

Regional Default and Migration in General Equilibrium*

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PRELIMINARY

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Abstract

Bonds are an important source of funding for cities. As financing for big budget construction projects or surprise shortfalls in tax revenue, bonds help smooth tax burden across time. There is good reason for this smoothing: if residents feel their tax burden is excessive, they can migrate. The ability of residents to migrate significantly hampers the ability of local governments to raise taxes, and, in the extreme, can lead to default. We document the relationship between debt, default, and migration across cities/states in the U.S. We then construct an islands model that captures these facts while allowing for endogenous migration, taxation, debt issuance, and default. We find that low productivity and low population push cities closer to default. Bailouts can prevent defaults while helping cities weather difficult times.

1 Introduction

On October 1, 2013, the city of Detroit defaulted on more than \$600 million of its general obligation bonds. This was the largest municipal bankruptcy in the U.S. history. But Detroit is just one example of a growing number of municipalities or regional entities facing financial headwinds in the aftermath of the Great Recession (Puerto Rico is a more recent example of this type of struggle). Aging population, rising pensions, and higher borrowing costs will likely put more pressure on cities' finances. As a consequence, Detroit may be just the tip of the municipal iceberg with more defaults down the road. In this paper, we study default by municipalities. To this end, we first document features of the data. We then propose a rich model capable of capturing some of these facts.

Detroit's recent experience raises several questions regarding cities' (financial) struggles. For example, its population has been shrinking since the 1950's. In the last decade alone, the city

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has lost roughly 25% of its inhabitants. However, Detroit's debt obligations did not go with it. In fact, their general obligations debt increased 40%, and this happened despite an *increase* in tax revenue, which was assessed on a smaller tax base. In 2005, Detroit's expenditures experienced a substantial spike, increasing 45%.

What drives a city to default? In the case of Detroit, it would appear to be a slow, steady decline in productivity followed by a large negative shock (the financial crisis). Then again, Detroit's large increase in expenditures eight years prior to default also contributed. Apart from why cities default, we want to understand how cities should manage large stocks of debt. Should they slash spending, raise taxes, default, or do some combination? Additionally, if one so desired, how should one bail them out in an unanticipated fashion, and how does that change if giving bailouts sets a precedent? Lastly, how does migration affects these answers, and is competition among cities welfare improving?

As evidenced by the drastic reduction in Detroit's population prior to its default, migration is a constraint on fiscal policy for cities. While many of the issues we look at have been addressed in the emerging economy literature ([Aguilar and Amador, 2013](#)), international migration is, from the point of view of workers, far more costly and time consuming than intercity migration. Moreover, migration matters because it changes labor force and hence revenues a city collects from labor taxes. Additionally, more productive workers are presumably more likely to migrate leaving the city with a population earning lower wages and demanding a higher provision of public services.¹ As a result, when studying defaults at the city level, we must factor in the impact that migration may have on the decision to default

In our work, we use the term municipality in a broad sense. It includes cities and counties as well. Furthermore, due to data limitations at local level, we will frequently rely on statistics at the state level as well. Hence, a municipality in our framework also encompasses states. The unifying theme among these *regional entities* is that they can issue sovereign debt, tax their residents/workers, and provide amenities (such as highways or parks) via public spending. To frame the discussion to follow, it is worth describing some facts about municipalities in the U.S.:

1. There are two types of municipal bonds, General Obligation (GO) bonds and non-GO bonds. According to Moody's report on Municipal Bonds, the difference between these two bonds is that a GO bond is backed "by the full faith credit and taxing power of the issuer."²
2. While both broadly used, GO and non-GO bonds have significantly different default rates.
3. The overall default rate is very low: Since 1970, there have only been 73 defaults.
4. The default rate has risen since 2008.
5. Recovery rates on defaulted debt range from 40% to 75% or even higher.

¹[Armenter and Ortega \(2010\)](#) examine to what extent government policy is undermined by migration in the context of workers with heterogeneous skills.

²Moody's Investor Service, US Municipal Bond Defaults and Recoveries, 1970-2012.

6. Aggregate municipal debt to Gross State Product (GSP) ratios are around 10% but can be as high as 20%.
7. There is substantial migration across municipalities.³

To capture these facts, our framework features a continuum of islands (in the spirit of [Lucas, 1972](#)). Each island has a planner (the local government) who chooses tax rates and issues debt benevolently and households who work for a firm, pay taxes, and receive transfers. Two critical features of our model are that 1) residents of an island can choose to leave it; and 2) sovereign islands can default on their obligations. We characterize the migration and labor-leisure decisions of households. In characterizing the island planner's problem, we establish that the sovereign's problem can be setup recursively and, importantly, is a contraction mapping taking bond prices and migration decisions as given.

To determine what drives cities to default, we consider default episodes along the lines of [Mendoza and Yue \(2012\)](#). Examining the default episodes shows defaults are triggered by long-run declines in productivity with a productivity *increase* in the period of default. As productivity declines below its long-run average, the planner borrows against expected future higher productivity. With the accumulated debt, the sovereign finds it optimal to default in response to a positive productivity shock because the associated higher wages cushion the blow of the default. Defaults themselves result in a sharp decline in government spending and increase in tax rates, but this result hinges on our assumption that default costs are proportional to tax revenue.

Our results suggest that the best bailout of a city, spending only enough resources to avoid a default, is to keep government services high and reduce debt slowly over a large number of years. While, absent a bailout, government services are slashed by as much as 50%, we find household welfare is not substantially affected because local tax revenue. This is because in our baseline calibration, government spending enters separably in the utility function. Since local tax revenue is small relative to consumption, the calibration then has households deriving only a small amount of utility from government services. We suspect that if government spending is non-separable this does not hold, and we are currently investigating this case.

One of our main findings is that migration effectively makes the planner more impatient. When thinking about how much debt to issue, migration essentially places a lower bound on household utility: If a city become an unpleasant place to live, they can move and receive some utility that is outside the planner's control. Consequently, by issuing lots of debt in the short run, consumption and utility can be substantially increased in the short with utility not much worse in the long run with households moving as taxes increase. This result hinges on our assumption that the planner seeks to maximize the welfare of households currently on the island. However, this is exactly what a political equilibrium should deliver as all the households currently on the island have the same objective function as the planner.

We also find migration makes the planner raise taxes slowly when trying to pay off debt. Without migration, the planner aggressively increases taxes to pay off debt. With migration, the

³We provide a lower bound by looking at interstate migration.

planner increases taxes much more slowly. While this is intuitive, the mechanism at work is a bit unintuitive. Our model’s timing has migration decisions made before the planner chooses taxes. Consequently, in any given period, the tax base is fixed. So, the planner could drastically raise taxes without eroding the tax base. Moreover, with lower debt in the future, the island would be more attractive and the tax base would actually increase. While this is feasible with migration, the planner’s effective impatience makes this policy—which sacrifices current consumption for higher future consumption—suboptimal. Our findings on the role of migration are reminiscent of [Blanchard and Katz \(1992\)](#), who find that adjustments to shocks to regional labor markets are predominantly driven by intranational migration.

Our work touches several areas in economics. Because we study intranational migration and municipalities, we are close to the literature on regional economics. The classical work by [Roback \(1982\)](#) has equilibrium wages and rents determined by an indifference condition. The indifference condition says that each island must deliver the same amount of utility since there is no moving costs and households can go to whatever island they want. However, that each island delivers the same utility means that there is absolutely no dynamics and no reason to issue debt, tax, or not tax (endogenously, one could have exogenous taxes however).⁴ Because we explicitly model the decision to default by municipalities, our paper is closely related to the large literature on sovereign default, which includes [Arellano \(2008\)](#), [Hatchondo and Martinez \(2009\)](#), [Chatterjee and Eyigungor \(2012\)](#), and [Mendoza and Yue \(2012\)](#), among others. We contribute to this literature by studying the interaction between immigration and taxation and the decision to default. Finally, since we assume the government cannot commit to future policies (like the decision to default and tax rates), our work is also tied to the literature on time-consistent policies, which includes [Krusell and Ríos-Rull \(1999\)](#) and [Klein, Krusell, and Ríos-Rull \(2008\)](#).

The rest of the paper is organized as follows. The next section presents some relevant features about cities/municipalities/states in the U.S. These salient features guide us in building our model in Section 3. Some theoretical results about the solution of our model are in Section 4. The calibration and results are reported in Sections 5 and 6. The final section provides some concluding remarks.

Except for the part on default, our main assumptions behind each city (or region) are broadly consistent with a large part of the literature in regional economics. Crucially, this literature often abstracts from important dimensions such capital accumulation and housing, not because they are uninteresting but for tractability (see for example, ⁵).

In our framework, housing ownership complicates the analysis because it creates two tiers of heterogeneity. The first level corresponds to the heterogeneity across cities due to workers holding different stocks of housing. When a worker migrates, it is important to know from what city the workers is moving. This is needed because housing determines the workers’ wealth. As a result, migration creates heterogeneity within cities. Namely, housing stock differs across

⁴[Armenter and Ortega \(2010\)](#) and [Coen-Pirani \(2010\)](#) provide a more recent discussion of migration in the U.S. [Alvarez and Veracierto \(2000\)](#) use Lucas’ island framework to study the impact of labor-market policies on the economy.

⁵[Armenter and Ortega \(2010\)](#), [Coen-Pirani \(2010\)](#), and)

recently relocated workers from those with more years in the city.

Yet, our formulation encompasses cases in which each city has a rental land market. If households consume land in fixed quantities (say one unit) and land supply is inelastic, real wages in our model can be reinterpreted as capturing effective wages. That is, the labor compensation minus the rental cost of land (see, for example, [Ferreyra \(2007\)](#)).

To the best of our knowledge, the introduction of default and hence dynamics into a regional model is novel. We view of the absence of capital/housing as a necessary tradeoff to focus in then very important issue of municipal borrowing.

We propose a very general framework that can be applied to important economic questions beyond those posed in this paper. For instance, a natural application of our framework is the recent European debt crisis. It is not difficult to appreciate the many commonalities that cities/regions in the U.S. share with the peripheral countries in Europe (such as a common currency and relatively easy migration). Another potential application is the study of inequality across cities/states/regions in the country. Finally, we can study tax competition among municipalities. Although this has been extensively studied elsewhere, we bring to the table the dynamics brought about by debt and migration. As will become clear, there is a nontrivial connection between taxes and migration, which makes the transition from one set of taxes to another one a crucial element of fiscal policy.

2 Empirical Features of Debt, Default, and Migration.

This section documents the empirical features of debt, default, productivity, and migration in the U.S. We begin with some case studies, focusing primarily on Detroit. We then look at the history of municipal defaults across the U.S.

2.1 Case Studies

It is worth discussing Detroit as an example of a recent case of a municipal default. To this end, we use data from Detroit's Comprehensive Annual Financial Reports. [Figure 2.1](#) displays the dynamics of population, total revenue, and total expenditures in Detroit before it defaulted (the variables are normalized to 1 in the default year). The figure portrays the picture of a struggling city. Expenditure and revenues were falling with the latter doing it on average at a faster pace. As a consequence, Detroit's deficit (expressed as a fraction of total revenues) averaged 10% in the years leading to default. Furthermore, the pressure on Detroit's finances can be seen as the amount of debt obligations piled up leading to the crisis in 2013. Indeed, between 2000 and 2012 debt grew by 40%. Over the same time period, the population was slowly declining until the financial crisis and GM bankruptcy in 2009 when it experienced a significant drop. In fact, Detroit's population has declined a staggering 60% since its peak in the 1950s of around 1.8 million people.

Plotting the default episodes for more than just Detroit, as is done in [Figure 2.2](#), reveals some stylized facts. First is that defaults are not always preceded by periods of population decline. In

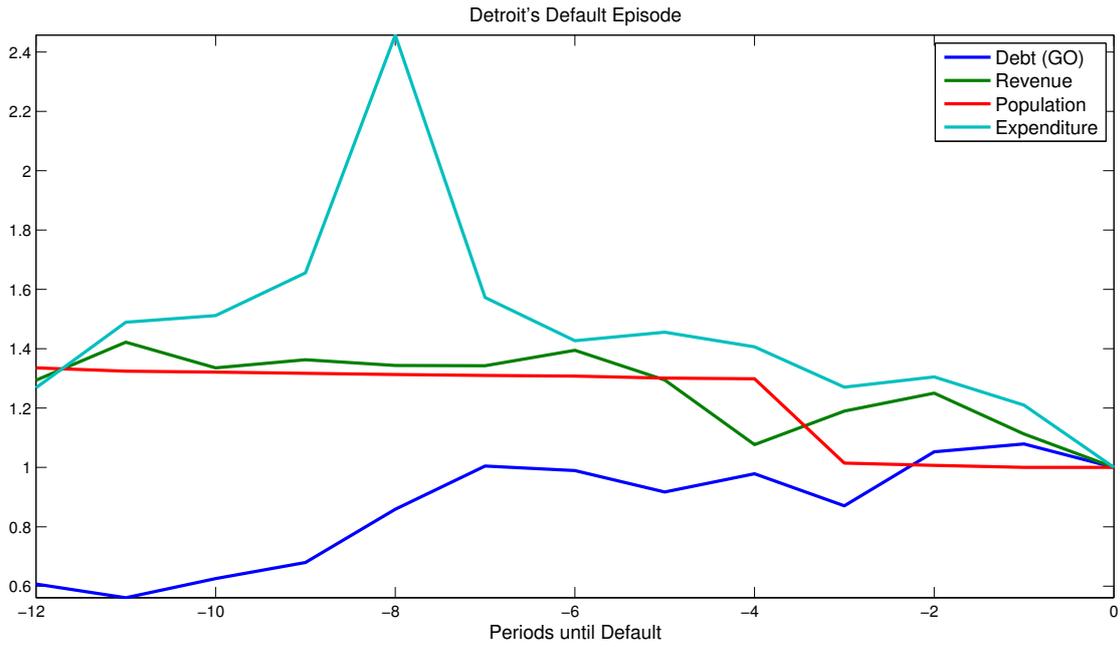


Figure 2.1: Detroit's Path to Default

fact, out of the seven default episodes we consider, three experience population growth, three experience decline, and one is stagnant. Second is that defaults are preceded by increases in tax revenue collection. Last is that, in contrast with Detroit, government expenditures tend to increase prior to a default.

