The hold-up problem with flexible unobservable investments

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Abstract

The paper studies the canonical hold-up problem with one-sided investment by the buyer and full ex-post bargaining power by the seller. The buyer can covertly choose any distribution of valuations at a cost and privately observes her valuation. The main result shows that in contrast to the well-understood case with linear costs, when investment costs are strictly convex in the buyer's valuation distribution, the buyer's equilibrium utility is positive, thus alleviating the hold-up problem. Moreover, the buyer's equilibrium utility and total welfare might be non-monotone in costs. The paper provides an equilibrium characterization in terms of the Gateaux derivative of the cost function.

Keywords: Information Design, Hold-Up Problem
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1 Introduction

Consider the canonical hold-up problem where a buyer can make a costly, relation-specific ex ante investment that increases her (default) valuation for the seller's good, and the seller has all bargaining power ex post. It is well known that if the buyer's valuation becomes public information ex post, the seller will extract all gains from trade, and in anticipation of this, the buyer will not invest. As a consequence, in equilibrium, the seller's profit is equal to the buyer's default valuation, and the buyer's utility is zero in equilibrium. Gul (2001) shows that the same welfare outcomes obtain even if the buyer's valuation remains her private information so that she can

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secure an information rent ex post: while some investment occurs in equilibrium, the buyer still gets zero utility overall because her ex post information rent is fully dissipated by her ex-ante investment.

An important assumption behind this result is that the buyer's cost of investment is linear in the probability distribution over investment outcomes.[1] In this paper, I consider the hold-up problem when investment costs are convex. My main result shows that if investment costs are strictly convex, the buyer's equilibrium utility (and thus total welfare) is strictly positive, thus alleviating the hold-up problem. Moreover, I show that both the buyer's utility and total welfare might increase in the magnitude of investment costs.

I adopt a framework similar to Condorelli and Szentes (2022) where the buyer can flexibly choose any distribution over an interval of valuations at a cost. The cost function is convex in the distribution, allowing for linear costs (as in Gul, 2001) as a special case. I further assume that the cost function is smooth in that it admits a functional derivative in the sense of a (linear) Gateaux differential. As is well known, the Gateaux differential is a directional derivative that captures the cost change when the buyer moves marginally from one distribution in the direction of another distribution.

As a methodological contribution, I provide an equilibrium characterization that states the conditions for a distribution by the buyer to be a best response to the seller's pricing strategy in terms of the Gateaux differential of the cost function. To do so, I draw on results in Luenberger (1997) that characterize the optima of a concave functional that admits a linear Gateaux differential. The best response condition formally corresponds to the familiar condition of a distribution (or, a mixed strategy) to be a best response: Any valuation in the support of the distribution must yield the same payoff, and any valuation outside the support must not yield a higher payoff. The novelty is that with non-linear cost, the payoff corresponding to a valuation depends on the entire distribution itself (not just on the valuation). This implies that already the problem to find a best response (not only the equilibrium) becomes a fixed point problem.[2]

Even though equilibria can, therefore, in general not be explicitly derived, it is remarkably simple to derive equilibrium utilities through the best response conditions. My main result that the buyer's equilibrium utility is zero if her costs are linear, but positive if her cost function is strictly convex, follows from basic marginal benefit versus marginal cost reasoning: One way to think about how the buyer's equilibrium utility comes about is to imagine that the buyer could choose only among the set of convex combinations of the default (zero cost) distribution, which arises if she does not invest, and the actual equilibrium distribution. The buyer's equilibrium

[1] Linear costs naturally arise in the case that the buyer can choose a deterministic valuations at a cost. A distribution over valuations then corresponds to a mixed strategy the cost of which is simply expected cost and thus linear.

[2] The characterization of best responses in terms of the Gateaux differential is quite general and can be used to study other applications with flexible pre-investments.

utility is then simply the aggregated differences between marginal benefits and marginal costs from increasing the weight put on the equilibrium distribution, aggregated over all weights from 0 to 1.

The marginal benefit of increasing the weight on the equilibrium distribution is constant. The reason is that the benefit from choosing a distribution is simply the buyer's expected trading surplus (which is linear in the distribution). On the other hand, the marginal cost of increasing the weight put on the equilibrium distribution can be shown to be the Gateaux differential of the cost function, evaluated at the respective convex combination.

The key observation is that strict convexity of the cost function implies that this marginal cost is strictly increasing in the weight put on the equilibrium distribution. Finally, equilibrium implies that at the equilibrium distribution (when the weight is one), marginal benefits are (weakly) larger than marginal costs for otherwise it would be profitable for the buyer to deviate in some direction. Therefore, marginal costs are strictly below marginal benefits whenever the buyer increases the weight on the equilibrium distribution. Hence, choosing the equilibrium distribution yields strictly positive overall utility. In contrast, when investment costs are linear, then both marginal benefits and marginal costs are constant, which implies that in equilibrium the buyer gets zero utility.

A direct corollary of these observations is that if the cost function is the sum of a linear and a scaled strictly convex part, then the buyer's utility increases if the convex part is scaled up from zero to positive, implying that the buyer's utility and thus total welfare locally increase with investment costs.

Explicit expressions for equilibrium strategies and utilities are hard to obtain in general. In a further part of the paper, I therefore impose more structure on the cost function. In particular, I impose assumptions on the Gateaux derivative that imply the cost function to be increasing or decreasing with respect to second order stochastic dominance.[3] In the first case, the buyer's valuation and the seller's pricing distribution have convex support. In the second case, these distributions are two-point distributions (or degenerate). In both cases, the valuation distributions are equal revenue distributions as in Condorelli and Szentes (2020), but in my case they emerge because they correspond to the equilibrium requirement that every price in the support of the pricing distribution must yield equal revenue for the seller. I also provide comparative static results for parametric examples that show that the buyer's utility and welfare are in increasing in costs over a non-degenerate set of parameters.

My paper is most closely related to the abovementioned papers by Gul (2001) and Condorelli and Szentes (2020). Like Gul (2001), I consider the hold-up problem with unobservable invest-

[3]This implies that costs are increasing or decreasing "in risk" in the terminology of Condorelli and Szentes (2020).

ments, but allow for convex investment costs.[4] Non-linear costs are also considered in Condorelli and Szentes (2020), who, in contrast to my paper, consider the case when the buyer can commit to an investment distribution (or, equivalently, the seller can observe the distribution but not the realized valuation). Under the commitment outcome, the buyer chooses a non-degenerate distribution over valuations, and the seller chooses a deterministic price. This outcome typically breaks down if the buyer's distribution is unobservable because for a deterministic price, the induced marginal benefits and marginal costs for the buyer differ so that she would want to deviate from the commitment distribution. When costs are linear (and marginal costs are constant), the deviation incentives are more pronounced than when when costs are convex and marginal costs are increasing. Thus, one interpretation of my result is that, with convex costs, the resulting equilibrium outcome is closer to the commitment outcome and thus more favorable to the buyer and overall welfare.

My paper shares with Ravid et al. (2022) the feature that the seller does not observe the distribution of the buyer's valuation. The key difference is that in Ravid et al. (2022), the buyer's ex ante choice is to acquire information about, rather than invest in, her valuation. Ravid et al. (2022) show that in the limit when information acquisition costs are small, the buyer is worse off than when information is for free. In contrast, I obtain welfare results away from the limit because the information acquisition constraint that the distribution be Bayesian consistent with a prior is missing from my framework and allows for more explicit equilibrium characterizations. In an extension, I show how my framework can be used to speak to information acquisition in the case that the buyer's true valuations can take on only two values.[5]

The paper is organized as follows. The next sections presents the model. Sections 3 and 4 contain the key equilibrium and welfare results. Section 5 derives explicit results for specific cost functions. Section 6 discusses a connection to information acquisition. Section 7 concludes.

2 Model

There is a seller who has a good, and there is a buyer who can invest in her valuation for the good by choosing a cumulative distribution function (cdf) F over the set of possible valuations $V = [\alpha, \omega]$, $0 \leq \alpha < \omega$, at cost $C(F)$. Let $\mathcal{F}$ denote the set of all cdf's over V. The timing is as follows: The seller chooses a price p, and the buyer simultaneously chooses a cdf $F \in \mathcal{F}$. Then

[4]Other papers that study versions of the hold-up problem with linear investment costs are Dilme (2019) and Lau (2008). Dilme (2019) considers a setting where the investment increases both parties' valuation, and the informed party makes a take-it or leave-it offer ex post, leading to endogenous adverse selection. He shows that the non-investing party gets less than when investment is observable. In Lau (2008), the seller obtains an truth-or-noise signal about the buyer's investment prior to making the take-it or leave-it offer. Lau (2008) shows that the information benefits the seller, while the buyer, as with no signal, still gets zero surplus.

