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Endogenous Fertility and the Henry George Theorem

by

Urs Schweizer[1]
Department of Economics
University of Bonn
Adenaueralle 24
D-53113 Bonn

Abstract

Models of endogenous demographic change deal with population size as an additional object of the welfare analysis. Hereby, the overlapping generations (OLG) model serves as the basic framework. In club theory, population size is also treated as an endogenous variable. In local public goods (LPG) models, the so-called Henry George Theorem which requires local public expenditures to be financed by a 100% tax on aggregate land rent is known as a condition for club efficiency. The present paper establishes and exploits the relation between steady states of the OLG model and allocations of the LPG model. In spite of Samuelson's fallacy concerning his goldenest golden rule, this rule as well as the Henry George Theorem as its LPG counterpart are shown to keep some meaning, not only as a necessary, but also as a sufficient condition for efficiency.

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Introduction

There is an ever growing literature on endogenous demographic change (see, e.g., Mirrlees (1972), Pazner and Razin (1980), Nerlove et al. (1984), Eckstein and Wolpin (1985), Cigno (1986), Lapan and Enders (1990), Wildasin (1990), Peters (1993) to mention only some of them). As far as normative aspects are concerned, this literature typically relies on the utility of the representative agent when population size is an additional object of the welfare analysis. Hereby, one or the other version of the overlapping generations (OLG) model serves as the basic setting. In many cases, the analysis focuses on steady states of the OLG-model.

Club theory is a different branch of economic theory where population size is treated as an endogenous variable (for a survey on club theory see Sandler and Tschirhart (1980)). Here, too, it is the utility per member which must be maximized in order to solve the club problem. Part of this literature deals with one or the other version of Tiebout's (1956) local public good (LPG) model. For such a model, Arnott and Stiglitz (1979) have established the so called Henry George Theorem which claims that, in order to reach the optimal size of population, aggregate land rent should be equal to local public expenditures.

In a formal sense, steady states in the OLG-model with endogenous fertility can be interpreted as allocations of a LPG-model with endogenous population size. Under this relation, it is consumption of the old in the OLG-model which corresponds to local public consumption in the LPG-model. Moreover, it is the number of offsprings per adult in the OLG-model which corresponds to the population size in the LPG-interpretation. The present paper establishes and systematically exploits the above relation.

As for the OLG-version, Samuelson (1975) has discerned what he calls the Serendipity Theorem. This theorem characterizes the goldenest golden rule state at which the utility of the representative agent achieves its maximum over all growth rates. Under the above relation, Samuelson's goldenest golden rule can safely be expected to correspond to the Henry George Theorem.

Unfortunately, however, this rule turns out to bear some risk. In fact, Deardorff (1976) has presented an overhelming class of examples for which Samuelson's goldenest golden rule fails to achieve its task because it characterizes a minimum rather than a maximum of well-being. Samuelson (1976) was quick enough to admit his fallacy.

Given the above relation, the Henry George Theorem might also be in danger to fail.

Both, the Serendipity Theorem and the Henry George Theorem, have in common that they give rise to simple and extremely nice rules of pricing. Therefore, it would be a pitty if they were without merit. No doubt, Deardorff's analysis is correct which means that we have to be careful. But, as I want to show, by exploiting the relation between the OLG- and the LPG-interpretation of the model, part of the message can be saved. Nevertheless, a strong caveat has to be kept in mind. While Samuelson's model is of an extremely simple type - it ignores for instance the pleasure of parenthood - , its basic mathematical difficulties must be taken into account in more elaborate versions of the model as well. It might be fun to derive Serendipity type and Henry George type results for general settings. The more tedious task, however, which consists of maximizing rather than minimizing utility should always be kept in mind. The present paper, hopefully, illuminates this important task.