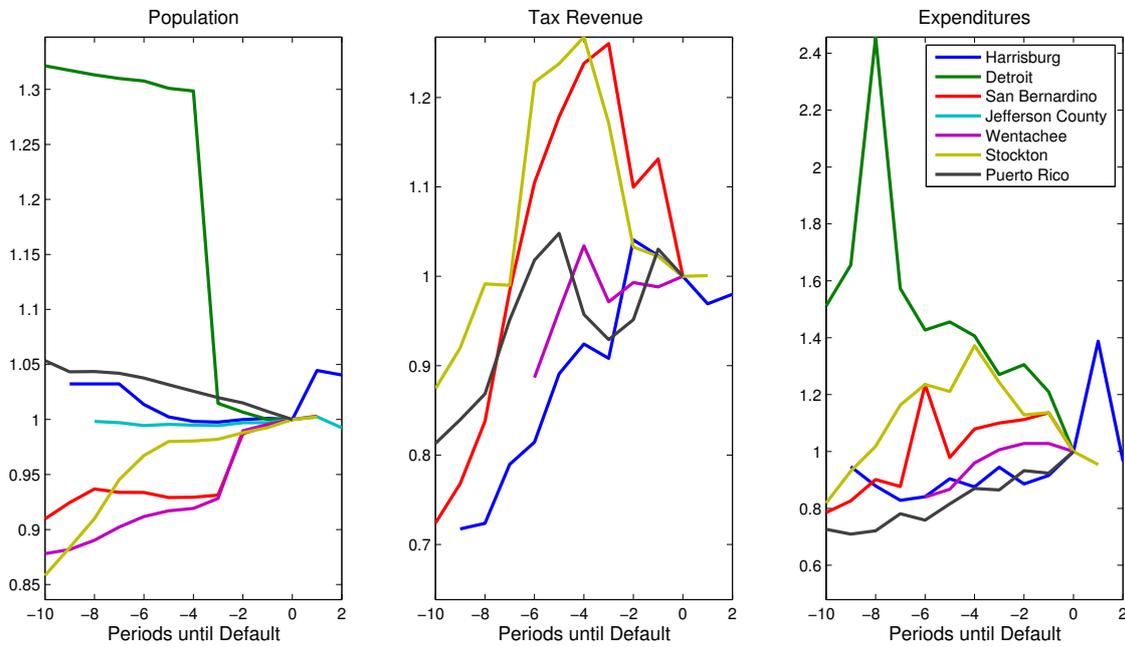


Figure 2.2: Paths to Default

It appears there is not a simple answer as to why a municipality defaults. In some cases,

like Detroit, it appears to have been brought on in by an eroded tax base due to declines in population. In others, population and tax revenues increase, but are matched with increased expenditures.

2.2 Default

For this section, we use data from Moody’s as reported in US Municipal Bond Defaults and Recoveries, 1970-2012. Municipalities issue two types of obligations. Moody’s defines General Obligation (GO) bonds are those obligations backed “by the full faith, credit and taxing power of the issuer”. While some GO bonds have limitations on the implied tax burden, others do not. Roughly half of municipal bonds, however, are non-GO and are not explicitly backed by taxing power. These bonds are meant to be repaid through proceeds from a revenue-producing venture like a utility facility, new tolls on a road or a bridge.

While municipalities have equally resorted to GO and non-GO bonds, (not surprisingly) a large fraction of defaults come from the second type of bonds. According to Moody’s, there have been 73 municipal bond defaults (of bonds rated by Moody’s) since 1970. Of those defaults, 93% (68 defaults) has been on non-backed debt (non-GO debt). These figures translates into a default rate of roughly one municipality defaulting every 4.3 years. Of the five GO defaults, three have occurred since 2008 (or a default rate of one municipality per year). These very low default rates imply that municipal bonds carry a very low interest rate.

More into the details, Moody’s reports that the twelve-month moving average default rate on municipalities with speculative grade (rating Ba1 and below) has doubled since the onset of the Great Recession (going from 1% during 1991-2007 to 2% during 2008-2012). The default rate is low compared to that from corporate defaults and sovereign default. Figure 2.2 shows the default rates for both municipalities and corporations over the past 40 years. Although the default rate of municipalities has declined in 2012, the lingering consequences of the recession will most likely keep the rates at values above their historical average.

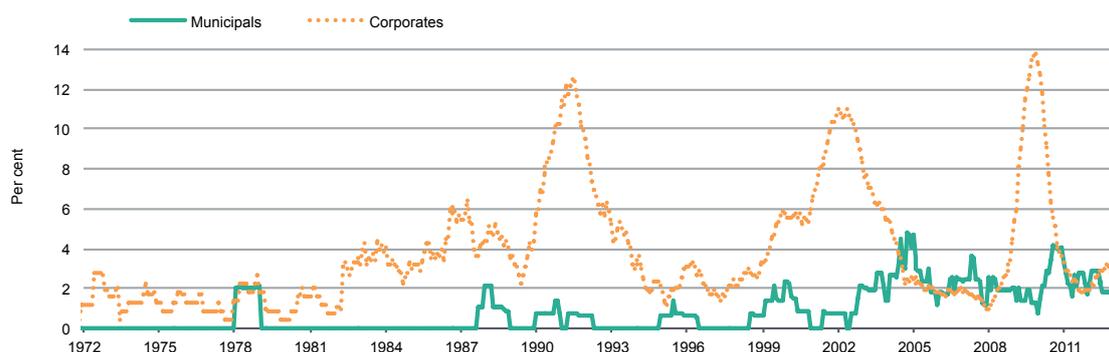


Figure 2.3: Default Rates Municipalities and Corporations (Source: Moody’s)

How much is recovered following a municipal default and how long does settlement take? The evidence varies over time. For example, [Hempel \(1971\)](#) found that in the 1930s municipal

defaults had a recovery rate of at least 84 percent. Yet recovery on defaulted obligations took several years. In table 1, we report more recent default events. These have exhibited on average lower recovery rates but vary greatly from as little as 40% to as much as 100%.⁶ There is also a large dispersion in the time to settle with some taking 15 years, most 2-3 years, and some only 3 months.

City / Institution	Date	Recovery*	Note
City of Wenatchee, WA	2012	100%	3 months to resolve.
Choate-Symmes Hospitals, MA	1990	61%	Recovered in 8 months.
Vanceburg, KY	1987	100%	Par+interest received in 1988.
Belfield, ND	1987	55%	Settled in 1991.
Washington Public Power Supply System, WA	1983	40%	
Cleveland, OH	1978	100%	\$14 million, resolved in 1980.
Chesapeake Bay Bridge	1970	?	Exited default after 15 years.

Recovery rates are approximate. Source: Moody's US Municipal Bond Defaults and Recoveries, 1970-2012.

Table 1: Select Municipal Bond Defaults

2.3 Debt

We now provide information on the burden of debt on municipalities using debt of local governments (towns, cities, and counties) as a proxy. The top left panel in Figure 2.4 reports the distribution of the ratio of local debt to state debt in the United States in 2010 (we shall consider state debt momentarily). Although the median (mean) ratio is 1.3 (1.6), local debt is more than twice statewide debt in a quarter of states in the country. Moreover, three states (Nevada, Tennessee, and Texas) have local debt five times larger than state debt (with ratios of 5.3, 5.6, and 5.0 respectively). When compared to output (right top panel), the median (mean) local-debt-to-GSP ratio is about 9.5% (9%), which is still lower than the ratios in emerging countries. For completeness, the bottom panels in Figure 2.4 display the burden of other liabilities at the state level. Clearly, a large source of financial stress inside states is coming from unfunded pensions (on average, it is about 6% of GSP).

According to the National League of Cities, between 20% to 25% of the operating revenue of municipalities comes from state transfers. Hence, state indebtedness should provide an insight into debt at a more disaggregated level. Table 1 reports the average and standard deviation of the (annual figures of) debt-to-Gross State Product (GSP) ratio and the debt service ratio in 2011 and 2012 in the U.S. states.⁷ Figure 2.2 in turn displays the distributions of those ratios. The striking result is that on average states are less indebted than emerging economies. Even for “highly” indebted states, the ratio barely reaches 10% (Massachusetts, 8.4%, and Hawaii,

⁶They may be gravitating towards the average corporate recovery rate of around 50% (according to Moody's).

⁷Moody's defines the debt service ratio as the fraction of “all state net tax-supported debt as a percentage of pledged revenues.”

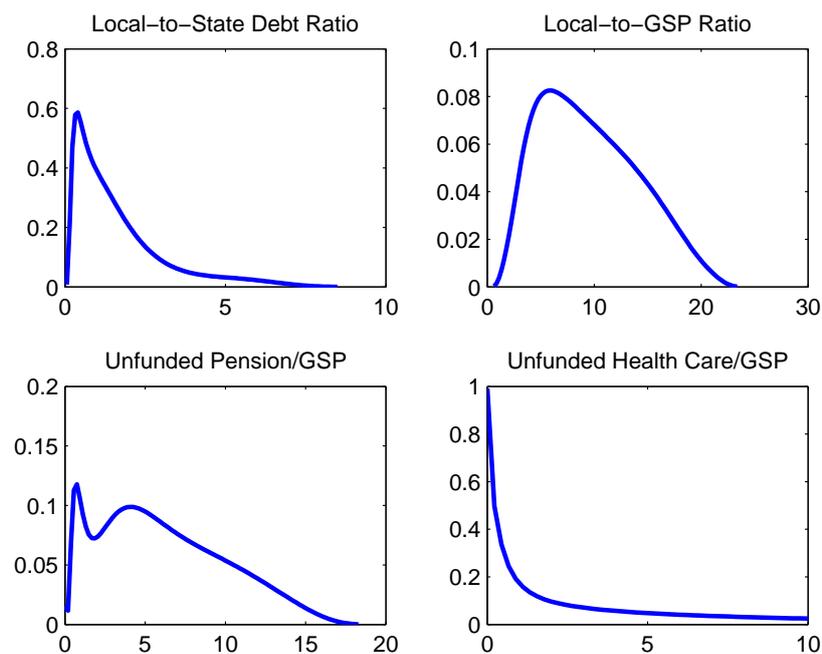


Figure 2.4: Local and State Liabilities

8.8%, have the highest debt/GSP ratios). In contrast, [Mendoza and Yue \(2012\)](#) report that the average debt/GDP ratio in a sample of emerging economies is about 35%.⁸

Annual D/GSP (%)		
	Mean	Std
2011	2.96	2.05
2012	2.92	2.12

Debt Service Ratio (%)		
	Mean	Std
2011	5.29	3.24
2012	5.24	3.20

Table 2: Debt, Income, and Debt Service

⁸For Argentina, the ratio is closer to 100% ([Chatterjee and Eyigungor, 2012](#)).

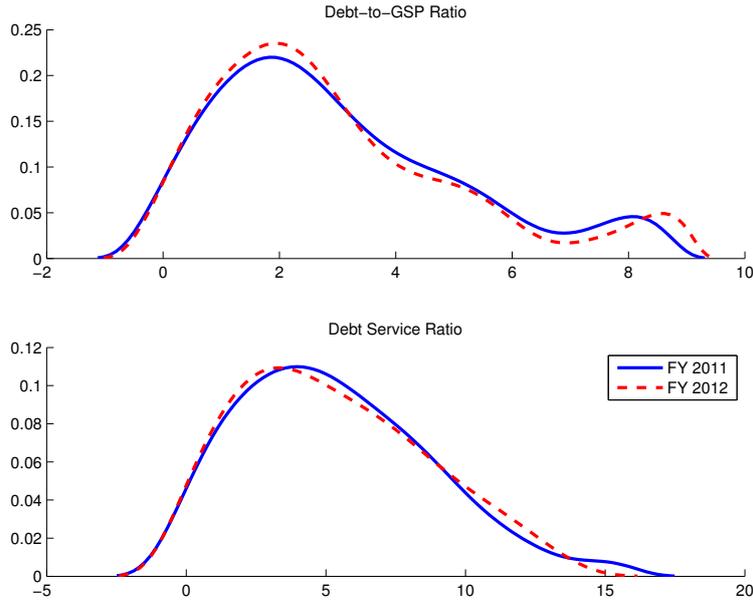


Figure 2.5: Debt, Gross State Product, and Debt Service

2.4 Taxes

An important aspect of municipalities finances is how they fund their expenditures. For this we rely on state data on taxes. Specifically, our measure of tax rate corresponds to the average of the state individual income tax rate for 2010.⁹ The average tax is computed as the average of the rates for each income bracket. For example, Alabama taxes income above 0 USD at 2 percent, income above 500 USD at 4 percent, and income above 3,000 USD at 5 percent. The average tax rate is then 3.7 percent. Table 3 presents some statistics for the average income tax rate across states in the U.S. There are states such as Alaska and Florida that do not charge income taxes. In the other side of the spectrum, Oregon imposes the largest average tax rate among the states.¹⁰ We omit other tax rates (such as corporate) because our model assumes perfect competition and labor is the only productive input.

	Mean	Std Dev	Skew	Min	Max
Tax Rate	4.2	2.1	-0.6	0.0	8.6

Table 3: Tax Rates in the States 2010

⁹As reported by the Tax Foundation: <http://taxfoundation.org/articles/state-individual-income-tax-rates-2000-2013>

¹⁰The tax rates (and the income brackets) are: 5 percent (0 USD), 7 percent (3,050 USD), 9 percent (7,650 USD), 10.8 percent (125,000 USD), and 11 percent (250,000 USD).

2.5 Productivity

How productive are municipalities? We rely on state data as a proxy. Figure 2.6 has the density of annual productivity growth for 2002 - 2007 from [Caliendo, Parro, Rossi-Hansberg, and Sarte \(2014\)](#).¹¹ The mean and standard deviation of productivity are 1.02 and 0.57, respectively. Clearly, productivity varies substantially across states. Productivity will be a key factor in our model since it drives the decisions of migrate and default.

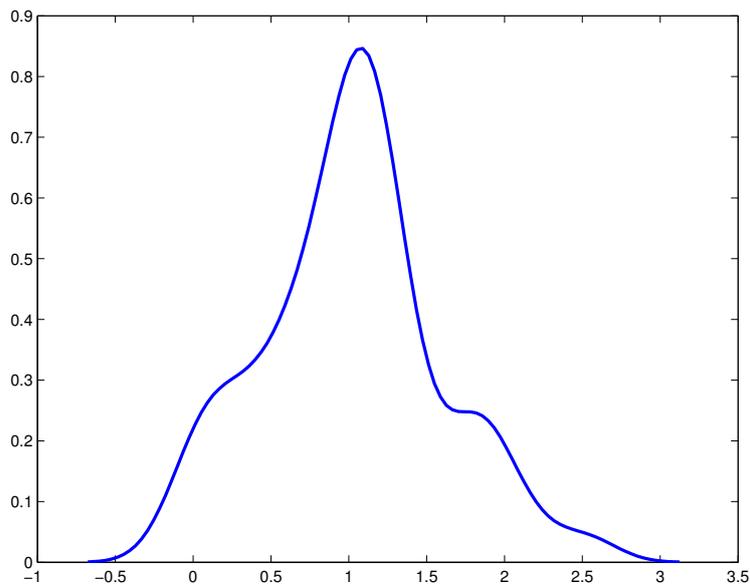


Figure 2.6: State Productivity from [Caliendo et al. \(2014\)](#)

Another measure of productivity comes from the growth rate of output per worker. we collect real gross state product and state employment for the period 1990 - 2011 from the BEA. Table 4 reports statistics for state productivity. The first two rows correspond to the case when we use yearly average productivity for two periods: 1991 - 2011 and 2002 - 2007. This second sample is used to facilitate comparison with the productivity reported in [Caliendo et al. \(2014\)](#) (third row in table 4).

	Mean	Std Dev	Skew	Min	Max
1991 - 2001	1.4	0.6	0.0	-0.8	3.7
2002 - 2007	1.3	0.6	1.1	0.2	3.6
Caliendo et al. (2014)	1.0	0.6	0.2	-0.1	2.5

Table 4: Productivity in the States - Some Statistics

¹¹We thank Lorenzo Caliendo for kindly sharing his productivity series with us.

2.6 Migration

As argued above, a critical difference between municipalities and sovereign countries is that migration is more likely in the former than in the latter. We therefore report data on migration flows based on the 2010 Census. Here, we follow the approach outlined in [Armenter and Ortega \(2010\)](#). Specifically, we compute population inflows and outflows for each state in 2010. The flows are computed using those participants in the Census that reported as living in a different state from a year ago. We consider respondents 25 year old and older.

