Web Appendix for "Exploiting Naivete about Self-Control in the Credit Market" — Proofs

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Proof of Lemma 1. ($\Rightarrow$) Suppose $(c, \mathcal{C})$ satisfies the condition of the lemma. Since only this contract is offered and it satisfies the borrower's PC, it is optimal for her to accept the contract and her choice between contracts is trivial. Thus Condition 1 of Definition 2 is satisfied. Conditions 2 and 4 hold by construction. The key part is to check Condition 3. Consider a contract $(c', \mathcal{C}')$ with incentive compatible repayment terms that the borrower strictly prefers. Incentive compatibility guarantees that the contract satisfies IC and PCC, and the fact that the borrower strictly prefers it implies that PC is satisfied when the outside option is $\underline{u}$. Hence, because $(c', \mathcal{C}')$ satisfies all constraints that $(c, \mathcal{C})$ does, and $(c, \mathcal{C})$ is optimal given these constraints and yields zero profits, $(c', \mathcal{C}')$ cannot yield positive expected profits.

($\Leftarrow$) Since there is only one $\hat{\beta}$ type, there can only be one contract. Let $(c, \mathcal{C})$ be that competitive-equilibrium contract. Condition 4 (non-redundancy) implies that there are only two repayment options in the contract, one for β_1 and one for β_2. Incentive compatibility implies that $(c, \mathcal{C})$ satisfies IC and PCC, and it trivially satisfies PC with $\underline{u}$ defined as the perceived utility from $(c, \mathcal{C})$. Now suppose by contradiction that $(c, \mathcal{C})$ does not maximize profits given these constraints. Then, there is a contract $(c', \mathcal{C}')$ that satisfies the same constraints and yields strictly positive profits. This means that for a sufficiently small $\epsilon > 0$, $(c' + \epsilon, \mathcal{C}')$ attracts all borrowers and yields strictly positive profits, violating Condition 3 of Definition 2.

Proof of Fact 1. It follows from Proposition 1 that she borrows $c = 2(k')^{-1}(1)$ and repays $(k')^{-1}(1)$ in each period in the unrestricted market, and from the proof of Proposition 3 that she borrows and repays the same amounts in long-term restricted market.

Proof of Proposition 1. A sophisticated borrower correctly foresees the repayment option she eventually chooses. Thus, a non-redundant contract (i.e. one that satisfies Condition 4 of Definition 2) has a single repayment option (q, r). Using this fact, Conditions 1 and 3 of Definition 2 imply that any competitive contract $(c, \mathcal{C})$ must solve

$$\max_{c,q,r} \quad q + r - c$$

$$\text{(PC)} \qquad \text{s.t.} \quad c - k(q) - k(r) \geq \underline{u},$$

where $\underline{u}$ is the perceived utility from accepting the competitive contract $(c, \mathcal{C})$. It is clear that in the maximization problem above PC is satisfied with equality; otherwise, the firm could increase profits by lowering c. Plugging PC into the maximand, we can rewrite the firm's problem as

$$\max_{q,r} \quad q + r - k(q) - k(r).$$

Solving this maximization yields $k'(q) = k'(r) = 1$ in any competitive contract. Furthermore, the zero-profit condition (Condition 2) implies that $c = q + r$, and this completely characterizes the unique competitive-equilibrium contract.

Proof of Proposition 2. We have established in the text that $\hat{q} > 0, \hat{r} = 0, k'(q) = 1, k'(r) = 1/\beta$, and Lemma 1 implies that $c = q + r$. Using Proposition 1, the sophisticated and non-sophisticated borrowers repay the same amount in period 1, but the non-sophisticated borrower repays more in period 2. Hence, the non-sophisticated consumer borrows more than the sophisticated one.

To show that $q + r > \hat{q}$, suppose by contradiction that $\hat{q} \geq q + r$. Then, notice that for a sufficiently small $\epsilon > 0$, self 0 strictly prefers the repayment schedule $(\hat{q}/2 + \epsilon, \hat{q}/2 + \epsilon)$ to $(\hat{q}, 0)$, the terms she thinks she is going to choose with the competitive-equilibrium contract. Hence, the firm could increase profits by offering a single repayment schedule $(\hat{q}/2 + \epsilon, \hat{q}/2 + \epsilon)$, a contradiction.

Finally, from the proof of Proposition 1 it is clear that the contract offered to a sophisticated borrower is the unique contract that maximizes period-0 welfare among all contracts that break even $(c = q + r)$. Since the borrower's contract also breaks even and differs from the sophisticated one, the borrower is strictly worse off than a sophisticated borrower.

Proof of Proposition 3. Let a restricted contract be described by the triplet (c, R, L), where c is consumption, R is the interest rate, and L is the present discounted value of total repayment from the perspective of period 1, using the interest rate R.

Consider sophisticated borrowers first. Notice that a contract with $R = 1/\beta$ will induce the borrower to repay in equal installments. This means that a contract that combines $R = 1/\beta$ with the ex-ante optimal consumption level c^* and the competitive L^* maximizes the borrower's utility subject to the constraint that consumption is equal to total repayment. Conversely, no other contract with which a firm breaks even maximizes the borrower's utility: for the borrower to repay according to $k'(q) = k'(r) = 1$, the contract must have $R = 1/\beta, L = L^*$, and then for the firm to break even consumption must be c^*. Hence, if this contract was not offered but firms made zero profits, for a sufficiently small $\epsilon > 0$ the contract $(c^* - \epsilon, 1/\beta, L^*)$ could be profitably introduced. Hence, $(c^*, 1/\beta, L^*)$ is the unique competitive-equilibrium contract.