The paper is organized as follows. Section 1 introduces the basic notation and it recalls the Henry George Theorem for a setting of, in essence, pure exchange. Section 2 establishes the relation between the OLG- and the LPG-interpretation. Hereby, the Henry George Theorem is shown to lead to a particular form of the goldenest golden rule. Section 3 deals with those aspects of the club problem which exhibit convexity. This section heavily draws from Schweizer (1983) who has established a link between Debreu's and Scarf's (1963) limit theorem on the core and the Henry George Theorem. The limit theorem serves well to highlight the partial convexity which the club problem and, hence, the problem of endogenous fertility involves. Section 4 deals with the missing link to full convexity which is at the heart of the difficulties. Moreover, it introduces a set of assumptions which ensure that the problem is a club problem in the strict sense, i.e. a problem which allows for a positive and bounded optimal size of population. In this case, the Serendipity Theorem and the Henry George Theorem give rise to conditions which are necessary for optimality. In general, unfortunately, these conditions still fail to be sufficient. Section 5, finally, revisits Samuelson's fallacy. It is shown that the pure exchange version of the OLG-model with only one commodity can never correspond to a club problem in the strict sense. It always features, however, a goldenest golden rule steady state. As a consequence, this steady state necessarily minimizes rather than maximizes wellbeing. Moreover, it is also shown that the OLG-model with capital accumulation introduces an additional non-convexity which, in the pure exchange version, is absent. Finally, the OLG-version including capital allows for a natural notion of equilibrium steady state. It is shown that this state corresponds, in the LPG-interpretation, to a Lindahl type equilibrium which tends to be distorted by a tax on the use of capital and by a subsidy on the supply of public goods. For this Lindahl type equilibrium to be free of distortions it turns out to be necessary and sufficient that the Henry George Theorem holds. Put differently, no matter whether the goldenest golden rule characterizes a minimum or a maximum over all growth rates, the corresponding equilibrium steady state is one which is free of distortions for the underlying growth rate.

1. The Henry George Theorem

In this section, a pure exchange version of the local public goods (LPG) model is introduced which will be used throughout the paper. For simplicity, all agents are assumed to be of the same type, that is they all have the same utility function and the same endowments. However, the reader is referred to concluding remarks where I comment on extensions of the results to the case involving different types of agents. All commodities serve, both, for private and for public use. The vector of private consumption x must belong to the private consumption set $X \subseteq \mathbb{R}_+^\ell$, the vector of public consumption must belong to the public consumption set $Y \subseteq \mathbb{R}_+^\ell$. It is assumed that these sets are of the form $X = g + \mathbb{R}_+^\ell$ and $Y = h + \mathbb{R}_+^\ell$ which means that they are translations of the nonnegative orthant. Consumptions sets may coincide with this orthant in which case the vectors of translation g and h, respectively, would have to be zero.

Endowments come in two varieties. First, there exists a vector $f \in \mathbb{R}_+^\ell$ of fixed endowments which corresponds to the land that is attached to a given local community. Second, each agent is endowed with a vector $e \in \mathbb{R}_+^\ell$ of mobile commodities that is attached to this agent. Let n denote the size of population of a given local community which is treated as a continuous variable. A triplet a = (n,x,y) is an equal treatment allocation if $n \in \mathbb{R}_{++}$, $x \in X$, $y \in Y$ and if markets are balanced, i.e. if

$$n \cdot x + y = n \cdot e + f =: F(n) \tag{1}$$

holds. More generally, the endowment function F(n) as defined by (1) could be thought of as being any concave function of population size n. The more general version allows for total endowments (productivity!) to depend on population size in a non-linear way (decreasing returns to scale!). Let A denote the set of all such allocations.

It proves useful to distinguish three notions of efficiency. An allocation is called <u>efficient</u> if it is Pareto efficient in the ordinary sense (fixed population). It is called population efficient if it maximizes the utility per head over all population sizes but at a fixed supply of public goods. Finally, it is called club efficient if it maximizes utility with respect to, both, population size as well as the supply of public goods. Put formally, an allocation $a = (n,x,y) \in A$ is called efficient if no other allocation $\tilde{a} = (n,\tilde{x},\tilde{y}) \in A$ exists for which $U(\tilde{x},\tilde{y}) > U(x,y)$; it is called population efficent if no other allocation $\tilde{a} = (\tilde{n},\tilde{x},y) \in A$ exists for which $U(\tilde{x},y) > U(x,y)$; it is called club efficient if no other allocation $\tilde{a} = (\tilde{n},\tilde{x},\tilde{y}) \in A$ exists for which $U(\tilde{x},\tilde{y}) > U(x,y)$. In order to derive, better to recall the necessary conditions let

$$L := U(x,y) + \sum_{i=1}^{l} \lambda_i \, [F_i(n) - nx_i - y_i]$$

denote the associated Lagrange function and let

$$p := \text{grad}_x \, U(x,y) \text{ and } q := \text{grad}_y \, U(x,y)$$

denote the vectors of partial derivatives of the utility function. By differentiating the Lagrange function L with respect to x, y and n, the following first order conditions for inner solutions are easily obtained. If the allocation $a \in A$ is efficient then

$$n \cdot q = p. \tag{2}$$

This, of course, is Samuelson's condition for the optimum supply of public goods. If the allocation $a \in A$ is population efficient then

$$p \cdot x = \sum_{i=1}^{l} p_i \cdot dF_i(n)/dn =: p \cdot F'(n). \tag{3}$$

Finally, if the allocation $a \in A$ is club efficient then the two conditions (2) and (3) must hold simultaneously.