[5]For the case with buyer commitment and information acquisition, see Roesler and Szentes (2017).

the buyer privately observes her realized valuation v and decides to accept or reject to trade at the price p. If she rejects, both parties get zero. If she accepts, the buyer's payoff is $v-p$, and seller's payoff is p.

The buyer's strategy specifies a cdf F and a decision to accept or reject, contingent on p. A (mixed) strategy for the seller is a cdf over prices. In a perfect Bayesian equilibrium (henceforth: equilibrium), the buyer accepts any price $p<v$ and rejects any price $p>v$, and her choice of cdf is optimal given the seller's pricing strategy, and the seller's strategy is optimal given the buyer's choice of cdf and acceptance/rejection decision.

It is a standard argument that in any equilibrium, the buyer accepts with probability 1 when indifferent ($p=v$) and that the seller never chooses a price strictly below the buyer's smallest possible valuation α. Moreover, it is weakly dominated for the seller to choose a price strictly above the buyer's largest possible valuation ω. To analyze stage 1 of the game, I therefore focus on seller strategies H that are cdf's from the set $\mathscr{F}$. The buyer's expected utility from valuation v is

$$\overline{H}(v)=\int_{\alpha}^{v}(v-p)\,dH(p)=\int_{\alpha}^{v}H(p)\,dp, \tag{1}$$

where the second equality follows from integration by parts. The stage 1 utilities for the buyer and seller are then given by[6]

$$U(F,H)=\int_{V}\overline{H}(v)\,dF-C(F), \qquad \Pi(H,F)=\int_{V}(1-F(p^{-}))p\,dH(p). \tag{2}$$

With abuse of language, I refer to a combination $(F,H)\in\mathscr{F}^2$ as an equilibrium when F and H are mutual best responses given U and Π.

I next state the assumptions on the cost function that I impose throughout the paper. To do so, recall first that a functional C defined on a vector space X is Gateaux differentiable at $F\in X$ in the direction $D\in X$ if the limit

$$\delta C(F;D)=\lim_{\epsilon\to 0}\frac{1}{\epsilon}[C(F+\epsilon D)-C(F)]=\frac{d}{d\epsilon}C(F+\epsilon D)\,|_{\epsilon=0} \tag{3}$$

exists. $\delta C(F;D)$ is referred to as "Gateaux differential" in the direction D. Intuitively, $\delta C(F;D)$ measures the change of C if one moves from F marginally in the direction D.

For technical reasons to facilitate the proof of Proposition 1 below, I take the domain of C to be the vector space BV of right-continuous functions of bounded variation on V of which the

[6] $F(p^{-})$ denotes the left limit $\lim_{q\uparrow p}F(q)$.

space of cdfs $\mathscr{F}$ is a strict subset.[7] Throughout, I assume:

A1: $C : BV \to \mathbb{R}$ is convex.

A2: C is Gateaux differentiable for all $F, D \in BV$, and there is a continuous "Gateaux derivative" $c_F : V \to \mathbb{R}$ so that

$$\delta C(F; D) = \int_V c_F(v)\, dD. \tag{4}$$

A3: $\arg\min_{F \in \mathscr{F}} C(F) \neq \emptyset$, and $\min_{F \in \mathscr{F}} C(F) = 0$.

Convexity is an economically natural assumption. A2 captures that costs are smooth. More precisely, expression (4) means that the Gateaux differential is linear (in D) so that the costs of local changes of the valuation distribution can be approximated linearly.[8] A3 says that C takes on a minimum. This is a simplifying assumption to ease the exposition. The assumption $\underline{C} = 0$ is a normalization which ensures that investment costs are non-negative. Moreover, if $\arg\min_{F \in \mathscr{F}} C(F) = \{F_{min}\}$ is a singleton, F_{min} can be interpreted as the default distribution that arises if the seller does "not invest".

An important special case is the class of linear cost functions. C is linear if and only if $C(F) = \int_V c(v)\, dF(v)$ for a continuous function $c : V \to \mathbb{R}$. In this case, the Gateaux derivative is $c_F = c$ at all points F. Linearity naturally arises in a setting where the buyer can choose a valuation $v \in V$ at cost $c(v)$, and V is the set of the buyer's pure strategies (see, e.g., Gul, 2001). $C(F)$ is then the cost of the mixed strategy that randomizes over V according to F. For a linear cost function, we have

$$\min_{v \in V} c(v) = 0. \tag{5}$$

To see this, note that $C(F) = \int_V c(v)\, dF(v)$ is minimized by any cdf F_{min} that places full mass on points v were c is minimal. Thus, $C(F_{min}) = \min_{v \in V} c(v)$ which is 0 by A3.

3 Equilibrium Analysis

My first proposition is the main equilibrium characterization of the paper.

Proposition 1 *(i) There is an equilibrium. (A2 and A3 are not needed for this.)*

[7] It is well known that a (right-continuous) function is of bounded variation if and only if it can be written as the difference between two (right-continuous) increasing functions.

[8] It is well-known that the Gateaux differential need, in general, not be linear, but only homogeneous.

(ii) (F,H) *is an equilibrium if and only if there are* λ *and* $\pi \geq \alpha$ *such that*

$$\overline{H}(v) - c_F(v) - \lambda \leq 0 \quad \forall v \in V, \tag{6}$$

$$\overline{H}(v) - c_F(v) - \lambda = 0 \quad \forall v \in supp(F), \tag{7}$$

$$(1 - F(p^-))p - \pi \leq 0 \quad \forall p \in V, \tag{8}$$

$$(1 - F(p^-))p - \pi = 0 \quad \forall p \in supp(H). \tag{9}$$

Part (i) follows from a standard fixed point argument along the same lines as the existence proof in Ravid et al. (2022, footnote 22). I omit the details. Part (ii), more precisely, the conditions (6) and (7) which characterize the buyer's best response, are, to my knowledge, novel. To shed light on part (ii), it is easiest to first consider the conditions (8) and (9). These conditions simply represent the familiar conditions for a (mixed) strategy by the seller to be a best response to F: Any price in the support of the strategy must yield the same profit $\pi = (1 - F(p^-))p$, and any price outside the support must not yield a higher profit.

The conditions (6) and (7) are analogous conditions for the buyer. In fact, consider the special case of linear C, and recall the interpretation of C as the cost of a mixed strategy when the buyer can choose a valuation v at cost $c(v)$. In this case, the buyer's utility from the pure strategy v is $\overline{H}(v) - c(v)$, and the conditions (6) and (7) therefore represent the conditions for a (mixed) strategy by the buyer to be a best response to H. The significance of part (ii) is that the same formal conditions characterize the buyer's best response even when the Gateaux derivative c_F is not constant in F.

Notice, however, that when c_F is not constant in F, (6) and (7) define a fixed point equation for F, because F appears on both sides. To see this more clearly, define for a cdf G the set $S(G)$ as the largest (closed) set $S \subseteq V$ so that $\overline{H}(v) - c_G(v) - \lambda = 0$ for all $v \in S$ and (6) holds. F is then a solution to (6) and (7) if and only if its support satisfies the fixed point property $supp(F) \subseteq S(F)$.

The proof strategy to establish (6) and (7) is to set up the buyer's best response problem as a constrained problem in which the buyer's choice variable is any right-continuous increasing function and the constraint requires that the function be a cdf (with λ being the respective multiplier). The corresponding Lagrangian is a concave functional over the set of right-continuous increasing functions. To maximize the Lagrangian, I use a lemma in Luenberger (1997, Lemma 1, p. 227) which characterizes the maximizers of concave functionals over this domain in terms of their Gateaux derivative. These conditions turn out to correspond to (6) and (7).[9]

[9] It is so I can apply Luenberger (1997)'s lemma why I impose assumptions A1 and A2. The lemma requires that the Gateaux differential of C is defined for all directions $D \in BV$ and is linear. Moreover, it requires the cost function to be convex on BV.

4 Welfare Analysis

This section contains the key welfare results of the paper.

