(\underline{P}_{0,0})$ and $\overline{\varphi}_{0,0} \in \Phi^l(\overline{P}_{0,0})$. Assume further that the prevailing price vector is $\overline{P}_{0,0}$. Consider shorting the contingent claim. Then the portfolio $\varphi^1_{0,0} := (-1, \underline{\varphi}^S_{0,0}, \underline{\varphi}^B_{0,0} + (\overline{C}_{0,0} - \underline{C}_{0,0})$ is admissible at going prices. So is the portfolio $\varphi^2_{0,0} := \overline{\varphi}_{0,0}$. Moreover, by the fact that $\underline{\varphi}_{0,0}$ and $\overline{\varphi}_{0,0}$ are both risk–free we have

$$\begin{aligned} -C_{1,j} + \underline{\varphi}^S_{0,0} S_{1,j} + \underline{\varphi}^B_{0,0} &\geq 0 \ \forall\, j \text{ and} \\ C_{1,j} + \overline{\varphi}^S_{0,0} S_{1,j} + \overline{\varphi}^B_{0,0} &\geq 0 \ \forall\, j\,. \end{aligned}$$

Hence, adding up the portfolios $\varphi^1_{0,0}$ and $\varphi^2_{0,0}$ produces a zero–cost portfolio that is independent of the contingent claim and has strictly positive payoffs in every state of period 1, i.e. is an arbitrage opportunity in the stock and the bond. This is, however, in contradiction to Assumption 3.2.

If the prevailing price vector had been $\underline{P}_{0,0}$ a similar argument could have been made based on risk–free portfolios $\underline{\varphi}_{0,0} \in \Phi^l(\underline{P}_{0,0})$ and $\overline{\varphi}_{0,0} \in \Phi^s(\overline{P}_{0,0})$ assuming that the contingent claim had been bought rather than sold. This completes the proof of the proposition. □

Let us now examine the properties of the intersection point between the MRS and MRL schedules in detail. To facilitate the discussion we shall introduce some new notation. By $C^*_{0,0}$ we denote the price for the contingent claim at which the MRS and MRL schedules intersect for given prices of the stock and the bond. The full price vector corresponding to this situation will be denoted $P^*_{0,0} := (C^*_{0,0}; S_{0,0}; 1)$. Finally, $\varphi^{*l}_{0,0}$ and $\varphi^{*s}_{0,0}$ respectively represent the long and short positions in the contingent claim by which the minimum risk from buying or selling the contingent claim can actually be attained at prices $P^*_{0,0}$.

Eventually we want to show that $C^*_{0,0}$ is indeed the unique pseudo–arbitrage free price for the contingent claim. Since by Proposition 3.6 we have that absence of arbitrage is a necessary condition for the absence of pseudo–arbitrage we had better be able to prove

Proposition 3.10
Under Assumption 3.2 at prices $P^*_{0,0}$ there are no arbitrage opportunities between the contingent claim, the stock and the bond.

Proof

Suppose to the contrary that there exists an arbitrage opportunity between the contingent claim, the stock and the bond at prices $P^*_{0,0}$. Then one of the following must hold:

a) $\exists\ \overline{\varphi}_{0,0} \in \Phi^s(P^*_{0,0})$ such that $R(\overline{\varphi}_{0,0}, P^*_{0,0}) = 0$ and $V(\overline{\varphi}_{0,0}, P_{1,j}) \geq 0\ \forall\ j$ with at least one strict inequality.

b) $\exists\ \underline{\varphi}_{0,0} \in \Phi^l(P^*_{0,0})$ such that $R(\underline{\varphi}_{0,0}, P^*_{0,0}) = 0$ and $V(\underline{\varphi}_{0,0}, P_{1,j}) \geq 0\ \forall\ j$ with at least one strict inequality.

Let us consider case a). Since MRS and MRL are equal by assumption at $P^*_{0,0}$ there must be some portfolio $\underline{\varphi}_{0,0} \in \Phi^l(P^*_{0,0})$ such that $R(\underline{\varphi}_{0,0}, P^*_{0,0}) = 0$. However, adding up $\overline{\varphi}_{0,0}$ and $\underline{\varphi}_{0,0}$ we obtain a risk–free portfolio that is independent of the contingent claim and produces a strictly positive payoff in at least one state j in period 1, i.e. is an arbitrage opportunity in the stock and the bond. Yet, this is in contradiction to Assumption 3.2. Obviously a similar argument could be made for case b), which completes the proof of the proposition. □

The following proposition now gives necessary conditions for a price system and positions in the contingent claim to give rise to an intersection point between the MRS and MRL schedules.

Proposition 3.11
Let there be given a price system $P^*_{0,0}$ such that $MRS(P^*_{0,0}) = MRL(P^*_{0,0})$ and denote by $\varphi^{*s}_{0,0}$ and $\varphi^{*l}_{0,0}$ the corresponding risk minimising short and long positions in the contingent claim. Then under Assumption 3.2 there is a permutation of the indices $j \in \{1, 2\}$ such that

$$\begin{aligned} V(\varphi^{*l}_{0,0}, P_{1,0}) &= 0\,, & V(\varphi^{*s}_{0,0}, P_{1,0}) &= 0\,,\\ V(\varphi^{*l}_{0,0}, P_{1,1}) &\leq 0\,, & V(\varphi^{*s}_{0,0}, P_{1,1}) &\geq 0\,,\\ V(\varphi^{*l}_{0,0}, P_{1,2}) &\geq 0\,, & V(\varphi^{*s}_{0,0}, P_{1,2}) &\leq 0\,, \end{aligned}$$

where strict inequalities hold if the payoff c of the contingent claim is not in the linear space spanned by the payoffs of the stock and the bond and equalities hold if c is spanned by the payoffs of the two traded assets.

Proof
see Appendix B

With this result in place we are now in a position to characterise fully the intersection point between the MRS and MRL schedules. That is we can state explicit formulas for $C^*_{0,0}$ and the risk minimising long and short positions in the contingent claim. Notice that for a full characterisation of these portfolios it is sufficient to determine the holdings in the stock.

Theorem 3.12
Let the states in period 1 be ordered as in Proposition 3.11. Then the following holds: There exists a unique martingale measure $q := (q_{1,j})\,, j \in \{0, 1, 2\}$ such that

$$C^*_{0,0} = E^q[\,C_{1,\cdot}\,]\,.$$

q is given by

$$\begin{aligned} q_{1,0} &= -\left(p_{1,1}\frac{\Delta S_{1,1}}{\Delta S_{1,0}} + p_{1,2}\frac{\Delta S_{1,2}}{\Delta S_{1,0}}\right) / N \\ q_{1,1} &= p_{1,1} / N \\ q_{1,2} &= p_{1,2} / N \\ N &= -\left(p_{1,1}\frac{\Delta S_{1,1}}{\Delta S_{1,0}} + p_{1,2}\frac{\Delta S_{1,2}}{\Delta S_{1,0}}\right) + p_{1,1} + p_{1,2}\,. \end{aligned}$$

At prices $P^*_{0,0}$ the stock holdings in the risk minimal long and short positions in the contingent claim are given by

$$\begin{aligned} \varphi^{*l,S}_{0,0} &= \frac{C^*_{0,0} - C_{1,0}}{\Delta S_{1,0}} \\ \varphi^{*s,S}_{0,0} &= -\varphi^{*l,S}_{0,0}\,. \end{aligned}$$

Proof
see Appendix C

It is worth pointing out that as a result of this theorem we can use any contingent claim that is not spanned by the payoffs of the traded assets to determine the martingale measure q. q is hence independent of the particular payoff of the contingent claim.