Figure 2.7 presents the densities for migration flows expressed as percentage of the state population. One can see that an average state saw about 2% of its population migrate (extreme values correspond to the District of Columbia). To put these numbers in context, annual net migration in Argentina for the period 1995 - 2014 was on average a mere -0.34 %. Even during the crisis years 2000 - 2005, net migration in that country barely reached (minus) half of a percentage point. This should not be surprising since intranational migration faces no legal restrictions and the economic cost is minimal compared to international migration. Table 5 presents additional statistics that confirm this.

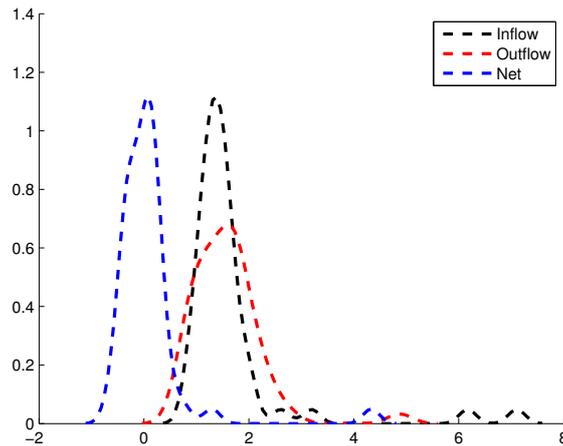


Figure 2.7: Migrations Flow in the U.S.

	Mean	Std Dev	Skew	Min	Max
Inflow	1.65	1.10	4.00	0.85	7.11
Outflow	1.57	0.68	2.39	0.68	4.88
Net	0.08	0.70	4.75	-0.64	4.34

Table 5: Migration Flows in the U.S.

2.7 Connections between Debt, Productivity, and Migration

Given that population is mobile across states, municipalities finances can be potentially susceptible to worker flows. Therefore, it is of interest to know how debt correlates with population.

At a state level, the two are weakly, but positively, correlated. This can be seen in Figure 2.8 that displays two measures of debt versus state population in 2012 in the states. The first measure (top panel) corresponds to net tax-supported debt in 2012 (NTSD) as percentage of gross state domestic product in 2011. In the bottom panel, we report debt service ratio (we also report a fitted line).¹² In general, we observe that states with larger population tend to sustain higher levels of debt. For example, whereas Wyoming with a population of 576,626 people had a NTSD-to-GSP ratio of less than 0.1%, Pennsylvania had a population about 24 times bigger and debt ratio of 2.7%.

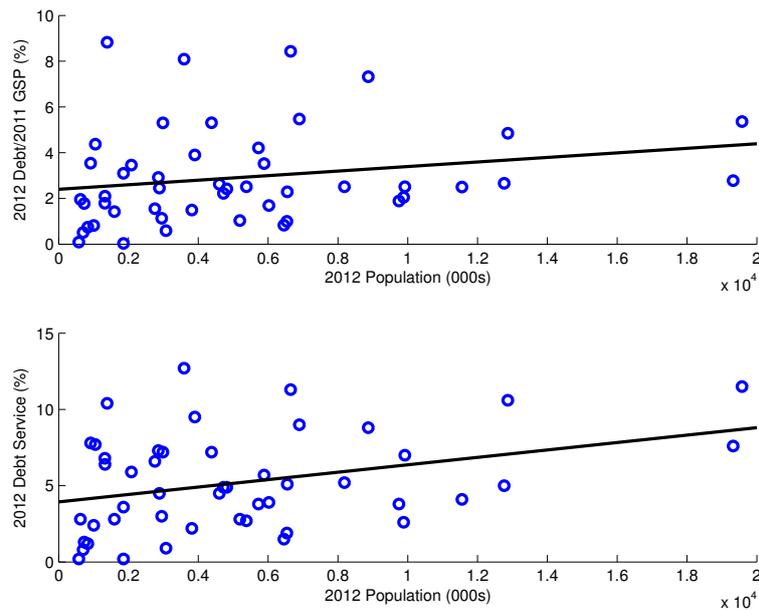


Figure 2.8: Debt and Population

A similar picture emerges if we use more disaggregated data. In particular, Moody’s reports that the overall debt burden increases with population in cities.¹³ For example, the median burden goes from 2.16% (2.9%) for cities with population less than 50,000 people and rating Aaa (Aa) to 2.9% (4.3%) for cities with population bigger than 500,000.

Debt, productivity, and debt are all associated with one another. This can be seen in Table 6. In agreement with Figure 2.8, the correlation between state population in 2012 and NTSD in 2012 is positive, albeit small. Population is a bit more positively correlated with productivity. Productivity is also positively correlated with debt.

¹²Debt data come from Moody’s 2013 State Debt Medians Report.

¹³Overall debt burden is defined as overall net debt outstanding divided by the fiscal year or more recent year estimated full-market value of all taxable property within the boundaries of the city.

	$\rho(\text{Pop,Debt})$	$\rho(\text{Pop,Productivity})$	$\rho(\text{Productivity,Debt})$
Correlation	0.17	0.30	0.37

Table 6: Some Correlations

3 Model

We model municipalities in the U.S. as a unit measure of islands. Each island consists of a continuum of households (not necessarily of measure one), a benevolent government, and a neoclassical firm. The government is a sovereign entity that issues non-contingent debt, taxes its citizens, provides government services, and transfers resources to households. Households consume, work, and crucially decide whether to stay on the island or migrate to another one. Finally, there is a continuum of risk neutral lenders willing to buy sovereign debt as long as they are compensated for default risk.

3.1 Household problem

Let \mathbf{x} be the state of the island. For now, we need not specify exactly what \mathbf{x} is, only that it has some possibly stochastic law of motion $\mathbf{x}' = \Gamma(\mathbf{x})$.

The moving decision is

$$V(\phi, \mathbf{x}) = \max_{m \in \{0,1\}} (1 - m)S(\mathbf{x}) + m(J - \phi)$$

where S is the value of staying on the island, J is an expected value from leaving the island, and ϕ is the utility cost of moving. We denote the optimal migration policy as $m(\phi, \mathbf{x})$. The utility conditional on staying is

$$\begin{aligned} S(\mathbf{x}) &= \max_{c,l} u(c, l, g(\mathbf{x})) + \beta \mathbb{E}_{\mathbf{x}} V(\phi', \mathbf{x}') \\ \text{s.t. } c &= (1 - \tau(\mathbf{x}))w(\mathbf{x})l + T(\mathbf{x}) + \pi(\mathbf{x}) \\ c &\geq 0, l \in [0, 1] \\ \mathbf{x}' &= \Gamma(\mathbf{x}) \end{aligned}$$

where π is the profit from the island's firm per person staying on the island, τ is a proportional tax on labor earnings wl , and T is a transfer from the sovereign. J is the expected utility from moving. We assume that if one decides to move, they are *randomly* assigned to an island and must stay at that island for at least one period. Consequently,

$$J = \int S(\mathbf{x})d\mu(\mathbf{x})$$

where $\mu(\cdot)$ is the distribution of islands over island states.

3.2 Firms

Each island has a firm operating the production function $z(L^d)^{1-\alpha}$ and solving

$$\dot{n}(\mathbf{x})\pi(\mathbf{x}) = \max_{L^d \geq 0} z(L^d)^{1-\alpha} - w(\mathbf{x})L^d$$

Let L^d denote labor demand. It is given by

$$w(\mathbf{x}) = (1 - \alpha)z(L^d(\mathbf{x}))^{-\alpha}.$$

Letting $n(\mathbf{x})$ be the number of households before any migration takes place and $\dot{n}(\mathbf{x})$ the number afterwards, total labor supply on the island is $\dot{n}(\mathbf{x})l(\mathbf{x})$. Then labor market clearing requires

$$L^d(\mathbf{x}) = \dot{n}(\mathbf{x})l(\mathbf{x}).$$

To reiterate, $\dot{n}(\mathbf{x})$ is labor after migration takes place and $l(\mathbf{x})$ is the optimal labor choice of households on an island with state \mathbf{x} . Note zero labor supply cannot be an equilibrium: Any finite wage results in strictly positive labor demand.

At the equilibrium wage, $\dot{n}\pi = 0$ if $\alpha = 0$. However, if $\alpha > 0$, then $\dot{n}\pi = \alpha z L^{d^{1-\alpha}} > 0$ which is greater than zero. In equilibrium, profit satisfies

$$\dot{n}\pi(\mathbf{x}) = \alpha z (\dot{n}(\mathbf{x})l(\mathbf{x}))^{1-\alpha}$$

or

$$\pi(\mathbf{x}) = \alpha z \dot{n}(\mathbf{x})^{-\alpha} l(\mathbf{x})^{1-\alpha}$$

which is decreasing in \dot{n} (for fixed l).

In the case of $\alpha < 0$, there is potentially a perverse incentive to reduce the population on the island as doing so increases both wages and profits. However, for cities and small regions, it is natural to think there is some diminishing returns to scale due to the fixed input land. Diminishing returns in labor also captures congestion in cities. Because of this, we do not interpret $(1 - \alpha)$ as the usual labor share on income with a value of roughly .64. Rather, we think of households as bringing capital with them when they move (or capital moves where there is more labor). So, we think of $(1 - \alpha)$ as being close to, but less than, one.¹⁴

The attentive reader may notice that there is no agglomeration effect in the production function or elsewhere in the model.¹⁵ Its absence could result in the model predicting a flat or negative relation between population and productivity. However, costly migration in our model introduces a sorting effect. Workers tend to leave cities with low productivity but stay in those with high productivity. As a

¹⁴One could incorporate land explicitly in the production function, but there is essentially no need to do this. Any differences in labor productivity due to land would be captured in the calibrated TFP process via the fixed effect.

¹⁵A simple way to introduce agglomeration is with the modified production function $y_t = f(L_t)N_t^\varpi$, where N is population and $\varpi > 1$. Duranton and Puga (2004) provide micro-foundations for this type of agglomeration.

consequence, the model does have the ability to generate the positive correlation between city density and productivity found in the data [Glasser \(2010\)](#).

3.3 The government, credit markets, and default

The benevolent government raises tax revenue using a proportional labor income tax $\tau \in [0, 1)$. It uses the tax revenue on services / amenities $g \geq 0$ and can pay out lumpsum transfers $T \geq 0$. The planner is able to issue long-term debt b from a finite set B . We use the convention $b < 0$ means debt.

In order to only have the stock of debt outstanding as a state variable rather than all previously issued amounts, we follow [Hatchondo and Martinez \(2009\)](#). In particular, we assume that a fraction λ of outstanding debt comes due each period. The amount of newly issued debt in a period is $b' - (1 - \lambda)b$. For this amount of debt, the planner receives a price of $q(b', \dot{n}, z)$.¹⁶ So, the amount of resources received in the current period is $q(b', \dot{n}, z)(b' - (1 - \lambda)b)$. Additionally, if choosing to not default, the planner must pay the λb that comes due.

Upon a default, the planner must forfeit γ fraction of tax revenue towards repayment for as long as the planner stays in default. We track the planner's default status using a state variable f with $f = 0$ meaning the planner is in good standing and $f = 1$ meaning the planner is in default. For as long as the planner is in default, he cannot borrow or save and continues to forfeit the γ fraction of tax revenue. The planner escapes default with probability $(1 - \delta) \in [0, 1]$.¹⁷

3.4 Equilibrium given planner instruments

Define $\mathbf{x} = (z, b, n, f)$ as an island's state. A steady state equilibrium is $c, l, m, L^d, \mu, T, \tau, g, b', d, \pi$ such that

1. (Household optimization) $c(\mathbf{x})$, $l(\mathbf{x})$, and $m(\phi, \mathbf{x})$ are optimal taking fiscal instruments, prices, and profit as given.
2. (Firm optimization) $L^d(\mathbf{x}) \geq 0$ is optimal taking w as given and generates profit $\pi(\mathbf{x})$.
3. (Labor market clearing) $L^d(\mathbf{x}) = \dot{n}(\mathbf{x})l(\mathbf{x})$.
4. (Government) $g \geq 0$, $T \geq 0$, $\tau \in [0, 1)$, $b' \in B$, $b' = 0$ if $d = 1$ or $f = 1$, $d = 0$ when $f = 1$, and the budget constraint is satisfied. The government budget constraint is

$$g + q(b', \dot{n}, z)(b' - (1 - \lambda)b) \leq w\tau\dot{n}l - T\dot{n} + \lambda b$$

¹⁶The price schedule could be conditioned on all information known at time t . However, all that matters for repayment probabilities is (b', \dot{n}, z) since this implies a distribution over (b', n', z', f') (note that $f = 0$ is assumed since $f = 1$ implies a planner cannot issue debt).

¹⁷It is possible that the recovery rate is greater than 100% because the sovereign always pays γ fraction of tax revenue for as long as he remains in default. However, it is unlikely that a sovereign would choose to default if this were the case. A recovery rate greater than 100% can be ruled out, but doing so entails adding a state variable.

if $d = 0$ and $f = 0$ and

$$g \leq (1 - \gamma)w\tau\dot{n}l - T\dot{n}$$

if $d = 1$ or $f = 1$.

We specify free disposal so that the planner may throw away resources if desired (in order to reduce the number of people attracted to his island).

Note that the way services appears in the budget constraint treats it like a public good: Every household enjoys the same amount of g , implying $g\dot{n}$ is enjoyed; however, only g is spent. This is not necessary. We could have the expenditures on amenities/services be $g\dot{n}$ or $\theta g\dot{n} + (1 - \theta)g$ controlling how many people benefit for a level of expenditure.

5. (Debt pricing) Since debt is issued to a continuum of islands, the return is known with certainty. Consequently, debt can be priced according to no arbitrage. The no arbitrage pricing is

$$q(b', n', z) = \bar{q}\mathbb{E}_z [(1 - d')[\lambda + (1 - \lambda)q(b'', n'', z')] + d'q^r(b_0 = b', b', n', z', f' = 0)]$$

where $d' = d(b', n', z', f' = 0)$ and $b'' = b'(b', n', z', f' = 0)$ and $n'' = \dot{n}(b', n', z', f' = 0)$. For $b' = 0$, any q is an equilibrium price. The term q^r is cents paid on the dollar for defaulted debt. It satisfies

$$q^r(b_0, \mathbf{x}) = \frac{\min\{\gamma w(\mathbf{x})\tau(\mathbf{x})\dot{n}(\mathbf{x})l(\mathbf{x}), -b_0(1 - \bar{q}(1 - \delta))\}}{-b_0} + \bar{q}\delta\mathbb{E}_z q^r(b_0, b' = 0, \dot{n}(\mathbf{x}), z', f' = 1).$$

The “min” term guarantees creditors are never made more than whole, i.e., $q^r \leq 1$.¹⁸ Note that the value of defaulted debt is only non-zero if $\gamma > 0$. This builds on the work of [Chatterjee and Gordon \(2012\)](#) who examine garnishment in a consumer default context.

6. (Consistency) μ and Γ are consistent with stochastic transitions and policy functions. Also, the law of motion for population is consistent:

$$\dot{n}(\mathbf{x}) := n - n \left(\int m(\phi, \mathbf{x})dF(\phi) \right) + \int \tilde{n} \left(\int m(\phi, \tilde{\mathbf{x}})dF(\phi) \right) d\mu(\tilde{\mathbf{x}}).$$

where \tilde{n} is a component of $\tilde{\mathbf{x}}$.