Proposition 2 *Let (F, H) be an equilibrium.*

(i) The buyer's equilibrium utility is[10]

$$U_B = \int_V c_F(v)\, dF(v) - C(F) - \min_{v \in V} c_F(v). \tag{10}$$

(ii) If C is linear, then $U_B = 0$.

(iii) If C is strictly convex and $C(F) \neq 0$, then $U_B > 0$.

The proof of part (i) shows that λ in part (ii) of Proposition 1 is equal to $-\min_{v \in V} c_F(v)$. Once this is established, the expression for U_B is immediate from (2) and (7).

Part (ii) is immediate from the definition of linear costs and the fact that $\min_{v \in V} c(v) = 0$ by (5).

The argument behind part (iii), a key result of my paper, is economically intuitive and it is instructive to elaborate on it. Imagine that, instead of choosing among all cdfs, the buyer chooses a uni-dimensional investment level $\tau \in [0, 1]$ by selecting a convex combination $F^\tau = \tau F + (1-\tau) F_{min}$ of the equilibrium distribution F and a default distribution F_{min}. Given the seller's equilibrium distribution H, the buyer's benefit from investing τ is

$$\Psi(\tau) = \int_V \overline{H}(v)\, dF^\tau = \tau \int_V \overline{H}(v)\, d(F - F_{min}) + \int_V \overline{H}(v)\, dF_{min}, \tag{11}$$

and her cost is $\Phi(\tau) = C(F^\tau)$.

Because the buyer's equilibrium choice F (in the equilibrium where she chooses among all cdfs) is simply F^1, her equilibrium utility is

$$U_B = \Psi(1) - \Phi(1). \tag{12}$$

I now use a standard marginal cost and benefit argument to show that this expression is strictly positive. Indeed, because the investment benefit $\Psi(\tau)$ is linear in τ, marginal benefits $\Psi'(\tau) = \int_V \overline{H}(v)\, d(F - F_{min})$ are constant. On the other hand, marginal costs at τ correspond exactly to

[10] Note that $\min_{v \in V} c_F(v)$ is well-defined since c_F is continuous by assumption.

the Gateaux differential at F^τ in the direction $F - F_{min}$:[11]

$$\Phi'(\tau) = \delta C(F^\tau, F - F_{min}) = \int_V c_{F^\tau}(v)\, d(F - F_{\min}). \tag{13}$$

Now, the fact that F is a best response implies that at $\tau = 1$, marginal benefits (weakly) exceed marginal costs, because otherwise, the buyer could profitably deviate by marginally lowering τ. Formally, the equilibrium conditions (6) and (7) imply that at $\tau = 1$:

$$\begin{aligned}
\Psi'(1) - \Phi'(1) &= \int_V \overline{H}(v) - c_F(v)\, d(F - F_{min}) &(14)\\
&= \int_V \lambda\, dF - \int_V \overline{H}(v) - c_F(v)\, dF_{min} &(15)\\
&= -\int_V \overline{H}(v) - c_F(v) - \lambda\, dF_{min} \geq 0. &(16)
\end{aligned}$$

Crucially, because C is strictly convex, so is Φ,[12] and thus Φ' is strictly increasing. Therefore, because marginal benefits are constant, marginal benefits strictly exceed marginal costs for all $\tau < 1$.

Finally, note that the buyer can guarantee herself a weakly positive utility by investing $\tau = 0$ which corresponds to choosing the default distribution $F_{min} = F^0$. This would result in utility $\Psi(0) - \Phi(0) = \int_V \overline{H}(v)\, dF_{min} - C(F_{min})$ which is weakly positive because $\overline{H}$ is positive and $C(F_{min}) = 0$ by A3.

Therefore, because the buyer's marginal utility is strictly positive up to $\tau = 1$, her overall utility is strictly positive. Formally:

$$U_B = \Psi(1) - \Phi(1) = \int_0^1 \Psi'(\tau) - \Phi'(\tau)\, d\tau + \Psi(0) - \Phi(0) > 0. \tag{21}$$

An immediate consequence of the proposition is that the buyer's equilibrium utility might be non-monotone in costs.

Corollary 1 *Consider a cost function that is a combination of a linear and a strictly convex cost*

[11]This observation is taken from Chew and Nishimura, 1992, page 300.

[12]To see this, let $\gamma \in [0,1]$, then

$$\begin{aligned}
\Phi(\gamma\tau + (1-\gamma)\tau') &= C(G + (\gamma\tau + (1-\gamma)\tau')(F - F_{min})) &(17)\\
&= C(\gamma(F_{min} + \tau(F - F_{min})) + (1-\gamma)(F_{min} + \tau'(F - F_{min}))) &(18)\\
&< \gamma C(F_{min} + \tau(F - F_{min})) + (1-\gamma)C(F_{min} + \tau'(F - F_{min})) &(19)\\
&= \gamma\Phi(\tau) + (1-\gamma)\Phi(\tau'). &(20)
\end{aligned}$$

function, that is, $C(F)=\eta\int_V \ell(v)\,dF+\kappa\tilde{C}(F)$, $\eta,\kappa\geq 0$, $\tilde{C}$ *strictly convex. If there is an equilibrium selection* (F_κ, H_κ) *with* $C(F_\kappa)\neq 0$ *for* $\kappa>0$*, then the buyer's equilibrium utility along the selection is increasing in* κ *at* $\kappa=0$.

Proposition 2 does not address the seller's equilibrium profits. Without further assumptions on c_F, equilibrium profits are not uniquely pinned down because there can be multiple equilibria with different profits. This can be illustrated already in the linear case. Suppose that $\alpha=0$, and that c is strictly positive on the interval (α,κ) and $c(\alpha)=c(\omega)=0$. In this case, all distributions that place mass only on α or ω are costless for the buyer. There are therefore multiple equilibria: the buyer places probability f on ω and $1-f$ on α for some $f\in[0,1]$, and the seller chooses $p=\omega$. The seller's profit is $f\omega$. Notice that equilibrium profit is between $\alpha=0$ and ω, the smallest and largest minimizer of c. This property generalizes.

Proposition 3 *The seller's profit* Π *in an equilibrium* (F,H) *is bounded by*[13]

$$\min\left(\arg\min_{v\in V} c_F(v)\right)\leq \Pi \leq \max\left(\arg\min_{v\in V} c_F(v)\right). \tag{22}$$

In many applications, it is natural that c_F is uniquely minimized at the lowest possible valuation α. In this case, Proposition 3 implies that equilibrium profit is α. Under the conditions of Corollary 1, then not only the buyer's utility but also total welfare is increasing in κ at $\kappa=0$.

5 Cost specifications

In this section, I impose more structure on the cost function to obtain sharper predictions. Throughout this section, I assume that $\alpha>0$.[14]

5.1 Convex c_F

In this section, I assume:

A4: c_F is strictly convex and differentiable for all $F\in\mathscr{F}$.

Convexity of c_F for all $F\in\mathscr{F}$ implies that C increases if F decreases in the sense of second order stochastic dominance (see Chew and Nishimura, 1992, Corollary 1). In particular, C is increasing in risk in the terminology of Condorelli and Szentes (2020).

[13]Note that $\arg\min_{v\in V} c_F(v)$ is compact, because it is closed (since c_F is continuous) and bounded (since V is bounded). Therefore, the min and the max of $\arg\min_{v\in V} c_F(v)$ exist.

[14]The case $\alpha=0$ is special in that the seller cannot guarantee himself a positive profit by charging a price equal to the lowest possible valuation. If the buyer's cheapest distribution places probability one on the lowest valuation, there is a "mis-coordination" equilibrium where the buyer does not invest and the seller charges a high price.

