Precautionary Reserves and the Interbank Market[1]

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1 Introduction

A recent paper by Bennett and Peristiani (2002) documents that reserve requirements are likely no longer binding for the largest banks due to the amount of vault cash held in vast ATM networks and through the use of sweep accounts. Recent changes to the discount window policy of the Federal Reserve would appear to further reduce the need of banks to hold reserves. So why do banks hold reserves at all?

We hypothesize that large banks have a precautionary demand for reserves related to the possibility that aggregate reserves become concentrated at the end of the day in the accounts of banks which are reluctant to lend. We start out with the view that financial constraints limit the ability of some banks to borrow, which in turn should motivate an unwillingness to lend. Ashcraft and Bleakley(2005) document that privately-held banks appear to face financial constraints when borrowing in the federal funds market. This paper develops a model in order to better understand the importance of this phenomenon and analyzes Fedwire data in order to document its empirical relevance.

In order to study this phenomenon, we examine a simple model of trading frictions in the interbank fed funds market. Banks have payment shocks at 3pm and 6pm. Large banks can lend or borrow fed funds at 3pm and 6pm after their shocks. We assume that

[1] The views expressed in this paper are those of the author and do not necessarily reflect the views of the Federal Reserve Bank of New York or the Federal Reserve System.

for credit and other trading friction reasons, small banks can lend but not borrow fed funds at 3pm after their shock, and cannot lend or borrow at 6pm. After 6pm, banks have to borrow at a penalty rate at the discount window to cover any overdrafts.

The friction from small banks produces the following results, where balances and borrowing and lending amounts are scaled by the size of the banks based on their payment shock size. Since small banks cannot lend at 6pm, large banks hold precautionary balances that they do not lend out at 3pm. These 3pm precautionary balances are held to self-insure against aggregate shocks from large to small banks at 6pm. Since small banks cannot borrow at 6pm, they also hold precautionary balances at 3pm to self-insure against shocks at 6pm. Because large banks can borrow at 6pm, their 3pm precautionary balances are smaller than that of small banks. This implies that the percentage of 3pm balances lent by large banks is larger than that of small banks. The precautionary balances held by banks at 3pm translate into the value of their expected overnight nonborrowed reserves (overnight reserves net of discount window borrowing), which are held in excess and are thus higher for small banks. Small banks also borrow on average a greater amount at the discount window.

Because small banks cannot borrow at 3pm, they hold very large clean balances, defined as overnight reserves plus net fed funds loans minus discount window borrowing. These large clean balances include both i) pre-3pm precautionary balances to allow small banks to self-insure against 3pm shocks, and ii) their 3pm precautionary balances. Each small bank every night lends to large banks strictly positive amounts of fed funds, which are the pre-3pm precautionary balances that the small bank holds plus or minus its 3pm shocks. Thus, the model uses the concept of precautionary balances to explain the stylized facts that small banks hold relatively large amounts of excess reserves overnight, while lending large amounts to large banks overnight, despite lending a lower percentage of available balances during the day than large banks lend.

The model also shows an increase in the volatility of the fed funds rate late in the day, and **predicts empirically** that fed funds lending increases with the fed funds rate. Furthermore, the model offers a new explanation for the phenomena of large amounts of fed funds lending that is multiples of aggregate bank reserves.

2 Motivation

This section outlines some motivating facts for the model. First, we highlight the importance of the federal funds market at the end of the busienss day. Figure 1 documents how the cross-sectional distribution of balances changes during the last 90 minutes of the business day. We focus on the top 100 accounts during all business days of 2005. At the start of this window (17:00), note that a significant fraction of banks have negative balances. These typically large institutions make use of intraday credit throughout the day. This credit is provided by the Federal Reserve at a below-market interest rate (30 basis points) in order to promote the timely sending of payments. As the end of the business day (18:30) nears, reserves are reallocated from institutions with positive balances to banks with negative balances, largely through federal funds loans.

Figure 2 documents that the last hour of the day can be a nervous time for banks. The graph plots the federal funds interest rate volatility measured by the time series standard deviation of the dollar-weighted average federal funds rate over the previous thirty minutes. The sample refers to loans between the top 100 banks during 2005. It is clear from the figure that volatility starts to increase around 17:30 and has a significant spike at 18:20 when banks seems fairly certain of their end-of-day balances. Banks in need of reserves during this time are subject to a severe hold-up problem, as the penalty on an overnight overdraft is the effective federal funds rate plus 400 basis points.

Figure 3 illustrates the average propensity that a bank lends or borrows at least once during the day is related to its size. Here the sample refers to the approximately 700 banks that ever lend or borrow during the first two months of 2007. We measure size using percentiles of the cross-sectional distribution of average daily Fedwire send for the bank over this time period. While the smallest banks lend about one out of every five days, they rarely borrow (about 5 percent of business days). On the other hand, the largest decile of banks lends on about 8.5 out of every 10 days, and borrows on about 7.5 out of every 10. The key takeaway is that smaller institutions are less likely to borrow and lend across all states of nature.

Figure 4 focuses on the average propensity of the smallest banks to lend across different states of nature measured by the actual balance during different windows of the day. For each bank, we measure the percentiles of the distribution of balance at a given minute of the day across all days of the sample period. The point of using bank-specific distributions is to take into account the fact that different banks have different stanards for what is normal at a given time of day. The figure documents that the smallest banks are most willing to lend in the 3pm to 5pm window, and that these institutes rarely lend during the last 90 minutes of the day. Moreover, the figure illustrates the natural phenomenon that banks are more likely to lend when faced when reserves are higher than normal. However, note that the willingness of these banks to lend is quite small, as only about 4 percent will lend during the 3pm to 5pm window when faced with the most favorable liquidity shock. These facts suggest that the smallest institutions withdraw from the federal funds market at the end of the day.

Figure 5 tells a much different story for the largest banks. While large banks are active lenders during the 3pm to 5pm window, they are also active lenders during the last 90 minutes of the day when faced with a favorable reserve position. The graph documents that in contrast to the smallest banks, more than 50 percent of the largest banks with the most favorable reserve position will lend during the last 90 minutes of the day. Moreover, note that 20 percent of the largest banks facing the most adverse reserve position are willing to lend during this late period. Together, these facts suggest that large banks are active lenders throughout the business day.

Figure 6 documents the average propensity of the smallest banks to borrow across percentiles of the balance distribution for different time windows. The smallest banks typically borrow during the 3pm to 5pm window when the reserve position is in one of the two most adverse deciles. However, small banks also borrow during the last 90 minutes of the day, but only when faced with the tail of the reserve balance distribution. Note that the mean probability of borrowing is quite low for small banks, suggesting that reserve management is largely accomplished by holding large precautionary reserves and not through borrowing.

The mean frequency of borrowing for the largest banks across percentiles of the balance distribution is illustrated in Figure 7. Large banks borrow throughout the day, but do borrow the most when hit with an adverse reserve balance at the end of the day. Note that the means are much higher for the large banks. For example, 85 percent of banks hit with the worst reserve position during he last 90 minutes borrow. This suggests that federal funds trading is a key component of the reserve management stratey of large banks throughout the day.

3 Model

Banks hold reserves for precautionary reasons to avoid being overdrawn at the end of the day. There are L large banks called type 'l' and S small banks called type 's'. There are four periods $t \in \{1pm, 3pm, 6pm, 9pm\}$, abbreviated as $t = \{1, 3, 6, 9\}$. Banks receive payments shocks at $t \in \{3, 6\}$ that they must pay during the period. A bank can make any amount of payments intraday regardless of its reserve balance, which abstracts from any fees or caps for intraday credit from the Fed. But if a bank is overdrawn at the end of the day, it must borrow from the discount window at a penalty rate.

Timeline $t = 1$: Bank $i \in \{l, s\}$, holds $b_1^i \in \mathbb{R}$ bonds and $m_1^i \in \mathbb{R}$ Federal Reserve account balances. The Fed conducts open market operations (equivalent to a repo market) by buying and selling bonds to banks at a price of one. The bank chooses $\Delta b_1^i \in \mathbb{R}$ bonds to buy. Bonds pay a gross return of $1 + R_1^b > 1$ at $t = 9$.

$t = 3$: Bank i holds $b_3^i = b_1^i + \Delta b_1^i$ bonds and $m_3^i = m_1^i - \Delta b_1^i$. (We could equivalently assume bank s does not trade during $t = 1$, and rather that m_3^s is its steady-state level of clean balances). Bank l has a payment shock of p_3^l to small banks and p_3^k to other large banks. Bank s has a payment shock of p_3^s to large banks. For simplicity, bank s has no payment shock to other small banks. (Bank l's shocks to other large banks at $t = 1$ and $t = 3$ below are not required for any results). Banks may then trade on the fed funds market, in which prices are taken as given. Bank s lends $f_3^s(R_3^s) \geq 0$ to large banks for a return due at $t = 9$ of R_3^s. Bank l borrows $-f_3^l(R_3^s) \geq 0$ from small banks and lends $f_3^k(R_3^k) \in \mathbb{R}$ to other large banks.

$t = 6$: Bank l has a payment shock of p_6^l to small banks and p_6^k to other large banks. Bank s has a payment shock of p_6^s to large banks. Bank l lends $f_6^k(R_6^k) \in \mathbb{R}$ in the fed funds market to other large banks. Bank $i \in \{l, s\}$ must borrow $w_6^i \geq 0$ from the Fed discount window for a return due at $t = 9$ of $R_6^w \geq R_1^b$, such that it's balance at the end of the period is non-negative.

$t = 9$: Period $t = 9pm$ can be considered as equivalent to occurring the next day before or at the beginning of the $t = 1pm$ period. Bank l has payment shocks of $-(p_3^l + p_6^l)$ to small banks and $-(p_3^k + p_6^k)$ to other large banks. Bank s has a payment shock of $p_9^s = -(p_3^s + p_6^s)$ to large banks. Bank l has a payment of $-(1 + R_3^s)f_3^l - (1 + R_3^k)f_3^k - (1 + R_6^k)f_6^k$, and bank s has a payment of $-(1 + R_3^s)f_3^s$, to repay fed funds. Bank i makes a payment of $(1 + R_6^w)w_6^i$ to the Fed to repay its discount window loan, and the Fed redeems bonds to bank i for $(1 + R_1^b)b_3^i$ in reserve balances (equivalent to trading longer-dated bonds for balances).