Now we consider non-sophisticated borrowers. For any R, L, there is a unique repayment schedule (q, r) the borrower follows, and hence a unique $c(R, L) = q + r$ with which a firm breaks even. Let $\mathcal{B}$ be the set of contracts $(c(R, L), R, L)$; this is the set of contracts that if accepted break even given the borrower's actual behavior, and is independent of $\hat{\beta}$. Furthermore, consider the borrower's perceived utility $\underline{U}_{\hat{\beta}}(c, R, L)$ as a function of (c, R, L) over $\mathcal{B}$; this is a function of $\hat{\beta}$. Notice that a competitive-equilibrium contract maximizes $\underline{U}_{\hat{\beta}}$ over $\mathcal{B}$; otherwise, a firm could find a contract that breaks even and gives the borrower higher perceived utility, and starting from this contract could decrease c slightly, attracting the borrower and earning positive profits. To see that competitive equilibrium exists, we first show that without loss of generality we can assume that $R \in [k'(0)/(\hat{\beta}k'(M)), k'(M)/(\beta k'(0))]$, and $L \in [0, M + M\hat{\beta}k'(M)/k'(0)]$. The borrower believes she will choose $\hat{q}$ to solve

$$\min_{\hat{q}} k(\hat{q}) + \hat{\beta}k(R(L - \hat{q})) \text{ s.t. } 0 \leq \hat{q} \leq M \text{ and } 0 \leq R(L - \hat{q}) \leq M, \tag{5}$$

and she actually chooses q to solve the above problem with q and β replacing $\hat{q}$ and $\hat{\beta}$. Hence, for any $R \geq k'(M)/(\beta k'(0))$ we have a corner solution in which $q = \hat{q} = M$ and hence the second-period repayment amounts are $\hat{r} = r = R(L - M)$. The firm can thus replicate the outcome of any contract (c, R, L) in which $R > k'(M)/(\beta k'(0))$ by one in which $R = k'(M)/(\beta k'(0))$ and L is appropriately adjusted. Similarly, if $R \leq k'(0)/(\hat{\beta}k'(M))$, then $q = \hat{q} = 0$, so that we can replace any contract featuring $R < k'(0)/(\hat{\beta}k'(M))$ with a contract featuring $R = k'(0)/(\hat{\beta}k'(M))$. Hence, without loss of generality we can restrict attention to contracts in which $R \in [k'(0)/(\hat{\beta}k'(M)), k'(M)/(\beta k'(0))]$. Since repayment amounts in each period are bounded from above by M and the interest rate from below by $k'(0)/(\hat{\beta}k'(M))$, we can furthermore restrict attention to $L \in [0, M + M\hat{\beta}k'(M)/k'(0)]$. Now since q, r (and hence $c = q + r$) and $\hat{q}, \hat{r}$ are continuous in R, L and R, L are chosen from compact sets, it follows that a contract exists that maximizes $\underline{U}_{\hat{\beta}}(c, R, L)$ over $\mathcal{B}$.

Now notice that given a contract (c, R, L), the borrower's perceived repayment behavior is continuous in $\hat{\beta}, R, L$, which in turn implies that $\underline{U}_{\hat{\beta}}(c, R, L)$ is continuous in $\hat{\beta}, c, R, L$. For $\hat{\beta} = \beta$, we have shown above that $\underline{U}_{\hat{\beta}}$ has a unique maximum at $(c^*, 1/\beta, L^*)$. We complete the proof by showing that as a result, if $\hat{\beta} \to \beta$, any selection of maximizers

$(c(\hat{\beta}), R(\hat{\beta}), L(\hat{\beta}))$ of $\underline{U}_{\hat{\beta}}$ over $\mathcal{B}$ must approach $(c^*, 1/\beta, L^*)$. This means that in the restricted market the welfare of a non-sophisticated borrower approaches that of a sophisticated borrower as $\hat{\beta} \to \beta$. In contrast, by Propositions 1 and 2, in the unrestricted market the welfare of a non-sophisticated borrower does not approach that of a sophisticated borrower as $\hat{\beta} \to \beta$, so for $\hat{\beta}$ sufficiently close to β the restricted market yields higher welfare.

Suppose by contradiction that there is some selection of maximizers $(c(\hat{\beta}), R(\hat{\beta}), L(\hat{\beta}))$ of $\underline{U}_{\hat{\beta}}$ over $\mathcal{B}$ that does not converge to $(c^*, 1/\beta, L^*)$ as $\hat{\beta} \to \beta$. Since the $(c(\hat{\beta}), R(\hat{\beta}), L(\hat{\beta}))$ are within a compact set, there must be a convergent subsequence with limit $(c, R, L) \neq (c^*, 1/\beta, L^*)$. Since $\mathcal{B}$ is closed, $(c, R, L) \in \mathcal{B}$. We know that $U_{\hat{\beta}}(c(\hat{\beta}), R(\hat{\beta}), L(\hat{\beta})) \geq U_{\hat{\beta}}(c^*, 1/\beta, L^*)$, so by continuity $U_\beta(c, R, L) \geq U_\beta(c^*, 1/\beta, L^*)$, contradicting that U_β has a unique maximum over $\mathcal{B}$ at $(c^*, 1/\beta, L^*)$.