3.5 Island planner problem

Each island has a planner who cannot commit to an optimal policy. Rather, the planner re-optimizes each time the state of the island changes (and takes into account that they will reoptimize in the future). We think time-consistent policies are a real constraint for municipal governments. The planner seeks to maximize the welfare of those currently on the island in a utilitarian fashion.

¹⁸If repayments in a period exceed $-b_0(1 - \bar{q}(1 - \delta))$, the extra is assumed to be dead-weight loss. At the expense of adding a state to the problem of a sovereign in default, one could make this not dead-weight loss.

The timing is as follows. Shocks z and ϕ are realized, as is whether an island recovers from a default. This gives the island state (b, n, z, f) and the household state (ϕ) . Then the household makes their migration decisions and move. After migration takes place, the planner announces his policies (d, τ, T, g, b') . The households make their consumption and leisure decisions (c, l) . That concludes the period.¹⁹

This timing where households choose migration decisions first has two main advantages. First, it gives a contraction mapping for the planner problem conditional on migration decisions and a price schedule. Second, it allows for a number of theoretical results concerning the provision of g and optimal taxation. We also think it not unrealistic that, for small changes in government policy, migration will respond very little on impact.

The planner's problem conditional on the island being in a state \mathbf{x} is

$$\begin{aligned} \max_{d(\mathbf{x}), \tau(\mathbf{x}), T(\mathbf{x}), g(\mathbf{x}), b'(\mathbf{x})} & \int ((1 - m(\phi, \mathbf{x}))S(\mathbf{x}) + m(\phi, \mathbf{x})(J - \phi)) dF(\phi) \\ \text{s.t. } & m(\phi, \mathbf{x}) \text{ given} \\ & d(\mathbf{x}), \tau(\mathbf{x}), T(\mathbf{x}), g(\mathbf{x}), b'(\mathbf{x}) \text{ satisfy the government budget constraint} \end{aligned}$$

Note that S is a function of d, τ, T, g, b' , but that this is not obvious in the notation. The planner always has a feasible option: Default, set tax rates to zero, and set services to zero.

The specified objective function is equivalent to saying the planner maximizes the utility of those staying on the island since J and ϕ are outside the planner's control. That is, a solution to the above problem is also a solution to

$$\begin{aligned} \max_{d(\mathbf{x}), \tau(\mathbf{x}), T(\mathbf{x}), g(\mathbf{x}), b'(\mathbf{x})} & S(\mathbf{x}) \\ \text{s.t. } & \\ & m(\phi, \mathbf{x}) \text{ given} \\ & d(\mathbf{x}), \tau(\mathbf{x}), T(\mathbf{x}), g(\mathbf{x}), b'(\mathbf{x}) \text{ satisfy the government budget constraint} \end{aligned}$$

and vice-versa. $m(\phi, \mathbf{x})$ is still needed because it determines \dot{n} which is inside S .

The assumption that the planner maximizes utility of those currently on the island or those staying on the island, rather than say the discounted utility of being on the island forever, has a dramatic consequence. In particular, suppose that the planner has very few resources (possibly because of a low TFP shock). Because of this, households on the island are likely to leave next period since the planner cannot promise them much in the way of transfers. But in that case, the planner anticipates this and so tries to move as many resources from the future into the present. This might involve huge amounts of borrowing in the current period. The fact that it has to be paid off tomorrow or defaulted upon is of little consequence to the planner and households.

¹⁹This can be formalized as a subgame perfect Nash equilibrium where households choose migration decisions, the planner chooses policies, a Walrasian auctioneer chooses wages, firms choose labor and make profits, and households choose their consumption and leisure.

Mathematically, one can see this as a low effective discount factor. If the probability of households leaving next period is p (conditional on the planner instruments), then the planner has a continuation utility that looks like $p\mathbb{E}[J - \phi | \phi < R] + (1 - p)S(\mathbf{x})$. In the extreme where everyone wants to leave, the planner solves a one-period optimization problem! This observation will help explain what we find later in the results section.

3.6 Political Equilibrium

A steady state political equilibrium is policies $d(\mathbf{x}), \tau(\mathbf{x}), T(\mathbf{x}), g(\mathbf{x}), b'(\mathbf{x})$ that are optimal for each island and induce J and μ when the initial distribution of islands is μ .

4 Theoretical Results

This section provides some theoretical characterizations. Additional characterizations for the household labor-leisure choice and labor market equilibrium are provided in the appendix.

4.1 Reservation strategy for moving

Note that since S and J are independent of ϕ , we immediately see m is characterized by a reservation level $R(\mathbf{x})$ in ϕ with $m(\phi, \mathbf{x}) = \mathbf{1}[\phi < R(\mathbf{x})]$ (with indifference if $\phi = R(\mathbf{x})$). and $R(\mathbf{x})$ given by

$$R(\mathbf{x}) = J - S(\mathbf{x}).$$

Using this, the measure leaving an island in any period is

$$n \int m(\phi, \mathbf{x}) dF(\phi) = nF(R(\mathbf{x}))$$

and the measure entering any island is

$$n^{in} := \int \tilde{n}(m(\phi, \tilde{\mathbf{x}}) dF(\phi)) d\mu(\tilde{\mathbf{x}}) = \int \tilde{n}F(R(\tilde{\mathbf{x}})) d\mu(\tilde{\mathbf{x}}).$$

So, the measure of households on island \mathbf{x} after migration has taken place is

$$\dot{n}(\mathbf{x}) = n(1 - F(R(\mathbf{x}))) + n^{in}$$

The continuation utility of a household is $\mathbb{E}_{\phi', \mathbf{x}' | \mathbf{x}} V(\phi, \mathbf{x}') = \mathbb{E}_{\mathbf{x}' | \mathbf{x}} \int V(\phi', x') dF(\phi')$. So, for reference we characterize

$$\int V(\phi, \mathbf{x}) dF(\phi) = \int_{-\infty}^{R(\mathbf{x})} (J - \phi) dF(\phi) + \int_{R(\mathbf{x})}^{\infty} S(\mathbf{x}) dF(\phi)$$

or equivalently

$$\int V(\phi, \mathbf{x}) dF(\phi) = S(\mathbf{x}) + F(R(\mathbf{x})) (J - S(\mathbf{x})) - F(R(\mathbf{x})) \mathbb{E}(\phi | \phi \leq R(\mathbf{x})).$$

4.2 A simplified planner problem

The planner's problem is

$$\begin{aligned} & \max_{d, \tau, T, g, b'} S(\mathbf{x}) \\ & \text{s.t. the government budget constraint} \\ & m(\phi, \mathbf{x}) \text{ given} \\ & \dot{n} = n(1 - F(R(\mathbf{x}))) + n^{in} \end{aligned}$$

Here, $S(\mathbf{x})$ is the value function of a household conditional on (d, τ, T, g, b') but holding fixed all the future policies of the planner, e.g. (d', τ', T', g', b'') . It can equivalently be written

$$\max_{d, \tau, T, g, b'} \left(\max_{c, l} u(c, l, g) + \beta \mathbb{E}_{z, f} V(\phi', b', \dot{n}, z', f') \right)$$

where the household decision is subject to $c \geq 0$, $l \in [0, 1]$, and $c = (1 - \tau)w(d, \tau, T, g, b')l + T$ and the planner is subject to the same constraints as before.

Since there are no intertemporal decisions by households, this then becomes

$$\max_{d, \tau, T, g, b'} \left(\max_{c, l} u(c, l, g) \right) + \beta \mathbb{E}_{z, f} V(\phi', b', \dot{n}, z', f')$$

Household decisions are only impacted directly by w, π, τ, T and, if not separable, g . This immediately implies the policy functions and indirect utility function are functions only of (w, π, τ, T, g) . So, call them $c(w, \pi, \tau, T, g)$, $l(w, \pi, \tau, T, g)$.

Assumption 1 *For any planner's instruments, there is a unique equilibrium wage (conditional on \dot{n}).*

Let the equilibrium wage and profits be denoted $w^*(\tau, T, g)$ and $\pi^*(\tau, T, g)$ showing they depend on τ, T, g (they also depend on \dot{n} and z but those will be held fixed conditional on \mathbf{x}). Additionally, define the equilibrium consumption and labor via $c^*(\tau, T, g) = c(w^*(\tau, T, g), \pi^*(\tau, T, g), \tau, T, g)$ and $l^*(\tau, T, g) = l(w^*(\tau, T, g), \pi^*(\tau, T, g), \tau, T, g)$. Lastly, let $U(\tau, T, g)$ denote the indirect utility function $u(c^*(\tau, T, g), l^*(\tau, T, g), g)$. Then the planner's problem is

$$\begin{aligned} \tilde{S}(\mathbf{x}) &= \max_{d, \tau, T, g, b'} U(\tau, T, g) + \beta \mathbb{E}_{z, f} V(\phi', b', \dot{n}, z', f') \\ & \text{s.t. the government budget constraint} \\ & R(\mathbf{x}) \text{ given} \\ & \dot{n} = n(1 - F(R(\mathbf{x}))) + n^{in} \end{aligned}$$

where I have defined \tilde{S} as the planner's value. In equilibrium, \tilde{S} and S agree (which is to say that the planner chooses the instruments that the household expects him to choose).

As is customary in the default literature, this problem can be broken up into two problems, one for default or being in default (D) and the other for no default (N):

$$\tilde{S}(b, n, z, f = 0) = \max_d S^N(b, n, z, f)(1 - d) + S^D(b, n, z, f)d \quad (4.1)$$

(if S^N is undefined because no feasible policy exists, $\tilde{S} = S^D$) and

$$\tilde{S}(b = 0, n, z, f = 1) = S^D(b = 0, n, z, f = 1) \quad (4.2)$$

where the value of repaying is

$$\begin{aligned} S^N(b, n, z, f) &= \max_{\tau, T, g, b'} U(\tau, T, g) + \beta \mathbb{E}_z V(\phi', b', \dot{n}, z', f' = 0) \\ \text{s.t. } g + q(b', \dot{n}, z)(b' - (1 - \lambda)b) &\leq \lambda b + \tau \dot{n} w^*(\tau, T, g) l^*(\tau, T, g) - T \dot{n} \\ g &\geq 0, \tau \in [0, 1), T \geq 0, b' \in B \\ \dot{n} &= n(1 - F(R(\mathbf{x}))) + n^{in} \end{aligned} \quad (4.3)$$

and the value of defaulting is

$$\begin{aligned} S^D(b, n, z, f) &= \max_{\tau, T, g} U(\tau, T, g) + \beta \mathbb{E}_z \left(\begin{array}{c} \delta V(\phi', b' = 0, \dot{n}, z', f' = 1) \\ +(1 - \delta)V(\phi', b' = 0, \dot{n}, z', f' = 0) \end{array} \right) \\ \text{s.t. } g &\leq (1 - \gamma)(\tau \dot{n} w^*(\tau, T, g) l(\tau, T, g)) - T \dot{n} \\ g &\geq 0, \tau \in [0, 1), T \geq 0, b' = 0 \\ \dot{n} &= n(1 - F(R(\mathbf{x}))) + n^{in}. \end{aligned} \quad (4.4)$$

Conjecture 1 *Fixing m , the iteration is a contraction mapping when one sets $S = \tilde{S}$ at each iteration (i.e., $V = \max\{\tilde{S}, J - \phi\}$).*

Apart from the default decision, there are four choice variables: τ, T, g, b' . One can be eliminated if the budget constraint binds. This is guaranteed if the budget constraint is increasing in g . However, this is not necessarily the case: If an increase in g causes w and l to move in opposite directions, then there is an ambiguous effect on tax revenues (holding fixed τ). Hence, unless g is separable, it is going to be very unlikely to show that the budget constraint holds with equality (at least by this line of reasoning). Thus to eliminate a choice variable, we make the following assumption:

Assumption 2 *g is separable in the sense that the equilibrium labor, consumption, wage, and profit are not functions of g .*

Proposition 2 *The budget constraint must hold with equality.*

Proof. Since g is separable, an increase in g leaves l unchanged, as well as the equilibrium wage and profit. Consequently, an increase in g increases the indirect utility function of the household, so the constraint must hold with equality. ■

If we are willing to assume no wealth effects in labor supply (GHH preferences in particular), then we can also effectively eliminate another choice variable. Namely, we can replace a choice of (g, T) with a choice of $g + T\dot{n}$ and a first order condition. If we are willing to go even further, then we can replace the FOC with a closed form solution for g and T as a function of $g + T\dot{n}$.

Proposition 3 *Suppose there is no wealth effect on labor so that l is invariant to changes in T (and the assumption g is separable is maintained).*

Assume the planner faces a potentially tighter constraint on transfers: $T \geq \underline{T}$ where $\underline{T} \geq 0$. Consider all (g, T) pairs such that $g + \dot{n}T = r$ for some $r \geq \dot{n}\underline{T}$ and fixed τ . The optimal (g, T) pair is either $(0, \frac{r}{\dot{n}})$, $(r - \dot{n}\underline{T}, \underline{T})$, or satisfies

$$u_c = u_g \dot{n}.$$

If $u(c, l, g) = v(c - G(l)) + \zeta v(g)$, then the optimal pair is either $(0, \frac{r}{\dot{n}})$, $(r - \dot{n}\underline{T}, \underline{T})$, or

$$g = r - \dot{n} \frac{G(l) - w(1 - \tau)l + v'^{-1}(\zeta \dot{n})r}{1 + \dot{n}v'^{-1}(\zeta \dot{n})}$$

with $T = (r - g)/\dot{n}$. The last choice is only feasible if $T \geq \underline{T}$ and $g \geq 0$.

If $u(c, l, g) = v(h(c - G(l), g))$ for h a CES aggregator

$$h(c - G(l), g) = \left((1 - \zeta)^{1/\nu} (c - G(l))^{1-1/\nu} + \zeta^{1/\nu} g^{1-1/\nu} \right)^{1/(1-1/\nu)},$$

then the optimal pair is either $(0, \frac{r}{\dot{n}})$, $(r - \dot{n}\underline{T}, \underline{T})$, or

$$g = \frac{r + \dot{n}((1 - \tau)w^*l^* + \pi^* - G(l^*))}{1 + \frac{1-\zeta}{\zeta}\dot{n}^{-\nu+1}}$$

with $T = (r - g)/\dot{n}$. The last choice is only feasible if $T \geq \underline{T}$ and $g \geq 0$.

Proof. Note that since there is no wealth effect from T and since g is separable, the optimal labor choice is invariant to changes in g and T . Consequently, the equilibrium wage and profit are also invariant as well as labor tax revenue for fixed τ . So, just write l^* , w^* , and π^* for $l^*(\tau, T, g)$, $w^*(\tau, T, g)$, $\pi^*(\tau, T, g)$ respectively.

The best (g, T) pair solves

$$\begin{aligned} & \max_T U(\tau, T, g(T)) \\ & \text{s.t. } g(T) = r - \dot{n}T \\ & \quad T \geq \underline{T}, T \leq \frac{r}{\dot{n}} \end{aligned}$$

where the second inequality constraint says $g(T) \geq 0$. A feasible choice only exists if $r \geq \dot{n}\underline{T}$. The indirect utility function satisfies

$$U(\tau, T, g) = u((1 - \tau)w^*l^* + T + \pi^*, l^*, g)$$

At an interior choice of T , the FOC

$$u_c + u_g g'(T) = 0$$

must hold. Equivalently,

$$u_c = u_g \dot{n}.$$

If $u(c, l, g) = v(c - G(l)) + \zeta v(g)$, then the FOC becomes

$$\dot{n}\zeta v'(g(T)) = v'((1 - \tau)w^*l^* + T + \pi^* - G(l^*))$$

Inverting v' , the equation is linear in g and solving gives

$$g = r - \dot{n} \frac{G(l^*) - w^*(1 - \tau)l^* - \pi^* + v'^{-1}(\dot{n}\zeta)r}{1 + \dot{n}v'^{-1}(\dot{n}\zeta)}.$$

From this, one also has $T = (r - g)/\dot{n}$. One must check $T \geq \underline{T}$ and $T \leq \frac{r}{\dot{n}}$ to ensure this is in fact an interior solution.