Strict convexity of c_F implies that c_F has a unique minimizer v_0^F, and that the function $v - c_F(v)$ has a unique maximizer v_1^F on $[\alpha, \omega]$. As in Condorelli and Szentes (2020), define for $\pi, B \in (0, \omega]$, the equal revenue distribution

$$\Gamma_\pi^B(v) = \begin{cases} 0 & if \quad v < \pi \\ 1 - \pi/v & if \quad v \in [\pi, B) \\ 1 & if \quad v \geq B \end{cases} . \tag{23}$$

Proposition 4 *Under A4, (F, H) is an equilibrium if and only if*[15]

$$F = \Gamma_{v_0^F}^{v_1^F}, \quad H(p) = c_F'(p)\mathbb{1}_{[v_0^F, v_1^F)}(p) + \mathbb{1}_{[v_1^F, \omega]}(p). \tag{24}$$

Proposition 4 says that the buyer's distribution is an equal revenue distribution and that the seller's pricing distribution is essentially equal to the derivative of the Gateaux derivative. In particular, the supports of both distributions are convex. The proof of Proposition 4 follows from the same arguments as in the proof of Proposition 1 in Gul (2001). The only difference is that Gul (2001) considers the case with linear cost C so that c_F does not depend on F. This does not, however, matter for the argument. I omit the details.[16]

Proposition 4 characterizes the equilibrium only implicitly because F depends on v_0^F and v_1^F which, in turn, depend on F, as they are minimizers and maximizers of $c_F(v)$ and $v - c_F(v)$. While this makes it difficult to derive equilibria explicitly, it is, however, straightforward to characterize the case when the distribution $F = \Gamma_\alpha^\omega$ that has full support $[\alpha, \omega]$ is chosen in equilibrium. This is the case if $c_{\Gamma_\alpha^\omega}(v)$ is minimized at α, and $v - c_{\Gamma_\alpha^\omega}(v)$ is maximized at ω, that is,

$$c_{\Gamma_\alpha^\omega}'(\alpha) \geq 0, \quad \text{and} \quad c_{\Gamma_\alpha^\omega}'(\omega) \leq 1. \tag{25}$$

I use this observation in the next corollary to extend the statement of Corollary 1 to parameters $\kappa > 0$.

Corollary 2 *Consider a cost function that is a combination of a linear and a strictly convex cost function, that is, $C(F) = \eta \int_V \ell(v)\, dF + \kappa \tilde{C}(F)$, where $\eta > 0, \kappa \geq 0$, $\tilde{C}$ strictly convex with Gateaux derivative $\tilde{c}_F$. Assume that $\eta\ell + \kappa\tilde{c}_F$ is differentiable and strictly convex with a minimum at α for all $\eta > 0, \kappa \geq 0$, $F \in \mathscr{F}$, and $\eta\ell'(\omega) < 1$.*

Then there is $\hat{\kappa} > 0$ so that the buyer's equilibrium utility U_B and total welfare is strictly increasing in κ for all $\kappa \in [0, \hat{\kappa})$.

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[15] $\mathbb{1}$ denotes the indicator function.

[16] As in Gul (2001), differentiability of c_F is not needed for the result. In this case, the derivative of c_F in (24) has to be replaced by the right derivative.

The intuition behind the corollary is that as κ increases in the range $[0,\hat{\kappa})$, the direct effect of facing higher investment costs is outweighed by the indirect strategic effect that the seller reduces the price (in the first order sense). More precisely, observe that the assumptions in the corollary ensure that for small values of $\kappa \leq \hat{\kappa}$ the conditions (25) are met so that the buyer chooses $F = \Gamma_\alpha^\omega$ in equilibrium. Therefore, as κ increases in the range $[0,\hat{\kappa})$, the buyer's distribution remains the same since Γ_α^ω is independent of κ. On the other hand, the Gateaux derivative $c_{\Gamma_\alpha^\omega}$ becomes steeper as κ increase. By Proposition 4, this means that the seller's pricing distribution decreases in the first order sense. This price effect outweighs the direct cost effect.

I next present a specific cost function that is a combination of a linear and a strictly convex cost function where equilibria can be calculated for all parameter values, including large κ.

5.1.1 Example

Consider the second moment $Q_F = \int_V v^2\, dF$, and define

$$C(F) = \frac{1}{2}\eta Q_F + \frac{1}{4}\kappa Q_F^2 - \left(\frac{1}{2}\eta\alpha^2 + \frac{1}{4}\kappa\alpha^4\right). \tag{26}$$

C is evidently convex, and it is easy to see that the cdf F_{min} that (uniquely) minimizes the function $\frac{1}{2}\eta Q_F + \frac{1}{4}\kappa Q_F^2$ is the cdf that places all mass on α. The term in the brackets in C thus ensures that $C(F_{min}) = 0$, in line with the normalization in A3.

The Gateaux derivative is

$$c_F(v) = \frac{1}{2}\eta v^2 + \frac{1}{2}\kappa Q_F v^2. \tag{27}$$

Lemma 1 *Let C be given by (26). Let $\hat{v}_1$ be the positive solution to the quadratic equation*

$$2\alpha\kappa v_1^2 + (\eta - \alpha^2\kappa)v_1 - 1 \;=\; 0. \tag{28}$$

and define the cutoffs $\kappa_v = \frac{1-\eta v}{\alpha v(2v-\alpha)}$ for $v \in \{\alpha,\omega\}$. Then the equilibrium values v_0^F and v_1^F in Proposition 4 are

$$v_0^F = \alpha, \quad v_1^F = \begin{cases} \omega & if \quad \kappa \leq \kappa_\omega \\ \hat{v}_1 & if \quad \kappa \in (\kappa_\omega, \kappa_\alpha) \\ \alpha & if \quad \kappa \geq \kappa_\alpha \end{cases}. \tag{29}$$

The example is sufficiently tractable so that welfare can be explicitly calculated. Notice first that since c_F is strictly increasing, it is uniquely minimized at α. Therefore, by Proposition 3, the seller's profit is $\Pi = \alpha$. The buyer's equilibrium utility can be calculated using (10).

Lemma 2 *Let C be as in (26). Then the seller's equilibrium profit is α, and the buyer's equilibrium utility is*

$$U_B = \kappa\alpha^2\left(v_1^F - \alpha\right)^2. \tag{30}$$

Figure 1 illustrates the typical shape of the buyer's utility as a function of κ. It increases linearly in the range $\kappa \in [0, \kappa_\omega]$ and then decreases. At κ_α, investment costs become prohibitive.

Note also that a straightforward calculation shows that U_B is strictly decreasing in η for $\kappa > 0$.

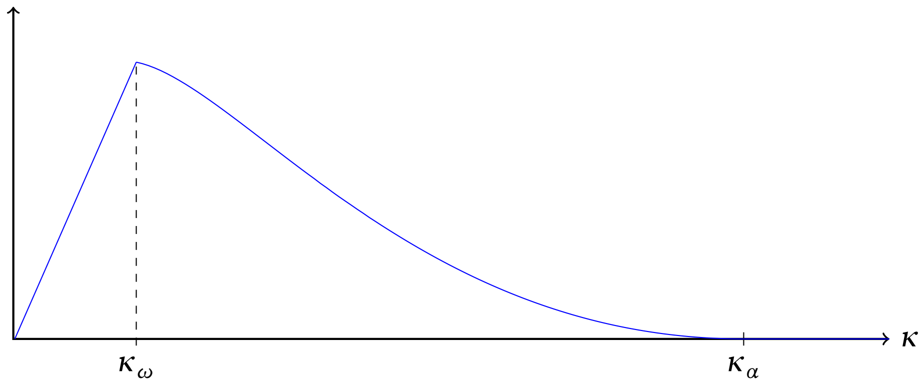

Figure 1: The figure shows U_B from Lemma 2 as a function of κ for the values $\alpha = 1$, $\omega = 3/2$, $\eta = 1/2$, $\kappa_\omega = 1/12$, $\kappa_\alpha = 1/2$.

5.2 Concave c_F

In this section, I assume:

A5: c_F is strictly concave for all $F \in \mathscr{F}$.

Concavity of c_F for all $F \in \mathscr{F}$ implies that C increases if F increases in the sense of second order stochastic dominance (see Chew and Nishimura, 1992, Corollary 1). In particular, C is decreasing in risk in the terminology of Condorelli and Szentes (2020).

Let

$$\sigma_F = \frac{c_F(\omega) - c_F(\alpha)}{\omega - \alpha}, \tag{31}$$

and denote by δ_v the cdf that places probability 1 on the point v.

Proposition 5 *Under A5, (F, H) is an equilibrium if and only if $F = (1-f)\delta_\alpha + f\delta_\omega$ and $H =$*

$(1-h)\delta_\alpha + h\delta_\omega$ *where*

$$(f,h) = \begin{cases} (0,0) & if \quad \sigma_F > 1 \\ (\leq \alpha/\omega, 0) & if \quad \sigma_F = 1 \\ (\alpha/\omega, 1-\sigma_F) & if \quad \sigma_F \in (0,1) \\ (\geq \alpha/\omega, 1) & if \quad \sigma_F = 0 \\ (1,1) & if \quad \sigma_F < 0 \end{cases} . \tag{32}$$

The proposition says that both the buyer's valuation and the seller's pricing distribution are supported on the points α and/or ω. The reason can be seen already in the linear case where c_F does not depend on F. The buyer's utility from placing probability weight on valuation v is then $\overline{H}(v) - c(v)$ which is convex because $\overline{H}$ is convex and c is concave. Therefore $\overline{H}(v) - c(v)$ is maximized at α and/or ω, and it follows that the support of the buyer's distribution is $\{\alpha, \omega\}$. Therefore, the seller's best response is to charge the price α or ω. More precisely, the price α is optimal if the probability f with which the buyer's valuation is ω, is sufficiently small, more precisely, if $f \leq \alpha/\omega$. And the price ω is optimal if $f \geq \alpha/\omega$.