Having thus dealt with the issue of risk minimisation we now turn to the question of expected payoffs of risk minimising strategies. A useful result in this direction is

Proposition 3.13
Let $P^*_{0,0}$, $\varphi^{*l}_{0,0}$ and $\varphi^{*s}_{0,0}$ be as in Theorem 3.12. Then we have

$$E^p[\,V(\varphi^{*l}_{0,0}, P_{0,\cdot})\,] = E^p[\,V(\varphi^{*s}_{0,0}, P_{0,\cdot})\,] = 0\,.$$

Proof
Assume that contrary to the proposition we had $E^p[\,V(\varphi^{*l}_{0,0}, P_{0,\cdot})\,] < 0$. This would imply

$$\begin{aligned} R(\varphi^{*l}_{0,0}, P^*_{0,0}) &= E^p[\,[\,V(\varphi^{*l}_{0,0}, P_{0,\cdot})\,]^-\,] \\ &> E^p[\,[\,V(\varphi^{*l}_{0,0}, P_{0,\cdot})\,]^+\,] \\ &= E^p[\,[\,V(-\varphi^{*l}_{0,0}, P_{0,\cdot})\,]^-\,] \\ &= R(\varphi^{*s}_{0,0}, P^*_{0,0})\,, \end{aligned}$$

contradicting that $\varphi^{*l}_{0,0}$ and $\varphi^{*s}_{0,0}$ give rise to an intersection between the MRS and MRL schedules at prices $P^*_{0,0}$. Clearly, a similar argument could be made for $E^p[\,V(\varphi^{*s}_{0,0}, P_{0,\cdot})\,]$ proving that both expectations are greater or equal to zero. Using this result twice and the fact that by Theorem 3.12 we have $\varphi^{*l}_{0,0} = -\varphi^{*s}_{0,0}$ we obtain

$$0 \le E^p[\,V(\varphi^{*l}_{0,0}, P_{0,\cdot})\,] = E^p[\,V(-\varphi^{*s}_{0,0}, P_{0,\cdot})\,] = -E^p[\,V(\varphi^{*s}_{0,0}, P_{0,\cdot})\,] \le 0\,,$$

which proves the proposition. □

This result is interesting in three respects. First, at prices $P^*_{0,0}$ a trader holding the risk minimising positions $\varphi^{*l}_{0,0}$ or $\varphi^{*s}_{0,0}$ in the given contingent claim will be hedged on average under the exogenous probability measure p. Second, $P^*_{0,0}$ being the prevailing price system there are no pseudo–arbitrage opportunities with respect to the (arbitrary) contingent claim. Third at prices $P^*_{0,0}$ there will consequently not be any arbitrage activity in the market for the contingent claim. Indeed, these prices prevailing a risk averse investor confined to taking risk minimal zero–cost positions in the contingent claim can only attain zero expected profit by engaging in the market for the contingent claim. However, in general, i.e. if the payoff of the contingent claim is not spanned by the traded assets, taking a risk minimal position in the contingent claim will imply taking some positive risk. Hence, no arbitrageur will engage in this market.

It remains to show that $P^*_{0,0}$ is indeed the only price system precluding pseudo–arbitrage opportunities. This is obvious if the payoff of the contingent claim is spanned by the payoffs of the traded assets. In this case there is in fact a unique arbitrage free price for the contingent claim. This is given by $C^*_{0,0}$. For any other price $C_{0,0} \neq C^*_{0,0}$ there would be arbitrage opportunities between the contingent claim, the stock and the bond implying the existence of pseudo–arbitrage opportunities between these assets by Proposition 3.6.

To see that $P^*_{0,0}$ is also the only price system precluding pseudo–arbitrage opportunities even if the payoff of the contingent claim is not spanned by the payoffs of the stock and the bond consider a price system $\overline{P}_{0,0}$, which is the same as $P^*_{0,0}$ apart from the price of the contingent claim being $\overline{C}_{0,0} > C^*_{0,0}$. In this case denote the risk minimising long and short positions in the contingent claim by $\overline{\varphi}^l_{0,0}$ and $\overline{\varphi}^s_{0,0}$. Suppose the stock holdings in $\overline{\varphi}^l_{0,0}$ and $\overline{\varphi}^s_{0,0}$ were given by

$$\begin{aligned} \overline{\varphi}^{l,S}_{0,0} &= \frac{\overline{C}_{0,0} - C_{1,0}}{\Delta S_{1,0}} \\ \overline{\varphi}^{s,S}_{0,0} &= \frac{C_{1,0} - \overline{C}_{0,0}}{\Delta S_{1,0}}. \end{aligned}$$

It is easy to see that as long as

$$\overline{C}_{0,0} < \frac{C_{1,2} - \frac{\Delta S_{1,2}}{\Delta S_{1,0}} C_{1,0}}{\frac{\Delta S_{1,2}}{\Delta S_{1,0}} - 1} \tag{1}$$

with these stock holdings the signs of the payoffs of these long and short positions in the different states in period 1 will still be as described in Proposition 3.11. Hence, the above stock holdings are indeed risk minimising under this condition. However, comparing the payoffs in period 1 with $\overline{P}_{0,0}$ the prevailing price system to the payoffs of the risk minimising positions in the case of $P^*_{0,0}$ we obtain

$$V(\overline{\varphi}^l_{0,0}, P_{1,0}) \quad = \quad V(\varphi^{*l}_{0,0}, P_{1,0}) = 0$$

$$
\begin{aligned}
V(\overline{\varphi}^l_{0,0}, P_{1,j}) &= C_{1,j} + \frac{\overline{C}_{0,0} - C_{1,0}}{\Delta S_{1,0}} \Delta S_{1,j} - \overline{C}_{0,0} \\
&< C_{1,j} + \frac{C^*_{0,0} - C_{1,0}}{\Delta S_{1,0}} \Delta S_{1,j} - C^*_{0,0}\,, j \in \{1,2\}
\end{aligned}
$$

and

$$
\begin{aligned}
V(\overline{\varphi}^s_{0,0}, P_{1,0}) &= V(\varphi^{*s}_{0,0}, P_{1,0}) = 0 \\
V(\overline{\varphi}^s_{0,0}, P_{1,j}) &= -C_{1,j} + \frac{C_{1,0} - \overline{C}_{0,0}}{\Delta S_{1,0}} \Delta S_{1,j} + \overline{C}_{0,0} \\
&> -C_{1,j} + \frac{C_{1,0} - C^*_{0,0}}{\Delta S_{1,0}} \Delta S_{1,j} + C^*_{0,0}\,, j \in \{1,2\}\,,
\end{aligned}
$$

where the inequality signs stem from the fact that $\Delta S_{1,j}/\Delta S_{1,0} < 0$ by Assumption 3.2. Since by Proposition 3.13 the expected payoffs of both the risk minimising long and short positions are zero at prices $P^*_{0,0}$ it follows from the above that for $\overline{C}_{0,0} > C^*_{0,0}$ but still sufficiently small to satisfy equation (1) we have

$$
\begin{aligned}
E^p[\,V(\overline{\varphi}^l_{0,0}, P_{1,\cdot}\,] &< 0 \\
E^p[\,V(\overline{\varphi}^s_{0,0}, P_{1,\cdot}\,] &> 0\,.
\end{aligned}
$$

Hence, due to the monotonicity properties of the MRS and MRL functions we have proved the existence of pseudo–arbitrage opportunities in the contingent claim. It is easy to see that if $\overline{C}_{0,0}$ exceeded the upper limit given in equation (1) there would be an arbitrage opportunity between the contingent claim, the stock and the bond implying again the existence of pseudo–arbitrage opportunities between these assets by Proposition 3.6.

