DISCUSSION PAPER B–263

ON THE STABILITY OF LOG-NORMAL INTEREST RATE MODELS AND THE PRICING OF EURODOLLAR FUTURES

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Abstract. The lognormal distribution assumption for the term structure of interest is the most natural way to exclude negative spot and forward rates. However, imposing this assumption on the continuously compounded interest rate has a serious drawback: expected rollover returns are infinite even if the rollover period is arbitrarily short. As a consequence such models cannot price one of the most widely used hedging instrument on the Euromoney market, namely the Eurofuture contract. The purpose of this paper is to show that the problem with lognormal models result from modelling the wrong rate, namely the continuously compounded rate. If instead one models the effective annual rate the problem disappears, i.e. the expected rollover returns are finite. The paper studies the resulting dynamics of the continuously compounded rate which is neither normal nor lognormal.

1. Introduction

Most models of the term structure of interest rate which start with modelling the short rate $r(t)$ are of the form

$$dr(t) = \mu(t,r)dt + \sigma(t,r)dw(t) \tag{1}$$

where $w(t)$ is a K–dimensional Brownian motion. For the purpose of this paper it suffices to consider $K = 1$. The basic differences in the various approaches stem from the specification of the volatility term $\sigma(t,r)$. There are three basic classes of volatility structures which can be specified as follows:
Define

$$\sigma(t,r) = \sigma(t) \cdot r^{\alpha}. \tag{2}$$

Then for $\alpha = 0$ we have the class of normal rate models, for which the Ho–Lee model is the prototype (cp. Ho–Lee [1986], Dybvig [1989]). The well–known disadvantage of this class is that rates – both short and forward – become negative with positive probability. For $\alpha = 1$ we obtain the class of lognormal models, which has been studied extensively in the literature (cp. Dothan [1978] , Black–Derman–Toy [1990], Black–Karasinsky [1991], Hull–White [1990]). These models guarantee positive rates. However, lognormal models have another serious drawback: expected rollover returns are infinite even if the rollover period is arbitrary short. One consequence of this clearly undesirable feature is that these models cannnot explain prices of Eurodollar future contracts, as shown by Hogan–Weintraub [1993]. For $\alpha = 0.5$ we obtain the class of so–called square–root processes first proposed and studied extensively by Cox–Ingersoll–Ross [1985]. The choice of $\alpha = 0.5$ can be viewed as a plausible compromise between the two extremes $\alpha = 0$ and $\alpha = 1$. But the choice $\alpha = 0.5$ alone does not rule out the possibility of infinite Eurodollar futures prices, it only reduces the probability of their occurence for "reasonable " parameter

Date. October 1994. This version: March 22, 1995.

Key words and phrases. Eurodollar Futures, Term Structure Models, Log-Normal Interest Rate.

The authors are grateful to David Heath, Ian Hofman, Bjarne Astrup Jensen, Marek Musiela and Jorgen Aase Nielsen for helpful discussions. Special thanks are due to Ian Hofman for providing a first proof of Theorem 5 in this paper. Financial support from the Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at Rheinische Friedrich-Wilhelms-Universität Bonn by both authors is gratefully acknowledged. Document typeset in LaTeX.

values.[1]

The purpose of the paper is to show that the problems with lognormal models result from modelling the wrong rate, namely the continuously compounded rate. Assuming that the continuous rate is lognormal results in "double exponential" expressions, i.e. expressions where the exponential function is itself an argument of an exponential, thus giving rise to infinite expectations under the martingale measure. The problem disappears if instead of the continuous rate one models the effective annual rate. To model the stochastic movements of the (discrete) effective short rate is natural in a discrete binomial setup. This model has been developed and studied – independently of Black–Derman–Toy – in a series of papers by Sandmann–Sondermann [1992, 1993].
Although this discrete model looks very similar to the Black–Derman–Toy model, its limit behaviour is quite different. In the limit of the BDT–model the continuous rate follows a lognormal diffusion. In the limit of the Sandmann–Sondermann model the continuous rate follows a diffusion which is neither normal nor lognormal, but a dynamic combination of both with the following properties (cp. Theorem 3): for small values of $r_c(t)$ the diffusion process approaches a lognormal diffusion [2], thus generating positive rates, whereas for large values of $r_c(t)$ the diffusion approaches a normal diffusion process, generating stable finite expected returns and futures values. It thus combines –in a very simple and straightforward setup – the strengths of the normal ($\alpha = 0$) and lognormal ($\alpha = 1$) model and avoids their shortcomings.