Hereby, condition (3) corresponds to what Arnott and Stiglitz (1979) have dubbed the Henry George Theorem for the following reason. Consider, first, the case of a linear endowment function $F(n) = n \bullet e + f$. In this case, condition (3) simply requires that the value $p \bullet x$ of private consumption must be equal to the value $p \bullet e$ of mobile endowments. Since markets have to be balanced (see (1)), it follows that the value of public consumption $p \bullet y$ must be equal to the value $p \bullet f$ of fixed endowments, i.e. to aggregate land rent. Put differently, in order to ensure population efficiency, local public expenditures should be financed through a 100 % tax on land rent. Since Henry George - though for reasons most likely other than population efficiency ! - has asked for such a tax, Arnott and Stiglitz refer to the result as the Henry George Theorem.

Consider, second, the case of a concave endowment function F(n). In this case, condition (3) requires that the value $p \bullet x$ of private consumption must be equal to the value of marginal contributions of each agent. It then follows from balancedness constraint (1) that

$$p \cdot y = p[F - n \cdot F'(n)], \tag{4}$$

i.e. that local public expenditures should be equal to total profits. In this case, it is a 100 % tax on profits which is necessary for an allocation to be population efficient. No matter what Henry George has asked for in terms of profits, I shall refer to (4) or its equivalent version (3) as the Henry George Theorem as well. In this sense, for an allocation to be club efficient it must satisfy, both, the Samuelson condition and the Henry George Theorem.

2. The Goldenest Golden Rule

This section considers a pure exchange version of the overlapping generations (OLG) model. Each generation lives for two periods. As for the generation born at time t - 1, endowments when young of each agent, i.e. endowments which are available at time t - 1, are denoted by $e \in \mathbb{R}_+^\ell$; endowments when old of each agent, i.e. endowments which become available at time t, are denoted by $f \in \mathbb{R}_+^\ell$; consumption per head when young and when old is denoted by $x_{t-1} \in X$ and $y_t \in Y$, respectively; the number of agents, finally, which are born at time t - 1 is denoted by N_{t-1}. It then follows that markets are balanced at time t if the following condition is met:

$$N_{t-1}\, y_t + N_t\, x_t = N_{t-1}\, f + N_t\, e.$$

In a steady state, the number $n = N_t/N_{t-1}$ of offsprings per adult, the consumption vector $x = x_t$ when young, and the consumption vector $y = y_t$ when old are all independent of time. For a steady state to be feasible, the condition

$$y + n \cdot x = f + n \cdot e =: F(n)$$

must hold. In a purely formal sense, this condition for steady states of the OLG-model looks exactly the same as the balancedness condition (1) of the LPG-model. To be sure, from the economic viewpoint, the two models are rather different. In particular, the above OLG-model does not include public consumption, whereas such consumption is at the heart of the LPG-model. From a mathematical viewpoint, however, the following relation can be established. The number of offsprings per adult in the OLG-model corresponds to population size in the LPG-model, whereas consumption when old in the OLG-model corresponds to public consumption in the LPG-model. Regarding the various notions of efficiency of steady states in the OLG-model, we must carefully distinguish between the definitions proper and their corresponding first order conditions. In this sense, let us call a steady state which maximizes utility over all steady states but at a given growth rate a golden steady state. The corresponding first order conditions are referred to as the golden rule. A steady state which maximizes utility over all steady states and all growth rates is called a goldenest golden steady state. Following Samuelson (1975), the corresponding first order conditions are referred to as goldenest golden rule.

Using the above concepts, the relations between the OLG-model and the LPG-model can be extended as follows. The golden rule of the OLG-model corresponds to Samuelson's condition (2) of the LPG-model whereas a golden steady state of the OLG-model corresponds to an efficient allocation of the LPG-model. Similarly, a goldenest golden state corresponds to a club efficient allocation whereas Samuelson's (1975) goldenest golden rule - his Serendipity Theorem - corresponds to Samuelson's public goods condition (2) paired with the Henry George Theorem (3) and (4). The term population efficiency, finally, will be used in either version of the model.