To shed light on expression (32), let h be the probability with which the seller chooses the high price ω. The buyer's utility from choosing valuation α is then $-c(\alpha)$, and her utility from choosing valuation ω is $(1-h)(\omega-\alpha)-c(\omega)$. Now suppose $\sigma \in (0,1)$ as in the third line of (32). By definition of σ, the buyer is then indifferent between α and ω if the seller chooses $h = 1-\sigma$. For this to be an equilibrium, the seller needs to be indifferent between charging α and ω which is the case if the buyer chooses $f = \alpha/\omega$. The other cases follow analogously.

The same reasoning goes through in the non-linear case in which c_F does depend on F. In this case, (32) characterizes an equilibrium only implicitly, as F appears on both sides. Next, I present an example.

5.2.1 Example

Let $M_F = \int_V v \; dF$ be the mean of F, and consider the cost function that corresponds to the negative variance:

$$C(F) = \frac{1}{4}\kappa(\alpha - \omega)^2 - \kappa \int_V (v - M_F)^2 \, dF(v). \tag{33}$$

Note first that C is convex, as can be easily verified.[17] Moreover, the cdf F_{min} that (uniquely) minimizes $-\kappa \int_V (v - M_F)^2 \; dF(v)$ (i.e. maximizes the variance) is the cdf that puts probability

[17]While the positive variance looks like a perfectly sensible cost function, as it captures the idea that adding noise is costly, it is actually concave and thus not a valid cost function in my framework.

one half each on α and ω. The first term in C thus ensures that $C(F_{min}) = 0$, in line with the normalization in A3.

The Gateaux derivative is

$$c_F(v) = -\kappa(v - M_F)^2. \tag{34}$$

Moreover, for $f \in (1/2, 1]$, define the function

$$\hat{\kappa}(f) = \frac{1}{(\omega - \alpha)(2f - 1)}. \tag{35}$$

Lemma 3 *Let C be given by (33). Then the equilibrium values of f and h in Proposition 5 are:*

(i) If $\alpha/\omega \leq 1/2$, then $f = 1/2$, $h = 1$;

(ii) If $\alpha/\omega > 1/2$, then:

(a) If $\kappa < \hat{\kappa}(\alpha/\omega)$, then $f = \alpha/\omega$, and $h = 1 - \kappa(2\alpha/\omega - 1)(\omega - \alpha)$;

(b) If $\kappa \geq \hat{\kappa}(\alpha/\omega)$, then $f = 1/2 \cdot \left(1 + \frac{1}{\kappa(\omega-\alpha)}\right)$, and $h = 0$.

To shed light on case (i), recall that the buyer's default distribution F_{min} puts probability $f = 1/2$ on α and ω. Because $\alpha/\omega \leq f = 1/2$, the seller chooses the high price ($h = 1$) against F_{min}. Note that there is no other equilibrium because at a lower price, the buyer's investment incentives would only be higher, thus making the lower price suboptimal.

In case (ii)(a), there is no equilibrium in which the seller chooses a deterministic price. The reason is that if the seller were to choose the high price ($h = 1$), the buyer would best respond with the default distribution ($F = 1/2$). But because $\alpha/\omega > 1/2$, the seller would then prefer the low price ($h = 0$). Moreover, if the seller were to choose the low price ($h = 0$), then because costs (κ) are fairly low, the buyer would best respond with increasing her investment beyond α/ω at which point it would be optimal for the seller to choose the high price. Thus, in equilibrium, the seller randomizes between the low and high price, and the buyer invests at the level $f = \alpha/\omega$ to keep the seller indifferent.

In case (b), costs are so high that even if the seller chooses the low price ($h = 0$), the buyer's investment incentives are fairly low and her best response is to invest at a level $f < \alpha/\omega$. This, in turn, renders the low price by the seller indeed optimal.

Next, I turn to welfare.

Lemma 4 *Let C be given by (33). Then the seller's equilibrium profit Π and the buyer's equilibrium utility U_B are:*

(i) If $\alpha/\omega \leq 1/2$, then $\Pi = 1/2 \cdot \omega$, and $U_B = 0$;

(ii) $\alpha/\omega > 1/2$, then:

(a) *If* $\kappa < \hat{\kappa}(\alpha/\omega)$*, then* $\Pi = \alpha$*, and* $U_B = \kappa(\omega - \alpha)^2(\alpha^2/\omega^2 - 1/4)$*;*

(b) *If* $\kappa \geq \hat{\kappa}(\alpha/\omega)$*, then* $\Pi = \alpha$*, and* $U_B = 1/2 \cdot (\omega - \alpha) + 1/(4\kappa)$*.*

Notice first that in case (i), the price is ω with probability 1 by Lemma 3. Thus, the buyer does not get any surplus from trade, and since she chooses the default distribution at zero cost, her utility is $U_B = 0$. This does not contradict part (iii) of Proposition 2, because the equilibrium distribution coincides with the distribution F_{min}. Moreover, because trade occurs with probability $f = 1/2$, the seller's profit is $\Pi = 1/2 \cdot \omega$.

In part (ii), equilibrium buyer utility is again non-monotone in κ, as it increases up to $\hat{\kappa}$ and decreases from then on (see Figure 2). The intuitive reason is that in case (a), as κ increases, the seller lowers the price in response. This price effect outweighs the direct effect of incurring higher investment costs. In case (b), the price is at its lowest level α for all κ. Thus, an increase in κ only increases costs but has no price effect. Finally, note that because the seller is indifferent or charges the low price, his profit is equal to α.

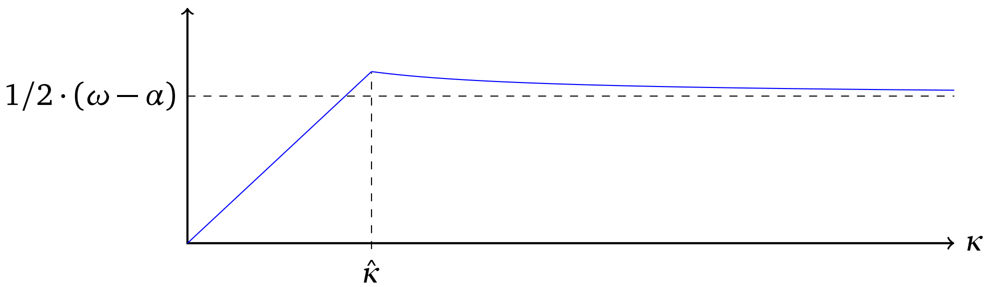

Figure 2: The figure shows U_B from Lemma 4, part (ii), as a function of κ for the values $\alpha = 1$, $\omega = 3/2$, $\alpha/\omega = 2/3$, $\hat{\kappa}(\alpha/\omega) = 6$.

6 Investment and information acquisition

My model can be extended to allow an interpretation where it is costly for the buyer not only to invest in, but also to learn about her valuation. Ravid et al. (2022) consider a framework where the buyer's true valuation is a value θ in a compact interval Θ. Initially, the buyer only has a prior belief $F_0 \in \mathscr{F}$ over Θ, but she can acquire a signal about her valuation at a cost before trading. Since the buyer's preferences for the good are linear in the valuation, a signal corresponds to a distribution $F \in \mathscr{F}$ of posterior means that is a mean preserving contraction (MPC) of the prior F_0. Optimizing over a functional subject to the MPC constraint is, by now, a well-studied problem when the functional is linear (Dworczak and Martini, 2019, Kleiner et al., 2021), but is difficult when the functional is non-linear such as when information acquisition costs are non-linear.

The approach presented in this paper can however be applied to include information acquisition when one restricts the space Θ of the buyer's true valuations to consist of only two possible values, as I now illustrate. Suppose that $\Theta \in \{\alpha, \omega\}$. A prior then corresponds simply to a mean $M_0 \in V = [\alpha, \omega]$, and any signal corresponds to a cdf $F \in \mathscr{F}$ over posterior means $v \in V$ with the simplified MPC constraint that F has mean M_0, that is, $\int_V v dF = M_0$.