2. Brief sketch of the discrete model of the effective short rate

Although akin to the BDT binomial model and already published elsewhere [1992, 1993], for the sake of clarifying the crucial difference between the two approaches we start with a brief sketch of our discrete model.
Denote by $\underline{\underline{T}} = \{0 = t_0 < t_1 < ... < t_N = T\}$ a set of trading dates which form a regular discretisation of the time axis. Suppose that the (annual) effective rate follows a path independent binomial process $r : \underline{\underline{T}} \times \Omega \to \mathrm{R}_{\geq 0}$ such that $r_e(i,n)$ is the (annual) effective rate at time $t_i \in \underline{\underline{T}}$ and state $n = 0, ..., i$. Let us assume that at time t_0 the prices of default free zero coupon bonds $B_0(t_i) = B(t_0, t_i)$ with maturity $t_i \in \underline{\underline{T}} \setminus \{t_0\}$ are known. The problem is to compute from these initial term structure the process of the effective rate in an arbitrage free way, such that negative effective rates are excluded and the initial term structure is consistent with the model. The initial effective rate $r_e(0,0)$ at time t_0 is determined by the zero bond with $B(t_0,t_1) = \left(\frac{1}{1+r_e(1,0)}\right)^{\Delta t}$ and $\Delta t = t_1 - t_0 = t_{i+1} - t_i \quad \forall\ i = 0, ..., N-1$.
Suppose that at time $t_i \in \underline{\underline{T}} \setminus \{t_N\}$ the effective rate is $r_{i,n}$ and denote by $p(i,n) \in\,]0,1[$ the (possible time and state dependent) transition probability such that either $r_e(i+1,n+1)$ (with probability $p(i,n)$) or $r_e(i+1,n)$ (with probability $1-p(i,n)$) are the following effective rates the next period.
Now we can define as the local risk measure of the model the volatility or more precisely the standard deviation of the logarithmic effective rate

$$
\begin{aligned}
\sigma^2_{(i,n)} &= V_{p(i,n)}[\log r_e(i+1,.)|r_e(i,n)] \\
\Leftrightarrow\quad r_e(i+1,n+1) &= \exp\left\{\frac{\sigma(i,n)}{\sqrt{p(i,n)(1-p(i,n))}}\right\} \cdot r_e(i+1,n) \qquad (3) \\
&=: f(\sigma(i,n),p(i,n)) \cdot r_e(i+1,n) \geq r_e(i+1,n)
\end{aligned}
$$

[1] As Hogan–Weintraub [1993] remark the square root diffusion has a "mild" case of infinite Eurodollar future values. Also the choice of $\alpha = 0.5$ cannot be based on empirical evidence. As Chan et al [1992] have shown historical short rate movements are best explained by choosing $\alpha = 1.5$.

[2] Indeed, if the continuous rate $r_c(t)$ becomes infinitesimal small, i.e. $r_c(t) = 0(dt)$, then the two dynamics coincide.

Thus future effective rates are proportional to each other and by induction we have:

$$r_e(i+1,n+1) = \left(\prod_{j=0}^{n} f(\sigma(i,j),p(i,j))\right) r_e(i+1,0) \tag{4}$$