Samuelson (1975) has established his goldenest golden rule for an OLG-model with capital accumulation (see section 5 below). To conclude the present section, the goldenest golden rule for the pure exchange version of the OLG-model can easily be derived by exploiting its relation to the LPG-model. For a steady state to be a goldenest golden steady state the value of consumption of each agent must be equal to the value of his endowments in each period. In other words, under the goldenest golden rule, no loans whatsoever will be involved. Let me mention that the same rule easily extends to economies with different consumer types. Therefore, put differently, the presence of loans means distortions with respect to the growth rate.

Intuitively, the result seems difficult to grasp. As a matter of fact, Samuelson's fallacy as pointed out by Deardorff (1976) should always be kept in mind. But does this mean that the goldenest golden rule and, hence, the Henry George Theorem are without any merit? Fortunately, the true verdict turns out to be less negative as the following section will show.

3. Partial Convexity and the Third Welfare Theorem

The LPG-model is known to involve partial convexity with respect to public goods as well as with respect to population size. The present section recalls some of those findings. The LPG-model and, hence, the OLG-model may fail, however, to exhibit global convexity as will be shown in the next section.

As for convexity with respect to public goods, let us assume that the utility function U(x,y) is quasi-concave in private and public consumption x and y, whereas the endowment function F(n) is assumed to be concave with respect to population size n. Suppose that Samuelson's condition (2) is satisfied. It then follows that the underlying allocation can be sustained as a Lindahl equilibrium and hence, by the first welfare theorem, the allocation must be efficient. In this sense, an allocation of the LPG-model which satisfies Samuelson's condition (2) necessarily has to be efficient. By exploiting the relation as established in the previous section, it follows that any steady state which satisfies the golden rule must be a golden steady state.

As for convexity with respect to population size, it follows from Schweizer (1983) that the Henry George Theorem is, both, necessary and sufficient for an allocation to be population efficient. I should mention that, in my earlier paper, the supply of public goods has been exogenously given. For that reason, there was no need to distinguish between club efficiency and population efficiency. For the present paper, however, the distinction turns out to be crucial. Moreover, I have shown that my version of the Henry George Theorem is essentially equivalent to Debreu's and Scarf's (1963) famous limit theorem on the core. In other words, the limit theorem can be interpreted as characterizing population efficient allocations and, as such, it deserves to be referred to as the third welfare theorem.

Let me briefly repeat that part of the argument, which is needed for the present purpose. Suppose that the utility function U(x,y) is quasi-concave with respect to private consumption x and that the endowment function F(n) is concave with respect to population size n. Moreover, for some given allocation $a = (n,x,y) \in A$, suppose that the Henry George Theorem, i.e. condition (3), is met. Let $w = p \bullet x$ denote the value of private consumption of each agent.

To establish population efficiency, take any allocation $\tilde{a} = (\tilde{n},\tilde{x},y) \in A$ with possibly different population size and possibly different private consumption but identical public consumption y. It then follows from the Henry George Theorem (3) that $pF'(n) = w$ and, by the concavity of F, that

$$pF(n) - wn \geq pF(\tilde{n}) - w\tilde{n}. \tag{5}$$

Given identical public consumption under the two allocations a and ã, it follows from the balancedness constraint (1) that

$$p[F(\tilde{n}) - \tilde{n}\tilde{x}] = py = p[F(n) - nx] = pF(n) - wn. \tag{6}$$

Equations (5) and (6) together imply that $\tilde{n}w \geq \tilde{n}p\tilde{x}$ and, hence, that $px = w \geq p\tilde{x}$. Since the utility function is assumed to be quasi-concave with respect to private consumption x and since p is the vector of partial derivatives of the utility function with respect to x it follows that $U(\tilde{x},y) \leq U(x,y)$. Hence, the allocation $a = (n,x,y)$ must be population efficient.

To summarize, no matter whether allocations of the LPG-model or steady states of the OLG-model (pure exchange version) are at stake, the Henry George Theorem is sufficient to ensure population efficiency. In the context of the OLG-model, this result can be stated as follows. Imagine that the consumption of the old is kept fixed at an arbitrary level and suppose that the growth rate is then adjusted such that no loans are needed to sustain the steady state allocation. In this case, the utility attains its maximum over all steady states and all growth rates which feature the same vector of consumption of the old. Therefore, in the context of population efficiency, the Henry George Theorem and its counterpart, the goldenest golden rule, have a role to play not only as a necessary but also as a sufficient condition for optimality.