If $u(c, l, g) = v(h(c - G(l), g))$, then the FOC becomes

$$h_1(c - G(l), g) = h_2(c - G(l), g)\dot{n}.$$

In the computational work, we use $v(x) = \frac{x^{\frac{1}{1-1/\nu}-\sigma}}{1-\sigma}$ and $h(c - G(l), g) = (1 - \zeta)^{1/\nu}(c - G(l))^{1-1/\nu} + \zeta^{1/\nu}g^{1-1/\nu}$ where ν is the elasticity of substitution between $c - G(l)$ and g . In this case, the FOC reduces to

$$\dot{n}^\nu \frac{\zeta}{1 - \zeta} = \frac{g}{c^* - G(l^*)}.$$

Using $T = (r - g)/\dot{n}$, one can find

$$g = \frac{r + \dot{n}((1 - \tau)w^*l^* + \pi^* - G(l^*))}{1 + \frac{1-\zeta}{\zeta}\dot{n}^{-\nu+1}}.$$

Again, one must check that $T \geq \underline{T}$ and $T \leq \frac{r}{\dot{n}}$.

■

The last proposition proves that it is never optimal to use a proportional tax $\tau > 0$ and lumpsum transfers $T > 0$ at the same time. While this is intuitive, actually proving it requires a good deal of work in part because one must account for the general equilibrium effects on wages and profits.

Proposition 4 Fix (b', d) and \mathbf{x} . Suppose g is separable with GHH preferences for c, l . In this case, the equilibrium wage and labor supply only depend on τ , call them $w^*(\tau)$ and $l^*(\tau)$. Suppose $l^*(0) < 1$ so that $0 < l^*(\tau) < 1 \forall \tau \in [0, 1)$ and $l'^*(\tau) < 0$.

Define $r = -q(b')(b' - (1 - \lambda)b) + \lambda b$. Whenever $\tau > 0$, one has $T = 0$ and $g = r + \tau w^*(\tau)\dot{n}l^*(\tau)$. Whenever $T > 0$, one has $\tau = 0$ and $g = r - T\dot{n}$ and $u_c = \dot{n}u_g$.

Proof. The planner's problem, for fixed (b', d) can be written

$$\begin{aligned}
& \max_{\tau, g} u(c^*(\tau, g), l^*(\tau), g) \\
& \text{s.t. } c^*(\tau, g) = (1 - \tau)w^*(\tau)l^*(\tau) + \pi^*(\tau) + T(\tau, g) \\
& T(\tau, g) = \frac{-g}{\dot{n}} + \frac{r}{\dot{n}} + \tau w^*(\tau)l^*(\tau) \\
& T(\tau, g) \geq 0 \quad (\lambda) \\
& \tau \geq 0 \quad (\theta) \\
& \tau \leq 1, \\
& g \geq 0
\end{aligned}$$

where notation x^* is the equilibrium value of x . The $\tau \leq 1$ constraint is non-binding, as is the $g \geq 0$ constraint. The Lagrangian is

$$\mathcal{L} = u(c^*(\tau, g), l^*(\tau), g) + \lambda T(\tau, g) + \theta \tau$$

The expression for c^* can be significantly simplified. Substituting in $T(\tau, g)$ gives

$$c^*(\tau, g) = w^*(\tau)l^*(\tau) + \pi^*(\tau) + \frac{-g}{\dot{n}} + \frac{r}{\dot{n}}.$$

Then, since $\dot{n}\pi = z\dot{n}^{1-\alpha}l^{1-\alpha} - w\dot{n}l$,

$$c^*(\tau, g) = z\dot{n}^{-\alpha}l^*(\tau)^{1-\alpha} + \frac{-g}{\dot{n}} + \frac{r}{\dot{n}} = \frac{y^*(\tau) - g + r}{\dot{n}}.$$

Define $REV(\tau) = \tau w^*(\tau)l^*(\tau)$ as per household tax revenue. Using the expression for equilibrium consumption, the Kuhn-Tucker conditions of the planner problem are

$$\begin{aligned}
FOC_\tau : & \quad u_c w^*(\tau)l^{*'}(\tau) + u_l l^{*'}(\tau) + \lambda REV'(\tau) + \theta = 0 \\
FOC_g : & \quad u_c + u_g(-\dot{n}) + \lambda = 0 \\
& \quad \lambda T(\tau, g) = 0, \theta \tau = 0 \\
& \quad \lambda \geq 0, \theta \geq 0
\end{aligned}$$

The household first order condition guarantees

$$u_c w^*(\tau)(1 - \tau) + u_l = 0$$

for an interior solution (which is equivalent to $l^{*'}(\tau) < 0$). Using this, the planner FOC_τ equation can be written

$$FOC_\tau : \quad \tau u_c w^*(\tau)l^{*'}(\tau) + \lambda REV'(\tau) + \theta = 0$$

Suppose $\theta = 0$, i.e., $\tau \geq 0$ is non-binding. Then from FOC_τ , it must be that both $\lambda > 0$ and $REV'(\tau) > 0$ (the latter says the planner is on the good side of the Laffer curve). Since $\lambda > 0$,

complementary slackness says $T = 0$.

Now, suppose $\lambda = 0$, i.e. $T \geq 0$ is non-binding. In this case, FOC_τ says $\theta > 0$. Complementary slackness then has $\tau = 0$. In this case, $u_c = u_g \dot{n}$ can be used to compute the optimal g (via the previous proposition using $\tau = 0$).

It cannot be the case that $\lambda = \theta = 0$ because then FOC_τ is violated unless $l^{*\prime}(\tau) = 0$ (or $\tau = 0$, in which case we check this case above). In this case, the planner is indifferent over all τ locally (i.e., in the region surrounding τ such that $l^{*\prime} = 0$).

The only other case to consider is $\lambda > 0$ and $\theta > 0$, i.e. both constraints are binding. In this case, $\tau = 0$ and $T = 0$. Moreover, $g = r$.

The appendix provides additional derivations for the functional form $u(c, l, g) = v(c - G(l)) + \zeta v(g)$. ■

To provide a further understanding of our model, we need to solve it numerically. We do so in the next sections.

5 Calibration

Productivity is central to our model, and so we estimate a process directly from the data. Specifically, we consider the econometric specification

$$\begin{aligned}\log z_{it} &= \alpha_i + u_{i,t} \\ u_{i,t} &= \rho u_{i,t-1} + \varepsilon_{it} \\ \alpha_i &\sim N(0, \sigma_\alpha^2), \varepsilon_{it} \sim N(0, \sigma_\varepsilon^2).\end{aligned}$$

For z , we use our annual state productivity measure. We think of α as capturing permanent (or very long-lived) differences in state productivity and as ε capturing persistent changes in productivity. We use exactly identified GMM to estimate $(\rho, \sigma_\varepsilon, \sigma_\alpha)$ and find $\rho = .7788$, $\sigma_\varepsilon = .0289$, and $\sigma_\alpha = .1588$.²⁰

We consider two utility functions,

$$u(c, l, g) = \frac{(c - G(l))^{1-\sigma}}{1-\sigma} + \zeta \frac{g^{1-\sigma}}{1-\sigma} + \kappa.$$

and

$$u(c, l, g) = \frac{\tilde{c}^{1-\sigma}}{1-\sigma}, \quad \tilde{c} = \left((1-\zeta)^{1/\nu} (c - G(l))^{1-1/\nu} + \zeta^{1/\nu} g^{1-1/\nu} \right)^{\frac{\nu}{\nu-1}}$$

where

$$G(l) = \eta \frac{l^{1+1/\theta}}{1+1/\theta} - \eta \frac{1}{1+1/\theta}.$$

Both utility functions satisfy $u_g > 0$, and $u_{gg} < 0$. The latter satisfies $u_{cg} > 0$ as long as $\nu < 1/\sigma$, i.e., the elasticity of substitution between $c - G(l)$ and g is less than the intertemporal

²⁰Note that the persistent shock has a persistence and standard deviation very similar to the U.S. values at an annual frequency.

elasticity of substitution (over the composite good \tilde{c}).²¹ For both utility functions, the optimal choice of labor is independent of both g and T .

We set $\sigma = 2$ and $\theta = .5$. The labor supply elasticity θ is possibly too low. We plan to conduct robustness to this parameter. The value for ζ and ν are calibrated jointly. The risk free borrowing rate \bar{q} is set to give a 2% risk-free return. The probability of exiting autarky is $\delta = .5$. We use $\lambda = 1$, i.e., short-term debt.

Current calibration targets are presented in Table 7. We have recently obtained better data but have not yet recalibrated to match the better targets. Also, at this time, the parameters are changing fairly frequently, so they are not reported.

Targeted Statistic	Parameter*	Data	Model
Fraction of Time Spent Working	η	.30	.30
Total Debt-Total Output Ratio	β	.03	.027
Average Default Rate $\times 100$	γ	.3	.13
Migration Rate Mean, $\mathbb{E}(F(R))$	$\mathbb{E}(\phi)$.016	.017
Migration Rate Stdev, $\sigma(F(R))$	$\sigma(\phi)$.007	.007
Mean island pop / median island pop	α	1.40	1.34
Average Tax Rate	ζ	.042	.040
Untargeted Statistic			
Correlation of n and $-b$			-0.73
Correlation of n and z			0.75
Correlation of z and $-b$			-0.81
Average debt-output ($-b/y$)			0.06
Average tax rate			0.04
Stdev of tax rate			0.02
Comovement of $\log y/n$ and $\log n$			0.43
Average net flow $((\dot{n} - n)/n)$			0.00
Average inflow (n_{in}/n)			0.02
Average outflow $((\dot{n} - n - n_{in})/n)$			-0.02
Total debt to total tax revenue $(-b/(\tau w l \dot{n}))$			0.92
Average of debt-tax revenue ratio $(-b/(\tau w l \dot{n}))$			1.12
Average taxes-output (taxes/y)			0.04

*Parameter listed has a strong influence on the select statistic.

Table 7: Calibration Targets

6 Quantitative Results

Some [very preliminary] results from our model.

²¹Note that the CES aggregator, as $\nu \rightarrow 1$, is Cobb-Douglas. In this case $\nu > 1/\sigma$ for $\sigma = 2$ which implies $u_{cg} < 0$. Thus we deviated from a typical Cobb-Douglas specification to let c and g be complements.

6.1 Cross-Section Implications

Our current calibration predicts a negative correlation between debt and population. This is quite natural in a sense and is what the [Roback \(1982\)](#) and Rosen models would predict: A model without default and no moving costs would imply $S(b, n, z, f) = J$. Since S is decreasing in n and $-b$, in order for the indifference condition to hold, they would necessarily be negatively correlated. This is counterfactual. However, our model is capable of delivering a positive correlation as it has in previous calibrations.

Figure 6.1 reports the probabilities of default in the next J periods conditional on population (first panel), debt normalized by average tax revenues (second panel), and productivity (last panel). As one would expect, more population reduces the likelihood of default. Similarly, islands with higher productivity have lower probabilities of default. Interestingly, the model predicts that the probability of default initially increases with more debt but then declines. Everything else equal, more debt leads to a higher probability of default, which explains the first half of the figure. But here there is a composition effect. The high indebted sovereigns tend to have higher productivity and higher populations. Consequently, they can carry high levels of debt with a low probability of default. The sovereigns with lower debt tend to be low productivity and so have higher probabilities of default.

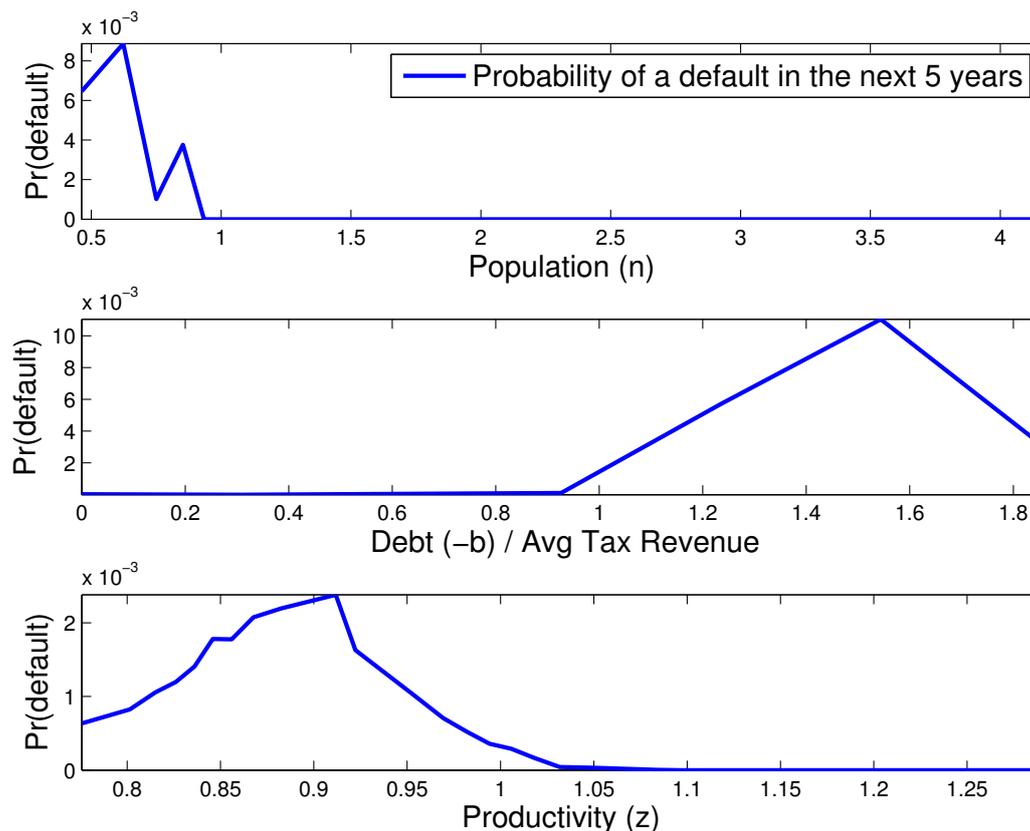


Figure 6.1: Probability of Default in Near Future

6.2 Typical Default Episode

Figures 6.2 and 6.3 display a typical default episode implied by our model. These events are computed by solving our model for a large number of periods and then taking the average across all those cases in which there are defaults. We report 10 years before and after default with the red dot corresponding to the period in which the sovereign defaults. Normalized variables are deflated by average tax revenue.

Some salient features are: 1) Similar to the experience in Detroit, default happens when both government spending and indebtedness are at high levels; 2) following default, the state balances its budget by cutting spending and transfers as well as by increasing taxes; 3) some workers choose to live the state, hence the decline in population; 4) default happens during periods of depressed productivity; 5) higher taxes translate into lower income driving consumption down and, via substitution effects, lower labor. Not surprisingly, wages increases following default. Figure 6.3 also reveals that as the state gets into higher debt, the likelihood of default rises quickly.