Suppose now that without investing, the buyer's valuation is equal to the lowest possible valuation α with probability 1. The buyer can invest to increase the mean M_F of the valuation distribution F at a cost $\Phi(M_F) - \Phi(\alpha)$. The mean M_F then corresponds to the prior, and the buyer can learn about the true valuation given the prior at a cost $\Psi(F)$. Specifically, consider a strictly convex function $k : V \to \mathbb{R}$ and let $\Psi(F) = \int_V k(v) - k(M_F)\, dF$ be the ("posterior-separable") cost of information acquisition. Since k is convex, Ψ increases in the mean preserving spread order, or equivalently, in Blackwell informativeness. Moreover, acquiring no information, which corresponds to choosing the degenerate distribution δ_M that places probability 1 on M, is costless. The cost function

$$C(F) = \Phi(M_F) - \Phi(\alpha) + \int_V k(v) - k(M_F)\, dF \tag{36}$$

therefore combines the cost of investing and the cost of information acquisition. The Gateaux derivative

$$c_F(v) = [\Phi'(M_F) - k'(M_F)]v + k(v) \tag{37}$$

is strictly convex, and therefore Proposition 4 applies, as long as C is convex. The latter is the case if $\Phi - k$ is convex.

As a specific example, consider the case that $\Phi(v) = \beta\varphi(v)$ and $k(v) = \gamma\varphi(v)$ with $\beta > \gamma$ so that

$$C(F) = (\beta - \gamma)\varphi(M_F) + \gamma \int_V \varphi(v)\, dF - \beta\varphi(\alpha). \tag{38}$$

If φ is strictly convex and minimized at α, the same arguments that establish Corollary 2 can be used to show that there is $\hat{\kappa} > 0$ so that the buyer's equilibrium utility and total welfare is increasing in $\beta - \gamma$ for all $\beta - \gamma \in (0, \hat{\kappa})$. Thus, in this range, the buyer's utility increases in investment costs (β) but decreases in information acquisition costs (γ).

7 Conclusion

In this paper, I reconsider the hold-up problem with unobservable investments when the buyer's investment costs are convex in the investment distribution. The main result is that in contrast to the case with linear costs, the buyer's utility is positive in equilibrium, and that buyer utility and total welfare might be locally increasing in costs. The equilibrium characterization I derive is portable to other applications with flexible pre-investments.

Appendix

Proof of Proposition 1 I only show part (ii). (Part (i) follows with the same arguments as the proof in footnote 22 in Ravid et al (2022).) Note first that it is a standard argument that the (mixed) strategy H is a best response to F for the seller if and only (8) and (9) hold, where π is the seller's best response profit. Because the seller can guarantee himself the profit α by choosing the price α with probability 1, we have $\pi \geq \alpha$.

Next, I show that given H, the buyer's best response is characterized by (6) and (7). Indeed, F is a best response if and only if F is a solution to

$$\max_{G \in \mathscr{F}} \int_V \overline{H}(v)\, dG - C(G). \tag{39}$$

Let $M^{\uparrow}$ be the set of bounded right-continuous increasing functions on V. Then problem (39) can be equivalently written as

$$\max_{G \in M^{\uparrow}} \int_V \overline{H}(v)\, dG - C(G) \quad s.t. \quad 1 - \int_V dG = 0, \tag{40}$$

where the constraint ensures that G is a cdf. By Luenberger (1997), F is a solution to (40) if and only if there is λ so that F maximizes the Lagrangian

$$L(G, \lambda) = \int_V \overline{H}(v)\, dG - C(G) + \lambda(1 - \int_V dG) \tag{41}$$

among all $G \in M^{\uparrow}$. Indeed, because there is only one (linear) equality constraint in (40), necessity follows from the statement in Problem 7 (p. 236), while sufficiency follows from Theorem 1 (p. 220) in Luenberger (1997).

I now apply Lemma 1 in Luenberger (1997, p. 227) which I restate in the form needed here.

Lemma 5 (Luenberger 1997, p. 227) *Let T be a concave, real-valued functional defined on the* *linear space X which has a linear Gateaux-differential $\delta T(F, D)$ for all $F, D \in X$. Let P be a convex cone in X. A necessary and sufficient condition that F maximize T over P is that*

$$\delta T(F;G) \leq 0 \text{ for all } G \in P, \quad \text{and} \quad \delta T(F;F) = 0. \tag{42}$$

To apply the lemma to my setting, let $X = BV$, and let T correspond to the Lagrangian $L(G, \lambda)$. Note that $P = M^{\uparrow}$ is a convex cone in BV. Since C is convex on BV by assumption A1, $L(\cdot, \lambda)$ is concave (in G). Moreover, the Gateaux-differential of the Langrangian,

$$\delta L(G, \lambda; D) = \int_V \overline{H}(v)\, dD - \int_V c_F(v)\, dD - \lambda \int_V dD \tag{43}$$

is well-defined for all $G, D \in BV$ and is linear in D by assumption A2. Thus, the lemma implies that F maximizes $L(\cdot, \lambda)$ over $M^{\uparrow}$ if and only if

$$\delta L(F, \lambda; G) \leq 0 \text{ for all } G \in M^{\uparrow}, \quad \text{and} \quad \delta L(F, \lambda; F) = 0. \tag{44}$$

With (43), this can be equivalently written as

$$\int_V \overline{H}(v) - c_F(v) - \lambda\, dG \quad \leq \quad 0 \quad \text{ for all } G \in M^{\uparrow}, \tag{45}$$

$$\int_V \overline{H}(v) - c_F(v) - \lambda\, dF \quad = \quad 0. \tag{46}$$

Because the G's and F are increasing functions, thus corresponding to positive measures, this is equivalent to (6) and (7), and this completes the proof. QED

Proof of Proposition 2 (i) I first show that $\lambda = -\min_{v \in V} c_F(v)$. Indeed, let $\underline{v} = \min supp(F)$, $\underline{p} = \min supp(H)$ be the lower support bounds.

Observe first that $\underline{v} \leq \underline{p}$. Otherwise, if $\underline{p} < \underline{v}$, then $F(\underline{p}^-) = 0$ so that the seller's profit at $\underline{p}$ is $(1 - F(\underline{p}^-))\underline{p} = \underline{p}$. But since $F(\underline{v}^-) = 0$ by definition of $\underline{v}$, the seller could strictly increase his profit by deviating to the price $p = \underline{v}$. By (8) and (9), this contradicts that $\underline{p}$ is in $supp(H)$.

Next, I show that $\underline{v} \in \arg\min_{v \in V} c_F(v)$. To the contrary, suppose $c_F(\underline{v}) > c_F(\hat{v})$ where $\hat{v} \in \arg\min_{v \in V} c_F(v)$. Because $\underline{v} \leq \underline{p}$, we have that $\overline{H}(\underline{v}) = 0$. Therefore, because (trivially) $\overline{H}(\hat{v}) \geq 0$, we have

$$\overline{H}(\underline{v}) - c_F(\underline{v}) - \lambda < \overline{H}(\hat{v}) - c_F(\hat{v}) - \lambda. \tag{47}$$

By (6) and (7), this contradicts that $\underline{v}$ is in $supp(F)$.

Therefore, because $\underline{v} \leq \underline{p}$ implies $\overline{H}(\underline{v}) = 0$, and because $c_F(\underline{v}) = \min_{v \in V} c_F(v)$, we infer from (7) that

$$0 = \overline{H}(\underline{v}) - c_F(\underline{v}) - \lambda = -\min_{v \in V} c_F(v) - \lambda, \tag{48}$$

as desired.

To see the expression for U_B, recall from (2) that $U_B(H, F) = \int_V \overline{H}(v)\ dF(v) - C(F)$. Thus, plugging in $\overline{H}$ from (7) yields (10).

(ii) Note that when C is linear, we have $\int_V c_F(v)\ dF = \int_V c(v)\ dF = C(F)$. Thus, $U_B = -\min_{v \in V} c(v) = 0$ by (5).

(iii) The proof is given in the main text.

Proof of Proposition 3 Let $\underline{v} = \min supp(F)$, $\underline{p} = \min supp(H)$ the lower support bounds.