By similar arguments one could obviously prove the existence of pseudo–arbitrage opportunities for all $\underline{C}_{0,0} < C^*_{0,0}$. Even if $\underline{C}_{0,0}$ was still sufficiently large so as to preclude arbitrage opportunities between the contingent claim, the stock and the bond we would have

$$
\begin{aligned}
E^p[\,V(\underline{\varphi}^l_{0,0}, P_{1,\cdot}\,] &> 0 \\
E^p[\,V(\underline{\varphi}^s_{0,0}, P_{1,\cdot}\,] &< 0\,.
\end{aligned}
$$

We have hence shown

Theorem 3.14
Under Assumption 3.2 $P^*_{0,0}$ is the unique pseudo–arbitrage free price system.

Moreover, a comparison of the expected payoffs of the risk minimising positions in the contingent claim shows that for $\underline{C}_{0,0} < C^*_{0,0}$ ($\overline{C}_{0,0} > C^*_{0,0}$) it will always be possible to find risk averse investors with monotonic preferences who will be ready to take long (short) but no short (long) positions in the contingent claim. In particular for $\underline{C}_{0,0} < C^*_{0,0}$ ($\overline{C}_{0,0} > C^*_{0,0}$) arbitrageurs not too risk averse would be long (short) in the contingent claim thus excerting some pressure on the price of this asset driving it in the direction of $C^*_{0,0}$.

4 The Multi–Period Model

The results obtained in the previous section would be of little interest if they were confined to the two–period model we have studied so far. However, it is obvious that this simple model can serve as a building block for a fully fledged multi–period model in which we can study interesting examples such as the pricing and hedging of options under stochastic volatility. The following assumption, which is modelled on Assumption 3.2 makes this idea precise.

Assumption 4.1

a) *The event tree consists of $N+1 \in \mathbb{N}$ periods, $i \in \{0, \ldots, N\}$. In period i there are 3^i states of the world labelled $j \in \{0, \ldots, 3^i - 1\}$. Each vertex (ij) in period $i < N$ has three successors labelled $(i+1, j^3)$, $(i+1, j^3+1)$ and $(i+1, j^3+2)$. The transition probabilities from state (i,j) to its successors in period $i+1$ are given exogenously and are assumed known.*

b) *There are two assets, a stock and a bond, which trade frictionlessly in all periods. The price process B is assumed constant and equal to 1. S is exogenous, stochastic and adapted to the event tree. $\Delta S_{i,j} \neq 0 \, \forall (i,j)$. Moreover, for every vertex (ij) and its three successors according to item a) the following holds: the $\Delta S_{i+1,\cdot}$ will not all have the same sign (absence of arbitrage between stock and bond); if $\Delta S_{i+1,j^3} < 0$ then $\Delta S_{i+1,j^3+k} > 0$, $k \in \{1,2\}$ and $p_{i+1,j^3}|\Delta S_{i+1,j^3}| > p_{i+1,j^3+k}\Delta S_{i+1,j^3+k}$, $k \in \{1,2\}$; if $\Delta S_{i+1,j^3} > 0$ then $\Delta S_{i+1,j^3+k} < 0$, $k \in \{1,2\}$ and $p_{i+1,j^3}\Delta S_{i+1,j^3} > p_{i+1,j^3+k}|\Delta S_{i+1,j^3+k}|$, $k \in \{1,2\}$.*

c) *There is a contingent claim with payoffs $(c_{i,\cdot})$ in periods $i \in \{1, \ldots, M\}$, where $M \leq N$ is the maturity of the contingent claim. For $i = M$ we obviously have $(c_{M,\cdot}) = C_{M,\cdot}$. In every vertex (ij) of the event tree with $i < M$ the contingent claim can be bought or sold at a price of $C_{i,j}$.*

In a nutshell this assumption means that in every vertex up to period $N-1$ of the multi–period event tree we repeat the structure of the two–period model that we have studied in the previous sections. Consequently all the definitions and results obtained in the two–period model can easily be carried over to the multi–period model by applying them to each vertex in the multi–period event tree separately. In fact they can be rephrased so that they fit the multi–period framework if we replace the index $(0,0)$ used in the previous sections by the general index (ij) and the indeces $(1,j)$ by the general indices $(i+1, j^3+k)$, where $k \in \{0,1,2\}$. Assuming this done we shall apply the results obtained in the two–period case to simplify the analysis of the multi–period model. For the precision of language it will, however, be useful to adapt Definition 3.5 to the multi–period framework.

Definition 4.2
Given the multi–period market model of Assumption 4.1 in a vertex (i,j), $i < M$ of the event tree there exists a local pseudo–arbitrage opportunity for the contingent claim iff at prices $P_{i,j}$ there is a portfolio $\varphi^*_{i,j}$ with the following properties:

a) $\varphi^*_{i,j} \in \Phi^l(P_{i,j}) \cup \Phi^s(P_{i,j})$,

b) $R(\varphi^*_{i,j}, P_{i,j}) \leq R(\varphi_{i,j}, P_{i,j}) \ \forall \ \varphi_{i,j} \in \Phi^l(P_{i,j}) \cup \Phi^s(P_{i,j})$,

c) $E^p[\, V_{i+1,\cdot}(\varphi^*_{i,j}, P_{i+1,\cdot}) \,|\, (i,j)\,] > 0$.

Since in the multi–period framework it makes sense to talk about the profits and losses from dynamic trading we now introduce the concept of a profit–and–loss process generated by a portfolio strategy.

Definition 4.3
The profit–and–loss process (P/L process) generated by a portfolio strategy φ given a price process P is a stochastic process defined by

$$PL_{i,\cdot}(\varphi, P) := \sum_{k=0}^{i-1} \varphi_{k,\cdot} \bullet \Delta P_{k+1,\cdot}\,, \ 0 \leq i < N\,.$$

As a consequence of this definition the P/L process generated by a long strategy in the contingent claim is in particular given by

$$PL_{i,\cdot}(\varphi^l, P) := C_{i,\cdot} + \sum_{k=0}^{i-1} \varphi^S_{k,\cdot} \Delta S_{i+1,\cdot}\,,$$

while the P/L process generated by a short strategy in the contingent claim is given by

$$PL_{i,\cdot}(\varphi^s, P) := -C_{i,\cdot} + \sum_{k=0}^{i-1} \varphi^S_{k,\cdot} \Delta S_{i+1,\cdot}\,.$$

With these definitions in place we can now state our central result on the existence and uniqueness of a locally pseudo–arbitrage free price process in the multi–period model.

Theorem 4.4
Under Assumption 4.1 the following holds for all $0 \leq i \leq M$.

a) There is a unique price process $P^* := (C^*, S, B)$ such that there are no local pseudo–arbitrage opportunities between the stock, the bond and the contingent claim.

b) There exists a unique martingale measure q such that C^* is given by

$$C^*_{ij} = E^q[\, C_{i+1,\cdot} \,|\, (ij)\,] + c_{ij}$$

for all vertices (ij) in the event tree with $i < M$ and

$$C^*_{Mj} = c_{Mj}$$

for all verices (Mj) in the event tree.