4. The Missing Link to Full Convexity

In this section, a class of examples is presented which allows to illuminate the missing link to full convexity in the LPG-model and its counterpart, the OLG-model. Given such nonconvexities, allocations in the LPG-model and steady states in the OLG-model may exist which satisfy both Samuelson's public goods condition (2) and the Henry George Theorem (3) and, hence, they must be efficient allocations and golden steady states, respectively. At the same time, they must be population efficient. They still may fail, however, to be club efficient allocations or goldenest golden steady states.

Worse, it is possible that allocations exist which satisfay both conditions (2) and (3) even though the problem at hand is not a club problem at all in the sense that, instead of being positive and finite, the optimum population size tends to zero or infinity. In fact, all problems of pure exchange with only one commodity ($\ell = 1$) belong to this class as will be shown in the next section.

There are, of course, other examples which fail to be club problems and which for that reason, fail to allow for allocations that satisfy Samuelson's public goods condition (2) and the Henry George Theorem (3) at the same time. If, however, the problem at hand is a club problem proper, then a club efficient allocation or a goldenest golden steady state does exist. In this case, the solution of the club problem has to satisfy, both, Samuelson's condition and the Henry George Theorem (or the Kuhn-Tucker version of these conditions if the solution is on the boundary). Yet, due to the inherent nonconvexity, further allocations may exist which satisfy both conditions but which fail to solve the club problem.

To confirm all these possibilities it is sufficient to consider the following class of examples which involves just two commodities ($\ell = 2$). As for the LPG-version, commodity 1 is the mobile commodity whereas commodity 2 is land. In the OLG-interpretation, the young generation is endowed with commodity 1 whereas the old generation is endowed with commodity 2. In any case, it is assumed that

$$e_1 > 0,\ e_2 = 0 \text{ whereas } f_1 = 0,\ f_2 > 0. \tag{7}$$

Moreover, for analytical simplicity, let us assume that there is no demand of land for public use and that the endowment of the old is of no use to the old generation itself, respectively. It then follows that

$$y_2 = 0. \tag{8}$$

The specific structure of the example as described by conditions (7) and (8) is such that the population part of the problem can easily be solved for. Given any level $y_1 > 0$ of public demand or of consumption of the old, then the corresponding population efficient solution is determined as follows. The Henry George Theorem (3) and (4) implies that

$$p_1 y_1 = p_2 f_2 \quad \text{and} \quad p_1 x_1 + p_2 x_2 = p_1 e_1 \tag{9}$$

where

$$p_i = \partial U(x,y_1) \, / \, \partial x_i \qquad (i=1,2).$$

For any given level $y_1 > 0$, let $(x_1^*,x_2^*) = (x_1^*,x_2^*)(y_1)$ denote the consumption bundle which maximizes utility $U(x_1,x_2,y_1)$ over the budget set (9). Hereby, notice that this budget set is uniquely determined by the level y_1 of public consumption. Finally, let the population size be $n^* = n^*(y_1) = f_2 \, / \, x_2^*(y_1)$. It then follows from the third welfare theorem (see section 3) that, for any $y_1 > 0$, the solution $[n^*(y_1), x_1^*(y_1), x_2^*(y_1), y_1]$ must be population efficient. To solve the club problem, utility

$$u(y_1) := U[x_1^*(y_1), x_2^*(y_1), y_1]$$

has now to be explored as a function of public consumption y_1 (or of consumption of the old). This is most easy to achieve if the utility function is assumed to be separable of the form $U(x,y_1) = A(x_1,x_2) + B(y_1)$. Let $V(p_2/p_1, I)$ denote the indirect utility function associated with $A(x_1,x_2)$. It then follows from (9) that

$$u(y_1) = V(y_1 \, / \, f_2, e_1) + B(y_1) .$$

The indirect utility function is falling in prices whereas, due to monotonicity and quasiconcavity, the second part $B(y_1)$ of the direct utility function is increasing and concave as a function of public consumption y_1. To simplify even further, suppose that $A(x_1,x_2) = x_1 x_2$ is of the symmetric Cobb-Douglas type. In this case, $u(y_1)$ can explicitely be calculated as

$$u(y_1) = (e_1 / 2)^2 f_2 / y_1 + B(y_1) .$$

Depending on the particular shape of $B(y_1)$, utility as a function of public consumption can easily be described. Figure 1 captures the two cases (i) : $B(y_1) = y_1$ and (ii) : $B(y_1) = - y_1^{-2}$. Therefore, in case (i), utility unboundedly increases as the level y_1 of public consumption approaches zero or infinity. The problem fails to be a club problem. Nevertheless, it allows for a solution at which, both, Samuelson's condition and the Henry George Theorem are met, namely the one where $u(y_1)$ attains its minimum. Case (ii), however, leads to a club problem proper. The club efficient solution requires that level of public consumption (consumption of the old) at which $u(y_1)$ attains its maximum. Moreover, in case (ii), Samuelson's condition paired with the Henry George Theorem is sufficient to ensure club efficiency.