Proof of Proposition 4. Let us call the restricted market in which the interest rate is zero (i.e. $R = 1$) the capped market. We begin by showing that the borrower's consumption is lower in the capped market than in the unrestricted market. Since self 0 thinks self 1's cost of repayment is $k(q) + \hat{\beta}k(r)$, she believes that for any L, self 1 will choose the repayment schedule by minimizing $k(q) + \hat{\beta}k(L - q)$ subject to $q, L - q \leq M$; let the solution be $\hat{q}$, and set $\hat{r} = L - \hat{q}$. In the competitive equilibrium of the capped market, the amount of credit c maximizes the borrower's perceived utility subject to $c = L$; otherwise, the firm could offer a contract that both has higher perceived utility and has $c < L$, attracting the borrower and making positive profits. We first observe that the competitive-equilibrium c is such that $\hat{q}, \hat{r} < M$. Suppose by contradiction that $\hat{q} \geq M$ or $\hat{r} \geq M$. Then, because $\hat{\beta} \leq 1$ implies $\hat{r} \geq \hat{q}$, we must have $\hat{r} = M$. Hence $k'(\hat{r}) = k'(M) \geq 1/\beta$, and using the perceived cost minimization of the borrower, $k'(\hat{q}) \geq \hat{\beta}k'(\hat{r}) \geq \hat{\beta}/\beta > 1$. Therefore, because the perceived marginal cost of repayment in both periods is strictly greater than the marginal utility of consumption, decreasing c and $L = c$ by a small amount increases the borrower's perceived utility independently of how she believes she will allocate the decreased L across periods 1 and 2, a contradiction. By a similar argument, we can show that competitive-equilibrium c is such that $\hat{q}, \hat{r} > 0$. Suppose by contradiction that this is not the case. Since $\hat{r} \geq \hat{q}$, this means that $\hat{q} = 0$. Then $k'(\hat{q}) = k'(0) < \beta$, and therefore $k'(\hat{r}) \leq k'(\hat{q})/\hat{\beta} < \beta/\hat{\beta} < 1$. Hence, because the perceived marginal cost of repayment in both periods is strictly lower than the marginal utility of consumption, increasing c and $L = c$ by a small amount increases the borrower's perceived utility independently of how she believes she will allocate the increased L across periods 1 and 2, a contradiction.

Because in a competitive equilibrium $0 < \hat{q}, \hat{r} < M$, the solution to the borrower's perceived repayment-cost minimization problem is described by the first-order condition $k'(\hat{q}) = \hat{\beta}k'(L - \hat{q})$. Let $\hat{q}(L)$ denote the unique solution to this first-order condition; this is the amount self 0 thinks self 1 will repay in period 1 if she owes L. Note that $\hat{q}(L)$ is a continuously differentiable function of L, with a derivative strictly between zero and one.

Again using that the competitive-equilibrium c maximizes the borrower's perceived utility subject to $L = c$, the competitive-equilibrium c solves

$$\max_{c} \quad c - k(\hat{q}(c)) - k(c - \hat{q}(c)),$$

yielding the first-order condition

$$1 = k'(\hat{q}(c))\hat{q}'(c) + k'(\hat{r}(c))(1 - \hat{q}'(c)).$$

Plugging in $k'(\hat{r}(c)) = k'(\hat{q}(c))/\hat{\beta}$ gives

$$1 = k'(\hat{q}(c))[\hat{q}'(c) + (1 - \hat{q}'(c))/\hat{\beta}].$$

Since the term in square brackets is greater than 1, $k'(\hat{q}(c)) \leq 1$, which implies that $k'(\hat{r}(c)) \leq 1/\hat{\beta} < 1/\beta$. Because $\hat{q}(c) + \hat{r}(c) = L = c$, we thus have $c < (k')^{-1}(1) + (k')^{-1}(1/\beta)$, which establishes that consumption is less than in the unrestricted market.

Now we use the fact that the borrower consumes more in the unrestricted market than in the capped market to show that she has lower welfare than in the capped market. Simple arithmetic yields the following lemma:

Lemma 2. *Suppose either (i) $k(x) = x^\rho$ for some $\rho > 1$; or (ii) $k(x) = (y - x)^{-\rho} - y^{-\rho}$ for some $y > 0, \rho > 0$. Then, in the capped market c is increasing in $\hat{\beta}$.*

PROOF:

We begin by establishing this for case (i). The borrower expects to repay c in a way such that $k'(\hat{q}) = \hat{\beta}k'(c - \hat{q})$, which in case (i) simplifies to

$$\hat{q}(\hat{\beta}, c) = \underbrace{\frac{\hat{\beta}^{\frac{1}{\rho-1}}}{1 + \hat{\beta}^{\frac{1}{\rho-1}}}}_{\equiv b(\hat{\beta})} c. \tag{6}$$

Thus, her perceived-period-zero utility is $c-(b(\hat{\beta})c)^{\rho}-((1-b(\hat{\beta}))c)^{\rho}$, which can be rewritten as $c-c^{\rho}\left[b(\hat{\beta})^{\rho}+(1-b(\hat{\beta}))^{\rho}\right]$. The borrower chooses c to maximize her perceived utility so that $1=\rho c^{\rho-1}\left[b(\hat{\beta})^{\rho}+(1-b(\hat{\beta}))^{\rho}\right]$. Since $b(\hat{\beta})$ is increasing and less than $1/2$, the term in square brackets is decreasing in $\hat{\beta}$, and thus c is increasing in $\hat{\beta}$.

In case (ii), let $W\equiv 2y-c$, $s\equiv y-\hat{q}$, and $t\equiv y-\hat{r}$. Hence in the capped market $t=W-s$. Rewriting $k'(\hat{q})=\hat{\beta}k'(c-\hat{q})$, yields

$$s(\hat{\beta},W)=\underbrace{\frac{\hat{\beta}^{\frac{-1}{1+\rho}}}{1+\hat{\beta}^{\frac{-1}{1+\rho}}}}_{\equiv b(\hat{\beta})}W. \tag{7}$$

Observe that $b(\hat{\beta})$ is decreasing and greater than $1/2$. The borrower's perceived period-zero utility is $c-\Big(b(\hat{\beta})W(c)\Big)^{-\rho}-\Big((1-b(\hat{\beta}))W(c)\Big)^{-\rho}-2y^{-\rho}$, which can be rewritten as $c-W(c)^{-\rho}\left[b(\hat{\beta})^{-\rho}+(1-b(\hat{\beta}))^{-\rho}\right]-2y^{-\rho}$. Since the power function with the exponent $-\rho$ is convex, and $b(\hat{\beta})$ decreasing and greater than $1/2$, an increase in $\hat{\beta}$ decreases the term in square brackets. Since at the perceived optimal c, $1=\rho W(c)^{-(\rho+1)}[b(\hat{\beta})^{-\rho}+(1-b(\hat{\beta}))^{-\rho}]$, an increase in $\hat{\beta}$ must lead to a decrease of $W(c)$ or—in other words—an increase in c.