Assumptions To summarize the notation, lowercase variables generally denote individual bank values. An 'l' or 's' superscript generally denotes a state variable for that bank type, a flow variable transaction from that bank type to the other bank type, or an interest rate R_t^i involving transactions of bank type. A 'k' superscript generally denotes a flow variable or interest rate for transactions among large banks. Subscripts denote the period $t \in \{1, 3, 6, 9\}$. Positive values of flow variables represent outflows from banks, while negative values of flow variables represent inflows, except for discount window loans, which are positive inflow values.

For economy of notation, the superscript 'l', 's' or 'k' that indicates a bank or transaction type is also used as the index number for summations, where $l \in \{1, ..., L\}$, $k \in \{1, ..., K\}$ and $s \in \{1, ..., S\}$. For each lowercase variable, its uppercase P_t^i, F_t^i or W_6^i denotes the sum for type i at period t. For instance, $P_t^s = \sum_{s=1}^{S} p_t^s$ and $P_t^l = \sum_{l=1}^{L} p_t^l$ for $t \in \{3, 6\}$. Banks are competitive, so they take prices and aggregate quantities F_t^i and W_t^i as given. The aggregate payment shocks from small banks to large banks equals the aggregate payment shocks from large banks to small banks, implying $P_t^s = -P_t^l$. Aggregate payment shocks among large banks must aggregate to zero, implying $P_t^k = 0$ for $t \in \{3, 6\}$.

Payments shocks have zero mean, with a uniform distribution $p_t^i \sim U[-\overline{p}^l, \overline{p}^l]$, $i \in \{l, s\}$, and an unspecified distribution for p_t^k, for $t \in \{3, 6\}$. $P_t^s = -P_t^l$, and for simplicity, we assume that P_t^i has a uniform distribution as well, where $P_t^i \sim U[-\overline{P}, \overline{P}]$, for $i \in \{l, s\}$ and $t = \{3, 6\}$. $\overline{P} = \gamma^i \overline{p}^i$ for $i \in \{l, s\}$, where $\gamma^l \in (0, L)$ and $\gamma^s \in (0, S)$, which implies that shocks for type $i \in \{l, s\}$ are not perfectly positively or negatively correlated.[2] Bank

[2]It is natural to think of unexpected payments as having zero mean, because any expected payments would typically be funded by repos or fed funds traded in the morning fed funds market. The uniform distribution of P_t^i is assumed for simplification and should not qualitatively effect the results. Consider the correlation of p_t^i across all banks of a particular type $i \in \{l, s\}$ and period $t \in \{3, 6\}$. If the correlation

i has combined liquid assets in the form of bonds and reserves greater that its potential payment shocks to other banks: $m^i + b_1^i \geq 2\overline{p}^i + \overline{p}^k \mathbf{1}_{i=l}$ for $i \in \{l, s\}$.

4 Bank Optimizations

The bank $i \in \{l, s\}$ optimization problem to maximize profits is as follows:

$$\max_{\boldsymbol{A}^i} \quad E[\pi^i] \tag{1}$$

$$\text{s.t.} \quad m_3^i \leq b_1^i \tag{2}$$

$$-f^l \mathbf{1}_{i=l} + f^s \mathbf{1}_{i=s} \geq 0 \tag{3}$$

$$w_6^i \geq 0 \tag{4}$$

$$m_9^i \geq 0. \tag{5}$$

For bank l,

$$m_6^l = m_3^l - p_3^l - p_3^k - f_3^l - f_3^k \tag{6}$$

$$m_9^l = m_6^l - p_6^l - p_6^k - f_6^k + w_6^l \tag{7}$$

$$\pi^l = (1 + R_1^b) b_3^l + m_3^l - R_6^w w_6^l + R_6^k f_6^k + R_3^s f_3^l + R_3^k f_3^k - b_1^l - m_1^l$$

$$\boldsymbol{A}^l = \{m_3^l, f_3^l, f_3^k, f_6^k, w_6^l\}.$$

For bank s,

$$m_6^s = m_3^s - p_3^s - f_3^s \tag{8}$$

$$m_9^s = m_6^s - p_6^s + w_6^s$$

$$\pi^s = (1 + R_1^b) b_3^s + m_3^s - R_6^w w_6^s + R_3^s f_3^s - b_1^s - m_1^s$$

$$\boldsymbol{A}^s = \{m_3^s, f_3^s, w_6^s\}.$$

Constraint (2) gives the maximum reserve balances that can be held at $t = 3$. Constraint (3), where $\mathbf{1}_{[\cdot]}$ represent the indicator function, gives the restriction that small banks

is negative one, P_t^i has a degenerate uniform distribution of $U[0, 0]$ and corresponds to the limiting case of $\gamma^i = 0$. If the correlation is one, P_t^i has a uniform distribution of $U[-L\overline{p}^i, L\overline{p}^i]$ for $i = l$ and $U[-S\overline{p}^i, S\overline{p}^i]$ for $i = s$, which corresponds to the limiting case of γ^i equal to L and S, respectively. If the correlation is zero, the central limit theorem implies that as L and S go to infinity, the distributions of P_t^l and P_t^s, would approach normal given by $N(0, \frac{L(\overline{p}^l)^2}{3})$ and $N(0, \frac{S(\overline{p}^s)^2}{3})$, respectively. Instead, the variance of P_t^i with its assumed uniform distribution is $\frac{(\gamma^i \overline{p}^i)^2}{3}$. For $\gamma^l = L^{\frac{1}{2}}$ and $\gamma^s = S^{\frac{1}{2}}$, P_t^i has the same variance as it would under the central limit theorem. The difference is that a uniform distribution implies P_t^i has much "fatter tails," or extremely lower kurtosis, than P_t^i would have under a normal distribution. This can be interpreted as a positive correlation of p_t^i, with a particularly high correlation among tail values of p_t^i.

cannot borrow from large banks. Constraint (4) restricts discount window loans to be non-negative, and constraint (5) requires that overnight reserve balances are non-negative.

We examine equilibria that are symmetric among type $i \in \{l, s\}$, and for which constraint (3) does not bind. As equilibrium conditions, aggregate interbank lending among large banks must net to zero each period, implying $F_t^k = 0$ for $t \in \{3, 6\}$, and aggregate interbank lending between large and small banks must satisfy $F_3^l(R_3^s) = -F_3^s(R_3^s)$.

We solve the model starting at $t = 6$. For bank l,

$$\pi^l = (b_1^l + m_1^l - m_3^l)R_1^b - R_6^w w_6^l + R_6^k f_6^k + R_3^s f_3^l + R_3^k f_3^k.$$

Bank l chooses discount window borrowing w_6^l and interbank lending f_6^k. Constraints (4) and (5) imply that

$$w_6^l = \max\{0, -m_6^l + p_6^l + p_6^k + f_6^k\}, \tag{9}$$

which is greater than zero if the bank cannot borrow enough on the interbank market to ensure its overnight balance m_9^l is not overdrawn. The first order condition for f_6^k gives

$$R_6^k = R_6^w \frac{dw_6^l}{df_6^k} = \begin{cases} 0 & \text{if } w_6^l = 0 \\ R_6^w & \text{if } w_6^l > 0, \end{cases} \tag{10}$$

except $m_9^l = w_6^l$, which implies $w_6^l = 0$ and $\frac{dw_6^l}{df_6^k}|_{w_6^l = m_9^l}$ is not defined. In order for the first order condition to hold for all large banks for which $m_9^l \neq w_6^l$, either they all borrow from the discount window or none do. This means that no large banks borrow at the discount window while others hold excess overnight balances. This allows for deriving the aggregate discount window borrowing $W_6^l = \sum_{l=1}^{L} w_6^l = \max\{0, -M_6^l + P_6^l\}$, where

$$M_6^l = M_3^l - P_3^l - F_3^l. \tag{11}$$

If $W_6^l = 0$, there is sufficient aggregate balances among large banks. No large banks borrow at the discount window, and those that need funds borrow from those with excess funds at $R_6^k = 0$. If $W_6^l > 0$, there is an aggregate shortage of balances among large banks, which requires borrowing at the discount window. The interbank lending rate equals the discount window rate, so it is arbitrary how large banks choose between w_6^l and f_6^k. For simplicity, we assume that all large banks borrow equally from the discount window according to

$$\begin{aligned} w_6^l &= \frac{1}{L} W_6^l \\ &= \max\{0, \frac{1}{L}(-M_6^l + P_6^l)\}, \end{aligned}$$

and trade in the interbank market to give themselves equal overnight balances. Banks are indifferent because if $R_6^k = 0$, then $w_6^l = 0$ and they borrow in the fed funds market at no cost. If $R_6^k = R_6^w$, then all large banks hold $m_9^l = 0$, and borrow at the same rate in the fed funds as at the discount window. This implies that for each large bank, $m_9^l = \frac{1}{L}M_9^l = \frac{1}{L}\sum_{l=1}^{L} m_9^l$. Substituting for m_9^l from (7) and simplifying,

$$m_6^l - p_6^l - p_6^k - f_6^k + w_6^l = \frac{1}{L}(M_6^l - P_6^l + W_6^l).$$

Substituting for $w_6^l = \frac{1}{L}W_6^l$ and solving for f_6^k gives

$$f_6^k = -\frac{1}{L}(M_6^l - P_6^l) + m_6^l - p_6^l - p_6^k,$$

to complete bank l's optimization at $t = 6$.