At each period t_i, $r(i,0)$ is the lowest possible effective rate and all other rates at that period are proportional related to this rate. Now let us construct the r_e process as a risk adjusted interest rate process which means that under the transition probability $p(i,n)$ the local expectation hypothesis is satisfied[3]. For the initial price $B(t_0,t_2)$ of a two period zero coupon bond

$$\begin{aligned} B(t_0,t_2) &= \left(\frac{1}{1+r_e(0,0)}\right)^{\Delta t} E_{p(0,0)}[B(t_1,t_2)] \\ &= \left(\frac{1}{1+r_e(0,0)}\right)^{\Delta t} [p(0,0)B_{1,1}(t_2)+(1-p(0,0))B_{1,0}(t_2)] \\ &= \left(\frac{1}{1+r_e(0,0)}\right)^{\Delta t} \left[p(0,0)\left(\frac{1}{1+r(1,1)}\right)^{\Delta t} + (1-p(0,0))\left(\frac{1}{1+r(1,0)}\right)^{\Delta t}\right] \end{aligned} \tag{5}$$

where $B_{1,n}(t_2);\ n=0,1$ are the possible values at time t_1 of a zero coupon bond with maturity t_2 . Given $p(0,0),\sigma(0,0)$and the proportional relationship (4) there exists a unique positive solution $r_e(1,0)$ for the lowest effective rate at time t_1 iff $B(t_0,t_1) > B(t_0,t_2)$. More generally we have the following theorem:

Theorem 1. Suppose $B(t_0,t_1) > B(t_0,t_2) > ... > B(t_0,t_N)$ are given zero bond prices. For any time and state dependent specification of transition probabilities and the volatilities with $p(i,n) \in \,]0,1[$ and $\sigma(i,n) > 0$ there exists a unique and positive binomial process of the effective rate $\{r_e(t_i)\}_{t_i \in \underline{\underline{T}}}$ such that the local expectation hypothesis is satisfied.

Proof. The proof is done by induction. Suppose the binomial effective rate process $\{r_e\}$ is already constructed for all $t_i \leq t_n \in \underline{\underline{T}}$. The local expectation hypothesis implies that the initial value of any zero coupon bond is equal to its expected discounted payoff under the transition probabilities $p(i,j)\quad i = 0,...,n-1;\ j=0,...,i$. In particular

$$B(t_0,t_n) = \left(\frac{1}{1+r(0,0)}\right)^{\Delta t} \cdot \sum_{i\in\kappa(n-1)} \left[\prod_{j=0}^{n-2} p(j,s(j,i))^{i_{j+1}}(1-p(j,s(j,i))^{1-i_{j+1}} \left(\frac{1}{1+r_e(j+1,s(j+1,i))}\right)^{\Delta t}\right]$$

[3] See e.g. Ingersoll [1987]. The absence of arbitrage opportunities implies that the excess return per unit risk $\lambda(i,n)$ defined as

$$\lambda(i,n) := \frac{E_q[B(t_{i+1},\tau)|B(t_i,\tau)] - (1+r(i,n))^{\Delta t} B(t_i,\tau)}{\sqrt{V_q[B(t_{i+1},\tau)|B(t_i,\tau)]}}$$

is independent of $\tau > t_i, \tau \in \underline{\underline{T}}$. The local expectation hypothesis holds if $\lambda(i,n) \equiv 0$. If $\lambda(i,n) \neq 0$ under some transition probability $q(i,n)$, it is easy to show that the following shift in the measure

$$p(i,n) := q(i,n) + \sqrt{q(i,n)(1-q(i,n))}\ \cdot \lambda(i,n)$$

implies that the local expectation hypothesis holds under p .

where $\kappa(n-1) \quad = \quad \{i = (0, i_1, ..., i_{n-1}) \in \{0\} \times \{0,1\}^{n-1}\}$ is the set of all possible paths

and $s(j,i) \quad := \quad \sum_{k=0}^{j} i_k \;\hat{=}\;$ number of up–movements of the path $i \in \kappa(n-1)$ at time $t_j \leq t_i$

Define $x = r_e(n, 0)$ and

$$\begin{aligned} g(n+1,x) \quad = \quad & \Bigg[\sum_{i \in \kappa(n-1)} \Bigg(\prod_{j=0}^{n-2} p(j,s(j,i))^{i_{j+1}} \left(1 - p(j,s(j,i))^{1-i_{j+1}}\right) \\ & \cdot \left(\frac{1}{1 + r_e(j+1, s(j+1,i))} \right)^{\Delta t} \Bigg) \cdot \left(\frac{1}{1 + r_e(0,0)} \right)^{\Delta t} \\ & \cdot \left(\frac{1}{1 + x \cdot \prod_{k=0}^{s(n,i)} f(\sigma(n,k), p(n,k))} \right)^{\Delta t} \Bigg] - B(t_0, t_{n+1}) \end{aligned}$$