To complete the proof, consider contracts in the capped market and restrict attention to contracts for which consumption is equal to total repayment ($c=L$). We show that for any $\beta,\hat{\beta}$, the actual repayment amounts satisfy $0<q(c)\leq r(c)<M$. The part $r(c)\geq q(c)$ is obvious. For $\hat{\beta}=\beta$, we have already established that $\hat{q}(c)>0$ and thus $q(c)>0$. Because by Lemma 2 c is increasing in $\hat{\beta}$, we also have $q(c)>0$ for all $\hat{\beta}\geq\beta$. For $\hat{\beta}=1$, $k'(\hat{q})=k'(\hat{r})=1$. Since $q(c)>0$ implies $k'(q(c))\geq\beta k'(r(c))$, we must have $k'(r(c))<1/\beta$, so that $r(c)<M$. Again using Lemma 2, since c is increasing in $\hat{\beta}$, for any $\hat{\beta}\leq 1$ we must have $r(c)<M$.

Since $0<q(c),r(c)<M$, replacing $\hat{\beta}$ by β in Equations 6 and 7 shows that the repayment amounts $q(c),r(c)$ increase linearly in c. Hence in the capped market the borrower's welfare is $c-k(a_1+bc)-k(a_2+(1-b)c)$ for some constants $a_1,a_2\in\mathbb{R}$, and $b\in(0,1)$. Twice differentiating with respect to c shows that for the utility functions in the proposition, among contracts where $R=1$ and $c=L$ the borrower's welfare is single-peaked in consumption. By revealed preference, the maximum occurs at the consumption level that the sophisticated borrower chooses in the capped market. Lemma 2 implies that a non-sophisticated borrower consumes more in the capped market than the sophisticated borrower, and we established above that she consumes even more than that

in the unrestricted market. This implies that she has lower welfare in the unrestricted than in the capped market.

Proof of Proposition 5. The firm's problem is

$$\max_{c,q,r,\hat{q},\hat{r}} \quad q + r - c$$

$$\text{(PC)} \qquad \text{s.t.} \quad c - k(\hat{q}) - \hat{\beta}k(\hat{r}) \geq \underline{u},$$

$$\text{(PCC)} \qquad -k(\hat{q}) - \hat{\beta}k(\hat{r}) \geq -k(q) - \hat{\beta}k(r),$$

$$\text{(IC)} \qquad -k(q) - k(r) \geq -k(\hat{q}) - k(\hat{r}).$$

The steps in the analysis are very similar to those in the time-inconsistent case. PC binds because otherwise the firm could increase profits by reducing c. In addition, IC binds because otherwise the firm could increase profits by increasing q. Given that IC binds and $\hat{\beta} > 1$, PCC is equivalent to $q \leq \hat{q}$, so conjecturing that $q \leq \hat{q}$ is optimal even without PCC, we ignore this constraint, and confirm our conjecture in the solution to the relaxed problem below.

The relaxed problem is

$$\max_{c,q,r,\hat{q},\hat{r}} \quad q + r - c$$

$$\text{(PC)} \qquad \text{s.t.} \quad c - k(\hat{q}) - \hat{\beta}k(\hat{r}) = \underline{u},$$

$$\text{(IC)} \qquad -k(q) - k(r) = -k(\hat{q}) - k(\hat{r}).$$

Notice that in the optimal solution, $\hat{r} = 0$: otherwise, the firm could decrease $k(\hat{r})$ and increase $k(\hat{q})$ by $\hat{\beta}$ times the same amount, leaving PC unaffected and creating slack in IC, allowing it to increase q. Using this, we can express $k(q)$ from IC and plug it into PC to get

$$c = k(q) + k(r) + \underline{u}.$$

Plugging c into the firm's maximand and solving yields all the statements in the proposition. Finally, using $\hat{r} = 0$ it follows from IC that $\hat{q} > q$, and thus the solution to the relaxed problem indeed satisfies PCC.

Proof of Proposition 6. Applying Lemma 1, we set up a firm's problem as choosing a type-independent consumption c and a menu of type-dependent repayment options $\{(q_1, r_1), (q_2, r_2)\}$ subject to participation, incentive, and perceived-choice constraints.

Notice that because both types initially believe they are the sophisticated type β_2 and the sophisticated borrower chooses the baseline repayment schedule, the non-sophisticated borrower's perceived-choice constraint is identical to the sophisticated borrower's incentive constraint. As in textbook models of screening (e.g. Bolton and Dewatripont 2005, Chapter 2), we solve a relaxed problem with only type 1's incentive constraint, and verify ex-post that the solution satisfies type 2's incentive constraint. Given these considerations, the firm's relaxed problem is

$$\max_{c,q_1,r_1,q_2,r_2} \quad p_1(q_1+r_1)+p_2(q_2+r_2)-c \tag{8}$$

$$\text{s.t.} \quad c-k(q_2)-k(r_2)\geq \underline{u}, \tag{PC}$$

$$-k(q_1)-\beta_1 k(r_1)\geq -k(q_2)-\beta_1 k(r_2). \tag{IC}$$

In the optimal solution, IC binds; otherwise, the firm could increase q_1 without violating IC or PC, increasing profits. In addition, PC binds; otherwise, the firm could decrease c and thereby increase profits. From the binding constraints, we get $k(q_2)=c-k(r_2)-\underline{u}$ and $k(q_1)=k(q_2)+\beta_1(k(r_2)-k(r_1))$.