For bank s,

$$\pi^s = (b_1^s + m_1^s - m_3^s)R_1^b - R_6^w w_6^s + R_3^s f_3^s.$$

Bank s chooses only discount window borrowing. Constraints (4) and (5) imply that bank s chooses

$$w_6^s = \max\{0, -m_3^s + p_3^s + f_3^s + p_6^s\}.$$

At $t = 3$, banks choose interbank lending. Bank l chooses interbank lending $f_3^l(R_3^s)$ to small banks (in negative amounts) and $f_3^k(R_3^k)$ to large banks. The first order conditions for f_3^l and f_3^k are

$$R_3^s = \frac{d}{df_3^l}E_3[R_6^w w_6^l - R_6^k f_6^k - R_3^k f_3^k] \tag{12}$$

$$R_3^k = \frac{d}{df_3^k}E_3[R_6^w w_6^l - R_6^k f_6^k - R_3^s f_3^l], \tag{13}$$

respectively. For solutions such that constraint (3) does not bind, $f_3^l < 0$ implies $R_3^k = R_3^s$. To show this, suppose $R_3^k < R_3^s$. Bank l would borrow infinitely from small banks to lend to other large banks, implying $f_3^k = \infty$. In aggregate, $F_3^k = \sum_{l=1}^{L} f_3^k = \infty$, a contradiction to the equilibrium condition of $F_3^k = 0$. Suppose instead $R_3^s > R_3^k$. Bank l would demand to borrow from other large banks and not from small banks, implying $f_3^l(R_3^s) = 0$ for all l, a contradiction to $f_3^l < 0$.

Since $R_3^k = R_3^s$, bank l is indifferent between lending to large or small banks, so its choice between f_3^l and f_3^k is arbitrary. We assume for simplicity that all large banks borrow equally from small banks according to $f_3^l = \frac{F_3^l}{L}$and then redistribute funds among themselves. This structure would also correspond to a model of small banks lending in a correspondent banking relationship to large banks, which then relend the funds on the interbank market.

Net borrowing at $t = 6$ is

$$R_6^w w_6^l - R_6^k f_6^k = \{ \begin{array}{ll} 0 & \text{if } W_6^l = 0 \\ R_6^w(-m_6^l + p_6^l + p_6^k) & \text{if } W_6^l > 0, \end{array} \tag{14}$$

found by substituting into the left-hand side of (14) for w_6^l from (9), and for R_6^k from (10), noting that $w_6^l > 0$ if and only if $W_6^l > 0$.

Expected net borrowing at $t = 6$ is

$$\begin{aligned} E_3[R_6^w w_6^l - R_6^k f_6^k] &= R_6^w \int_{-\overline{P}}^{\overline{P}} \int_{-\overline{p}^l}^{\overline{p}^l} \int_{-\overline{p}^k}^{\overline{p}^k} (-m_6^l + p_6^l + p_6^k) \mathbf{1}_{W_6^l > 0} \psi(p_6^k, p_6^l, P_6^l) dp_6^k dp_6^l dP_6^l \\ &= R_6^w \int_{-\overline{P}}^{\overline{P}} \int_{-\overline{p}^l}^{\overline{p}^l} \int_{-\overline{p}^k}^{\overline{p}^k} (-m_6^l + p_6^l + p_6^k) \mathbf{1}_{P_6^l > M_6^l} \psi(p_6^k, p_6^l, P_6^l) dp_6^k dp_6^l dP_6^l \\ &= R_6^w \int_{M_6^l}^{\overline{P}} \int_{-\overline{p}^l}^{\overline{p}^l} \int_{-\overline{p}^k}^{\overline{p}^k} (-m_6^l + p_6^l + p_6^k) \psi(p_6^k, p_6^l, P_6^l) dp_6^k dp_6^l dP_6^l, \end{aligned} \tag{15}$$

where $\psi(\cdot)$ is a uniform (joint where appropriate) p.d.f. Substituting the right-hand side for the left-hand side of (15) into (12), substituting for m_6^l from (6), noting $R_3^k = R_3^s$ and simplifying gives

$$\begin{aligned} R_3^s &= (1 + \frac{df_3^k}{df_3^l}) R_6^w \int_{M_6^l}^{\overline{P}} \int_{-\overline{p}^l}^{\overline{p}^l} \int_{-\overline{p}^k}^{\overline{p}^k} \psi(p_6^k, p_6^l, P_6^l) dp_6^k dp_6^l dP_6^l - R_3^s \frac{df_3^k}{df_3^l} \\ &= R_6^w \int_{M_6^l}^{\overline{P}} \int_{-\overline{p}^l}^{\overline{p}^l} \int_{-\overline{p}^k}^{\overline{p}^k} \psi(p_6^k | p_6^l, P_6^l) \psi(p_6^l | P_6^l) \psi(P_6^l) dp_6^k dp_6^l dP_6^l \\ &= R_6^w \int_{M_6^l}^{\overline{P}} \frac{1}{2\overline{P}} dP_6^l \\ &= \frac{R_6^w (\overline{P} - M_6^l)}{2\overline{P}}. \end{aligned}$$

Substituting similarly as above into (13) and simplifying gives the same solution:

$$\begin{aligned} R_3^s &= (1+\frac{df_3^l}{df_3^k})R_6^w \int_{M_6^l}^{\overline{P}} \int_{-\overline{p}^l}^{\overline{p}^l} \int_{-\overline{p}^k}^{\overline{p}^k} \psi(p_6^k, p_6^l, P_6^l) dp_6^k dp_6^l dP_6^l - R_3^s \frac{df_3^l}{df_3^k} \\ &= \frac{R_6^w(\overline{P} - M_6^l)}{2\overline{P}}. \end{aligned}$$

Substituting for M_6^l from (11) gives

$$R_3^s = R_6^w \frac{(\overline{P} + P_3^l + F_3^l - M_3^l)}{2\overline{P}}. \tag{16}$$

Solving for $-F_3^l$ gives the large banks' aggregate demand for borrowing from small banks:

$$-F_3^l(R_3^s) = -2\frac{R_3^s}{R_6^w}\overline{P} - M_3^l + P_3^l + \overline{P}.$$

To interpret this, first note that

$$\begin{aligned} E_3[R_6^k] &= R_6^w E[\mathbf{1}_{W^C>0}] \\ &= R_6^w \int_{M_6^l}^{\overline{P}} \frac{1}{2\overline{P}} dP_6^l \\ &= R_3^s, \end{aligned}$$

where we substitute for R_6^s on the left-hand side from (10). Since $E_3[R_6^k] = R_3$ and (16) are independent of f_3^l and f_3^k, bank l is indifferent to borrowing/lending at $t = 3$ versus at $t = 6$. For simplicity, we assume large banks trade at $t = 3$ to hold equal balances: $m_3^l = \frac{M_3^l}{L}$. The individual bank l first order conditions for f_3^s and f_3^k require that in (16), aggregate large bank borrowing F_3^l equates the return on a marginal unit of fed funds borrowed by large banks in aggregate, R_3^s, with the expected cost of large banks needing to borrow a marginal unit from the discount window, which is the return on discount window borrowing, R_6^w, multiplied by the probability that large banks have to borrow from the discount window based on F_3^l, which is the last factor on the right-hand side of (16). Substituting for m_6^l from (6) into $m_6^l = \frac{M_6^l}{L}$, simplifying and solving for f_3^k,

$$f_3^k = -\frac{M_6^l}{L} + m_3^l - p_3^l - p_3^k - f_3^l. \tag{17}$$

For bank s, the first order condition for f_3^s is

$$R_3^s = R_6^w \frac{d}{df_3^s} E_3[w_6^s],$$

where

$$\begin{aligned} E[w_6^s] &= E[w_6^s | p_6^s > m_6^s] \Pr(p_6^s > m_6^s) \\ &= \left(\frac{\overline{p}^s - m_6^s}{2\overline{p}^s}\right)\left(\frac{\overline{p}^s - m_6^s}{2}\right). \end{aligned}$$

In the second line, the first factor is the probability of being overdraft, and the second factor is the expected discount window borrowing given that the bank is overdraft. Taking the derivative with respect to f_3^s gives

$$\begin{aligned} E_3[w_6^s] &= \int_{-\overline{p}^s}^{\overline{p}^s} (p_3^s + p_6^s + f_3^s - m_3^s)\mathbf{1}_{p_6^s > m_3^s - p_3^s - f_3^s} \psi(p_6^s) dp_6^s \\ &= \int_{m_3^s - p_3^s - f_3^s}^{\overline{p}^s} (p_3^s + p_6^s + f_3^s - m_3^s)\psi(p_6^s) dp_6^s \\ &= \frac{(p_3^s + f_3^s - m_3^s + \overline{p}^s)^2}{4\overline{p}^s}, \end{aligned} \tag{18}$$

giving

$$R_3^s = R_6^w \left[\frac{\overline{p}^s - (m_3^s - p_3^s - f_3^s)}{2\overline{p}^s}\right].$$

This first order condition for f_3^s shows that bank s chooses f_3^s to equate its return on a marginal unit of fed funds lending, R_3^s, with its expected cost of needing to borrow a marginal unit from the discount window, which is the return on discount window borrowing, R_6^w, multiplied by the probability bank s has to borrow based on f_3^s.

Solving for f_3^s,

$$f_3^s(R_3^s) = 2\overline{p}^s \frac{R_3^s}{R_6^w} - p_3^s + m_3^s - \overline{p}^s. \tag{19}$$

The aggregate supply of interbank loans by small banks is

$$\begin{aligned} F_3^s(R_3^s) &= \sum_{s=1}^{S} f_3^s(R_3^s) \\ &= S[2\overline{p}^s \frac{R_3^s}{R_6^w} + m_3^s - \overline{p}^s] - \sum_{s=1}^{S} p_3^s, \end{aligned}$$

where $\sum_{s=1}^{S} m_3^s = Sm_3^s$ since banks of type $i \in \{l, s\}$ are ex-ante identical and choose the same m_3^i at $t = 1$. Solving for R_3^s gives

$$R_3^s = \frac{R_6^w(F_3^s + P_3^s - M_3^s + S\overline{p}^s)}{2S\overline{p}^s}.$$

The competitive market equilibrium for fed funds, determined by $F_3^s(R_3^s) = -F_3^l(R_3^s)$, is

$$
\begin{aligned}
F_3^s &= -P_3^s + \frac{\overline{P}M_3^s - S\overline{p}^s M_3^l}{S\overline{p}^s + \overline{P}} && (20) \\
R_3^s &= \tfrac{1}{2}R_6^w\{1 - \frac{M_3^s + M_3^l}{S\overline{p}^s + \overline{P}}\}. && (21)
\end{aligned}
$$

R_3^s does not depend on P_3^s. An early payment shock P_3^s shifts the aggregate small banks' supply curve and large banks' demand curve in equal amounts to the right, so the fed funds amount increases but the price is unchanged.