The martingale measure q is characterised by transition probabilities from arbitrary vertices (ij) to their successors. The transition probabilities are given by

$$\begin{aligned}
q_{i+1,j^3} &= -\left(p_{i+1,j^3+1} \frac{\Delta S_{i+1,j^3+1}}{\Delta S_{i+1,j^3}} + p_{i+1,j^3+2} \frac{\Delta S_{i+1,j^3+2}}{\Delta S_{i+1,j^3}} \right) / \, N_{ij} \\
q_{i+1,j^3+1} &= p_{i+1,j^3+1} \, / \, N_{ij} \\
q_{i+1,j^3+2} &= p_{i+1,j^3+2} \, / \, N_{ij} \\
N_{ij} &= -\left(p_{i+1,j^3+1} \frac{\Delta S_{i+1,j^3+1}}{\Delta S_{i+1,j^3}} + p_{i+1,j^3+2} \frac{\Delta S_{i+1,j^3+2}}{\Delta S_{i+1,j^3}} \right) \\
&\quad + p_{i+1,j^3+1} + p_{i+1,j^3+2} \, .
\end{aligned}$$

c) *At prices P^* there are unique long and short strategies in the contingent claim, φ^{*l} and φ^{*s}, such that for every vertex (ij), $0 \le i < M$ we have*

$$R(\varphi^{*l}_{ij}, P^*_{i,j}) < R(\varphi^{l}_{ij}, P^*_{i,j}) \;\; \forall \; \varphi^l_{ij} \in \Phi^l(P^*_{ij})$$

and

$$R(\varphi^{*s}_{ij}, P^*_{i,j}) < R(\varphi^{s}_{ij}, P^*_{i,j}) \;\; \forall \; \varphi^s_{ij} \in \Phi^l(P^*_{ij}) \, ,$$

where φ^{*l}_{ij} (φ^{*s}_{ij}) is the long (short) position in the contingent claim assigned to vertex (ij) by the strategy φ^{*l} (φ^{*s}). For the stock holdings in these strategies we have

$$\begin{aligned}
\varphi^{*l,S}_{ij} &= \frac{C^*_{ij} - C_{i+1,j^3}}{\Delta S_{i+1,j^3}} \\
\varphi^{*s,S}_{ij} &= -\varphi^{*l,S}_{ij} \, .
\end{aligned}$$

d) *$PL_{\cdot,\cdot}(\varphi^{*l}, P^*)$ and $PL_{\cdot,\cdot}(\varphi^{*s}, P^*)$ are martingales under the exogenous probability measure p.*

PROOF

a) Proceed by backward induction from period $i = M$ to period $i = 0$ and apply Theorem 3.14 to each vertex.

b)-c) Proceed by backward induction from period $i = M$ to period $i = 0$ and apply Theorem 3.12 to each vertex.

d) Proceed by backward induction from period $i = M$ to period $i = 0$ and apply Proposition 3.13 to each vertex to obtain

$$\begin{aligned}
E^p[\, PL_{i+1,\cdot}(\varphi^{*l}, P^*) \,|\, (ij) \,] &= E^p[\, PL_{i+1,\cdot}(\varphi^{*l}, P^*) - PL_{ij}(\varphi^{*l}, P^*) \,|\, (ij) \,] \\
&\quad + PL_{ij}(\varphi^{*l}, P^*) \\
&= E^p[\, C^*_{i+1,\cdot} + \varphi^{*l,S}_{ij} \Delta S_{i+1,\cdot} - C^*_{ij} \,|\, (ij) \,] \\
&\quad + PL_{ij}(\varphi^{*l}, P^*) \\
&= PL_{ij}(\varphi^{*l}, P^*) \, .
\end{aligned}$$

Since a similar argument could obviously be made for φ^{*s} this completes the proof of the theorem. $\square$

While this theorem is an easy consequence of the results obtained in the two–period model the economic rationale behind it deserves some additional comments.

First, the locally pseudo–arbitrage free price process P^* is compatible with rational expectations. Indeed, in period $M-1$ we may apply the same reasoning as in the two–period model to argue that if the price of the contingent claim differs from the locally pseudo–arbitrage free price there will be scope for arbitrage activity in broad sense as we have described it in Section 2. For this reason an investor following the theory proposed in this paper will expect that in period $M-1$ the locally pseudo–arbitrage free price will prevail. Consequently in period $M-2$ just as in period $M-1$ he will argue that any deviation from the locally pseudo–arbitrage free price will invoke countervailing supply and demand from arbitrageurs that will also in period $M-2$ drive the price of the contingent claim towards the locally pseudo–arbitrage free price. Continuing in this way up to period 0 we obtain that the locally pseudo–arbitrage free price process is indeed consistent with rational expectations.

Second it is worthwhile studying the relations that exist between the concept of a locally pseudo–arbitrage free price process and the concept of local risk minimisation introduced by Schweizer ([Sch88]). At first sight the two concepts bear little resemblance due to the fact that Schweizer uses a quadratic objective function whereas the measure of risk used in this paper is based on the $[\cdot]^-$ function. However, Schweizer's central concepts can easily be adapted to the framework of this paper. We need to define what we mean by "remaining risk". Then we need to show that the strategies φ^{*l} and φ^{*s} are "locally risk minimising" in Schweizer's sense for the price process P^* with respect to the concept of remaining risk to be defined. Before we can do that we need to define what in our framework is an "admissible local variation" of a portfolio strategy.

Definition 4.5
Let φ be a portfolio strategy at prices P. An admissible local variation of φ at some trading date i is a portfolio strategy $\Delta\varphi$ such that $\Delta\varphi_{k,\cdot} = (0;0;0)\,\forall\, k \neq i$ and $\Delta\varphi_{i,\cdot} = (0;\sigma_{i,\cdot};\beta_{i,\cdot})$.

Notice that unlike what is the case in Schweizer's definition ([Sch88], p. 18) in our definition $\beta_{i,\cdot}$ can never be chosen freely but is automatically determined by the choice of $\sigma_{i,\cdot}$ and the requirement that $\Delta\varphi$ should be a portfolio strategy. Next we introduce our concept of remaining risk.

Definition 4.6
At a trading date i the remaining risk at prices P up to trading date $M > i$ of a portfolio strategy φ is given by

$$\overline{R}_{i,\cdot}(\varphi, P) := E^p\left[\left.\sum_{k=0}^{M-1} R(\varphi_{k,\cdot}, P_{k,\cdot}) - \sum_{k=0}^{i-1} R(\varphi_{k,\cdot}, P_{k,\cdot})\,\right|\,(i,\cdot)\right].$$

Closely following Schweizer we posit

Definition 4.7
A portfolio strategy φ is called locally risk minimising on a set of trading dates $\{0,\ldots,M-1\}$ at prices P if for any trading date $i \in \{0,\ldots,M-1\}$ and any admissible local variation $\Delta\varphi$ of φ we have

$$\overline{R}_{i,\cdot}(\varphi+\Delta\varphi, P) \geq \overline{R}_{i,\cdot}(\varphi, P) \quad p-a.s..$$

Notice that under Assumption 4.1 this concept of a locally risk minimising trading strategy is void of meaning if the holdings in the contingent claim are zero. For then due to the absence of arbitrage opportunities between the stock and the bond the only locally risk minimising portfolio strategy is the strategy that is identically equal to zero. However, this makes sense in that there is no point in considering risk minimal hedging or financing strategies if only holdings in the stock and the bond are taken into account. Having thus transferred Schweizer's concepts to our framework we obtain

Proposition 4.8
Under Assumption 4.1 at prices P^*, φ^{*l} and φ^{*s} are locally risk minimising portfolio strategies for all trading dates $i \in \{0,\ldots,M-1\}$.