We first establish uniqueness of the competitive equilibrium. Based on the above arguments, the firm's problem reduces to

$$\max_{c,q_1,r_1,q_2,r_2} p_1(q_1+r_1)+p_2(q_2+r_2)-c$$

$$c-k(q_2)-k(r_2)=\underline{u} \tag{PC}$$

$$k(q_2)+\beta_1 k(r_2)=k(q_1)+\beta_1 k(r_1). \tag{IC}$$

We prove that $r_1<r_2$ is suboptimal. Supposing by contradiction that $r_1<r_2$, using IC we have $k(q_2)+k(r_2)=k(q_1)+\beta_1 k(r_1)+(1-\beta_1)k(r_2)>k(q_1)+k(r_1)$. Then, if $q_1+r_1\geq q_2+r_2$, the firm could eliminate the repayment option (q_2,r_2) without decreasing profits, creating slack in PC and thereby allowing it to decrease c. And if $q_1+r_1<q_2+r_2$, the firm would be strictly better off not offering (q_1,r_1), yielding the desired contradiction.

Now, substituting PC into the maximand gives

$$\begin{aligned} \max \quad & p_1(q_1+r_1)+p_2(q_2+r_2)-k(q_2)-k(r_2) \\ k(q_2)+\beta_1 k(r_2) \quad = \quad & k(q_1)+\beta_1 k(r_1) \quad \text{(IC)}. \end{aligned}$$

Let $A = k(q_2), B = k(r_2), D = k(r_1) - k(r_2)$. Then, $k(r_1) = B + D$ and using the IC constraint $k(q_1) = A - \beta_1 D$. Let $f = k^{-1}$. Since k is strictly increasing and strictly convex, f is strictly increasing and strictly concave, and our assumptions on k furthermore ensure that $\lim_{x\to\infty} f'(x) = 0$. Then, the firm's maximization problem can be written as

$$\max_{A\geq 0, B\geq 0, 0\leq D\leq A/\beta_1} \quad p_1(f(A-\beta_1 D) + f(B+D)) + (1-p_1)(f(A)+f(B)) - A - B \tag{9}$$

with no constraints. The first-order conditions are:

$$p_1 f'(A-\beta_1 D) + (1-p_1)f'(A) = 1, \tag{FOC$_A$}$$

$$p_1 f'(B+D) + (1-p_1)f'(B) = 1, \tag{FOC$_B$}$$

$$f'(B+D) - \beta_1 f'(A-\beta_1 D) = 0. \tag{FOC$_D$}$$

Notice that there is a lower bound T such that if $A, B \geq T$, then $p_1(f(A-\beta_1 D) + f(B+D)) + (1-p_1)(f(A)+f(B)) - A - B \leq 0$ for any permissible D. Since the maximand is strictly positive if the firm offers the optimal committed contract (for which $D = 0$ and $A = B = A - \beta_1 D = k[(k')^{-1}(1)]$), this means that there is a global maximum that either satisfies the above first-order conditions or is at a corner. We show that for $k'(0) < 1-p_1, \beta_1$, or equivalently $f'(0) > 1/(1-p_1), 1/\beta_1$, the global maximum is not at a corner. It is clear from the derivatives of the maximand with respect to A and B that the firm's objective function does not obtain a maximum when $A = 0$ or $B = 0$. If $D = A/\beta_1$, either FOC$_B$ does not hold, in which case the maximum is not attained, or FOC$_B$ holds, in which case $f'(B+D) < 1$ and thus $f'(0) > 1/\beta_1$ implies that the derivative of the maximand with respect to D is negative, ruling out such a corner solution. For $D = 0$, either FOC$_A$ and FOC$_B$ do not both hold, in which case the maximum is not attained, or FOC$_A$ and FOC$_B$ both hold, in which case $f'(A) = f'(B) = 1$ and hence the derivative of the maximand with respect to D is positive, ruling out such a corner solution as well.

We have established that a global maximum must satisfy the system of first-order conditions. To prove that the competitive equilibrium is unique, we next show that the solution to the system of first-order conditions is unique. Because $k'(0) < p_1$ and hence $f'(0) > 1/p_1$, for any $D \geq 0$ there is a unique $A > \beta_1 D$ satisfying FOC$_A$; call this $\alpha^A(D)$. Since $\alpha^A(D)$ is strictly increasing in D, $\alpha^A(D) - \beta_1 D$ must be strictly decreasing in D. Also, notice that if $B \geq 0$ is fixed, then for any $D \geq 0$ there is either a unique $A \geq \beta_1 D$ satisfying FOC$_D$ or—in case $f'(B+D) > \beta_1 f'(0)$—there exists no solution to this first-order condition; if a solution exists for some B and D, one also exists for higher B and D. If the solution exists, we refer to it as $\alpha_B^D(D)$ and otherwise we set $\alpha_B^D(D) = \beta_1 D$. Note also that if $\alpha_B^D(D) > \beta_1 D$, $\alpha_B^D(D) - \beta_1 D$ is strictly increasing in D.

Since f is strictly concave, f' and f'^{-1} are strictly decreasing. Consider the range of B given by $B \leq f'^{-1}(\beta_1)$, or equivalently $f'(B) \geq \beta_1$. If for fixed B and $D = 0$ there is an A satisfying FOC$_D$, then $\alpha_B^D(0) = f'^{-1}(f'(B)/\beta_1)$; and otherwise $\alpha_B^D(0) = 0$. In either case, $\alpha_B^D(0) \leq f'^{-1}(1) = \alpha^A(0)$. Using the implicit function theorem,

$$\frac{d\alpha^A(D)}{dD} = \beta_1 \frac{p_1 f''(\alpha^A(D) - \beta_1 D)}{p_1 f''(\alpha^A(D) - \beta_1 D) + (1-p_1) f''(\alpha^A(D))} < \beta_1,$$

and whenever $\alpha_B^D(D) > \beta_1 D$,

$$\frac{d\alpha_B^D(D)}{dD} = \frac{f''(B+D) + \beta_1^2 f''(\alpha_B^D(D) - \beta_1 D)}{\beta_1 f''(\alpha_B^D(D) - \beta_1 D)} > \beta_1.$$