Since $f(\ ,\ ) \geq 1$ the function $g(n+1, .)$ is strictly decreasing for $x \geq 0$ and

a) $g(n+1, 0) = B(t_0, t_n) - B(t_0, t_{n+1}) \quad > 0$ by assumption
b) $\lim_{x \to +\infty} g(n+1, x) = -B(t_0, t_{n+1}) \quad < 0$

Thus there exist a unique solution x^* such that $g(n+1, x^*) = 0$

□

By construction the process $\{r_e\}$ depends on both the specification of the volatility and the transition probability. To study its the dependence on the transition probability we consider the limit distribution of the effective rate process.

Theorem 2. Suppose the transition probability is constant over time, $p \in]0,1[$ and the volatility $\sigma : \underline{\underline{T}} \rightarrow \mathrm{R}_{>0}$ is at most a function of time, so that $\sigma^2(t_i)$ is proportional to the length of the time interval with $\sigma^2(t_i) = h(t_i) \cdot \Delta t$ where $h(.)$ converges to a bounded function on $[0, T]$, then

$$\left\{ x(t_i) := \ln \left(\frac{r_e(t_i)}{r_e(t_{i-1})} \right) \right\}_{t_i \in \underline{\underline{T}}(N)} \tag{6}$$

satisfies the Central Limit Theorem[4] .
Given the construction principle we know that for constant p

$$V_p\left[x(t_i) | x(t_{i-1})\right] = \sigma^2(i-1)$$

and thus in the limit the annual effective interest rate is lognormally distributed with

$$dr_e \quad = \quad \mu(t) r_e dt + \sigma(t) r_e dw \tag{7}$$

where $\mu(t)$ is some function of the inital zero coupon bonds and the volatility such that the local expectation hypothesis is satisfied.

3. Limit Dynamics of the Continuous Rate

Theorem 3. For the limit model (7) the continuous rate $r_c(t)$ follows the following diffusion process:

$$dr_c(t) = (1 - e^{-r_c(t)}) \left\{ \left(\mu(t) - \frac{1}{2}(1 - e^{-r_c(t)}) \sigma^2(t) \right) dt + \sigma(t) dw(t) \right\} \tag{8}$$

[4]The proof is given in Sandmann,Sondermann [1993].

Proof. The connection between the (annual) continuous rate $r_c(t)$ and the (annual) effective rate $r_e(t)$ is $r_c(t) = \ln x(t)$ with $x(t) = 1 + r_e(t)$. Hence, the dynamics (7) and Ito's Lemma imply

$$\begin{aligned} dr_c &= \frac{1}{x}dx - \frac{1}{2}\frac{1}{x^2}d\langle x\rangle \\ &= \frac{1}{1+r_e}\left(\mu r_e dt + \sigma r_e dw(t)\right) - \frac{1}{2}\frac{1}{(1+r_e)^2}\sigma^2 r_e^2 dt \end{aligned}$$

Using the relation $r_e/(1+r_e) = 1 - e^{-r_c}$ gives (8).

□

Remark.

(i) For $r_c(t) \to \infty$ the dynamics (8) converges to the normal diffusion $dr_c(t) = \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right) dt + \sigma(t)dw(t)$

(ii) For $r_c(t) = o(dt)$ it follows from $1 - e^{-r_c(t)} = r_c(t) + o(dt^2)$ and $r_c^2(t) = o(dt^2)$ that (8) becomes

$$\frac{dr_c(t)}{r_c(t)} = \mu(t)dt + \sigma(t)dw(t)$$

Hence only for infinitesimal small values the continuous rate $r_c(t)$ follows a lognormal diffusion with same drift and volatility as the effective rate $r_e(t)$. But $r_c(t)$ is the annual continuous rate and for most paths far away from $o(dt)$.