Fig. 1 - The missing link to full convexity -

Moreover, mixing cases (i) and (ii) in an appropriate way allows to construct examples where utility $u(y_1)$ as a function of public consumption features (local) minima as well as (local) maxima. In such cases, there exist several solutions at which, both, Samuelson's condition and the Henry George Theorem do hold. If utility unboundedly increases as y_1 approaches zero or infinity, none of these solutions corresponds to a club efficient one. If, however, utility remains bounded then its global maximum gives rise to a club efficient solution or to a goldenest golden steady state. Hereby, maxima which are only local as well as local (and global) minima lead to solutions where both conditions are met but which still fail to be club efficient allocations or goldenest golden steady states.

To conclude this section, let me point out that there is a simple way to ensure that the problem becomes a club problem in the proper sense. For that purpose it is useful to admit translations of the nonnegative orthant as the consumption sets (see section 1). Let us assume that there is a minimum requirement of land per person or a minimum consumption level of the endowment of the old per member of the young generation $g_2 > 0$ such that, for any feasible solution, it must hold that $x_2 \geq g_2$. Similarly, let us assume that there exists a minimum requirement of the mobile commodity for public use or a minimum consumption level of the endowment of young per member of the old generation $h_1 > 0$ such that, for any feasible solution, it holds that $y_1 \geq h_1$. It then follows that the admissible population size n is bounded from below by h_1/e_1 whereas it is bounded from above by f_2/g_2. Therefore the set of feasible solutions is compact and, hence, a club efficient allocation or a goldenest golden steady state does exist. Moreover, if the gradient of the utility function at the minimum consumption bundle (g,h) is steep enough, consumption will exceed these minimum levels such that none of the minimum requirements become binding. In this case, the club efficient solution has to be an inner solution such that, both, Samuelson's condition (2) and the Henry George Theorem (3) must hold. Due to the potential nonconvexity, however, these two conditions may still fail to be sufficient for club efficiency.

5. Samuelson's Fallacy Revisited

The final section deals with the one commodity case ($\ell = 1$). Hereby, we have the OLG-version in mind such that, for the pure exchange model, e corresponds to each agent's endowment when young, whereas f denotes his endowment when old. In this case, Samuelson's condition (2) reduces to $U_y(x,y)/U_x(x,y) = 1/n$ whereas, since $\ell = 1$, the Henry George Theorem simply requires that each agent consumes his endowments, i.e. x = e and y = f. In other words, at any population efficient solution, the number n of offsprings has to be adapted such that the condition

$$U_y(e,f) \;/\; U_x(e,f) \;=\; 1/n$$

is met. Figure 2 depicts this solution.

<u>Fig. 2:</u> - The population efficient solution at y = f

The solution which satisfies both conditions clearly fails to lead to a maximum of well-being. In fact, by changing the growth rate, any consumption bundle in the shaded area of Fig. 2 - they all lead to higher utility levels - can be obtained within a feasible steady state. Moreover, since there is a single commodity ($\ell = 1$), y = f is the only level of consumption when old which is compatible with a population efficient solution at all.

The above example adapts Deardorff's (1976) findings to a setting of pure exchange. While, in the presence of capital, the Serendipity Theorem has a remote chance of being correct for particular examples, this is definitely not true for the one commodity pure exchange case. Recall, however, that such an overall negative outcome is confined to the one commodity case as follows from the findings of the previous section. In any case, it seems worthwhile to elaborate on the one commodity case involving capital.

For ease of notation, let us assume that capital does not depreciate. The steady state constraint can then be written as

$$nx + y \;=\; n[f(k) - (n-1)k] \;=:\; F(n,k) \tag{10}$$

where (n-1) is the growth rate, k is capital per head and f(k) is the neoclassical production function. The endowment function F(n,k) as defined by (10) is introduced in order to show the formal similarity between the setting with capital and the pure exchange setting which has been studied in the paper so far.

Of course, first order conditions can still not generally be expected to be sufficient for a solution of (10) to be a goldenest golden steady state. Therefore, taking the insights of section 3 into account, the analysis is confined to the partial aspects of efficient (golden steady states) and of population efficient solutions.