Since at any crossing point of the two curves $\alpha^A(D) = \alpha_B^D(D) > \beta_1 D$, this means that at any crossing point α_B^D is steeper. In addition, since $\lim_{y\to\infty} f'(y) = 0$, it follows from FOC$_D$ that as $D \to \infty$, $\alpha_B^D(D) > \beta_1 D$ and $f'(\alpha_B^D(D) - \beta_1 D) \to 0$ while FOC$_A$ implies that $f'(\alpha^A(D) - \beta_1 D) > 1$ for any $D > 0$. Hence $\alpha_B^D(D) > \alpha^A(D)$ for sufficiently large D. Summarizing, since $\alpha_B^D(0) \leq \alpha^A(0)$, $\alpha_B^D(D)$ is steeper than $\alpha^A(D)$ at any crossing point, both curves are continuous, and for a sufficiently high D we have $\alpha_B^D(D) > \alpha^A(D)$, for this range of B there is a unique A and D satisfying first-order conditions FOC$_A$ and FOC$_D$. Call these solutions $A^*(B)$ and $D^*(B)$, respectively. If $B > f'^{-1}(\beta_1)$ then $\alpha_B^D(0) > \alpha^A(0) > \beta_1 D$ and since $\alpha_B^D(D)$ is steeper than $\alpha^A(D)$ at any crossing point no solution to the first-order conditions FOC$_A$ and FOC$_D$ exists in this range of B.

To complete the proof, notice that since $\alpha^A(D)$ is independent of B and $\alpha_B^D(D)$ is increasing in B, $A^*(B)$ and $D^*(B)$ are decreasing in B; by FOC$_A$, this means that $A^*(B) - \beta_1 D^*(B)$ is increasing in B, which by FOC$_D$ means that $B + D^*(B)$ is increasing in B. Hence, the function $p_1 f'(B + D^*(B)) + (1-p_1) f'(B)$, which is continuous in B, is strictly decreasing in B. Furthermore, because $k'(0) < 1 - p_1$, $f'(0) > 1/(1-p_1)$, so $p_1 f'(0 + D^*(0)) + (1-p_1) f'(0) > 1$. Since for $B = f'^{-1}(\beta_1)$, $\alpha_B^D(0) = \alpha_A(0)$, one has $\beta_1 = f'(B) = f'(B + D^*(B))$ for this value of B. Hence for $B = f'^{-1}(\beta_1)$, one has $p_1 f'(B + D^*(B)) + (1-p_1) f'(B) < 1$. Since $p_1 f'(B + D^*(B)) + (1-p_1) f'(B)$ is strictly decreasing in B, this implies there exists a unique $B \in (0, f'^{-1}(\beta_1))$ for which $B, D^*(B)$ satisfies FOC$_B$. Because for $B \leq f'^{-1}(\beta_1)$, $A^*(B), D^*(B)$ characterize a solution to

FOC_A and FOC_B, we have shown that $B, A^*(B), D^*(B)$ is the unique solution to the system of first-order conditions. Thus we have shown that the competitive equilibrium is unique.

To characterize the optimal installment plan, we invert the expressions for $k(q_1)$ and $k(q_2)$ found above and plug them into the principal's objective function, yielding

(10)

$$\max_{c,r_1,r_2} \quad p_1\left[k^{-1}\big(c - k(r_2) - \underline{u} + \beta_1\left(k(r_2) - k(r_1)\right)\big) + r_1\right] + p_2\left[k^{-1}\big(c - k(r_2) - \underline{u}\big) + r_2\right] - c.$$

The first-order-conditions with respect to r_1 and r_2 are:

$$\begin{aligned} p_1\left[1 - \beta_1 \frac{k'(r_1)}{k'(q_1)}\right] &= 0, \\ p_2\left[1 - \frac{k'(r_2)}{k'(q_2)}\right] - p_1(1-\beta_1)\frac{k'(r_2)}{k'(q_1)} &= 0. \end{aligned}$$

Rewriting these first-order conditions gives the equations in the proposition, which in turn imply that $q_1 < r_1$ and $q_2 > r_2$. It remains to establish that $q_1 + r_1 > q_2 + r_2$. Suppose by contradiction that $q_1 + r_1 \leq q_2 + r_2$. Then the firm would be at least as well off offering a single repayment option (q_2, r_2): the resulting contract satisfies PC and, since there is no choice in period 1, it also satisfies PCC and IC, and yields at least as high profits. This, however, contradicts the fact that in any optimal contract $q_1 < r_1$ and $q_2 > r_2$.

Finally, we show that borrowers overborrow on average. Taking the first-order condition of the maximization problem 10 with respect to c gives

$$p_1 \frac{1}{k'(q_1)} + p_2 \frac{1}{k'(q_2)} = 1.$$

By Jensen's inequality, the left-hand side is greater than

$$\frac{1}{p_1 k'(q_1) + p_2 k'(q_2)},$$

which gives $p_1 k'(q_1) + p_2 k'(q_2) > 1$.

To show the analogous inequality for r_1 and r_2, we solve for $k(r_1)$ and $k(r_2)$ from the binding constraints (instead of solving for $k(q_1)$ and $k(q_2)$), invert these, and plug them into the principal's objective function to get

$$\max_{c,q_1,q_2} \quad p_1\left[q_1+k^{-1}\big(c-k(q_2)-\underline{u}+\left(k(q_2)-k(q_1)\right)/\beta_1\big)\right]+p_2\left[q_2+k^{-1}\big(c-k(q_2)-\underline{u}\big)\right]-c.$$

Again taking the first-order condition with respect to c and using Jensen's inequality completes the proof.