Proof
The proof follows by backward induction. At trading date $M-1$ we have $\overline{R}_{M-1,\cdot}(\varphi, P^*) = R(\varphi_{M-1,\cdot}, P^*_{M-1,\cdot})$. Hence, by Theorem 4.4 φ^{*l} and φ^{*s} are locally risk minimising. Assuming the proposition to hold up to some trading date $0 < i+1 < M-1$ we have for trading date i

$$\overline{R}_{i,\cdot}(\varphi^{*l}, P^*) = E[\,\overline{R}_{i+1,\cdot}(\varphi^{*l}, P^*)\,|\,(i,\cdot)\,] + R(\varphi^{*l}_{i,\cdot}, P^*_{i,\cdot})\,.$$

Using the induction assumption and the monotonicity of the conditional expectation as well as the fact that by Theorem 4.4 $\varphi^{*l}_{i,\cdot}$ minimises $R(\varphi_{i,\cdot}, P^*_{i,\cdot})$ over all $\varphi \in \Phi^l(P^*_{i,\cdot})$ it is obvious that the proposition holds for trading date i as well. Since a similar argument could be made for φ^{*s} this completes the proof of the proposition. □

Finally assume that the actual price process in the market was arbitrary (apart from satisfying Assumption 4.1) but the profit and loss of a trading strategy was still calculated using the locally pseudo–arbitrage free price process P^*. Then whenever the actual price of the contingent claim was above (below) the locally pseudo–arbitrage free price there would be an incentive for a trader to go short (long) in the contingent claim. For, if he did so he could enter into the locally risk minimising short (long) strategy φ^{*l} (φ^{*s}) — knowing by Theorem 4.4 that the profit–and–loss process resulting from this strategy would be a martingale — and pocket the difference between the costs (payoff) of this portfolio and the actual price of the contingent claim. Hence, if the deviation of the actual price process of the contingent claim from the locally pseudo–arbitrage free price process persisted the total profit–and–loss process that would result from entering repeatedly into locally risk–minimising portfolio strategies in the contingent claim as described above would be a sub–martingale. Apparently this is the closest analogue of an arbitrage strategy we can hope to find in our incomplete markets' framework. Notice that using the price process P^* rather than the actual price process to determine the profits and losses does make sense in that in a complete markets' framework one would naturally use the arbitrage free price process to determine hedging portfolios and to calculate the result from strategies designed to exploit arbitrage opportunities.

This discussion shows that the concept of locally pseudo–arbitrage free prices does indeed give an economically sensible meaning to the idea of "mispricing" even when markets are incomplete. The concept itself is consistent in that it agrees with rational expectations. Moreover, using a duly adapted concept of a locally risk–minimising portfolio strategy we were able to outline a normative trading theory by which in our incomplete marktes' framework deviations from the locally pseudo–arbitrage free price process can be exploited while taking risk that is minimal in a well–defined sense.

5 An Application to Stochastic Volatility

In this section we shall apply our theory to the pricing and hedging of options in a stochastic volatility framework. Among the numerous stochastic volatility models that have been suggested in the literature (for an overview see Frey (1997) [Fre97]) we shall concentrate on a modification of the model proposed by Hull and White (1987) [HW87]. In contrast to these authors we shall allow for correlation between the stock price and the volatility. In particular we shall take as a starting point the following continuous–time specifications of the processes of the (discounted) stock price S and the square of its volatility η

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dW_t^1 \\ d\eta_t &= \kappa \eta_t dt + \gamma \eta (\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2)\,, \end{aligned} \tag{2}$$

where $\eta = \sigma^2$, while μ, κ, γ as well as ρ are constants and W^1 and W^2 are independent standard Brownian motions under the given exogenous physical probability measure. With these specifications the increments in the stock price and the square of its volatility are instantaneously correlated with a correlation factor of ρ. This type of a stochastic volatility model is particularly interesting in that it can be viewed as the continuous–time limit of the NGARCH model suggested by Engle and Ng [EN93] (see Frey 1997 [Fre97]).

Our aim is to find a discrete–time model that approximates this continuous time specification and at the same time satisfies Assumption 4.1. This is easily achieved along the lines suggested in He (1990) [He90]. Following He's method we approximate the increments in the two Brownian motions by two uncorrelated random variables w^1 and w^2, which we specify as follows

$$\begin{aligned} w^1 &= (-1)^{1_{[\mu\geq 0]}}\,1/2\,((-1-\sqrt{3});\ (-1+\sqrt{3});\ 2) \\ w^2 &= 1/2\,((-1+\sqrt{3});\ (-1-\sqrt{3});\ 2)\,. \end{aligned}$$

These two random variables give rise to three realisations

$$\begin{aligned} \omega^0 &= ((-1)^{1_{[\mu\geq 0]}}\,1/2\,(-1-\sqrt{3});\ \frac{1}{2}(-1+\sqrt{3})) \\ \omega^1 &= ((-1)^{1_{[\mu\geq 0]}}\,1/2\,(-1+\sqrt{3});\ \frac{1}{2}(-1-\sqrt{3})) \\ \omega^2 &= ((-1)^{1_{[\mu\geq 0]}};\ 1)\,, \end{aligned}$$

which occur with an equal probability of 1/3.

With the help of w^1 the increments of the stock price can be approximated by

$$\Delta S_{i+1,\cdot} = \max\{1_{[S_{ij}>0]}(\mu S_{ij}\Delta t + \sigma_{ij}S_{ij}w^1(\cdot)\sqrt{\Delta t});\ -S_{ij}\}\,.$$

This specification ensures that even in a discrete–time model the stock price is always nonnegative. Moreover, zero is an absorbing boundary for S, which is necessary for the absence of arbitrage in our model. Furthermore, in order to preclude arbitrage we need to make sure that the $\Delta S_{i+1,\cdot}$ are not all of the same sign. In particular to satisfy Assumption 4.1 as long as $S_{ij}\neq 0$ we must have

$$\operatorname{sign}(\Delta S_{i+1,j^3}) \neq \operatorname{sign}(\Delta S_{i+1,j^3+1})\,. \tag{3}$$

If this is satisfied, by the choice of w^1 it immediately follows that $\operatorname{sign}(\Delta S_{i+1,j^3}) \neq \operatorname{sign}(\Delta S_{i+1,j^3+2})$. Moreover, it is easy to see that if condition (3) is satisfied we also have that $|\Delta S_{i+1,j^3}| > |\Delta S_{i+1,j^3+k}|$, $k\in\{1,2\}$ so that the conditions in Assumption 4.1 on the relative size of the increments of the stock price process at each point of time are satisfied as long as they are different from zero. A simple calculation reveals that condition (3) is satisfied if for σ_{ij} we have

$$\sigma_{ij} > \frac{2}{(-1)^{1_{[\mu\geq 0]}}(1-\sqrt{3})}\mu\sqrt{\Delta t}\,. \tag{4}$$

Notice that as $\Delta t\to 0$ this condition simply requires σ to be nonnegative, which is guaranteed by the specification of the model in equation (2). In order to satisfy condition (4) we propose the following discretisation of the η process.

$$\begin{aligned} \Delta\eta_{i+1,\cdot} = \max\Big\{ & 1_{[\frac{2\mu}{1-\sqrt{3}}\sqrt{\Delta t}<\sqrt{\eta_{ij}}]}\left(\kappa\eta_{ij}\Delta t + \gamma\eta_{ij}(\rho w^1(\cdot)+\sqrt{1-\rho^2}w^2(\cdot))\sqrt{\Delta t}\right)\ ; \\ & \sqrt{\eta_{ij}} - \frac{2\mu}{1-\sqrt{3}}\sqrt{\Delta t}\Big\} \end{aligned}$$