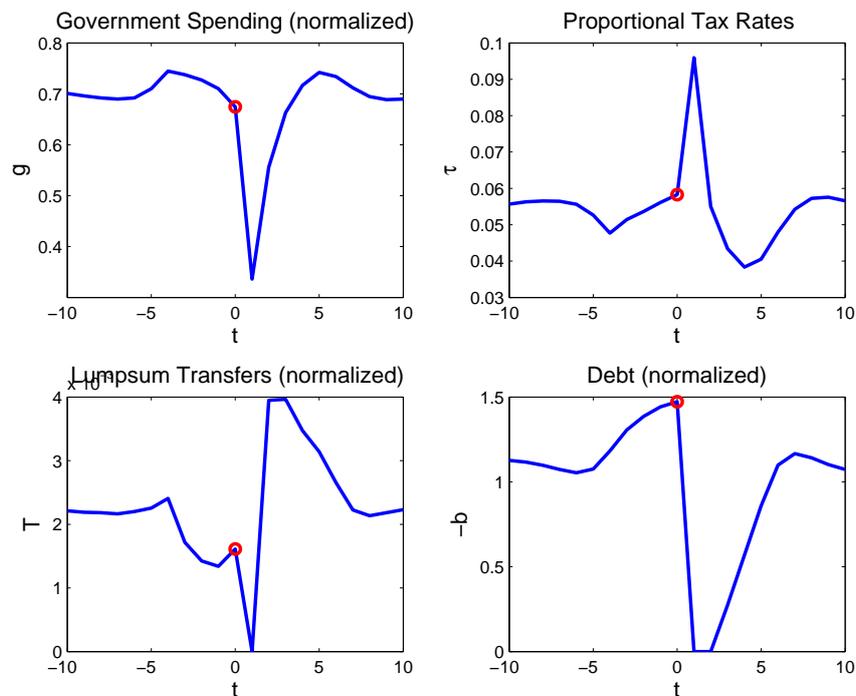


Figure 6.2: Default Episode: Fiscal Instruments

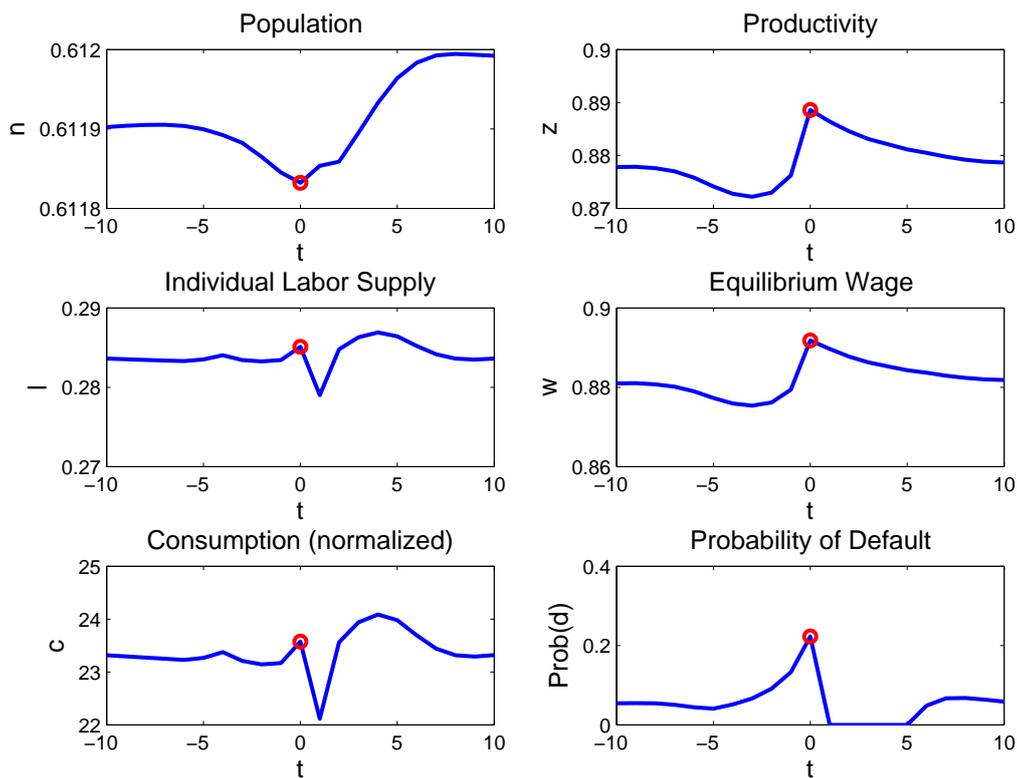


Figure 6.3: Default Episode: Other Variables

6.3 A Bailout Exercise

With our benchmark model in hand, we can now study the important issue of bailing out a city. Since every island is small, the cost of such a bailout is effectively zero. However, even if they were not small, one could imagine that a substantial increase in municipal default rates could lead to contagion: Creditors, suddenly fearful of default, withdraw funds. To avoid this, the federal or state government (or perhaps even other local governments) may want to bailout a city.

How do we model this scenario? Suppose that the municipality initially has some state $(b, n, z, f = 0)$. That is conditional on debt, population, and productivity, the sovereign is in good standing with creditors. Furthermore, let's assume that the stock of debt is sufficiently large that the sovereign will default. If the federal government wants to "bailout" the city, it is in the form of some extra cash. That is, the state variable $b = b + \varepsilon$, where ε is large enough to avoid a default (in that period). Then we look at the time series from two simulations: One with an initial state $(b, n, z, f = 0)$ and another with $(b + \varepsilon, n, z, f = 0)$. Since the sovereign is benevolent, the difference between the two paths for government choice variables is the ideal way to bailout a city by only spending ε units of the consumption good (on that island).

Figures 6.4 and 6.5 display the transition paths from our thought experiment. For comparison purposes, we also include a simulation in which the state is in good standing without debt, i.e.

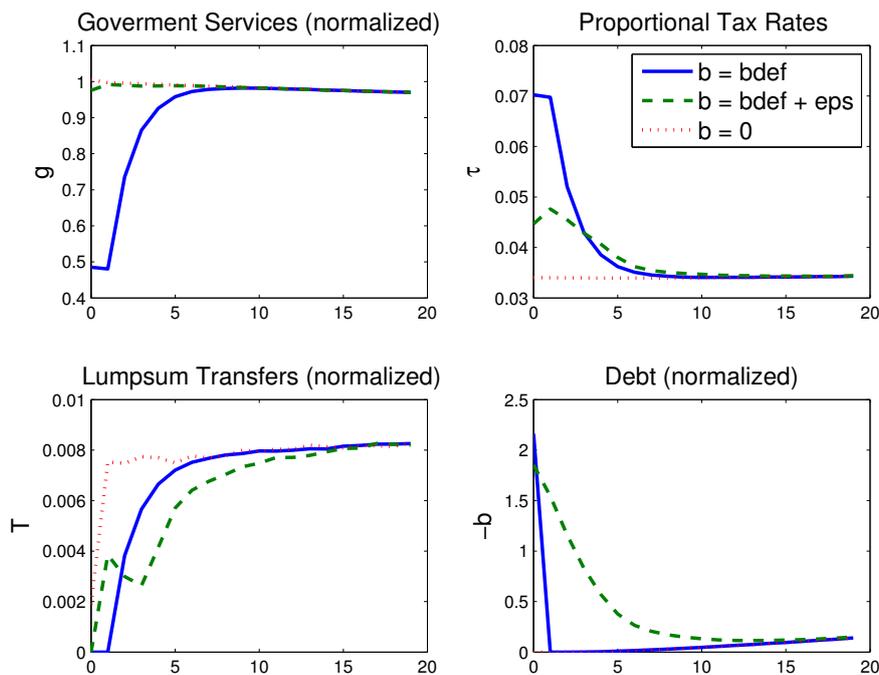


Figure 6.4: Bailout: Fiscal Instruments

($b = 0, n, z, f = 0$). Without external aid, the state defaults and its transitional dynamics are similar to those described in the previous section (blue solid line): Default, raise taxes, and cut transfers and government spending. By construction, the aid package moves the state away from default (green dashed line). Instead of defaulting, they slowly pay down debt (rather than abruptly reneging) and keep government services much higher than in default. The fact that tax rates are higher if the sovereign defaults is a consequence of the default costs which garnish tax revenue: In order to prevent government services from falling even further, the sovereign must raise tax revenue sharply sending a large portion of revenue to creditors. While there are some differences in lumpsum transfers, all the series are very close to zero.

Figure 6.5 reveals some peculiar features about our calibration. In particular, the secular decline in population is not affected by default. Consumption is noticeably different at impact, initially 4% lower without a bailout, the net present value of consumption is barely changed. This, combined with a calibrated utility function having $u(c - G(l)) + (1e - 5)u(g)$, means a default does not affect utility much. While not easily interpretable since presented in utils, the bottom right panel confirms this: Utility is not affected by a default when government services are separable.

6.4 Migration, Default, and Bailouts

From our discussion above, migration seems to limit the policies that can be implemented by municipalities. To understand the impact of migration, we compare our benchmark model with one in which the cost of migrating is prohibitively high so no migration takes place. Table

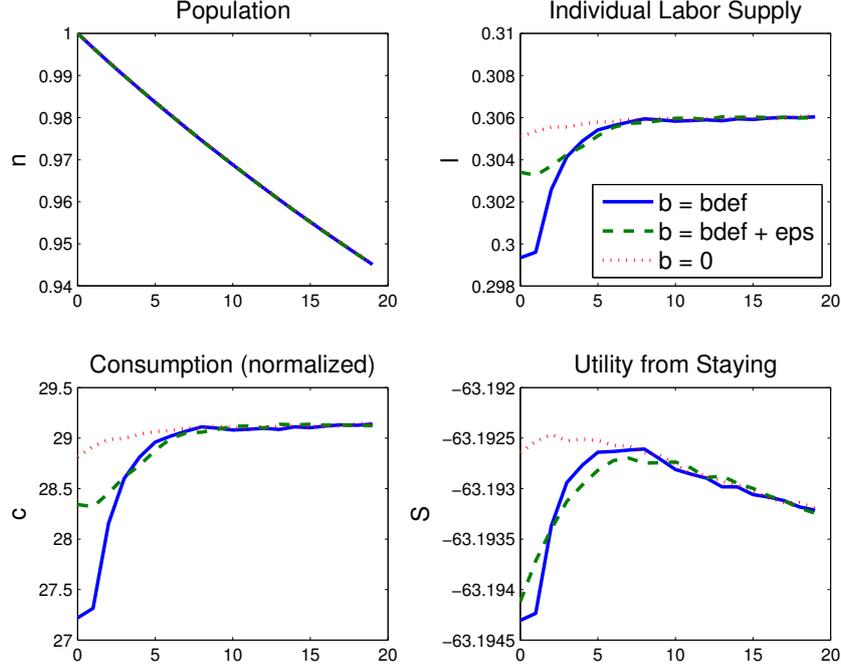


Figure 6.5: Bailout: Other Variables

8 shows that the planner finds it optimal initially to increase the tax rate on labor (by 2 percentage points). The planner can do so because people don't have other option than to stay put in their location. The default rate increases slightly because in some adverse scenarios the planner chooses to default and use those resources to increase spending to compensate workers for the higher taxes. With additional resources from taxation, the government eventually retires all debt. Consequently, the default rate goes to zero and the tax rate goes back to 4 % in the long run.

	No Migration		
	Benchmark	Short Run	Long Run
Debt-output ratio	.03	.03	.0000
Average default rate	.001	.002	.0000
Average tax rate	.04	.06	.04

Table 8: Impact of Migration on Default

The planner's willingness to increase in taxes in the short run is that in the absence of migration the planner becomes more patient. Let's assume for a moment that there is no default and productivity is constant. Under these conditions, the planner's continuation utility is

$$\beta \int \max\{S', J - \phi\} dF(\phi) = \beta(1 - F(R'))S' + \beta F(R')\mathbb{E}[J - \phi | \phi \leq R'].$$

Here, R' is the reservation cost of moving. Note that how much the planner discounts the utility of those in the island tomorrow, S' , depends on β as well as on the endogenously determined

reservation cost. Crucially, the effective discount factor, $\beta(1 - F(R'))$, is less than β . Hence, eliminating migration makes the planner discount the future at a lower rate. This *additional* patience results in not only the tax increase in the short run, but also the retiring of all debt in the long run.

Let's consider once again the bailout experiment but now without migration. In Figure 6.6, we plot the default episode (which happens absent a bailout) with and without migration. Migration has only a small effect in the short run. In fact, in the very short run, there is no difference because for as long as the sovereign is in default (i.e., has $f = 1$), their problem is static: The discount factor does not matter. However, in the long run with migration, the debt stock increases and government services start to decline; in the long run without migration, the debt stock goes to zero and stays there, and government services remain high. All these features are consistent with migration making the planner effectively impatient.

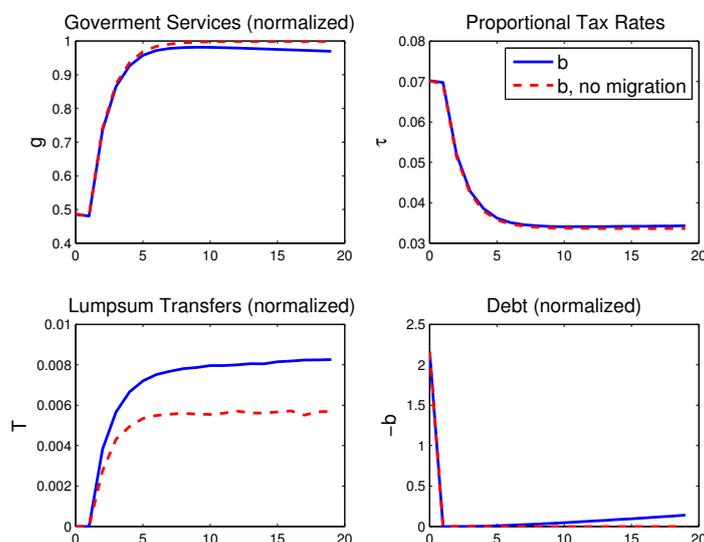


Figure 6.6: Default with and w/o Migration

Figure 6.7 displays the paths of the bailout exercise with (blue line) and without (red dashed line) migration. We also include the path without bailout in the benchmark model (the gray lines). The critical finding from this exercise is that the planner finds it optimal to use the bailout funds to pay off debt rapidly. This is possible because the planner can simultaneously raise taxes to keep sustain high government services. In contrast, a planner who faces migration has to repay debt at a slower pace. This in turn implies that taxes do not to increase sharply following default.