Let $B(t,T)$ denote the zero coupon bond price at time t with maturity T. Then No–Arbitrage resp. the Local Expectation Hypotheses implies

$$B(t,T) = E\left[\exp\left\{-\int_t^T r_c(s)ds\right\}\Big|\mathbb{F}_t\right] , \tag{9}$$

where $E[\cdot|\mathbb{F}_t]$ is the conditional expectation under the equivalent martingale measure. Let

$$\beta(t,T) = \exp\left\{\int_t^T r_c(s)ds\right\}$$

denote the accumulation factor for 1\$ rolled over the period $[t,T]$. Then (9) is equivalent to the condition that the process

$$Z(t,T) = B(t,T)/\beta(0,t)$$

is a martingale.

The following Theorem was proved by Hogan and Weintraub [1993].

Theorem 4. Consider the following two short rate processes for the continuous rate r_c:

$$dr_c = \alpha r_c dt + \sigma r_c dw(t) \tag{10}$$

$$d\ln r = \kappa(\theta - \log r)dt + \sigma dw(t) \tag{11}$$

where κ and θ are non–negative. Then, for any $0 \leq \tau < t < T$,

$$E_\tau[B(t,T)^{-1}] = \infty$$

Remark. Equation (10) is the model of Dothan [1978] and the limit of the BDT binomial model [1990]. Equation (11) is the model of Black–Karasinsky [1991].

The Theorem of Hogan–Weintraub has two consequences

(1^0) Expected accumulation factors over any positive time interval are infinite

(2^0) The models (10) and (11) attach negative infinite values to Eurodollar future contracts.

To see (1^0) observe that by Jensen's inequality

$$B(t,T)^{-1} = E_t\left[\exp\left\{-\int_t^T r_c(s)ds\right\}\right]^{-1} < E_t[\beta(t,T)]$$

where $E_t[\cdot]$ denotes conditional expection w.r.t. $\mathbb{F}_t$. Hence, for any $\tau < t$, $E_\tau[\beta(t,T)] > E_\tau[B(t,T)^{-1}] = \infty$.

For (2^0) observe how Eurodollar futures are quoted and settled. E.g. a quotation of 90% on a 3–month instrument implies an interest of 2.5% and a price of 97.5% times the face value of the contract[5]. At time t the contract is settled at price $100 - .25 \times r_L(t)$ percent, where $r_L(t)$ is the 3–month LIBOR rate valid at t. More generally, a Eurodollar future contract for the period $[t, T = t+\delta]$ is settled at t at the %–price

(12) $$X = 100 - \delta r_L(t,\delta)\ ,$$

where $r_L(t,\delta)$ is the δ–LIBOR rate ($\delta \leq 1$) valid at t. Cox–Ingersoll–Ross [1981] have shown that with continuous resettlement the future price $F_\tau(t,T)$ at time $\tau < t$ of a contract which settles at the amount X_t at time t is $E_\tau[X_t]$. Since at t

(13) $$B(t,T) = (1 + \delta \cdot r_L(t,\delta)/100)^{-1}$$

(12) and (13) imply

(14) $$X = 100 \times (2 - B(t,T)^{-1})$$

Hence by Theorem 4 $F_\tau(t,T) = E_\tau[X] = -\infty$.

Theorem 5. Let the annual effective rate r_e follow the lognormal diffusion

$$\frac{dr_e(t)}{r_e(t)} = \mu(t)dt + \sigma(t)dw(t)$$

where $\mu(\cdot)$ and $\sigma^2(\cdot)$ are integrable and bounded functions on $[0,T]$. Then, for any $0 \leq \tau < t < T$,

$$E_\tau[B(t,T)^{-1}] \quad \text{and} \quad E_\tau[\beta(t,T)].$$

are finite.