Using Theorem 4.4 we can now easily determine the martingale measure q corresponding to the locally pseudo–arbitrage free price process as long as S is sufficiently far away from the lower boundary of zero so that $\Delta S_{i+1,\cdot} \neq -S_{ij}$. Under this condition the martingale measure is given by

$$\begin{aligned}
q_{i+1,j^3} &= \frac{1}{3}\left((-1)^{1_{[\mu\geq 0]}}\frac{4}{1+\sqrt{3}}\frac{\mu}{\sigma_{ij}}\sqrt{\Delta t}+1\right) \\
q_{i+1,j^3+1} &= \frac{1}{3}\left((-1)^{1_{[\mu\geq 0]}}\frac{-2}{1+\sqrt{3}}\frac{\mu}{\sigma_{ij}}\sqrt{\Delta t}+1\right) \\
q_{i+1,j^3+2} &= \frac{1}{3}\left((-1)^{1_{[\mu\geq 0]}}\frac{-2}{1+\sqrt{3}}\frac{\mu}{\sigma_{ij}}\sqrt{\Delta t}+1\right) .
\end{aligned} \tag{5}$$

Notice that q is strictly positive as long as condition (4) is satisfied. If S was zero the martingale measure would be arbitrary. If S was so close to the lower boundary of zero that it could reach this boundary with the next increment we would have to calculate the martingale measure in the respective vertex according to the formulae in Theorem 4.4 using the actual increments of S in this vertex.

For the time being it is, however, more interesting to compare the martingale measure in equation (5) to the martingale measure proposed by Hull and White ([HW87]) in their model of stochastic volatility. They suggested that as long as the volatility process was uncorrelated with the gross consumption process the market price of volatility risk should be zero. Assuming like Hull and White that η, the process of squared volatility, was uncorrelated with the stock price process in our framework this would mean that the expected value of w^2 under the martingale measure q sould be zero. This is easily seen not to be the case. In fact, in our framework the martingale measure satisfying the assumptions of Hull and White would have to solve the following system of equations.

$$\begin{aligned}
(-1-\sqrt{3})\tilde{q}_{i+1,j^3} + (-1+\sqrt{3})\tilde{q}_{i+1,j^3+1} + 2(1-\tilde{q}_{i+1,j^3}-\tilde{q}_{i+1,j^3+1}) &= (-1)^{1_{[\mu\geq 0]}}\frac{-2\mu\sqrt{\Delta t}}{\sigma_{ij}} \\
(-1+\sqrt{3})\tilde{q}_{i+1,j^3} + (-1-\sqrt{3})\tilde{q}_{i+1,j^3+1} + 2(1-\tilde{q}_{i+1,j^3}-\tilde{q}_{i+1,j^3+1}) &= 0
\end{aligned}$$

As long as S is sufficiently far away from the lower boundary of zero $\tilde{q}$ is given by

$$\begin{aligned}
\tilde{q}_{i+1,j^3} &= \frac{1}{3}\left((-1)^{1_{[\mu\geq 0]}}\frac{-1}{1-\sqrt{3}}\frac{\mu}{\sigma_{ij}}\sqrt{\Delta t}+1\right) \\
\tilde{q}_{i+1,j^3+1} &= \frac{1}{3}\left((-1)^{1_{[\mu\geq 0]}}\frac{-1}{1+\sqrt{3}}\frac{\mu}{\sigma_{ij}}\sqrt{\Delta t}+1\right) \\
\tilde{q}_{i+1,j^3+2} &= \frac{1}{3}\left((-1)^{1_{[\mu\geq 0]}}(-1)\frac{\mu}{\sigma_{ij}}\sqrt{\Delta t}+1\right) .
\end{aligned} \tag{6}$$

Again it is easy to see that this martingale measure is strictly positive as long as condition (4) is satisfied. Comparing $\tilde{q}$ and q shows that our theory leads to a martingale measure that is clearly distinct from the one suggested by Hull and White.

This means that Hull/White prices of options imply the existence of local pseudo–arbitrage opportunities. In particular if the market followed the prices suggested by Hull and White instead of entering into a delta hedging strategy on the basis of their prices it would be possible to buy or sell options and enter into locally risk minimising hedging strategies as defined in the previous section on the basis of locally arbitrage free prices. Whenever such a position was entered into this would lead to an increase in the trader's wealth instead of leaving it constant as would be the case if the trader delta hedged Hull/White prices. The ensuing profit and loss process from this strategy would be a martingale. Hence, the trader would be hedged in mean. Repeatedly entering into such positions would yield a trader's wealth process that would be a sub–martingale while the trader's risk at any point in time would be minimal in the sense of Definition 4.7. Hence, a trader following the theory proposed in this paper could exploit a trader following Hull and White's theory.

Finally, we should like to point out that the results presented in this section are compatible with the interpretation that Lamourex and Lastrapes [LL93] give to their empirical results on the predictive power of implied volatilities. In fact calculating the expectation of w^2 under q we obtain

$$E^q[\, w^2 \,|\, (ij)\,] = (-1)^{1_{[\mu \geq 0]}} \frac{\mu}{\sigma_{ij}} \frac{1 + 3\sqrt{3}}{3(1+\sqrt{3})} < 0\,.$$

This shows that if we leave possible correlations between the increments in the stock price and the increments in volatility out of the picture and concentrate solely on the source of risk proper to volatility under the martingale measure q the expected increment in volatility will be less than under the physical probability measure. Hence we have a positive market price of volatility risk. Moreover, since σ_{ij} is in the denominator of the above expected value the market price of risk is decreasing in the voaltility of the stock price. Both agrees well with the findings of Lamourex and Lastrapes.

6 Conclusion

In this paper we have developed a new approach to pricing and hedging contingent claims in incomplete markets. It distinguishes itself from the previous literature in that the prices we suggest are justified by a theory of hedging. Our aim was to mimic as closely as possible in an incomplete markets framework the no–arbitrage arguments that have been developed in complete markets. This has lead us to define the concept of pseudo–arbitrage. Building on this concept we were able to extend the no–arbitrage idea to a world of incomplete markets in such a way that based on a concept of risk compatible with the axioms of Artzner et al. [ADEH96] we could derive unique prices and corresponding optimal hedging strategies without invoking specific assumptions on the risk–return preferences of investors (other than monotonicity and risk aversion) or even fully fledged general equilibrium models. Our price processes can be represented as martingales under a unique martingale measure. A comparison to a version of the Hull/White stochastic volatility model has shown that in contrast to their approach explicitly taking into account optimal hedging strategies may well lead to non–zero market prices of risk for volatility even if the latter is instantaneously uncorrelated with the stock price process. This was shown to be in agreement with the findings of Lamoureux and Lastrapes [LL93].

A driving assumption behind our results is clearly the measure of risk that we have adopted. Although economically a strong assumption it may well be considered a weak assumption from a practical perspective since in view of the ongoing discussion about measuring an controlling the risks of banks it does not seem unreasonable to expect that a measure of risk as we have used it will in the future be stipulated by the regulatory bodies to replace VaR. Moreover, our measure of risk is well founded on the axiomatic basis laid by Artzner et al. [ADEH96]. As it stands we believe that our theory is better suited than the existing literature to support pricing and trading decisions of individual traders who can only base their decisions on market data but not on the observation of preferences.