Proof of Proposition 7. First, we show that the borrower strictly prefers the unrestricted market over the restricted one by showing that the perceived utility $\underline{u}$ generated by the competitive-equilibrium contract in the unrestricted market is higher than the borrower's perceived utility in the restricted market. Suppose by contradiction that this is not the case. Then, a contract with the consumption and repayment terms the two types of borrowers choose in the restricted market satisfies the constraints PC, IC, and PCC in Lemma 1, and breaks even, and is therefore a competitive-equilibrium contract. But this is impossible since a competitive equilibrium identified in Proposition 6 does not replicate outcomes in the restricted market: for the condition $k'(q_1)=\beta_1 k'(r_1)$ to hold, the firm needs to set $R=1$, and at this interest rate sophisticated borrowers will not repay more in period 1 than 2.

Since sophisticated borrowers understand their behavior, the fact that their perceived utility is higher than in the restricted market implies that their actual welfare is also higher.

We next consider social welfare. The same steps as in Proposition 3 establish that as $\beta_1 \to \beta_2$, the competitive-equilibrium contract approaches $(c^*, 1/\beta_2, L^*)$, so that both types' outcomes approach the welfare-maximizing outcome (the only difference in the argument is that the break-even $c(R,L)$ must be defined in expectation). Since in the unrestricted market $k'(q_1)=\beta_1 k'(r_1)$ for any $\beta_1<\beta_2$, total welfare remains bounded away from the welfare-maximizing level as $\beta_1 \to \beta_2$. Hence, for β_1 sufficiently close to β_2 the restricted market yields higher social welfare. Finally, since a non-sophisticated borrower has lower welfare than a sophisticated borrower, the fact that total welfare remains bounded away from optimal as $\beta_1 \to \beta_2$ implies that the non-sophisticated borrower's welfare also does. Since her welfare in the restricted market approaches the optimum as $\beta_1 \to \beta_2$, for β_1 sufficiently close to β_2 the restricted market yields higher welfare for her.

Proof of Proposition 8. We begin by establishing that there is a competitive equilibrium in which the same contracts are offered as when $\hat{\beta}$ is known and each borrower selects the contract designed for her belief $\hat{\beta}$. We first show borrower optimality (Condition 1 of Definition 2). Since a borrower of type $\hat{\beta}$ expects to choose the baseline repayment option in a contract intended for any $\hat{\beta}' \leq \hat{\beta}$, among these contracts she prefers the one intended for her because by Condition 1 it gives her the highest perceived period-0 utility. Second, while from a period-0 perspective the borrower prefers the baseline option in the contract for $\hat{\beta}' > \hat{\beta}$ to the baseline option in the contract for her own type, she also believes that she will switch away from this option ex post. Once she takes this into account, the period-0 utility from the contract designed for $\hat{\beta}' > \hat{\beta}$ is lower. To see this last point, suppose by contradiction that a type $\hat{\beta}$ preferred to select the contract designed for $\hat{\beta}' > \hat{\beta}$. Then, the contract for $\hat{\beta}$ is suboptimal when $\hat{\beta}$ is known: the contract designed for $\hat{\beta}'$ both attracts $\hat{\beta}$ types and induces all of them to choose the non-sophisticated repayment option, which by Proposition 6 is strictly profitable, and thus this contract guarantees positive profits when $\hat{\beta}$ is known. Since the contracts are identical to the ones in which $\hat{\beta}$ is observable they satisfy the zero-profit and non-redundancy requirements (Conditions 2 and 4). Furthermore, Condition 3 is satisfied because any other contract that gives a borrower of type $\hat{\beta}$ a higher perceived utility makes losses on this type of borrowers, since otherwise this contract could also be profitably introduced when $\hat{\beta}$ is observable.

Now we argue that this competitive equilibrium is unique. Consider any purported equilibrium in which not all $\hat{\beta}$ types are offered the competitive-equilibrium contract for the case in which $\hat{\beta}$ is known. Let $\underline{u}_i'$ be the perceived utility of $\hat{\beta}_i$ in this situation. First, we show that there is some i such that $\underline{u}_i' < \underline{u}_i$. Suppose by contradiction that $\underline{u}_i' \geq \underline{u}_i$ for all i. Then, even if $\hat{\beta}$ was observable, a firm could only break even on each type, and do so only using the competitive-equilibrium contract for each type—contradicting that not all $\hat{\beta}$ types get the same contract as when $\hat{\beta}$ is known.

Now consider the highest i such that $\underline{u}_i' < \underline{u}_i$. For a sufficiently small $\epsilon > 0$, a contract that is optimal for type $\hat{\beta}_i$ with the outside option $\underline{u}_i' + \epsilon$ attracts $\hat{\beta}_i$ and makes positive expected profits on this type. Furthermore, since for any $j > i$, $\underline{u}_i' < \underline{u}_i \leq \underline{u}_j \leq \underline{u}_j'$, the contract does not attract higher $\hat{\beta}_j$. If it attracts $\hat{\beta}_j$ for some $j < i$, it makes strictly positive profits on these borrowers, since they all select the non-sophisticated repayment option in the contract. Hence, the contract makes positive expected profits.

Proof of Proposition 9. As in the case of degenerate borrower beliefs, the notion of competitive equilibrium is based on the notion of incentive compatible maps determining what a borrower expects to choose for each possible $\hat{\beta}$ and what she actually chooses (similarly to Definition 1). Accordingly, we think of a firm's problem as selecting $(\hat{q}(\hat{\beta}), \hat{r}(\hat{\beta}))$

the borrower thinks she will choose for each possible $\hat{\beta}$, as well as a $(q, r) = (\hat{q}(\beta), \hat{r}(\beta))$ the borrower actually chooses, where $\hat{q}(\cdot)$ and $\hat{r}(\cdot)$ must be incentive compatible.