To begin with, suppose that the endowment function F(n,k) is concave with respect to k (which it is if f(k) is concave!), and the utility function U(x,y) is quasi concave with respect to x and y, then the first order conditions

$$U_y(x,y)/U_x(x,y) \;=\; 1/n \quad \text{and} \quad F_k(n,k) \;=\; 0 \tag{11}$$

are sufficient for the solution to be a golden steady state. Similarly, if F(k,n) is globally concave as a function of both arguments and if the utility function U(x,y) is quasi-concave with respect to x then the first order conditions

$$F_n(n,k) \;=\; x \quad \text{and} \quad F_k(n,k) \;=\; 0 \tag{12}$$

are sufficient for the steady state to be population efficient. The above two statements can be proved in a very similar way as the related results have been established in section 3. Condition (11) corresponds to Samuelson's public goods condition whereas condition (12) corresponds to the Henry George Theorem.

The additional difficulty which capital brings into the picture stems from the fact that the endowment function F(n,k) as defined by (10) need not be globally concave. To be sure, since $F_{kk} = nf''(k) < 0$ and $F_{nn} = -\,2k < 0$ it follows that the endowment function is concave as a function of either of its arguments if taken separately. As a function of its two arguments jointly, however, it may fail to be concave. In fact, even in the class of Cobb-Douglas production functions $f(k) = A\,k^{\alpha}$ (where $0 < \alpha < 1$) examples can be found for which the Henry George Theorem (12) is not sufficient to ensure population efficiency. Put differently, if the endowment function F(n,k) fails to be globally concave then the third welfare theorem no longer needs to be true. In this sense, capital adds nonconvexities which are absent in the pure exchange case.

Nevertheless, surprisingly enough, the Henry George Theorem keeps some meaning as a sufficient condition for efficiency as we now want to show. For the OLG-model including capital, a steady state (n,x,y,k) which is feasible (i.e. the constraint (10) is met) is called an equilibrium steady state if, for $r = f'(k)$ and $w = f(k) - kf'(k)$ it holds that

$$U_y(x,y) \;/\; U_x(x,y) = 1 \;/\; (1+r) \quad \text{and} \quad x + nk = w\,. \tag{13}$$

For the definition and the existence of such equilibrium steady states, the reader is referred to Galor and Ryder (1989). The remaining part of this section is devoted to the reinterpretation of such equilibrium steady states in the LPG-model. Since consumption y of the old corresponds to public consumption in the LPG-interpretation, it follows from (13) that 1/(1+r) is the personalized Lindahl price for the public good as expressed in terms of the private good. Moreover, let $\alpha := F_n(n,k)$ and $\beta := F_k(n,k)$ denote the marginal product of labor and capital, respectively. Then the net profit of the firm producing the private commodity amounts to $F(n,k) - \alpha n - \beta k$. Notice, however, that capital is produced by means of the private commodity. Moreover, the endowment function F(n,k) as defined by (10) is listed net of the output that is devoted to capital. Therefore, β, rather than being the price for capital, should better be viewed as the rate at which the use of capital is taxed. The Lindahl firm, finally, supplies the public good. Hereby, the rate of transformation between private and public good is unity. Since the Lindahl firm sells the public good to each agent at Lindahl price 1/(1+r), its profit amounts to [n/(1+r) - 1]y. In order to compensate for potential losses, the Lindahl firm must be subsidized at rate $\tau = 1 - n/(1+r)$ per unit of the public good. To summarize, an equilibrium steady state if reinterpreted in the LPG-model gives rise to a Lindahl type equilibrium which exhibits a tax on the use of capital and a subsidy on the supply of the public good.

The notion of equilibrium steady state commonly takes the growth rate as given.

Therefore, reinterpreted in the LPG-model, the proper welfare criterion is that of efficiency. Since the endowment function F(n,k) as defined by (10) is concave with respect to k, the first and the second welfare theorem both hold provided, of course, that the utility function is quasiconcave. Put differently, Lindahl equilibria are efficient, and any efficient allocation of the LPG-model can be sustained as a Lindahl equilibrium with redistribution. To be sure, the notion of Lindahl equilibrium which enters the welfare theorems is in the strict sense which does not allow for taxes on the use on capital nor for subsidies on the supply of public goods. Rather, if such taxes and subsidies actually occur, then they must be distortive. This is well in line with the fact that equilibrium steady states typically fail to be golden steady states.