The amount borrowed from small banks is equal across large banks by assumption from above. By (19), bank lending across small banks is equal except for the p_3^s term. Thus, in equilibrium $-f_3^l = \frac{F_3^s}{L}$ and $f_3^s = -p_3^s + \frac{F_3^s - P_3^s}{S}$, which gives

$$
\begin{aligned}
-f_3^l &= \frac{P_3^l}{L} + \frac{\overline{P}M_3^s - S\overline{p}^s M_3^l}{L\left(S\overline{p}^s + \overline{P}\right)} && (22) \\
f_3^s &= -p_3^s + \frac{\overline{P}M_3^s - S\overline{p}^s M_3^l}{S\left(S\overline{p}^s + \overline{P}\right)}. && (23)
\end{aligned}
$$

At $t = 0$, bank i chooses m_3^i by buying Δb_1^i bonds according to their first order condition for m_3^i. For bank l, this is

$$
R_1^b = \frac{d}{dm_3^l}E_1[-R_6^w w_6^l + R_6^k f_6^k + R_3^s f_3^l + R_3^k f_3^k].
$$

Substituting for R_3^k with R_3^s, for $-R_6^w w_6^l + R_6^k f_6^k$ from (14), for f_3^k from (17) and simplifying gives

$$
\begin{aligned}
R_1^b &= \frac{d}{dm_3^l}E_1[R_6^w(\frac{M_6^l}{L} - p_6^l - p_6^k)\mathbf{1}_{W_6^l>0} - R_3^s(\frac{M_6^l}{L} - m_3^l + p_3^l + p_3^k)] \\
&= E_1[R_3^s].
\end{aligned}
$$

For bank s, the first order condition is

$$
\begin{aligned}
R_1^b &= \frac{d}{dm_3^s}E_1[-R_6^w w_6^s + R_3^s f_3^s] \\
&= \frac{d}{dm_3^s}E_1[E_3[-R_6^w w_6^s + R_3^s f_3^s]]
\end{aligned}
$$

Substituting for w_6^s from (18) and for f_3^s from (19) and simplifying gives the same result,

$$
\begin{aligned}
R_1^b &= \frac{d}{dm_3^s}E_1[-R_6^w\overline{p}^s(\frac{R_3^s}{R_6^w} + 1)^2 + R_3^s[2\overline{p}^s\frac{R_3^s}{R_6^w} - p_3^s + m_3^s - \overline{p}^s]] \\
&= E_1[R_3^s] \\
&= R_3^s.
\end{aligned}
$$

Substituting R_1^b for R_3^s into (21) and solving for the aggregate clean balances gives

$$M_3^s + M_3^l = (1 - \frac{2R_1^b}{R_6^w})(S\overline{p}^s + \overline{P}). \tag{24}$$

From the equilibrium solution for f_3^s in (23) and f_3^l in (22), if

$$\overline{P}M_3^s - S\overline{p}^s M_3^l > p_3^s S(S\overline{p}^s + \overline{P}) \text{ for all } s, \tag{25}$$

then $f_3^s > 0$ for all s, and $f_3^l < 0$ for all l, since $f_3^l = -\frac{S}{L}F_3^s$, so constraint (3) holds and does not bind.

The inequality (25) always holds if

$$\gamma^s M_3^s - SM_3^l > S\overline{p}^s(\gamma^s + S), \tag{26}$$

and implies that

$$F_3^s = \sum_{s=1}^{S} f_3^s > S\overline{p}^s - \overline{P} > 0. \tag{27}$$

This shows that when each bank s holds optimal balances so that its borrowing constraint is not binding, their precautionary reserves imply that there is always aggregate strictly positive lending to large banks. For solutions satisfying (24) and (26),

$$\begin{aligned} M_3^l &< \overline{P}(1 - \frac{2R_1^b}{R_6^w}) - S\overline{p}^s < 0 \\ M_3^s &> 2S\overline{p}^s(1 - \frac{R_1^b}{R_6^w}) > 0 \end{aligned}$$

which imply

$$m_3^l < \frac{\overline{P}}{L}(1 - \frac{2R_1^b}{R_6^w}) - \frac{S}{L}\overline{p}^s < 0 \tag{28}$$

$$m_3^s > 2\overline{p}^s(1 - \frac{R_1^b}{R_6^w}) > 0. \tag{29}$$

To satisfy constraint (2), $m_3^s < 2\overline{p}^s$, which implies $m_3^l \geq \frac{\overline{P}}{L}(1 - \frac{2R_1^b}{R_6^w}) - \frac{S}{L}\overline{p}^s(1 + \frac{2R_1^b}{R_6^w})$. Thus, to satisfy constraints (2) and (3),

$$\begin{aligned} m_3^l &\in \left(\frac{\overline{P}}{L}(1 - \frac{2R_1^b}{R_6^w}) - \frac{S}{L}\overline{p}^s(1 + \frac{2R_1^b}{R_6^w}), \frac{\overline{P}}{L}(1 - \frac{2R_1^b}{R_6^w}) - \frac{S}{L}\overline{p}^s\right) \\ m_3^s &\in \left(2\overline{p}^s(1 - \frac{R_1^b}{R_6^w}, 2\overline{p}^s\right), \end{aligned}$$

subject to (24).

5 Precautionary Balances and Bank Lending

We compare the percentage of available balances that large and small banks lend on the interbank market at $t = 3$. We show that for a given bank reserve balance, controlling for the size of the bank by scaling by the maximum $t = 6$ shock size, large banks lend a greater percentage of available reserve balances than small banks. The net amount that bank l lends at $t = 3$ is

$$f_3^k + f_3^l = -m_3^l + \frac{P_3^l}{L} + \frac{F_3^l}{L} + m_3^l - p_3^l - p_3^k \quad (30)$$

$$= m_3^l - p_3^l - p_3^k - \frac{\overline{P}}{L}(1 - \frac{2R_1^b}{R_6^w}), \quad (31)$$

which is found by substituting on the right-hand side of (30) for $\frac{F_3^l}{L} = f_3^l$ from (22), solving for M_3^s in (24) and substituting for it, then simplifying. The reserve balances that bank l has available to lend at $t = 3$ are

$$m_3^l - p_3^l - p_3^k. \quad (32)$$

The net amount that bank s lends at $t = 3$ is

$$f_3^s = m_3^s - p_3^s - \overline{p}^s(1 - \frac{2R_1^b}{R_6^w}), \quad (33)$$

which is found by solving for M_3^l in (24) and substituting for it in (23). The reserve balances that bank s has available to lend at $t = 3$ are

$$m_3^s - p_3^s. \quad (34a)$$

To compare lending percentage between bank l and s when their scaled bank balances are equal, set the right-hand side of (32) divided by $\overline{p}^l + \overline{p}^k$ equal to the right-hand side of (34a) divided $\overline{p}^s$:

$$\frac{m_3^l - p_3^l - p_3^k}{\overline{p}^l + \overline{p}^k} = \frac{m_3^s - p_3^s}{\overline{p}^s}. \quad (35)$$

We want to show that bank l lends a greater percentage of available balances at $t = 3$ than bank s:

$$\frac{m_3^l - p_3^l - p_3^k - \frac{\overline{P}}{L}(1 - \frac{2R_1^b}{R_6^w})}{m_3^l - p_3^l - p_3^k} > \frac{m_3^s - p_3^s - \overline{p}^s(1 - \frac{2R_1^b}{R_6^w})}{m_3^s - p_3^s}, \quad (36)$$

where the percentage of balances lent by bank l is on the left-hand side and by bank s is on the right-hand side.

Considering the case of strictly positive available reserve balances, substituting from (35) and for $\overline{P} = \gamma^l \overline{p}^l$ and simplifying gives the inequality condition as

$$L > \frac{\overline{p}^l}{\overline{p}^l + \overline{p}^k} \gamma^l,$$

which always holds. The precautionary balances held are found by subtracting balances lent from balances available, and are equivalent to m_6^i balances held at the end of period $t = 3$. Banks target to hold the same amount of precautionary balances m_6^i across their type at the end of $t = 3$. The amount of precautionary balances that they do not lend out during $t = 3$ is m_6^i. Bank l holds (scaled) precautionary balances at $t = 3$ of

$$\begin{aligned} \frac{m_6^l}{\overline{p}^l + \overline{p}^k} &= \frac{\overline{P}}{L(\overline{p}^l + \overline{p}^k)}(1 - \frac{2R_1^b}{R_6^w}) \\ &< (1 - \frac{2R_1^b}{R_6^w}), \end{aligned} \tag{37}$$

which is less than that of bank s, which holds

$$\frac{m_6^s}{\overline{p}^s} = (1 - \frac{2R_1^b}{R_6^w}). \tag{38}$$

Bank i holds fixed precautionary balances at $t = 3$ (and bank l will borrow if necessary to acquire them) regardless of the amount of reserve balances the bank has available to lend at $t = 3$. Hence, the percentage of balances that large or small banks lend increases with their available balances.