It seems worthwhile considering the extension of the concepts proposed in this paper in a number of directions. From an applications point of view it would be interesting to take the discrete–time model in this paper to a continuous–time framework and employ more efficient numerical techniques to solve the optimisation and pricing problems. Furthermore, one might be interested in relaxing the assumption that the true stochastic processes are actually known. The ideas in this paper could also be used to tackle the issue of developing a normative theory for volatility trading without necessarily assuming that observed option prices are correct. Moreover, it may be worthwhile trying to apply the ideas in this paper to other situations of market incompleteness such as credit risk.

Finally, the paper clearly considers a special situation in that in our incomplete markets model there is only one stock and the markets can be completed by introducing a single non–redundant asset. Relaxing these assumptions will raise a number of interesting issues that will also be left to future research.

Appendix

A

First we prove the convexity of MRL. For simplicity we skip the index $(0,0)$ where no ambiguities can arise.

In vertex $(0,0)$ take two prices $\overline{C} > \underline{C}$ for the contingent claim. Denote by $\overline{P}$ the price vector in this vertex with the high price for the contingent claim and by $\underline{P}$ the otherwise identical vector with the low price for the contingent claim.

Let $\overline{\varphi} \in \arg\min_{\varphi\in\Phi^l(\overline{P})}$ and $\underline{\varphi} \in \arg\min_{\varphi\in\Phi^l(\underline{P})}$. Take $\lambda \in (0,1)$. Then we have

$$\begin{aligned}
\mathrm{MRL}(\lambda\underline{P} + (1-\lambda)\overline{P}) &\leq R((\lambda\underline{\varphi} + (1-\lambda)\overline{\varphi}), (\lambda\underline{P} + (1-\lambda)\overline{P})) \\
&\leq \lambda R(\underline{\varphi}, \underline{P}) + (1-\lambda)R(\overline{\varphi}, \overline{P}) \\
&= \lambda\mathrm{MRL}(\underline{P}) + (1-\lambda)\mathrm{MRL}(\overline{P})\,,
\end{aligned}$$

where the first inequality follows from the definition of MRL and the fact that $(\lambda\underline{\varphi} + (1-\lambda)\overline{\varphi}) \in \Phi^l(\lambda\underline{P} + (1-\lambda)\overline{P})$, while the second inequality is an immediate consequence of the subadditivity of the $[\cdot]^-$ function implying the subadditivity of $R(\cdot)$. This proves the convexity and hence the continuity of MRL in C. A similar argument can obviously be made for MRS.

To prove the monotonicity of MRL suppose that $\mathrm{MRL}(\overline{P}) > 0$. We have

$$\begin{aligned}
\mathrm{MRL}(\overline{P}) = R(\overline{\varphi}, \overline{P}) &= E^p[\,[\,C_{1,\cdot} + \overline{\varphi}\Delta S_{1,\cdot} - \overline{C}_{0,0}]^- \,|\, (0,0)\,] \\
&> E^p[\,[\,C_{1,\cdot} + \overline{\varphi}\Delta S_{1,\cdot} - \underline{C}_{0,0}]^- \,|\, (0,0)\,] \\
&\geq E^p[\,[\,C_{1,\cdot} + \underline{\varphi}\Delta S_{1,\cdot} - \underline{C}_{0,0}]^- \,|\, (0,0)\,] \\
&= \mathrm{MRL}(\underline{P})\,,
\end{aligned}$$

proving that MRL is strictly monotonically increasing if it is not zero. We also conclude from this calculation that had $\mathrm{MRL}(\overline{P})$ been equal to zero in agiven vertex it would have remained so for all $\underline{C} < \overline{C}$ because by construction of the measure of risk it is non–negative for every P.

The proof of the monotonicity of MRS follows the same line starting out with the assumption that $\mathrm{MRS}(\underline{P}) > 0$. This completes the proof of the lemma. □

B

Assume first that c is not in the linear space spanned by the payoffs of the stock and the bond. Then since the MRS and MRL schedules intersect by assumption and since Assumption 3.2 precludes arbitrage opportunities between the stock and the bond we must have $R(\varphi^{*s}_{0,0}) > 0$ and $R(\varphi^{*l}_{0,0}) > 0$. Now, consider the different states in period 1 in turn.

Assume that $V(\varphi^{*l}_{0,0}, P^*_{1,0}) < 0$. Then since $p_{1,0}|\Delta S_{1,0}| > p_{1,j}|\Delta S_{1,j}|\ j \in \{1,2\}$ by part b) of Assumuption 3.2 $\varphi^{*l}_{0,0}$ could only be a risk minimising position if we had that $V(\varphi^{*l}_{0,0}, P^*_{1,j}) \leq 0$ for both $j = 1$ and $j = 2$. However, then $R(-\varphi^{*l}_{0,0}, P^*_{0,0}) = 0$ and — contrary to assumption — we could not have an intersection between the MRS and MRL schedules. Hence, $V(\varphi^{*l}_{0,0}, P^*_{1,0}) \geq 0$. Yet, we cannot have $V(\varphi^{*l}_{0,0}, P^*_{1,0}) > 0$ either. To see this, notice that since $R(\varphi^{*l}_{0,0}, P^*_{0,0}) > 0$ we must have $V(\varphi^{*l}_{0,0}, P^*_{1,j}) < 0$ for at least one $j \in \{1,2\}$. Since by Assumption 3.2 both $\Delta S_{1,j}\ j \in \{1,2\}$ are of opposite sign as $\Delta S_{1,0}$ we could if $V(\varphi^{*l}_{0,0}, P^*_{1,0}) > 0$ change $\varphi^{*l}_{0,0}$ so as to reduce risk. Hence, since $\varphi^{*l}_{0,0}$ is risk minimal by assumption we must have $V(\varphi^{*l}_{0,0}, P^*_{1,0}) = 0$.

By a similar argument it is easy to see that we must also have $V(\varphi^{*s}_{0,0}, P^*_{1,0}) = 0$.

Moreover, since $R(\varphi^{*l}_{0,0}, P^*_{0,0}) > 0$ for at least one $j \in \{1,2\}$ we must have $V(\varphi^{*l}_{0,0}, P^*_{1,j}) < 0$. However, this inequality cannot hold for both $j = 1$ and $j = 2$ at the same time since then we would have $R(-\varphi^{*l}_{0,0}, P^*_{0,0}) = 0$ and we could not have an intersection between the MRS and MRL schedules. Hence, we may assume without loss of generality that $V(\varphi^{*l}_{0,0}, P^*_{1,1}) < 0$ and $V(\varphi^{*l}_{0,0}, P^*_{1,2}) > 0$.

By a similar reasoning it is easy to see that we must have $V(\varphi^{*s}_{0,0}, P^*_{1,\underline{j}}) < 0$ for exactly one state $\underline{j} \in \{1,2\}$ and $V(\varphi^{*s}_{0,0}, P^*_{1,\overline{j}}) > 0$ for the remaining state in period 1.

It remains to show that assuming $V(\varphi^{*l}_{0,0}, P^*_{1,1}) < 0$ and $V(\varphi^{*l}_{0,0}, P^*_{1,2}) > 0$ implies that $\underline{j} = 2$ and $\overline{j} = 1$. However, since the fact that $V(\varphi^{*s}_{0,0}, P^*_{1,0}) = 0$ and $V(\varphi^{*l}_{0,0}, P^*_{1,0}) = 0$ implies that $\varphi^{*s}_{0,0} = -\varphi^{*l}_{0,0}$ the above tenet immediately follows.