To see the left inequality in (22), recall from the proof of Proposition 2 that $\underline{v} \in \arg\min_{v \in F} c_F(v)$. Moreover, the seller can can guarantee himself a profit $\underline{v}$ by charging the price $\underline{v}$. Thus, $\min(\arg\min_{v \in F} c_F(v)) \leq \underline{v} \leq \Pi$.

To see the right inequality in (22), let $\hat{v} = \max(\arg\min_{v \in V} c_F(v))$, and suppose to the contrary that $\hat{v} < \Pi$. Since $\underline{p} \in supp(H)$, we have that $\Pi \leq \underline{p}$. Therefore, $\hat{v} < \underline{p}$. This implies that $\overline{H}(\hat{v}) = 0$ and $c_F(\hat{v}) < c_F(\underline{p})$ by definition of $\hat{v}$. Hence,

$$\overline{H}(\hat{v}) - c_F(\hat{v}) - \lambda = 0 - c_F(\hat{v}) - \lambda > 0 - c_F(\underline{p}) - \lambda = \overline{H}(\underline{p}) - c_F(\underline{p}) - \lambda. \tag{49}$$

Therefore, (6) and (7) imply that $\underline{p} \notin supp(F)$.

Now distinguish two cases:

(a) $F(\underline{p}^-) < 1$. Since $F(\omega) = 1$ and $\underline{p} \notin supp(F)$, this implies that $\underline{p} < \omega$. Therefore, since $\underline{p} \notin supp(F)$, there is $q > \underline{p}$ with $F(\underline{p}^-) = F(q)$, and hence the seller could increase profits by increasing the price from $\underline{p}$ to q, contradicting that $\underline{p} \in supp(H)$.

(b) $F(\underline{p}^-) = 1$. Then the seller's profit from price $\underline{p}$ is zero, and hence, since $\underline{p} \in supp(H)$, we have that $\Pi = 0$. This contradicts that $\hat{v} < \Pi$. QED

Proof of Proposition 4 The proof is the same as the proof of Proposition 1 in Gul (2001). QED

Proof of Corollary 2 Let $\Gamma = \Gamma_\alpha^\omega$. Since $\eta\ell'(\omega) < 1$, and $\eta\ell + \kappa\tilde{c}_F$ is strictly convex, there is a unique $\hat{\kappa}$ so that

$$c'_\Gamma(\omega) = \eta\ell'(\omega) + \kappa\tilde{c}'_\Gamma(\omega) < 1 \text{ for all } \kappa < \hat{\kappa} \text{ with equality at } \hat{\kappa}. \tag{50}$$

By assumption, α minimizes c_Γ, and (50) implies that ω maximizes $v - c_\Gamma(v)$ for all $\kappa \leq \hat{\kappa}$.

Therefore, Proposition 4 implies that in equilibrium, the buyer chooses Γ for all $\kappa \leq \hat{\kappa}$. By part (i) of Proposition 2, her equilibrium utility is thus

$$\begin{aligned} U_B &= \int_V c_\Gamma(v)\, d\Gamma(v) - C(\Gamma) - \min_{v\in V} c_\Gamma(v) && (51)\\ &= \int_V \eta\ell(v) + \kappa\tilde{c}_\Gamma(v)\, d\Gamma(v) - \int_V \eta\ell(v)\, d\Gamma(v) - \kappa\tilde{C}(\Gamma) - \min_{v\in V}[\eta\ell(v) + \kappa\tilde{c}_\Gamma(v)] && (52)\\ &= \kappa\left(\int_V \tilde{c}_\Gamma(v)\, d\Gamma(v) - \tilde{C}(\Gamma) - \min_{v\in V} \tilde{c}_\Gamma(v)\right) - \eta\ell(\alpha), && (53)\end{aligned}$$

where the third equality follows because $\eta\ell(v) + \kappa\tilde{c}_\Gamma(v)$ is minimized at α for all $\eta > 0, \kappa \geq 0$ by assumption.

Now note that it follows with the same arguments as in part (iii) of Proposition 2 that the term in brackets is strictly positive. Thus, since Γ is independent of κ, U_B is strictly increasing in κ for all $\kappa \leq \hat{\kappa}$.

Finally note that since α uniquely minimizes c_Γ, equilibrium profits are $\Pi = \alpha$ by Proposition 3. Thus, also total welfare, $U_B + \alpha$, strictly increasing in κ for all $\kappa \leq \hat{\kappa}$. QED

Proof of Lemma 1 Because c_F in (27) is increasing, it is minimized at α. Thus, by Proposition 4, $v_0^F = \alpha$ and $F = \Gamma_\alpha^{v_1^F}$. On the other hand, v_1^F is given as the maximizer of $v - c_F(v)$ over $[\alpha, \omega]$. I now determine v_1^F under the assumption that it is interior and thus a solution to the first order condition $c_F'(\hat{v}_1) = 1$. I then check when this is indeed the case. We have:

$$c_F'(\hat{v}_1) = \eta\hat{v}_1 + \kappa Q_F \hat{v}_1 = 1 \quad \Leftrightarrow \quad \hat{v}_1 = \frac{1}{\eta + \kappa Q_F}. \tag{54}$$

I now determine $Q_{\hat{F}}$ for $\hat{F} = \Gamma_\alpha^{\hat{v}_1}$. Note that $\hat{F}$ has density α/v^2 on the interval $[\alpha, \hat{v}_1)$ and a mass point of size $\alpha/\hat{v}_1$ at $\hat{v}_1$. Thus,

$$Q_{\hat{F}} = \int_V v^2\, d\Gamma_\alpha^{\hat{v}_1} = \int_\alpha^{\hat{v}_1} v^2 \frac{\alpha}{v^2}\, dv + \frac{\alpha}{\hat{v}_1}(\hat{v}_1)^2 = 2\alpha\hat{v}_1 - \alpha^2. \tag{55}$$

Inserting this into (54) and re-arranging terms yields:

$$\hat{v}_1 = \frac{1}{\eta + \kappa(2\alpha\hat{v}_1 - \alpha^2)} \quad \Leftrightarrow \quad 2\alpha\kappa(\hat{v}_1)^2 + (\eta - \alpha^2\kappa)\hat{v}_1 - 1 = 0. \tag{56}$$

The quadratic equation has one positive solution

$$\hat{v}_1 = \frac{1}{4\alpha\kappa}\left(\alpha^2\kappa - \eta + \sqrt{(\eta - \alpha^2\kappa)^2 + 8\alpha\kappa}\right). \tag{57}$$

I now check when this solution is interior, that is, $\hat{v}_1 \in (\alpha, \omega)$. A straightforward calculation delivers

$$\alpha < \hat{v}_1 \quad \Leftrightarrow \quad \kappa < \frac{1-\eta\alpha}{\alpha^3} = \kappa_\alpha, \tag{58}$$

$$\hat{v}_1 < \omega \quad \Leftrightarrow \quad \kappa > \frac{1-\eta\omega}{\alpha\omega(2\omega-\alpha)} = \kappa_\omega. \tag{59}$$

Thus, v_1^F is interior with $v_1^F = \hat{v}_1$ if $\kappa \in (\kappa_\omega, \kappa_\alpha)$. Moreover, if $\kappa \geq \kappa_\alpha$, then the same considerations imply that $v - c_{\Gamma_\alpha^\alpha}(v)$ is maximized at α. Thus, $v_1^F = \alpha$ and $F = \Gamma_\alpha^\alpha$. Finally, if $\kappa \leq \kappa_\omega$, then $v - c_{\Gamma_\alpha^\omega}(v)$ is maximized at ω. Thus $v_1^F = \omega$ and $F = \Gamma_\alpha^\omega$. This is what was to show. QED

Proof of Lemma 2 I calculate U_B from (10). By (27):

$$\int_V c_F(v)\,dF(v) \;=\; \frac{1}{2}(\eta+\kappa Q_F)\int_V v^2\,dF(v) = \frac{1}{2}(\eta Q_F + \kappa Q_F^2), \tag{60}$$

and

$$\min_{v\in V} c_F(v) = c_F(\alpha) = \frac{1}{2}(\eta+\kappa Q_F)\alpha^2. \tag{61}$$

Hence, by (10),

$$\begin{aligned} U_B &= \int_V c_F(v)\,dF(v) - C(F) - \min_{v\in V} c_F(v) && (62)\\ &= \frac{1}{2}(\eta Q_F + \kappa Q_F^2) - \frac{1}{2}\eta Q_F - \frac{1}{4}\kappa Q_F^2 + \left(\frac{1}{2}\eta\alpha^2 + \frac{1}{4}\kappa\alpha^4\right) - \frac{1}{2}(\eta+\kappa Q_F)\alpha^2 && (63)\\ &= \frac{1}{4}\kappa\left(Q_F^2 + \alpha^4 - 2Q_F\alpha^2\right) && (64)\\ &= \frac{1}{4}\kappa\left(Q_F - \alpha^2\right)^2. && (65) \end{aligned}$$

Plugging in Q_F from (55) yields the claim. QED

Proof of Proposition 5 To simplify notation, I suppress the subindex F. Define by $B(v) = \overline{H}(v) - c(v)$ the expression from the left hand side of the buyer's best response conditions (6) and (7). I refer to B as the buyer's "pseudo-payoff". Now, let (F, H) be an equilibrium. By (6) and (7), any v in the support of F must yield the same pseudo-payoff (equal to λ), and any v outside the support must not yield a higher pseudo-payoff.