To elaborate on the distortions which may be present in equilibrium steady states, it should be observed that

$$\beta = F_k = n[f' - (n-1)] = n[r + 1 - n] = n(1+r)\tau \ .$$

Therefore, if the use of capital is actually taxed, i.e. if $\beta \neq 0$, then the subsidy is actually paid, i.e. $\tau \neq 0$, and vice versa. On the other hand, for the equilibrium steady state to be free of distortions it is sufficient to show that no subsidies are granted, i.e. that $\tau = 0$. There is another way to express the same condition. Since, by (13),

$$x = w - nk = f - kf' - nk = f - (n+r)k = F_n + (n-r-1)k = F_n - \tau(1+r)k$$

it follows that $\tau = 0$ if, and only if, $x = F_n$. Therefore, the equilibrium steady state is free of distortions if, and only if, the Henry George Theorem (see (12)) does hold. This is the normative content of the Henry George Theorem which remains valid if capital is added to the OLG-model.

Concluding Remarks

There exists a formal relation between steady states of overlapping generations (OLG) models with endogenous fertility and allocations of local public goods (LPG) models with endogenous population. Hereby, consumption of the old corresponds to public consumption, consumption of the young corresponds to private consumption, endowments of the old correspond to land, endowments of the young correspond to mobile endowments, and the number of offsprings per adult corresponds to population size. The paper has exploited this relation in a systematic way.

As long as the endowment function is globally concave, the condition which, in the realm of LPG-models, has been dubbed the Henry George Theorem and which, in essence, is driven by the limit theorem on the core can be shown to be necessary and sufficient for a solution to be population efficient. Under this notion of efficiency, the growth rate and the population size is adjusted to the level of consumption of the old and of public consumption, respectively, such that the utility per agent attains its maximum. The solution is called a goldenest golden steady state and a club efficient allocation if, in addition to population, the maximum is taken with respect to consumption of the old and public consumption as well. The goldenest golden rule and the Henry George Theorem paired with Samuelson's condition on the optimum supply of public goods, while remaining necessary for an inner solution to be efficient in the overall sense, generally fail to be sufficient. Therefore, it is the notion of population efficiency where the Henry George Theorem keeps its meaning as a sufficient condition. Notice, however, for that to be true the endowment function must be globally concave which it happens to be in the pure exchange case but which it may fail to be in the case arising from the OLG-model including capital. Yet, here, too, the Henry George Theorem keeps some meaning as a welfare statement. It has been shown that an equilibrium steady state is free of distortions, i.e. it is a golden steady state if, and only if, the condition of the Henry George Theorem is met.

The analysis has been carried out for economies with a single type of agents. It can be extended to the case involving several types as follows. As for the LPG-model, the reader is referred to Schweizer (1983). For steady states of the OLG-model, the mixture of population has to be distinguished from the growth rate which might also be different for different consumer types. Let N^s denote the number of consumers of type s living in a given period and let n^s denote the number of offsprings per adult of type s. Hereby, for simplicity, it is assumed that offsprings are of the same type as their parents. If x^s and e^s denote consumption and endowments of an agent of type s when young and y^s and f^s denote consumption and endowments of such an agent when old, then the steady state constraint amounts to

$$\sum_s N^s y^s + \sum_s n^s N^s x^s = \sum_s N^s f^s + \sum_s n^s N^s e^s.$$

For an (inner) solution of this constraint to be a goldenest golden steady state, a price vector p and scalars γ^s must exist such that

$$\text{grad}_x U^s = n^s \cdot \text{grad}_y U^s = \gamma^s \cdot p$$

holds. This condition corresponds to the golden rule of the single type economy. Moreover, as far as budget constraints are concerned, it must hold that

$$px^s = pe^s \quad \text{and} \quad py^s = pf^s \quad \text{for all } s.$$

This, of course, is the extension of the Henry George Theorem to the case with different consumer types. The condition requires that the value of consumption must be equal to the value of endowments for each type and for each period. In other words, not only, no loans between different periods of time are involved, but also, there is no scope for redistribution. Given that the limit theorem is at work, this does not come as a surprise: an allocation of a pure exchange economy which remains in the core under replication can be sustained as a Walrasian equilibrium without redistribution!

As a final remark, it should be pointed out that the limit theorem and, hence, the Henry George Theorem are, in essence, results on economies with a continuum of agents. To be sure, the limit theorem avoids the continuum. Rather, it replicates the finite economy sufficiently enough. Yet, if the present paper takes derivatives with respect to the number of agents of a given type, strictly speaking, this requires that there exists a continuum of agents of each type. In club theory as well as in the theory on endogenous growth, however, such an assumption is commonly imposed.

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