Taking the derivative of the left-hand side (right-hand side) of (36) with respect to the left-hand side (right-hand side) of (35) shows that the lending percentage of bank l (s) is concave in its scaled balances, as shown in Figure 1. Figure 1 shows the percentage of $t = 3$ balances lent as a function of scaled balances. The x-intercept of the bank's lending percentage curve gives the bank's precautionary balances, shown to be greater for bank s. The lending curve for bank s lies below that for bank l, showing that bank s lends a lower percentage of its balances. The lending percentage increases for bank s and l with scaled balances, and the difference of lending percentage between bank s and l decreases with scaled balances.

Rewriting (37) and (38) as

$$R_6^w(\frac{\overline{P} - M_6^l}{2\overline{P}}) = R_3^s \tag{39a}$$

$$R_6^w(\frac{\overline{p}^s - m_6^s}{2\overline{p}^s}) = R_3^s, \tag{39b}$$

respectively, shows that these precautionary balances equalize the expected marginal cost of having to borrow from the discount window due to $t = 6$ shocks R_6^w times the probability of discount window borrowing, with the marginal opportunity cost $R_3^s = R_1^b$ of holding excess precautionary balances at $t = 3$. When $R_1^b = \frac{1}{2}R_6^w$, banks hold zero precautionary balances to give a one-half probability of borrowing at the discount window with a one-half probability of holding excess $t = 3$ precautionary balances. When $R_1^b < \frac{1}{2}R_6^w$, banks hold strictly positive precautionary balances since the cost of excess balances is less than the cost of the discount window. Bank s holds greater scaled precautionary balances because it cannot borrow at $t = 6$. Bank l can borrow from other large banks, so it only has to borrow at the discount window if the aggregate shock to large banks at $t = 6$ is greater than the aggregate balances held. This is why (39a) is written with the probability of overdraft of large banks in aggregate as a factor, whereas (39b) is written with the probability of overdraft of an individual small bank.

These precautionary balance and lending percentage results are derived assuming that large banks hold equal balances at the end of $t = 3$. However, large banks are indifferent to the relative balances held among themselves. The rate R_3^k at which they trade among themselves at $t = 3$ is equal to the expected rate they trade at $t = 6$. If there were a cost of trading, they would trade less at $t = 3$, which could possibly show that they lend a lower percentage of balances than small banks lend. However, if large banks were slightly risk averse, or if there were any trading frictions at $t = 6$, they would strictly prefer this amount of trading.

We also examine lending by large banks at $t = 6$. The percentage of available balances that is lent is

$$\frac{f_6^k}{m_6^l - p_6^l - p_6^k} = \frac{m_6^l - p_6^l - p_6^k - \frac{1}{L}(M_6^l - P_6^l)}{m_6^l - p_6^l - p_6^k}.$$

For $W_6^l = 0$, this is less than one since $M_6^l - P_6^l \geq 0$. Since there are excess balances, banks do not lend them all. The fed funds rate R_6^k is zero. As reserve balances increase for bank l, the percentage lent increases toward one. This **predicts empirically** that there is a lower lending percentage when the fed funds rate is lower. For $W_6^l > 0$, $M_6^l - P_6^l < 0$, so the lending percentage is actually greater than one. This is because we assume large banks borrow equally from the discount window. Anticipating this, banks who need the least amount (or zero) borrowing at the discount window lend to others at the fed funds rate of $R_6^k = R_6^w$. A more natural assumption may be that banks with $m_6^l - p_6^l - p_6^k \geq 0$ do not borrow from the discount window, and only banks with $m_6^l - p_6^l - p_6^k < 0$ do borrow from the discount window. This still implies that banks with available balances lend all of them at a rate of $R_6^k = R_6^w$. This **predicts empirically** that there is a higher ratio of available balances lent when the fed funds rate is high.

The model also gives more general implications when there is any market friction that prevents a random positive epsilon amount of reserves from being tradable efficiently at the end of the day, such that the segment of the market that is trading at the end of the day is always in aggregate long or short of reserves. If this segment trades efficiently, then R_6^k is either zero or R_6^W. Greater end-of-day rate volatility implies greater market efficiency given that the full market does not trade. This also holds true if the random long or short for the market is due to "misses" by the Fed's open market operations desk that targets the supply of reserves in the market and if this "miss" information is only revealed throughout the day.

The average (or expected) amount of discount window borrowing, scaled for size, is larger for small banks than for large banks. For bank s,

$$\begin{aligned} E[\frac{w_6^s}{\overline{p}^s}] &= \left(\frac{p_3^s + f_3^s - m_3^s + \overline{p}^s}{2\overline{p}^s}\right)^2 \\ &= \left(\frac{R_1^b}{R_6^w}\right)^2, \end{aligned}$$

found by substituting for $E[w_6^s]$ from (18) and then for f_3^s from (33), whereas for bank l,

$$\begin{aligned} E[\frac{w_6^l}{\overline{p}^l + \overline{p}^k}] &= E[\frac{(-M_6^l + P_6^l)^+}{L(\overline{p}^l + \overline{p}^k)}] \\ &= \frac{1}{L(\overline{p}^l + \overline{p}^k)} \int_{-\overline{P}}^{-M_6^l} (-M_6^l + P_6^l) \frac{1}{2\overline{P}} dP_6^l \\ &= \left(\frac{\gamma^l \overline{p}^l}{L(\overline{p}^l + \overline{p}^k)}\right) \left(\frac{R_1^b}{R_6^w}\right)^2 \\ &< \left(\frac{R_1^b}{R_6^w}\right)^2. \end{aligned}$$

An **empirical prediction** is that discount window borrowing for small banks should be less correlated and occur more often than for large banks.

The average amount of nonborrowed reserves held overnight, scaled for size, is simply equal to m_6^i, the precautionary reserves held at $t = 3$, since banks' shocks (and large banks' fed funds lending) is zero on average at $t = 6$. Thus the scaled amount of nonborrowed reserves is also larger for small banks than large banks. For bank s,

$$\begin{aligned} E[\frac{m_9^s - w_6^s}{\overline{p}^s}] &= \frac{m_6^s}{\overline{p}^s} \\ &= (1 - \frac{2R_1^b}{R_6^w}), \end{aligned} \tag{40}$$

whereas for bank l,

$$\begin{aligned} E[\frac{m_9^l - w_6^l}{\overline{p}^l + \overline{p}^k}] &= \frac{m_6^l}{\overline{p}^l + \overline{p}^k} \\ &= \frac{\overline{P}}{L(\overline{p}^l + \overline{p}^k)}(1 - \frac{2R_1^b}{R_6^w}) \\ &< (1 - \frac{2R_1^b}{R_6^w}). \end{aligned} \tag{41}$$

Note that while we include the shock size $\overline{p}^k$ for payments between large banks, all results hold for $\overline{p}^k = 0$. The term $\overline{p}^k$ shows that the results hold even more strongly as the amount of payments shocks among large banks increases.

The clean balances held by banks from (8) is

$$\begin{aligned} m_3^s &= m_6^s + p_3^s + f_3 \\ &> \overline{p}^s(1 - \frac{2R_1^b}{R_6^w}) + \overline{p}^s, \end{aligned}$$

where the second line is from (29) and (38). The first term of the second line is the $t = 3$ precautionary balances of bank s. The second term is the bank's pre- $t = 3$ precautionary balances to self-insure against p_3^s. Any excess $f_3^s = m_3^s - m_6^s - p_3^s$ is lent at $t = 3$. Thus, bank s always lends a strictly positive amount , even when it ends up borrowing at the discount window at day's end. The clean balances held by bank l is shown by (28) to be negative. In expectation, bank l rolls-over overnight fed funds borrowing every day to hold $t = 3$ precautionary balances during the day and positive balances overnight. Since bank s has to choose its lending before $t = 6$ shocks, it has to lend every day, whereas bank l can borrow on the aggregate market after $t = 6$ shocks, which explains why aggregate fed funds lending (27) from small to large banks is strictly positive

$$F_3^s = S\overline{p}^s - \overline{P} > 0.$$

The model offers a partial explanation for the large amount of interbank lending relative to bank reserves. The interbank market lends for an overnight term multiples of the amount of aggregate reserve balances held by banks. At first, this phenomena appears to imply that banks must lend the same funds multiple times among banks. However, this model offers a different explanation. In this model, large banks have negative clean balances, $M_3^l < 0$, and rely on borrowing from small banks to achieve non-negative overnight reserves. The amount of funds lent F_3^s may exceed the net supply of reserve balances $M_3^s + M_3^l$, even if there is no relending of reserves. The model also explains why fed funds lending that acts as a large source of financing from small to large banks is primarily of overnight term. Since the lending is a way for small banks to self-insure against daily shocks, the small banks require daily repayment for its potential liquidity needs.

The aggregate amount of clean balances equals the aggregate amount of nonborrowed reserves, and also equals the aggregate amount of $t = 3$ precautionary balances:

$$\begin{aligned} M_3^l + M_3^s &= (M_9^l - W_6^l) + (M_9^s - W_6^s) \\ &= M_6^l + M_6^s, \end{aligned}$$

found by substituting (41) and (40) into the right-hand side of (24). In aggregate, the only purpose for reserves is for precautionary reasons at $t = 3$, because the aggregate pre-$t = 3$ precautionary balances held by small banks that are not used for $t = 3$ shocks are lent to large banks. Anticipating this lending, large banks hold negative clean balances. Aggregate reserves can also be interpreted in the context of an interest rate corridor, with a deposit facility rate of zero and a lending facility rate of R_6^w. If $R_3^s = \frac{1}{2} R_6^w$, (24) shows aggregate reserves equal zero. The marginal opportunity cost depositing excess reserves and borrowing needed reserves are equal since banks have a one-half probability of either occurring. As R_1^b decreases below the corridor midpoint, overnight shortages are costlier than overnight excesses, so aggregate reserves increase.

6 Further Empirical Predictions and Institutional Questions

1) Is there evidence about frictions to limits on small banks' clean balance sizes that can be used as precautionary reserves?