Note first since $\overline{H}$ is convex and c is strictly concave, B is strictly convex. Therefore, B is maximal at α or ω which implies that $supp(F) \subseteq \{\alpha, \omega\}$. Let f be the probability that the valuation is ω.

This, in turn, implies that $supp(H) \subseteq \{\alpha, \omega\}$ because a price $p \notin \{\alpha, \omega\}$ can never be optimal for the seller if $supp(F) \subseteq \{\alpha, \omega\}$. In particular, it follows that $\overline{H}(\alpha) = 0$ and $\overline{H}(\omega) = (1-h)(\omega-\alpha)$, where $1-h$ is the probability that the seller chooses price α. This also implies that

$$B(\alpha) = -c(\alpha), \quad B(\omega) = (1-h)(\omega-\alpha) - c(\omega). \tag{66}$$

I now consider the various possible cases for σ and characterizes when (F, H) is indeed an equilibrium.

(a) Let $\sigma > 1$, that is, $c(\alpha) < c(\omega) - (\omega - \alpha)$. Then B is uniquely maximized at $v = \alpha$, because $B(\alpha) > B(\omega)$ by (66). Thus, we have $supp(F) = \{\alpha\}$, that is, $f = 0$ in equilibrium. The seller's best response is therefore to charge $p = \alpha$ with probability 1, that is, $h = 0$. This establishes the first line in (32).

(b) Next, let $\sigma < 0$, that is, $c(\alpha) > c(\omega)$. Then B is uniquely maximized at $v = \omega$, because $B(\alpha) < B(\omega)$ by (66). Thus, we have $supp(F) = \{\omega\}$, that is, $f = 1$ in equilibrium. The seller's best response is therefore to charge $p = \omega$ with probability 1, that is, $h = 1$. This establishes the fifth line in (32).

(c) Next, let $\sigma \in (0, 1)$, that is, $c(\alpha) = c(\omega) + \sigma(\omega - \alpha)$. Therefore,

$$B(\alpha) = 0 - c(\alpha) = \sigma(\omega - \alpha) - c(\omega). \tag{67}$$

I now argue that $h = 1 - \sigma$ in equilibrium. Indeed, suppose to the contrary that $h > 1 - \sigma$. Then, by (66) and (67), we have that $B(\alpha) > B(\omega)$ so that $supp(F) = \{\alpha\}$ and $f = 0$ in equilibrium. The seller's best response is therefore to charge $p = \omega$ with probability 1, that is, $h = 0$, a contradiction to $h > 1 - \sigma$. A similar contradiction can be constructed if $h < 1 - \sigma$.

Because $h = 1 - \sigma \in (0, 1)$, the seller must be indifferent between charging α and ω in equilibrium, that is, $\alpha = f\omega$, and hence $f = \alpha/\omega$. Moreover, since $h = 1 - \sigma$, we have that $B(\alpha) = B(\omega)$ by (66) and (67). Thus, $f = \alpha/\omega$ is indeed a best response for the buyer. This establishes the third line in (32).

Finally, the cases (d) $\sigma = 1$ and (e) $\sigma = 0$ can be dealt with analogously to the previous cases. I omit the details. QED

Proof of Lemma 3 The proof consists of checking when the conditions of Proposition 5 are satisfied. For F from Proposition 5, we have $M_F = (1-f)\alpha + f\omega$. Inserting this into (34) yields:

$$c_F(\alpha) = -\kappa f^2(\omega - \alpha)^2, \quad c_F(\omega) = -\kappa(1-f)^2(\omega - \alpha)^2, \tag{68}$$

so that

$$\sigma_F = \frac{c_F(\omega) - c_F(\alpha)}{\omega - \alpha} = -\kappa((1-f)^2 - f^2)(\omega - \alpha) = \kappa(2f-1)(\omega - \alpha). \tag{69}$$

I now go through the five possible cases in (32).

(a) Note that $\sigma_F < 0$ if and only if $f < 1/2$. Therefore, by the fifth line in (32), there is no equilibrium with $\sigma_F < 0$, as this would require $f = 1$.

(b) Note that $\sigma_F > 1$ only if $f > 1/2$. Thus, by the first line in (32), there is no equilibrium with $\sigma_F > 1$, as this would require $f = 0$.

(c) Note that $\sigma_F = 0$ if and only if $f = 1/2$. Thus, by the fourth line in (32), there is an equilibrium with $\sigma_F = 0$ and $f = 1/2$ and $h = 1$ if and only if $f = 1/2 \geq \alpha/\omega$. This establishes part (i) of the lemma.

(d) Note that $\sigma_F \in (0,1)$ if and only if $f > 1/2$ and $\kappa < \frac{1}{(2f-1)(\kappa-\alpha)} = \hat{\kappa}(f)$. Thus, by the third line in (32), there is an equilibrium with $\sigma_F \in (0,1)$ with $f = \alpha/\omega$ and $h = 1 - \sigma_F$ if and only if $\alpha/\omega > 1/2$ and $\kappa < \hat{\kappa}(\alpha/\omega)$. This establishes part (ii)(a) of the lemma.

(e) Note that $\sigma_F = 1$ if and only if $f > 1/2$ and

$$\kappa = \frac{1}{(2f-1)(\kappa - \alpha)} \quad \Leftrightarrow \quad f = \frac{1}{2}\left(1 + \frac{1}{\kappa(\omega - \alpha)}\right) = \hat{f}. \tag{70}$$

Thus, by the second line in (32), there is an equilibrium with $\sigma_F = 1$, $f \leq \alpha/\omega$, and $h = 0$ if and only if $\hat{f} > 1/2$ (which is always the case) and

$$\hat{f} \leq \frac{\alpha}{\omega} \quad \Leftrightarrow \quad \kappa \geq \frac{1}{(\omega - \alpha)(2\alpha/\omega - 1)} = \hat{\kappa}(\alpha/\omega). \tag{71}$$

This establishes part (ii)(b) of the lemma, and completes the proof. QED

Proof of Lemma 4 Part (i) is shown the main text. To calculate the buyer's equilibrium utility in part (ii), note that the buyer receives a positive trade surplus $\omega - \alpha$ only if the price is α and her valuation is ω, which occurs with probability $(1-h)f$. Thus, the buyer's expected trade surplus is $(1-h)f(\omega - \alpha)$.

By (33), the buyer's cost of the two point distribution $F = (1-f)\delta_\alpha + f\delta\omega$ is:

$$\begin{aligned} C(F) &= \frac{1}{4}\kappa(\alpha - \omega)^2 - \kappa\left[(1-f)(\alpha - \{(1-f)\alpha - f\omega)\} + f(\omega - \{(1-f)\alpha - f\omega)\}\right] & (72) \\ &= \kappa(\alpha - \omega)^2\left(\frac{1}{4} - f(1-f)\right). & (73) \end{aligned}$$

Thus, the buyer's equilibrium utility is

$$U_B = (1-h)f(\omega-\alpha) - \kappa(\alpha-\omega)^2\left(\frac{1}{4} - f(1-f)\right). \tag{74}$$

Inserting the values for f and h from part (ii)(a) and a tedious calculation yields

$$U_B = \kappa(\omega-\alpha)^2\left(\frac{\alpha^2}{\omega^2} - 1/4\right). \tag{75}$$

And inserting the values for f and h from part (ii)(b) and a tedious calculation yields

$$U_B = \frac{1}{2}(\omega-\alpha) + \frac{1}{4\kappa}. \tag{76}$$

QED

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