Proof. Since by Jensen's inequality

$$B(t,T)^{-1} = E_\tau^{-1}\left[\exp\left\{-\int_t^T r_c(s)ds\right\}\right] < E_\tau\left[\exp\left\{\int_t^T r_c(s)ds\right\}\right]$$

[5] See e.g. Aftalion–Poncet [1993].

it suffices to prove that the second expression is finite. Again Jensen's inequality and the relation $r_c(t) = \ln(1 + r_e(t)) \geq 0$ imply

$$
\begin{aligned}
E_\tau[\beta(t,T)] &= E_\tau\left[\exp\left\{\int_t^T \ln(1+r_e(s))ds\right\}\right] \\
&= E_\tau\left[\exp\left\{\int_t^T \frac{1}{T-t}\ln(1+r_e(s))^{T-t}ds\right\}\right] \\
&\leq E_\tau\left[\exp\left\{\ln\int_t^T \frac{1}{T-t}(1+r_e(s))^{T-t}ds\right\}\right] \\
&= \frac{1}{T-t}\int_t^T E_\tau\left[(1+r_e(s))^{T-t}\right]ds \\
&\leq \frac{1}{T-t}\int_t^T E_\tau\left[(1+r_e(s))^{k}\right]ds \quad \text{for } k = \min\{i \in \mathbb{N} | i \geq T-t\}
\end{aligned}
$$

which is finite, since the above assumption implies in particular that $\forall i \leq k$

$$
\int_t^T \exp\left\{\int_\tau^s \left(\mu(u) + \frac{i}{2}\sigma^2(u)\right)du\right\}^i ds \quad < \infty \quad .
$$

□

Hence with lognormal effective short rates both expected returns and Eurodollar future prices are finite. Note that for $B(t,T) < \frac{1}{2}$ (14) becomes negative and hence also $F_\tau(t,T)$. But this is in accordance with the quotation convention of futures. E.g. $B(t,T) < \frac{1}{2}$ occurs for $\delta = 1$ and r_L greater 100%, but then future quotes will also be negative.

Appendix

Some remarks on the p.d.f. of the continuously compounded rate and the discount factor.

Proposition 1. Suppose that the annual effective interest rate is according to (7) lognormally distributed. Then the probability density function of the continuously compounded interest rate r_c is given by

$$
f(x;t,\bar\mu,\bar\sigma) = \begin{cases} \frac{1}{\sqrt{2\pi}\bar\sigma}\,\frac{e^x}{e^x-1}\,\exp\left\{-\frac{(\ln(e^x-1)-\bar\mu)^2}{2\bar\sigma^2}\right\} & x \in [0,\infty] \\ 0 & \text{otherwise} \end{cases} \tag{15}
$$

with

$$
\begin{aligned}
\bar\mu &= E[\ln r_e(t)] = \ln r_0 + \int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)ds \\
\bar\sigma^2 &= V[\ln r_e(t)] = \int_0^t \sigma^2(s)ds
\end{aligned}
$$

Proof. By definition we have $r_c = \ln\ 1 + r_e$. Let $0 < a < b$ be two real numbers. Since r_e is lognormal distributed we have

$$
\begin{aligned}
&\text{prob}[a < r_c \leq b] \\
= \ &\text{prob}\left[a' := \frac{\ln(e^a-1)-\bar\mu}{\bar\sigma} < \frac{\ln r_c - \bar\mu}{\bar\sigma} \leq \frac{\ln(e^b-1)-\bar\mu}{\bar\sigma} =: b'\right] \\
= \ &\int_{a'}^{b'} \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}dz
\end{aligned}
$$

Substitution $x(z) := \ln(\exp\{\bar\sigma \cdot z + \bar\mu\} + 1)$ gives (15)

□

Remark.

a) For the expected value of the continuously compounded interest rate r_c the following boundary conditions apply:

i) By Jensens inequality:

$$\begin{aligned} E[r_c(t)] &= E_{N(\bar{\mu},\bar{\sigma}^2)}[\ln(e^x+1)] \\ &\geq \ln\left(1+\exp\{E_{N(\bar{\mu},\bar{\sigma}^2)}[x]\}\right) \\ &= \ln\left(1+\exp\{\bar{\mu}\}\right) \quad = \quad \ln\left(1+r_0\exp\left\{\int_0^t \mu(s)-\frac{1}{2}\sigma^2(s)ds\right\}\right) \end{aligned}$$

ii) By Jensens inequality:

$$\begin{aligned} E[r_c(t)] &= E[\ln(1+r_e)] \\ &\leq \ln E[1+r_e] = \ln\left(1+\exp\left\{\bar{\mu}+\frac{1}{2}\bar{\sigma}\right\}\right) \\ &= \ln\left(1+r_0\exp\left\{\int_0^t \mu(s)ds\right\}\right) \end{aligned}$$

b) Using the continuous time approach the drift function $\mu(t)$ of the effective interest rate is endogenously determined by the initial term structure via

$$B(0,\tau) \quad = \quad E\left[\exp\left\{-\int_0^\tau r_c(s)ds\right\}|\mathbb{F}_0\right] \quad \forall\tau\leq T$$

Simulations of the discrete time version suggest that

$$\int_0^\tau \mu(s)ds = \ln f(0,\tau) - \ln r(0) + \frac{1}{2}\int_0^t \sigma^2(s)ds \tag{16}$$

is a reasonable approximation[6] for the drift factor, where $f(0,t)$ is the t_0 forward rate. Under the assumption (16) the following figure of the p.d.f. of the continuously compounded interest rate and the effective summarize the present discussion. Especially the difference between these two distribution asumptions is clearified.

[6] So far an exact proof of the relationship (16) is missing.

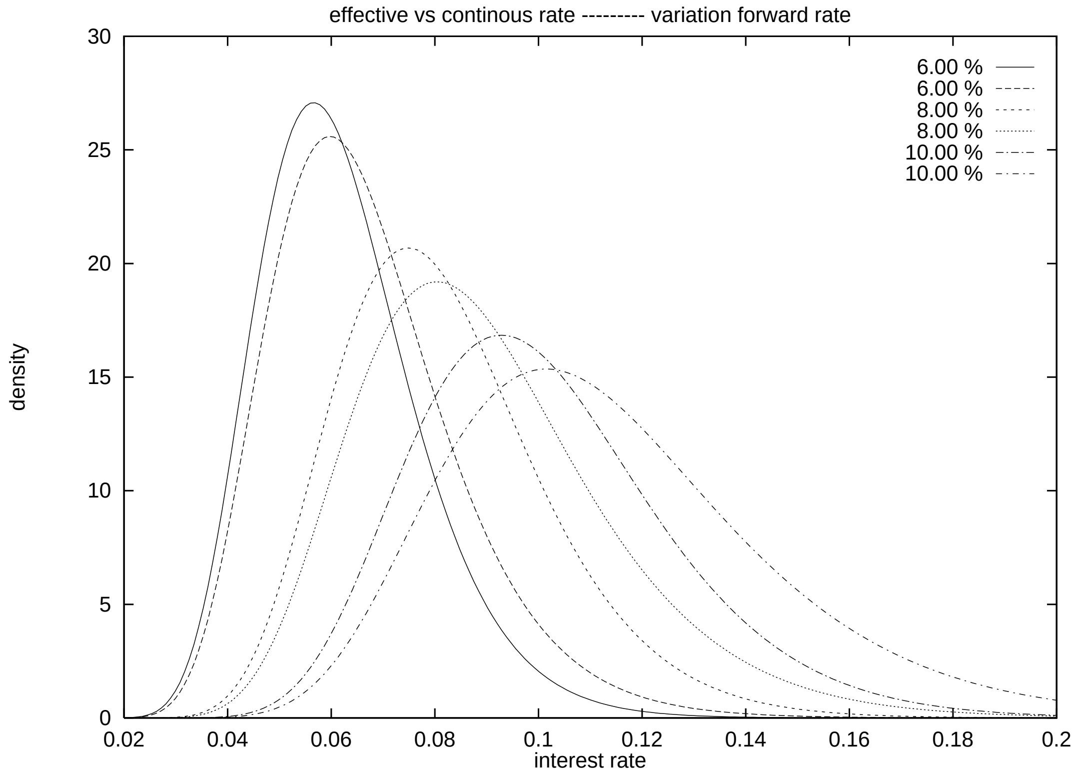

Figure 1 : p.d.f. of r_c and r_e for different initial forward rates ($T = 3, f(0,3) = 6, 8, 10\%$ and $\sigma = 15\%$). The solid lines represent the continuously compounded interest rate according to (15) wheras the doted lines are the corresponding lognormal p.d.f.