1a) In particular, is there a limit to how much small banks can lend to large banks, perhaps based on large banks ability to repay the fed funds in the morning and have potential large intraday overdrafts before being able to reborrow the fed funds in the afternoon?

1b) How often does a small or large bank who has lent fed funds during the day ever end up needing to borrow at the discount window overnight? This would indicate the bank (especially if it's small banks but also for large banks) has a risk of late day payment shocks, which it has to hold precautionary balances against and has to balance that risk against lending more funds out earlier in the day.

2) Supply and demand of fed funds are perfectly elastic at $t = 6$, yet R_6^k is very volatile due to volatile up and down shifts in supply and demand curves, while supply and demand are less elastic at $t = 3$, and R_3^k is not volatile because supply and demand curves do not shift up and down. But the curves do shift left and right, so the volume of fed funds is very volatile at $t = 3$. The level of volatility at $t = 6$ is unclear but apparently could be high or low with perfectly elastic supply and demand curves shifting up and down. Since $R_3^k = E[R_6^k]$, are there any other implications for supply and demand curves?

Are there empirical implications regarding that f_3^s is determined, but that $f_3^l + f_3^k$ are not, only F_3^l is? Large banks are individually indifferent between borrowing/lending at $t = 3$ versus at $t = 6$; it only matters in aggregate. Large banks would strictly prefer to have equal balances at $t = 3$ to the extent (outside the model) they are risk averse or their trading costs are greater at $t = 6$.

3) Does the open market operations desk achieve its exact targeted rate for its repos in the morning? How do they know if fed funds rate misses are due to their miscalculation of daily reserve supply in the market, or if there are demand shifts or trading frictions?

7 Extensions

The results may give insight beyond small banks to medium and large banks that are not market makers of fed funds, require brokers, or have other trading frictions, giving rise to similar qualitative results. In particular, the model shows how trading frictions in lending versus borrowing and in earlier versus later periods give different implications.

I believe we can show that the equilibria for constraint (3) not binding are optimal. The reason for the suboptimal outcome is because bank s cannot trade at $t = 6$. Given that, bank s is indifferent to self-insuring at $t = 3$. An extension would be to consider additional frictions for which constraint (3) binds. However, the results in the model appear to explain many stylized facts and so constraint (3) no binding may be approximately correct. Moreover, analyzing constraint (3) binding has been difficult to try to solve and interpret analytically. With constraint (3) binding, p_3^s shocks should effect the realization of R_3^s and the determination of m_3^i.

Assuming m_3^i or b_1^i are small enough relative to $\overline{p}^i$ would imply constraint (3) binds. But other papers in the literature such as Ennis and Weinberg (2007) assume as we currently do that banks have the feasibility to cover their shocks, and instead focus on the problem of the optimal choice of whether to hold enough reserves to cover shocks at the expense of ex-post excess balances. This questions what is the friction that would drive the restriction on the size of m_3^i. There are many possible frictions that intuitively could drive this restriction, but here we should be careful what path we take. The ability for small banks to hold large clean balances is key to the result that small banks lend large amounts on average to large banks overnight. In order to study this result carefully, we should look to the data or institutional facts for what friction ultimately limits the extent of this result by limiting clean balance holdings m_3^s by small banks. Another approach would be to model the balance sheet of the banks more fully to understand how b_1^i is chosen versus other assets (loans and money) and relative to the size of potential shocks p_t. Looking at the balance sheet and size of b_1^i could also allow for examining collateral available for discount window loans. If banks can run out of collateral, they cannot borrow at the discount window and are overdraft overnight, implying nonlinear rates on discount window borrowing, which should imply that R_3^s varies with p_3^s. However, given the broad allowances for collateral, is it likely binding?

An interesting reason to look at constraint (3) binding is to examine the distributional effects of shocks among small banks. If two small banks held medium balances at $t = 3$, they would lend in aggregate less to large banks than if one small bank held large balances and another small balances. This should imply that i) small banks would hold even greater precautionary balances at $t = 0$ and perhaps at $t = 3$ than at present, ii) small banks would perhaps show even greater reluctance to lend at $t = 3$ than large banks, and iii) the fed funds rate would vary at $t = 3$ with payments shocks. $E[R_3^s]$ may even be less than R_1^b, reflecting the aggregate excess supply F_3^s is strictly positive.

TO DO: Banks choose lowercase variables, then caps are established in equilibrium. Check through all cap variables, particularly with bank l derivatives, to see that choice then equilibrium conditions imposed, and that banks take caps as given. For example, $\frac{W_6^l}{L} - w_6^l$ does cancel in equilibrium. Write out () to think this issue out. Interpret what it means that bank l chooses w_6^l and m_3^l but $w_6^l = \frac{W_6^l}{L}$ and $m_3^l = \frac{M^l}{L}$ in equilibrium. Should I keep all $\frac{W_6^l}{L}$ as such (and other bank l and s caps such as M_3^s) instead of writing as w_6^l to be clear? Or use some notation such as $\overline{w}_6^l$ to show choice variable versus equilibrium/given value $w_6^l = \frac{W_6^l}{L}$?

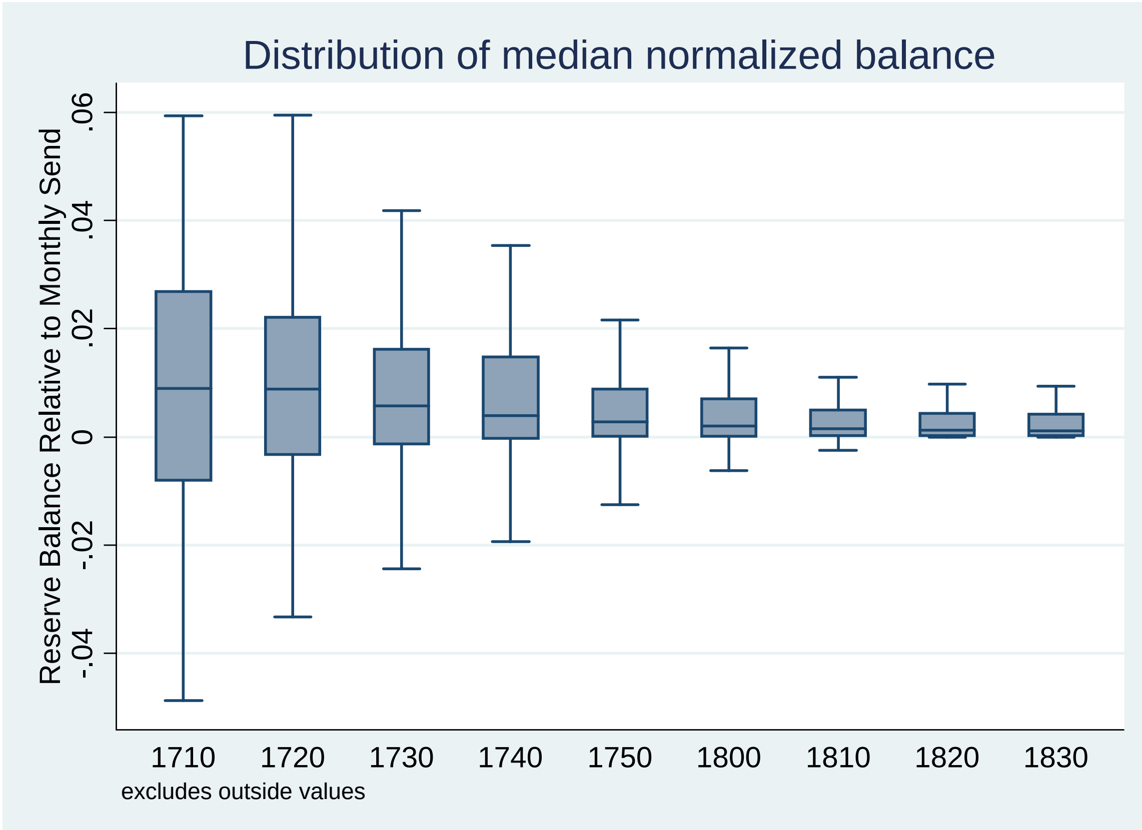

Figure 1: Distribution of reserves across banks over the day. Normalized balance is defined as the actual balance for that bank at that time of day divided by the amount sent by that institution using Fedwire over the month. The x-axis documents time of day for the last 90 minutes of the business day. The graph documents the massive redistribution of reserves which occurs within the top 100 institutions over the last 90 minutes of the day. Note that many institutions (typically the largest) have large negative balances throughout the day, making generous use of intra-day credit from the Federal Reserve, and rely on their ability to unwind these positions through Federal Funds borrowing quickly before the close of Fedwire at 6:30 pm.