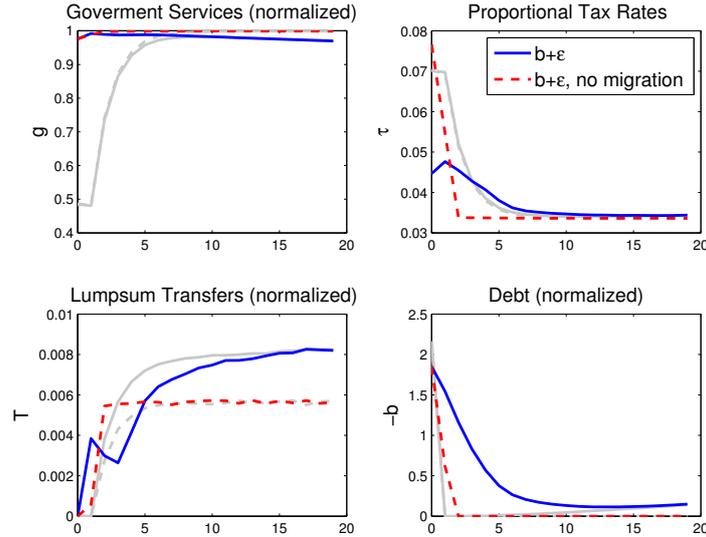


Figure 6.7: Default with and w/o Migration

7 Conclusion

In this paper, we have documented some salient features of municipalities in the U.S. Chief among them is the fact that cities do default but quite infrequently when compared to emerging economies. A second important feature is that intercity migration significantly restrains the policy options that municipalities can implement to weather difficult times. In our framework, cities default during periods of persistently low productivity and low populations. Upon default, cities optimally cut government services while sharply increasing taxes. We find that bailouts can help cities to avert default. Crucially, these bailouts help cities to maintain the provision of government services while containing the increase of taxes during tough economic times. Finally, we find that migration induces impatience in local governments. As a consequence, they are less inclined to resort to tax hikes to resolve fiscal imbalances.

Although we propose a rich framework to study migration and default across cities in the U.S., we have abstracted from important dimensions like housing and pensions. Similarly, we omit the possibility of directed migration in which households move to cities with high productivities. These missing aspects in our model are motivated by a search of tractability and they will be tackled in future work.

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A Solution Method

Assume GHH preferences of the form

$$u(c, l, g) = v(c - G(l)) + \zeta v(g).$$

In the code, one can use

$$u(c, l, g) = \frac{\left(c - \eta \frac{l^{1+1/\theta}}{1+1/\theta} + \eta \frac{1}{1+1/\theta}\right)^{1-\sigma}}{1-\sigma} + \zeta \frac{g^{1-\sigma}}{1-\sigma}$$

$\sigma = 2$ is best because then only division is needed. θ , the elasticity of labor supply, should probably be around .5. η should be set eventually so $l \approx .3$.

A.1 Overview

The overall algorithm works as follows:

1. Guess on $S(b, n, z, f)$, $\dot{n}(b, n, z, f)$, $q(b', \dot{n}, z)$, J , and n^{in} .
2. Given S , compute an update on S holding fixed \dot{n} , q , J , and n^{in} .
3. Compute the implied \dot{n} , say $\tilde{\dot{n}}$.
4. Compute the implied q , say \tilde{q} .
5. Update \dot{n} and q using a relaxation method:

$$\dot{n} := \zeta_n \tilde{\dot{n}} + (1 - \zeta_n) \dot{n} \text{ and } q := \zeta_q \tilde{q} + (1 - \zeta_q) q$$

6. Iterate until the change in S is less than 1d-7 and the change in q and n are less than 1d-5.
7. Compute the invariant distribution μ .
8. Compute the implied $\tilde{J} = \int S d\mu$ and $\tilde{n}^{in} = \int n F(R) d\mu$ using the invariant distribution.
9. If J and n^{in} have not changed much, stop. Otherwise, update the guess on J and n^{in} using a relaxation method:

$$J := \zeta_J \tilde{J} + (1 - \zeta_J) J \text{ and } n^{in} := \zeta_{n^{in}} \tilde{n}^{in} + (1 - \zeta_{n^{in}}) n^{in}.$$

A.2 Given S , compute an update on S

To compute $S^N(b, n, z, f)$ and $S^D(b, n, z, f)$ for some (b, n, z, f) , grid search over b', τ pairs to find the best value (for S^D , force $b' = 0$). S^D must be computed for the following states:

$$(b, n, z, f = 0) \forall b, n, z \text{ and } (b = 0, n, z, f = 1) \forall n, z.$$

S^N must be computed for the following states:

$$(b, n, z, f = 0) \forall b, n, z.$$

Conditional on (b', τ) , compute the period utility, g , and T as follows:

1. Get \dot{n} from $\dot{n} = n(1 - F(R(b, n, z, f))) + n^{in}$.
2. Compute the (unique) equilibrium wage, profit, and labor choice. If $\alpha = 0$, then $w^* = z$ and $\pi^* = 0$. If $\alpha > 0$, then w^* and l^* must be found as the solution to

$$\begin{aligned} l^* &= \min\{1, G'^{-1}(w^*(1 - \tau))\} \\ w^* &= z(1 - \alpha)(\dot{n}l^*)^{-\alpha} \end{aligned}$$

Given w^* , equilibrium labor supply l^* is found from the first equation (even if $\alpha = 0$). Given l^* , equilibrium profit is found from

$$\pi^* = \alpha z \dot{n}^{-\alpha} l^{*1-\alpha}$$

For the specific functional form of GHH we use, w^* is either

$$w^* = \left(\frac{1}{\dot{n}} \eta^\theta (1 - \tau)^{-\theta} z^{1/\alpha} (1 - \alpha)^{1/\alpha} \right)^{1/(\theta+1/\alpha)}$$

and $l^* < 1$ or $l^* = 1$ and $w^* = z(1 - \alpha)\dot{n}^{-\alpha}$ (where “ $\theta = 0$ ”—i.e., the supply curve is perfectly inelastic).

3. Compute r , the slackness in the government budget constraint contingent on b', d , and τ . For the S^N problem,

$$r = -q(b', \dot{n}, z)(b' - (1 - \lambda)b) + b(\lambda + (1 - \lambda)z) + \tau \dot{n} w l.$$

For the S^D problem,

$$r = (1 - \gamma)\tau \dot{n} w l.$$

4. If $r < 0$, the chosen (b', τ) combination is not feasible. Go to the next one. If $r = 0$, then the only feasible policy is $(g, T) = (0, 0)$. This won't be optimal unless it's the only feasible policy. If $r > 0$, go to 4.
5. Define $(g_1, T_1) = (0, \frac{r}{\dot{n}})$, $(g_2, T_2) = (r, 0)$, and (g_3, T_3) by

$$T_3 = \frac{G(l) - w(1 - \tau)l - \pi + v'^{-1}(\zeta \dot{n})r}{1 + \dot{n}v'^{-1}(\zeta \dot{n})}$$

and $g_3 = r - T_3$. Check that $T_3 > 0$. If not, don't consider (g_3, T_3) as a choice below.

6. Choose the best (g_i, T_i) evaluating them according to $v(w(1-\tau)l + T_i + \pi - G(l)) + \zeta v(g_i)$. Let the corresponding utility and policies be denoted $U^N(b', \tau)$, $g^N(b', \tau)$ and $T^N(b', \tau)$ for S^N and similarly for S^D .

Given U^N and U^D , the discounted lifetime utility from a choice of (b', τ) is

$$U^N(b', \tau) + \beta \mathbb{E}_z V(\phi', b', \dot{n}, z', f' = 0)$$

in repaying and

$$U^D(b' = 0, \tau) + \beta \mathbb{E}_z \left(\begin{array}{l} \delta V(\phi', b' = 0, \dot{n}, z', f' = 1) \\ +(1 - \delta)V(\phi', b' = 0, \dot{n}, z', f' = 0) \end{array} \right)$$

if defaulting or in default. S^N and S^D are just the maximums of these lifetime utilities:

$$S^N(b, n, z, f) = \max_{b', \tau} U^N(b', \tau) + \beta \mathbb{E}_z V(\phi', b', \dot{n}, z', f' = 0)$$

and

$$S^D(b, n, z, f) = \max_{\tau} U^D(b' = 0, \tau) + \beta \mathbb{E}_z \left(\begin{array}{l} \delta V(\phi', b' = 0, \dot{n}, z', f' = 1) \\ +(1 - \delta)V(\phi', b' = 0, \dot{n}, z', f' = 0) \end{array} \right)$$

The expectations in the continuation utility can be computed using the reservation strategy, S , and J (as well as transitions in z).

The assumptions here significantly reduce the difficulty of the problem and increase accuracy. Rather than searching over a coarse grid for T or g and computing the other one as a residual, this finds the optimal T, g pair exactly (conditional on b', τ). Additionally, T and g both don't have a priori upper bounds while τ does.

Note that one does not need to search over all $\tau \in [0, 1]$, but only the τ to the left of the Laffer curve peak (the Laffer curve for these preferences is only a function of τ). The peak, if not known analytically, can be computed.

With S^D and S^N , one can recover \tilde{S} via

$$\begin{aligned} \tilde{S}(b, n, z, f = 0) &= \max_d S^N(b, n, z, f = 0)(1 - d) + S^D(b, n, z, f = 0)d \\ \tilde{S}(b = 0, n, z, f = 1) &= S^D(b = 0, n, z, f = 1) \end{aligned} \tag{A.1}$$

(if S^N is undefined because no feasible policy exists, $\tilde{S} = S^D$).

A.3 Use updated S to get update on \dot{n}

Compute R using S , J , and ϕ . Then use the law of motion for n :

$$\dot{n} = n(1 - F(R)) + n^{in}$$

A.4 Use updated S to get update on q

Use the policies and the pricing equation from the equilibrium conditions.

Let the current price schedule be thought of as next period's price schedule, i.e. q_{t+1} . The objective is to compute the implied price schedule today, q_t , taking q_{t+1} as given. In the case of short-term debt, q_{t+1} is irrelevant. Similarly, let q_{t+1}^r be next period's price for defaulted debt and q_t^r be the current period's price.

The formulas for debt pricing are

$$q_t(b', n', z) = \bar{q} \mathbb{E}_z [(1 - d')[\lambda + (1 - \lambda)(\kappa + q_{t+1}(b'', n'', z'))] + d' q_{t+1}^r(b_0 = b', b', n', z', f' = 0)]$$

where $d' = d(b', n', z', f' = 0)$ and $b'' = b'(b', n', z', f' = 0)$ and $n'' = \dot{n}(b', n', z', f' = 0)$. The price of defaulted debt $-b_0$ is

$$q_t^r(b_0, \mathbf{x}) = \frac{\min\{\gamma w(\mathbf{x})\tau(\mathbf{x})\dot{n}(\mathbf{x})l(\mathbf{x}), -b_0(1 - \bar{q}(1 - \delta))\}}{-b_0} + \bar{q}\delta \mathbb{E}_z q_{t+1}^r(b_0, b' = 0, \dot{n}(\mathbf{x}), z', f' = 1).$$

Using the policy functions, it is straightforward to compute these updates.

A.5 Compute the invariant distribution

We approximate the invariant distribution with a mass density function (mdf) which we call $\mu(b, n, z, f)$. Since it is an mdf, $\sum_{b,n,z,f} \mu(b, n, z, f) = 1$ and $\mu(b, n, z, f) \in [0, 1]$ for all (b, n, z, f) . The measure of islands is one, but also the measure of households is one. The measure of households, according to the mdf, is $\sum_{b,n,z,f} n\mu(b, n, z, f) = 1$.

We always compute an invariant distribution conditional on the same starting distribution. We do not know if there are multiple invariant distributions, so we try to be consistent in selecting it in the same way. The initial distribution has zero mass except where $b = 0$, $f = 0$ and $n = 1$. The distribution for $\mu(b = 0, n = 1, z, f = 0)$ is the invariant distribution of z . Note that this initial distribution both sums to one and has a household population of one.

Let the initial distribution be denoted μ_0 . Let $t \geq 0$ and suppose μ_t is known. We compute μ_{t+1} in the following way. Take $R = J - S$ as given.

1. Set $\mu_{t+1}(b, n, z, f) = 0$ for all (b, n, z, f) .
2. Choose a state $\mathbf{x} = (b, n, z, f)$.
3. Compute $n^{in} = \sum_{\mathbf{x}} nF(R(\mathbf{x}))\mu_t(\mathbf{x})$. n^{in} gives the measure of households migrating given the reservation strategy if the distribution of islands is μ_t . Note that it changes on each iteration.
4. Define $m_t = \mu_t(\mathbf{x})$. If $m_t > 0$, go to the next step. If $m_t = 0$, go to the final step.
5. Compute $\dot{n} = n(1 - F(R(\mathbf{x}))) + n^{in}$. Clearly, \dot{n} is not likely to lie on a grid. The next step handles this.

6. For an increasing n -grid of $\{n_i\}_{i=1}^N$, let i be 1 if $\dot{n} < n_1$, $N - 1$ if $\dot{n} \geq n_N$, or the i such that $\dot{n} \in [n_i, n_{i+1})$ if $\dot{n} \in [n_1, n_N)$.

Now, compute $\omega_i = \min\{1, \max\{0, (n_{i+1} - \dot{n})/(n_{i+1} - n_i)\}\}$.

7. Distribute m_t into μ_{t+1} as follows:

- (a) If $f = 0$ and $d = 0$, then $f' = 0$. So

$$\begin{aligned}\mu_{t+1}(b'(b, n, z, f), n_i, z', 0) &:= \mu_{t+1}(b'(b, n, z, f), n_i, z', 0) + F(z'|z)\omega_i m_t \\ \mu_{t+1}(b'(b, n, z, f), n_{i+1}, z', 0) &:= \mu_{t+1}(b'(b, n, z, f), n_{i+1}, z', 0) + F(z'|z)(1 - \omega_i)m_t\end{aligned}$$

for all z' .

- (b) If $(f, d) = (0, 1)$ or $f = 1$, then $f' = 1$ w.p. δ and 0 w.p. $1 - \delta$. So,

$$\begin{aligned}\mu_{t+1}(0, n_i, z', 0) &:= \mu_{t+1}(0, n_i, z', 0) + (1 - \delta)F(z'|z)\omega_i m_t \\ \mu_{t+1}(0, n_i, z', 1) &:= \mu_{t+1}(0, n_i, z', 1) + \delta F(z'|z)\omega_i m_t \\ \mu_{t+1}(0, n_{i+1}, z', 0) &:= \mu_{t+1}(0, n_{i+1}, z', 0) + (1 - \delta)F(z'|z)(1 - \omega_i)m_t \\ \mu_{t+1}(0, n_{i+1}, z', 1) &:= \mu_{t+1}(0, n_{i+1}, z', 1) + \delta F(z'|z)(1 - \omega_i)m_t\end{aligned}$$

for all z' .

8. Go to the next state if any have not yet been visited. Otherwise, stop: μ_{t+1} is the next period distribution.

Using this procedure, not only does μ_{t+1} sum to one, but also the expected value of n (i.e., the population of households) also equals one if $\dot{n} \in [n_1, n_N]$ for every state. The reason is using the weights ω_i distribute mass in such a way that the mean is unchanged (as long as the max and min statements don't bind). See Young (2010) for details in a similar context.

Given the next period distribution μ_{t+1} , one must determine whether μ_t and μ_{t+1} are close enough. For this, we use the supnorm with a tight stopping criteria. Namely, stop if $\max_{\mathbf{x}} |\mu_{t+1}(\mathbf{x}) - \mu_t(\mathbf{x})| < 10^{-12}$. If this is not meant, t is incremented and the process repeats.

B Extended Model

To model the endowment given to the island government, assume the endowment takes the form $\omega(b)e$ where $\omega(b)$ is some deterministic function of b and $e \in E$ follows a Markov chain $F(e'|e)$. This nests the cases we wish to consider, and allows flexibility for future expansion:

1. There is no endowment (the benchmark): $\omega(b) = 0$, $E = \{0\}$, and $F(0|0) = 1$.
2. The endowment is not a function of debt, is unanticipated, and disappears next period. In this case, $\omega(b) = \omega$, $E = \{0, 1\}$, and $F(0|e) = 1, F(1|e) = 0$. This way, if a sovereign happens to have $e = 1$, then they get ω for one period only.