First suppose that firms know $F(\cdot)$. Denote the support of F by $\overline{F}$. Rewriting 1, the firm's problem is

$$\max_{c,q,r,\hat{q}(\hat{\beta}),\hat{r}(\hat{\beta})} \quad q + r - c$$

$$\text{(PC)} \quad \text{s.t.} \quad \int \left[c - k(\hat{q}(\hat{\beta})) - k(\hat{r}(\hat{\beta}))\right] dF(\hat{\beta}) \geq \underline{u},$$

$$\text{(PCC)} \qquad -k(\hat{q}(\hat{\beta})) - \hat{\beta}k(\hat{r}(\hat{\beta})) \geq -k(\hat{q}(\hat{\beta}')) - \hat{\beta}k(\hat{r}(\hat{\beta}')) \text{ for any } \hat{\beta} \in \overline{F}, \hat{\beta}' \in \overline{F} \cup \{\beta\},$$

$$\text{(IC)} \qquad -k(q) - \beta k(r) \geq -k(\hat{q}(\hat{\beta})) - \beta k(\hat{r}(\hat{\beta})) \text{ for any } \hat{\beta} \in \overline{F}.$$

As before, PC binds because otherwise a firm could raise profits by decreasing c. Notice that for any $\hat{\beta} \leq \beta$, for PCC and IC to both hold we must have $\hat{q}(\hat{\beta}) \leq q$ and $\hat{r}(\hat{\beta}) \geq r$. Hence, the IC constraint $k(q) + \beta k(r) \leq k(\hat{q}(\hat{\beta})) + \beta k(\hat{r}(\hat{\beta}))$ implies that $k(q) + k(r) \leq k(\hat{q}(\hat{\beta})) + k(\hat{r}(\hat{\beta}))$, with a strict inequality if $(q, r) \neq (\hat{q}(\hat{\beta}), \hat{r}(\hat{\beta}))$. Hence, given PC it is optimal to set $(\hat{q}(\hat{\beta}), \hat{r}(\hat{\beta})) = (q, r)$ for all $\hat{\beta} \leq \beta$, and in any optimal contract the set of $\hat{\beta} \leq \beta$ for which this equality does not hold must have measure zero under the agent's beliefs $F(\cdot)$.

Next consider $\hat{\beta} > \beta$. We ignore PCC for these $\hat{\beta}$; it is obvious to check that the resulting contract satisfies it. It is optimal to set $\hat{r}(\hat{\beta}) = 0$ for all $\hat{\beta} > \beta$: for any $\hat{\beta}$ with $\hat{r}(\hat{\beta}) > 0$, we can decrease $k(\hat{r}(\hat{\beta}))$ by some amount and increase $k(\hat{q}(\hat{\beta}))$ by β times the same amount, leaving IC unaffected and weakly increasing the left-hand side of PC. Furthermore, in any optimal contract the set of $\hat{\beta} > \beta$ for which $\hat{r}(\hat{\beta}) > 0$ must have measure zero; otherwise, these steps would create a slack in PC, allowing the firm to decrease c. With $\hat{r}(\hat{\beta}) = 0$ for all $\hat{\beta} > \beta$ (other than a measure zero set under $F(\cdot)$), it is optimal to set $\hat{q}(\hat{\beta}) = \hat{q}$ at the level such that IC binds, and the set of $\hat{\beta} > \beta$ for which this is not the case must have measure zero under $F(\cdot)$.

Given these simplifications, the firm's problem becomes

$$\max_{c,q,r,\hat{q},\hat{r}} \quad q + r - c$$

$$\text{(PC)} \qquad \text{s.t.} \quad F(\beta)\left[c - k(q) - k(r)\right] + (1 - F(\beta))\left[c - k(\hat{q})\right] = \underline{u},$$

$$\text{(IC)} \qquad -k(q) - \beta k(r) = -k(\hat{q}).$$

Expressing $k(\hat{q})$ from IC, plugging it into PC, and solving for c and plugging it into the maximand yields that the firm wants to maximize

$$q + r - k(q) - [F(\beta) + (1 - F(\beta))\beta]\, k(r).$$

Solving this yields Equation 4. That $q < \hat{q} < q + r$ follows from the fact that IC binds.

Finally, we argue that the above (essentially unique) contract is the competitive-equilibrium contract for a borrower with beliefs $F(\cdot)$ even if firms do not observe borrowers' beliefs. The argument is in two parts.

I. Offering these contracts is a competitive equilibrium. To see this, notice first that the profits a firm earns from an accepted contract are independent of the borrowers' beliefs. Suppose by contradiction that a borrower with beliefs $F(\hat{\beta})$ strictly prefers the contract $(c', \mathcal{C}')$ to a contract $(c, \mathcal{C})$ we have solved for above. Then, the firm could offer a contract $(c' - \epsilon, \mathcal{C}')$ for some $\epsilon > 0$ when $F(\hat{\beta})$ is known and earn strictly positive profits, contradicting the no-profitable-deviation condition of competitive equilibrium.

II. There is no other competitive equilibrium. Let $(c, \mathcal{C}) = (q + r, \{(\hat{q}, 0), (q, r)\})$ be the (essentially unique) competitive-equilibrium contract when $F(\cdot)$ is known (for which we have solved above). Suppose by contradiction that there is a competitive equilibrium in which a borrower with beliefs $F(\cdot)$ accepts a contract $(c', \mathcal{C}')$ that does not satisfy the conditions specified in the proposition. Let $\underline{u}'$ be her perceived utility from $(c', \mathcal{C}')$, and let $\underline{u}$ be her perceived utility from $(c, \mathcal{C})$. Notice that $\underline{u}$ maximizes the borrower's perceived utility among contracts that earn zero profits given the borrower's actual behavior. Since $(c', \mathcal{C}')$ is not a competitive equilibrium when $F(\cdot)$ is known but earns zero profits, this implies that $\underline{u} > \underline{u}'$. Therefore, a firm can offer $(c - \epsilon, \mathcal{C})$, and for a sufficiently small $\epsilon > 0$ both attract the borrower with beliefs $F(\cdot)$ and make positive profits from her. Since a borrower's behavior is independent of her beliefs, a firm still makes positive profits if it also attracts other borrowers, contradicting the no-profitable-deviation condition.