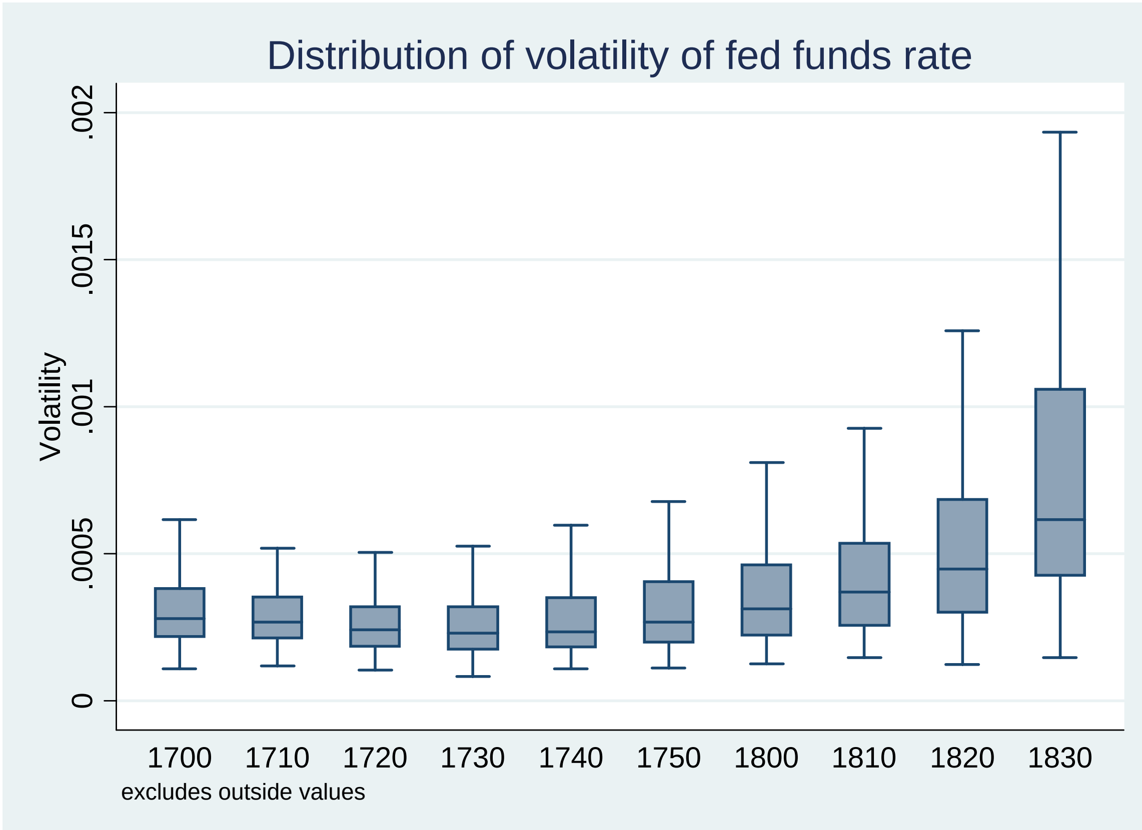

Figure 2. Distribution across days of federal funds interest rate volatility. The graph documents the time-series volatility the interest rate on federal funds loans between banks in the top 100 across the last 90 minutes of the day. The interest rate is a dollar-weighted average of all federal funds loans in a particular minute of the day. The figure illustrates a significant increase in interest rate volatility during the last 60 minutes of the day.

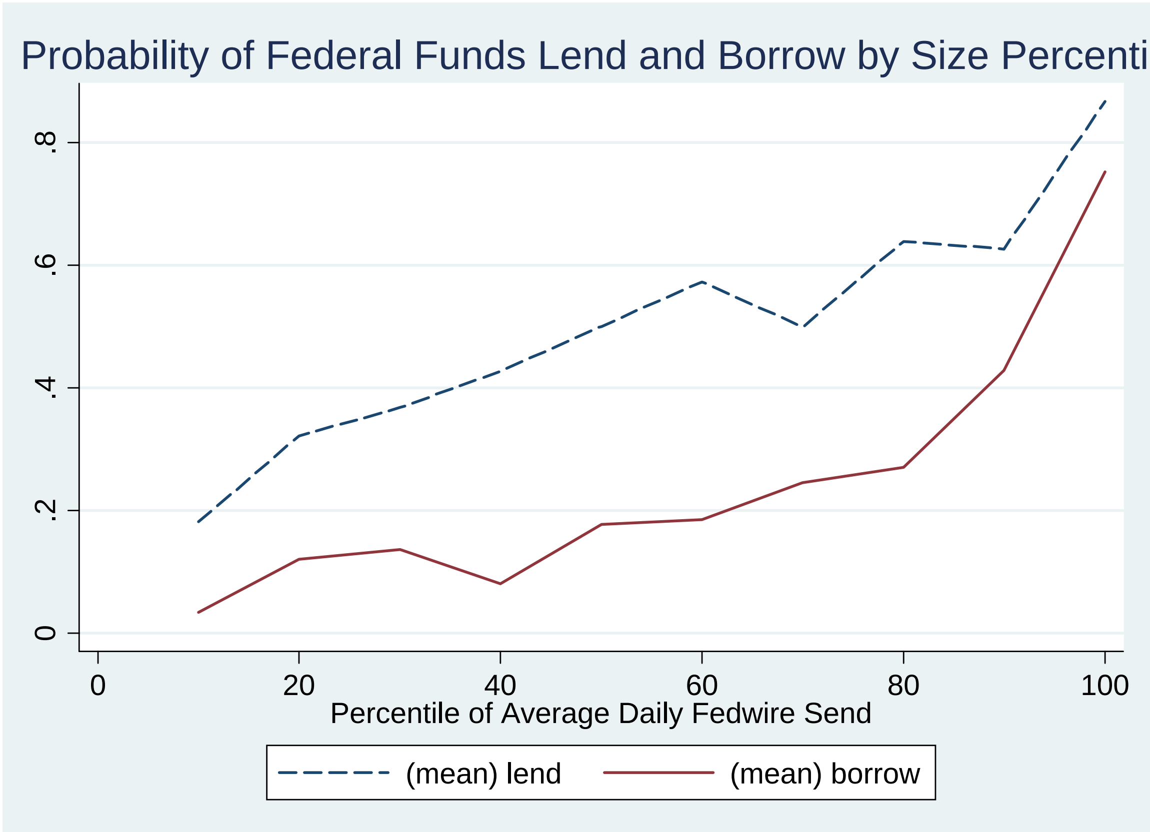

Figure 3: The propensity to borrow and lend across bank size. The graph documents the probability that a bank either borrows or lends in the federal funds market at least once during the day across institution size. Bank size is defined by the percentile of cross-sectional distribution of the average dollar amount sent over Fedwire. The sample is limited to approximately 700 banks which ever borrow or lend during January through February 2007. The picture illustrates that smaller banks are generally less likely to lend and borrow.

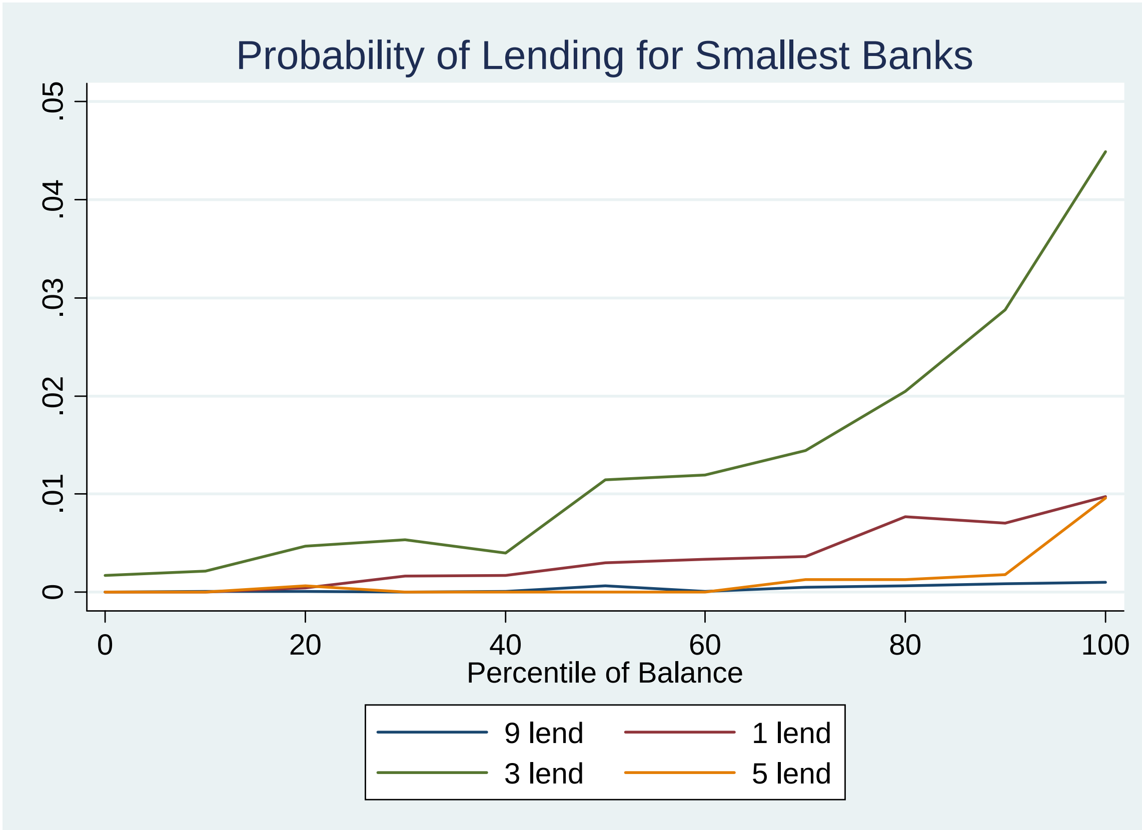

Figure 4: The propensity of small banks to lend. This picture documents the propensity of the smallest decile of banks to lend across the percentile of balance during four different time windows of the day: 9pm-1pm; 1pm-3pm; 3pm-5pm; and 5pm-6:30pm. The percentile of balance is measured for each institution at a given time of day across all days. The graph illustrates that the propensity of small banks to lend is maximized during the 3pm-5pm window, and that small banks are reluctant to lend even when hit with favorable liquidity shocks. At the highest percentile of reserve balance, small banks only lend at a frequency of about 4.5%.