3. The endowment is a deterministic function of debt and is anticipated. In this case, $\omega(b)$ is set to the desired schedule, $E = \{1\}$, and $F(1|1) = 1$.

Additionally, let the repayment rate γ be stochastic and iid with some mdf π_γ . Moreover, let it only be known after default is declared, and let it be drawn new each period.

Finally, allow for pensions by having a stochastic lower bound on transfers \underline{T} that follows a Markov chain $\pi_{TT'}$. Of course this nests the benchmark case of $\underline{T} = 0$ and $\pi_{TT'} = 1$. We assume that when in default, $\underline{T} = 0$. The interpretation is that pension obligations are discharged through default. In fact, one must assume that this lower bound on transfers only applies in the absence of default: Otherwise, there may not be a feasible choice for the planner.

Abusing notation, let \mathbf{z} be a vector of (z, e, \underline{T}) . Then \mathbf{z} follows some Markov chain $\pi_{\mathbf{z}\mathbf{z}'}$.

$$\tilde{S}(b, n, \mathbf{z}, f = 0) = \max_d S^N(b, n, \mathbf{z}, f)(1 - d) + S^D(b, n, \mathbf{z}, f)d \quad (\text{B.1})$$

(if S^N is undefined because no feasible policy exists, $\tilde{S} = S^D$) and

$$\tilde{S}(b = 0, n, \mathbf{z}, f = 1) = S^D(b = 0, n, \mathbf{z}, f = 1) \quad (\text{B.2})$$

where the value of repaying is

$$\begin{aligned} S^N(b, n, \mathbf{z}, f) &= \max_{\tau, T, g, b'} U(w^*, \pi^*, \tau, T, g) + \beta \mathbb{E}_{\mathbf{z}} V(\phi', b', \dot{n}, \mathbf{z}', f' = 0) \\ \text{s.t. } g + q(b', \dot{n}, \mathbf{z})(b' - (1 - \lambda)b) &\leq b(\lambda + (1 - \lambda)\kappa) + \tau \dot{n} w^* l(w^*, \pi^*, \tau, T, g) - T \dot{n} + \omega(b)e \\ g &\geq 0, \tau \in [0, 1), T \geq \underline{T}, b' \in B \\ w^* &= w^*(\tau, T, g) \\ \pi^* &= \pi^*(\tau, T, g) \\ \dot{n} &= n(1 - F(R(\mathbf{x}))) + n^{in} \end{aligned} \quad (\text{B.3})$$

and the value of defaulting is

$$S^D(b, n, \mathbf{z}, f) = \sum_{\gamma} \pi_{\gamma} S_{\gamma}^D(\gamma, b, n, \mathbf{z}, f) \quad (\text{B.4})$$

where

$$\begin{aligned}
S_\gamma^D(\gamma, b, n, \mathbf{z}, f) &= \max_{\tau, T, g} U(w^*, \pi^*, \tau, T, g) + \beta \mathbb{E}_{\mathbf{z}} \left(\begin{array}{l} \delta V(\phi', b' = 0, \dot{n}, \mathbf{z}', f' = 1) \\ +(1 - \delta)V(\phi', b' = 0, \dot{n}, \mathbf{z}', f' = 0) \end{array} \right) \\
&\text{s.t.} \\
g &\leq (1 - \gamma)(\tau \dot{n} w^* l(w^*, \pi^*, \tau, T, g)) - T \dot{n} \\
g &\geq 0, \tau \in [0, 1), T \geq 0, b' = 0 \\
w^* &= w^*(\tau, T, g) \\
\pi^* &= \pi^*(\tau, T, g) \\
\dot{n} &= n(1 - F(R(\mathbf{x}))) + n^{in}
\end{aligned} \tag{B.5}$$

C Additional Derivations

C.1 Household optimization and the labor market equilibrium

C.1.1 GHH preferences

The most convenient preferences to use, and the ones we adopt in the benchmark, are [Greenwood, Hercowitz, and Huffman \(1988\)](#) (GHH). We will consider two specifications,

$$u(c - G(l)) + v(g) \text{ and } u(c - G(l), g)$$

where $G(l)$ satisfies $G(1) \geq 0$, $G' > 0$, $G'' > 0$, and $G'(l) = 0$ and u is increasing, concave, and defined for all positive arguments.²² In either specification, an interior choice of labor is

$$G'(l) = w(1 - \tau),$$

the constraint $l \geq 0$ never binds, and the $l \leq 1$ binds if $G'^{-1}(w(1 - \tau)) > 1$. In other words, the optimal labor choice is

$$l = \min\{1, G'^{-1}(w(1 - \tau))\}.$$

Note that since $G'(0) = 0$ and G' is an increasing function, G'^{-1} is a strictly increasing continuous function that begins at 0. Consequently, the labor supply curve is an increasing continuous function beginning from zero and going to infinity. Additionally the labor demand function is a strictly decreasing continuous function with a limit at ∞ as l goes to zero and a limit at 0 as l goes to ∞ . Hence, there is a unique equilibrium with a wage $w^* > 0$ and $l^* > 0$.

Here, the level of transfers is irrelevant for the intensive margin of labor supply (although it is still relevant for the extensive margin).

If $\alpha = 0$, there is a unique equilibrium with $w^* = z$. If $\alpha > 0$, we can say the equilibrium wage must be one of two values. The first is when $l^* = 1$. In this case, $w^* = z(1 - \alpha)\dot{n}^{-\alpha}$. The

²²Without the assumption $G(1) \geq 0$, one needs an additional constraint that $c - G(l) \geq 0$ if u is only defined for positive values.

second is when $l^* < 1$. In this case, w^* must satisfy

$$\frac{1}{\dot{n}} \left(\frac{w}{z(1-\alpha)} \right)^{-1/\alpha} = G'^{-1}(w(1-\tau)).$$

This is a one-dimensional root-finding problem that is not difficult to solve.

In the quantitative work, we use the function forms

$$u(c, l, g) = \frac{\left(c - \eta \frac{l^{1+1/\theta}}{1+1/\theta} + \eta \frac{1}{1+1/\theta} \right)^{1-\sigma}}{1-\sigma} + v(g) \text{ and } u(c, l, g) = \frac{\left(h(g, \left(c - \eta \frac{l^{1+1/\theta}}{1+1/\theta} + \eta \frac{1}{1+1/\theta} \right)) \right)^{1-\sigma}}{1-\sigma}$$

where h is a CES aggregator and $\theta > 0$ the elasticity of labor supply. The optimal labor-leisure choice is then given by

$$l = \eta^{-\theta} ((1-\tau(\mathbf{x}))w(\mathbf{x}))^\theta$$

when $l < 1$. Note that θ is the labor supply elasticity: $d \log l / d \log w = \theta$. The equilibrium labor and wage are either $l^* = 1$ and $w^* = z(1-\alpha)\dot{n}^{-\alpha}$ or $l^* < 1$ and

$$w^* = \left(\frac{1}{\dot{n}} \eta^\theta (1-\tau)^{-\theta} z^{1/\alpha} (1-\alpha)^{1/\alpha} \right)^{1/(\theta+1/\alpha)}.$$

C.1.2 CRRA preferences

This section details the labor market equilibrium for household preferences with income effects.

Choosing a utility function for the household of

$$u(c, l, g) = \frac{(c^\theta (1-l)^{1-\theta})^{1-\sigma}}{1-\sigma} + v(g)$$

we can get closed form solutions for c^* and l^* (note $l^* = 1$ is never optimal with these preferences unless it is the only feasible choice). They are

$$l^* = \max\left\{0, \theta - \frac{(1-\theta)(T+\pi)}{(1-\tau)w}\right\}$$

and

$$c^* = (1-\tau)wl^* + T + \pi$$

Since there is no heterogeneity, if $l^* = 0$, there is also no output on an island. This may be optimal for the planner since he can borrow.

If the optimal taxes are such that $l^* = 0$, then $c^* = T$ and τ is irrelevant. Moreover, the indirect utility function $u(c^*, l^*) = T^{\theta(1-\sigma)} / (1-\sigma)$ (we must impose that the planner guarantees the household budget constraint is non-empty).

If the optimal taxes are such that $l^* > 0$, then we have

$$c^* = \theta((1-\tau)w + T + \pi),$$

and

$$l^* = \theta - \frac{(1-\theta)(T+\pi)}{(1-\tau)w}.$$

The equilibrium wage, labor choice, and profit must satisfy three equations:

$$l^*(\mathbf{x}) = \max\left\{0, \theta - \frac{(1-\theta)(T+\pi(\mathbf{x}))}{(1-\tau)w}\right\}$$

$$w(\mathbf{x}) = z(1-\alpha)(\dot{n}l^*(\mathbf{x}))^{-\alpha}$$

$$\pi(\mathbf{x}) = \alpha z \dot{n}(\mathbf{x})^{-\alpha} l^*(\mathbf{x})^{1-\alpha}$$

If $\alpha = 0$, there is a unique equilibrium. If $\alpha > 0$, then note the second equation can be written

$$l^*(\mathbf{x}) = \frac{1}{\dot{n}} \left(\frac{w(\mathbf{x})}{z(1-\alpha)} \right)^{-1/\alpha}$$

which begins at ∞ and falls to 0 as w goes from 0 to ∞ . However, because of wealth effects, there are three cases to consider.

There cannot be an equilibrium where $l^* = 0$ since the second equation would not be satisfied (i.e., there does not exist a wage such that the firm would choose zero labor demand). So focus on $l^* > 0$. Also, take the three equations and eliminate π to bring it down to two. Then one has

$$l^*(\mathbf{x}) = \theta - \frac{(1-\theta)(T + \alpha z \dot{n}(\mathbf{x})^{-\alpha} l^*(\mathbf{x})^{1-\alpha})}{(1-\tau)w}$$

$$w(\mathbf{x}) = z(1-\alpha)(\dot{n}l^*(\mathbf{x}))^{-\alpha}.$$

The first equation is

$$l^*(\mathbf{x}) = \theta - \frac{(1-\theta)T}{(1-\tau)w} - (1-\theta) \frac{\alpha z \dot{n}(\mathbf{x})^{-\alpha} l^*(\mathbf{x})^{1-\alpha}}{(1-\tau)w}$$

or

$$l^*(\mathbf{x}) = \theta - \frac{(1-\theta)T}{(1-\tau)w} - (1-\theta) \frac{\alpha z \dot{n}(\mathbf{x})^{-\alpha} l^*(\mathbf{x})^{1-\alpha}}{(1-\tau)z(1-\alpha)(\dot{n}l^*(\mathbf{x}))^{-\alpha}}$$

or

$$l^*(\mathbf{x}) = \theta - \frac{(1-\theta)T}{(1-\tau)w} - (1-\theta) \frac{\alpha l^*(\mathbf{x})}{(1-\tau)(1-\alpha)}$$

or

$$\left(1 + \frac{\alpha(1-\theta)}{(1-\tau)(1-\alpha)}\right) l^*(\mathbf{x}) = \theta - \frac{(1-\theta)T}{(1-\tau)w}$$

or

$$l^*(\mathbf{x}) = \gamma \left(\theta - \frac{(1-\theta)T}{(1-\tau)w} \right)$$

where $\gamma := \left(1 + \frac{\alpha(1-\theta)}{(1-\tau)(1-\alpha)}\right)^{-1}$ is a strictly positive constant.

First is $T = 0$. In this case $l^* = \gamma\theta$ and there is a unique equilibrium. Second is $T > 0$. In that case, l^* is increasing in the wage and so there is a unique equilibrium. Third is $T < 0$. In that case, l^* is decreasing in the wage and is bounded below by $\gamma\theta$. So, an equilibrium exists,

but it is not necessarily unique (at least I have not shown that).

Our restriction that the planner have $T \geq 0$ guarantees a unique equilibrium in the labor market. It is also equivalent to saying the substitution effect dominates the income effect (i.e., l^* is increasing in the wage).

Regardless of preferences, when $\alpha > 0$, a necessary condition for a labor market equilibrium is $l^* > 0$ which requires $\tau < 1$.

C.2 Proposition 4 with explicit formulas.

For the computation, we now provide more explicit formulas. Take $u(c, l, g) = v(c - G(l)) + \zeta v(g)$.

1. In the case of $\lambda = 0$ (implying $\tau = 0$),

$$v' \left(z \dot{n}^{-\alpha} l^*(0)^{1-\alpha} + \frac{-g}{\dot{n}} + \frac{r}{\dot{n}} - G(l^*(0)) \right) = \zeta v'(g) \dot{n}$$

Solving for g and calling the solution \hat{g} , one has

$$\hat{g} = \frac{y^*(0) + r - \dot{n}G(l^*(0))}{1 + \dot{n}v'^{-1}(\zeta \dot{n})}.$$

Hence, $(\tau, T, g) = (0, \frac{r-\hat{g}}{\dot{n}}, \hat{g})$ if $r - \hat{g} \geq 0$.

2. In the case of $\theta = 0$ (implying $T = 0$), solving for λ from FOC_g and substituting into FOC_τ , one has two equations characterizing the solution for (τ, g) :

$$\begin{aligned} u_c \tau w^*(\tau) l^{*'}(\tau) + (u_g \dot{n} - u_c) REV'(\tau) &= 0 \\ g &= r + \dot{n} \tau w^* l^*. \end{aligned}$$

Substituting the budget constraint in and suppressing dependence on τ ,

$$v'(c^* - G(l^*)) (\tau w^* l^{*'} - REV') + \zeta v'(g) \dot{n} REV' = 0.$$

One can show $c^* = \left(\frac{1}{1-\alpha} - \tau \right) w^* l^*$ using the relations $z \dot{n}^{-\alpha} l^{*(1-\alpha)} = w^* l^*/(1-\alpha)$ and $g = r + \dot{n} \tau w^* l^*$.

This is a one-dimensional root-finding problem (unfortunately, a closed-form solution does not seem to exist for general v). However, rather than finding the root, it is computationally faster, and less error prone, to find the root by solving

$$\hat{\tau} = \arg \max_{\tau} v \left(\left(\frac{1}{1-\alpha} - \tau \right) w^* l^* - G(l^*) \right) + \zeta v(r + \dot{n} \tau w^* l^*).$$

For reference, when $G(l)$ is $\eta(l^{1+1/\theta} - 1)/(1 + 1/\theta)$,

$$w^*(\tau) = \left(\frac{1}{\dot{n}} \eta^\theta (1-\tau)^{-\theta} z^{1/\alpha} (1-\alpha)^{1/\alpha} \right)^{1/(\theta+1/\alpha)}.$$

$$l^*(\tau) = \eta^{-\theta}(1 - \tau)^\theta w^*(\tau)^\theta$$

Then $(\tau, T, g) = (\hat{\tau}, 0, r + \dot{n}\hat{\tau}w^*(\hat{\tau})l^*(\hat{\tau}))$.

3. In the case of $\lambda > 0$ and $\theta > 0$, $(\tau, T, g) = (0, 0, r)$. Since $c^*(\tau, g) = \frac{y^*(\tau) - g + r}{\dot{n}}$, this has $c^* = y^*/\dot{n}$.

We note that the optimal policy is the max of the three points, $(0, \frac{r - \hat{g}}{\dot{n}}, \hat{g})$, $(\hat{\tau}, 0, r + \dot{n}\hat{\tau}w^*(\hat{\tau})l^*(\hat{\tau}))$, and $(0, 0, r)$, which is potentially useful for computation.