This proves the proposition for the case that c is not in the linear space spanned by the payoffs of the stock and the bond in period 1.

If on the other hand c is in the linear space spanned by the payoffs of the stock and the bond we have $R(\varphi^{*s}_{0,0}, P^*_{0,0}) = 0$ and $R(\varphi^{*l}_{0,0}, P^*_{0,0}) = 0$. It immediately follows that $V(\varphi^{*s}_{0,0}, P^*_{1,j}) \geq 0$ and $V(\varphi^{*l}_{0,0}, P^*_{1,j}) \geq 0\ \forall j$. However, since a strict inequality for one j would — in contradiction to Assumption 3.2 — imply the existence of an arbitrage opportunity in the stock and the bond we must have $V(\varphi^{*s}_{0,0}, P^*_{1,j}) = 0$ and $V(\varphi^{*l}_{0,0}, P^*_{1,j}) = 0\ \forall j$.

This completes the proof of the proposition. $\square$

C

We need to show that for an arbitrary contingent claim with payoff c we obtain $\mathrm{MRS}(P^*_{0,0}) = \mathrm{MRL}(P^*_{0,0})$ if and only if $C^*_{0,0}$, $\varphi^{*l,S}_{0,0}$ and $\varphi^{*s,S}_{0,0}$ are as given in the theorem.

By the zero value conditions in state 0 in period 1 according to Proposition 3.11 we necessarily have for the stock holdings in the long and short positions in the contingent claim at an intersection point between the MRS and MRL schedules

$$\begin{aligned}\varphi^{*l,S}_{0,0} &= \frac{C^*_{0,0} - C_{1,0}}{\Delta S_{1,0}} \\ \varphi^{*s,S}_{0,0} &= -\varphi^{*l,S}_{0,0}\,.\end{aligned}$$

Moreover, by definition of an intersection between the MRS and MRL schedules the risk of the long and short positions in the contingent claim needs to be equal. By Proposition 3.11 we, hence, need to equate the expected payoff of a short position in vertex $(1,1)$ with the expected payoff of a long position in vertex $(1,2)$. Plugging $\varphi^{*l}_{0,0}$ and $\varphi^{*s,S}_{0,0}$ into the respective equations we obtain

$$p_{1,1}\left(-C_{1,1} + \frac{C^*_{0,0} - C_{1,0}}{\Delta S_{1,0}}\Delta S_{1,1} + C^*_{0,0}\right) = p_{1,2}\left(C_{1,2} - \frac{C^*_{0,0} - C_{1,0}}{\Delta S_{1,0}}\Delta S_{1,2} - C^*_{0,0}\right)\,.$$

Solving for $C^*_{0,0}$ we obtain

$$C^*_{0,0} = q_{1,0}C_{1,0} + q_{1,1}C_{1,1}q_{1,2}C_{1,2}\,, \tag{7}$$

where the $q_{1,j}$ are as given in the theorem for $j \in \{0,1,2\}$.

To see that $C^*_{0,0}$, $\varphi^{*l}_{0,0}$ and $\varphi^{*s,S}_{0,0}$ thus obtained do give rise to an intersection between the MRS and MRL schedules we need to show that at prices $P^*_{0,0}$ $\varphi^{*l}_{0,0}$ and $\varphi^{*s,S}_{0,0}$ are indeed risk minimising positions in the contingent claim. To this end consider the values of the two positions in the different states in period 1.

Clearly, $V(\varphi^{*s}_{0,0}, P_{1,0}) = V(\varphi^{*l}_{0,0}, P_{1,0}) = 0$ by construction. Now, assume for the moment that

$$\begin{aligned}V(\varphi^{*l}_{0,0}, P_{1,1}) &= C_{1,1} + \frac{\sum_{j=0}^{2} q_{1,j}C_{1,j} - C_{1,0}}{\Delta S_{1,0}}\Delta S_{1,1} + \sum_{j=0}^{2} q_{1,j}C_{1,j} \\ &= \left[C_{1,0}\left(\frac{\Delta S_{1,2} - \Delta S_{1,1}}{\Delta S_{1,0}}\right) + C_{1,1}\left(1 - \frac{\Delta S_{1,2}}{\Delta S_{1,0}}\right) + C_{1,2}\left(\frac{\Delta S_{1,1}}{\Delta S_{1,0}} - 1\right)\right]\frac{p_{1,2}}{N} \\ &\leq 0\,.\end{aligned} \tag{8}$$

Then by construction we also have

$$V(\varphi^{*l}_{0,0}, P_{1,2}) \leq 0\,.$$

Finally by a simple calculation we obtain

$$\begin{aligned} V(\varphi^{*l}_{0,0}, P_{1,2}) &= -\left[C_{1,0}\left(\frac{\Delta S_{1,2}-\Delta S_{1,1}}{\Delta S_{1,0}}\right) + C_{1,1}\left(1-\frac{\Delta S_{1,2}}{\Delta S_{1,0}}\right) + C_{1,2}\left(\frac{\Delta S_{1,1}}{\Delta S_{1,0}}-1\right)\right]\frac{p_{1,1}}{N} \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} V(\varphi^{*s}_{0,0}, P_{1,1}) &= -\left[C_{1,0}\left(\frac{\Delta S_{1,2}-\Delta S_{1,1}}{\Delta S_{1,0}}\right) + C_{1,1}\left(1-\frac{\Delta S_{1,2}}{\Delta S_{1,0}}\right) + C_{1,2}\left(\frac{\Delta S_{1,1}}{\Delta S_{1,0}}-1\right)\right]\frac{p_{1,2}}{N} \\ &\geq 0 \end{aligned}$$

Since by Assumption 3.2 we have $p_{1,0}|\Delta S_{1,0}| > p_{1,j}|\Delta S_{1,j}|$, $j \in \{1,2\}$ in this situation it is not possible to adjust $\varphi^{*l}_{0,0}$ or $\varphi^{*s}_{0,0}$ (i.e. more precisely the holdings in the stock in the two portfolios) so as to reduce risk. The long and short positions as given in the theorem are hence risk minimal at prices $P^*_{0,0}$ provided equation (8) is satisfied. If equation (8) was not satisfied for a given contingent claim with payoff $(c_{1,\cdot}) = (C_{1,\cdot})$ we would have

$$V(\varphi^{*l}_{0,0}, P_{1,2}) \leq 0\,,$$

and it would suffice to interchange the numbering of states 1 and 2 in period 1 to see that the above argument would go through unaltered. Hence we have shown that the long and short positions as given in the theorem are indeed risk minimal at prices $P^*_{0,0}$.

Finally, we need to show that the $(q_{1,\cdot})$ as given in the theorem actually constitute a martingale measure. As it is easy to see that we have

$$\sum_{j=0}^{2} q_{1,j} = 1 \quad \text{and} \quad \sum_{j=0}^{2} q_{1,j}\Delta S_{1,j} = 0\,,$$

it suffices to show that $q_{1,j} > 0\ \forall\, j \in \{0,1,2\}$. However, since equation (7) holds for arbitrary contingent claims it holds in particular for the Arrow–Debreu securities. Obviously, the prices for these securities in vertex $(0,0)$ would just be the $(q_{1,\cdot})$. Now, since by Proposition 3.10 at prices $P^*_{0,0}$ there are no arbitrage opportunities between any arbitrary contingent claim, the stock and the bond it follows that $q_{1,j} > 0\ \forall\, j \in \{0,1,2\}$. This completes the proof of the theorem. $\square$

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