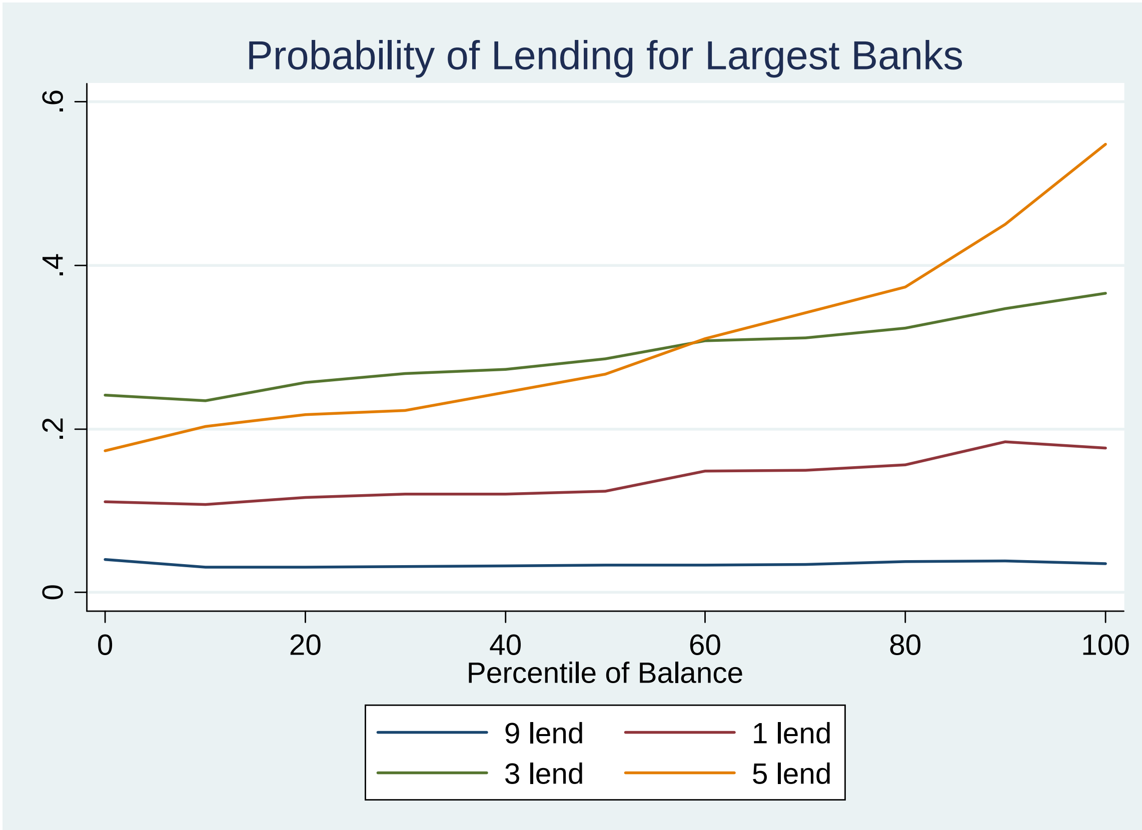

Figure 5: The propensity of large banks to lend. This picture documents the propensity of the largest decile of banks to lend across the percentile of balance during four different time windows of the day: 9pm-1pm; 1pm-3pm; 3pm-5pm; and 5pm-6:30pm. The percentile of balance is measured for each institution at a given time of day across all days. The graph illustrates that the propensity of large banks to lend is maximized during the 5pm-6:30pm window when balances are high. Moreover, large banks appear eager to lend during the late period even when hit with adverse liquidity shocks. At the lowest percentile of reserve balance, large banks still lend at a frequency of about 18%.

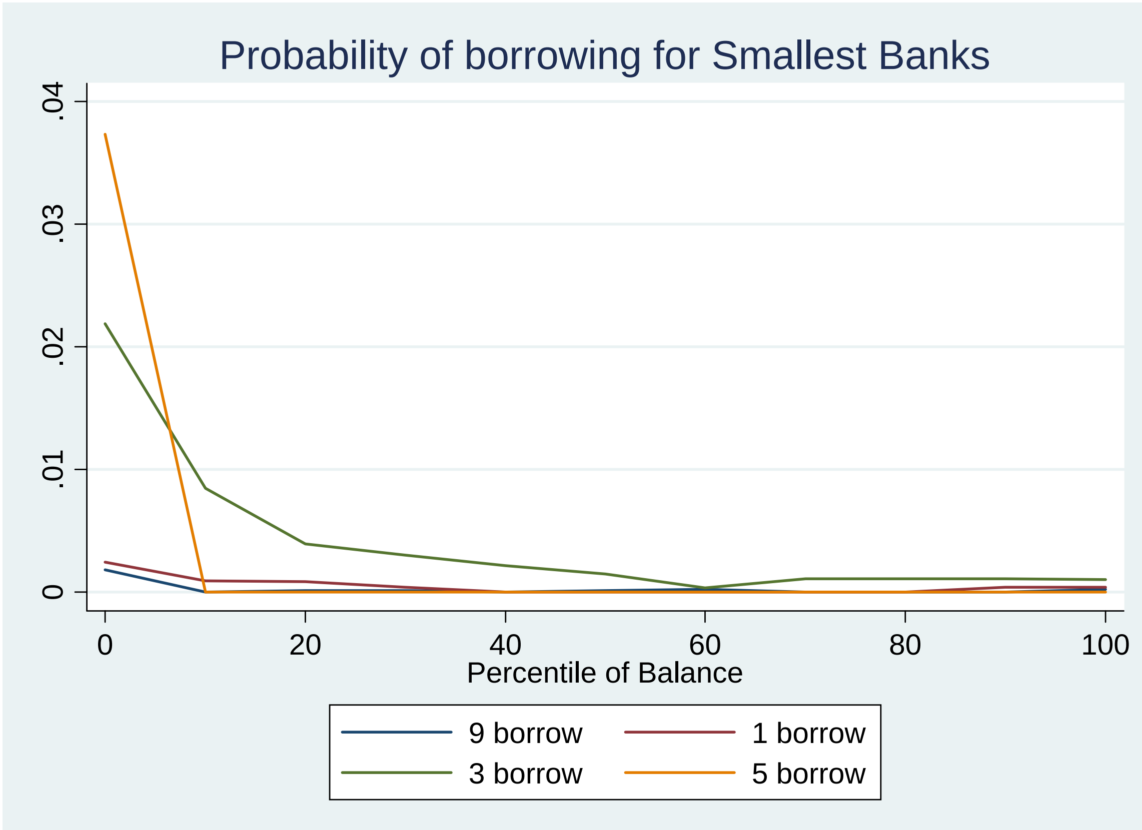

Figure 6: The propensity of small banks to borrow. This picture documents the propensity for the smallest decile of banks to borrow across the percentile of balance during four different time windows of the day: 9pm-1pm; 1pm-3pm; 3pm-5pm; and 5pm-6:30pm. The percentile of balance is measured for each institution at a given time of day across all days. The graph illustrates that the propensity of small banks to borrow is maximized during the 5pm-6:30pm window in the face of the most adverse liquidity shock, but that this figure is less then 4 percent. In other words, the vast majority of small banks survive the most adverse liquidity shocks by holding a high reserve balance.

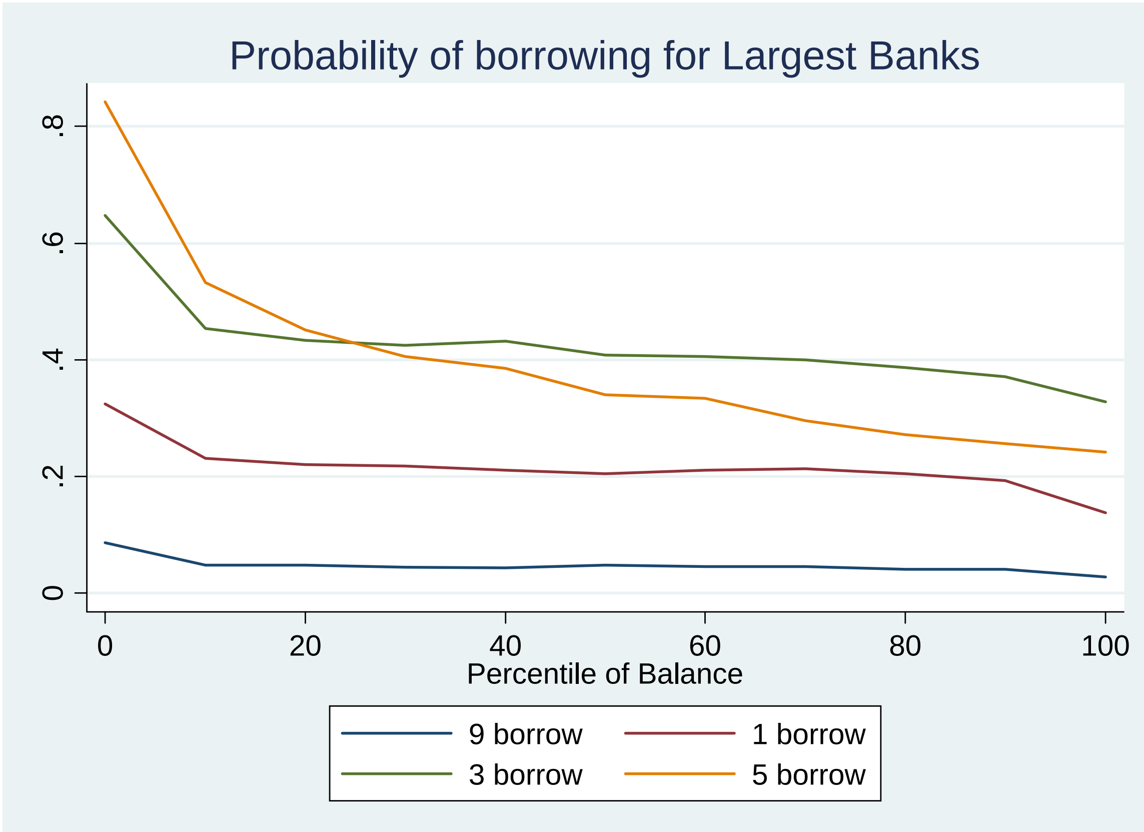

Figure 7: The propensity of large banks to borrow. This picture documents the propensity for the largest decile of banks to borrow across the percentile of balance during four different time windows of the day: 9pm-1pm; 1pm-3pm; 3pm-5pm; and 5pm-6:30pm. The percentile of balance is measured for each institution at a given time of day across all days. The graph illustrates that the propensity of small banks to borrow is maximized during the 5pm-6:30pm window in the face of the most adverse liquidity shock, where this figure is most than 80 percent. In other words, large banks rely extensively on the federal funds market in order to manage their reserve balance.