One object of the paper is to clearify the difference between the dynamic of the annual effective interest rate and the continuously compounded one. Since discount factors play a principle role in most applications the following theorem looks at their behaviour more closely.

Theorem 6. Let $\alpha \in]0,1]$ be a fixed real number and define by

$$y_\alpha := (1 + r_e)^{-\alpha} = \exp\{-r_c \cdot \alpha\} \tag{17}$$

the discount factor for a period of length α. Suppose that the annual effective rate r_e is lognormally distributed according to (7) then

a) we have

$$dy_\alpha = \alpha y_\alpha \left(1 - y_\alpha^{\frac{1}{\alpha}}\right) \left\{ \left(\frac{1}{2}(1+\alpha)(1 - y_\alpha^{\frac{1}{\alpha}})\sigma^2(t) - \mu(t) \right) dt - \sigma(t) dw \right\} \tag{18}$$

b) the p.d.f. of y_α is given by

$$g(x; t, \bar{\mu}, \bar{\sigma}) = \begin{cases} \frac{1}{\sqrt{2\pi}\ \bar{\sigma}} \ \frac{1}{\alpha \cdot (1 - x^{\frac{1}{\alpha}}) x} \exp\left\{ -\frac{\left(\ln\left(x^{\frac{-1}{\alpha}} - 1\right) - \bar{\mu}\right)^2}{2\bar{\sigma}^2} \right\} & x \in]0,1] \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

Proof.

ad a) Ito's lemma implies

$$\begin{aligned} dy_\alpha &= -\alpha \exp\{-r_c\alpha\} dr_c + \frac{1}{2}\alpha^2 \exp\{-r_c\alpha\} d\langle r_c \rangle \\ &= \alpha y_\alpha (1 - y_\alpha^{\frac{1}{\alpha}}) \left(\frac{1}{2}(1+\alpha)(1 - y_\alpha^{\frac{1}{\alpha}})\sigma^2(t) - \mu(t) \right) dt \\ &\quad -\alpha y_\alpha (1 - y_\alpha^{\frac{1}{\alpha}}) \sigma(t) dw \end{aligned}$$

ad b) Let $0 < a < b \leq 1$ be two real numbers. Due to theorem 4 we have:

$$\begin{aligned}\text{prob}[a < y_\alpha \leq b] &= \text{prob}\left[-\frac{1}{\alpha}\ln b \leq r_c < -\frac{1}{\alpha}\ln a\right] \\ &= \int_{-\frac{1}{\alpha}\ln b}^{-\frac{1}{\alpha}\ln a} f(u;t,\bar{\mu},\bar{\sigma})du\end{aligned}$$

Consider the substitution $x(u) = \exp\{-u \cdot \alpha\}$

$$\Rightarrow \quad \int_{-\frac{1}{\alpha}\ln b}^{-\frac{1}{\alpha}\ln a} f(u;t,\bar{\mu},\bar{\sigma})du = -\int_b^a \frac{1}{\alpha \cdot x} \cdot f\left(-\frac{1}{\alpha}\ln x; t, \bar{\mu}, \bar{\sigma}\right) dx$$

$$= \quad \int_a^b \frac{1}{\sqrt{2\pi}\ \bar{\sigma}}\ \frac{1}{\alpha \cdot x(1 - x^{\frac{1}{\alpha}})} \exp\left\{-\frac{\left(\ln\left(x^{-\frac{1}{\alpha}} - 1\right) - \bar{\mu}\right)^2}{2\bar{\sigma}^2}\right\} dx$$

□

Remarks.

i) The distribution of y_α belongs to a class of distributions studied already by Johnson [1946, 1949]. Let z be a standard normal distributed variable then the transformation $z \longmapsto (\gamma + \delta \exp\{\theta z\})^{-1}$ defines a class of propability distributions which Johnson denotes by S_B.

ii) Applying again the relationship (16) to the endogenous drift $\int_0^T \mu(s)ds$, the p.d.f. of the discount factor y_α is given by the following picture

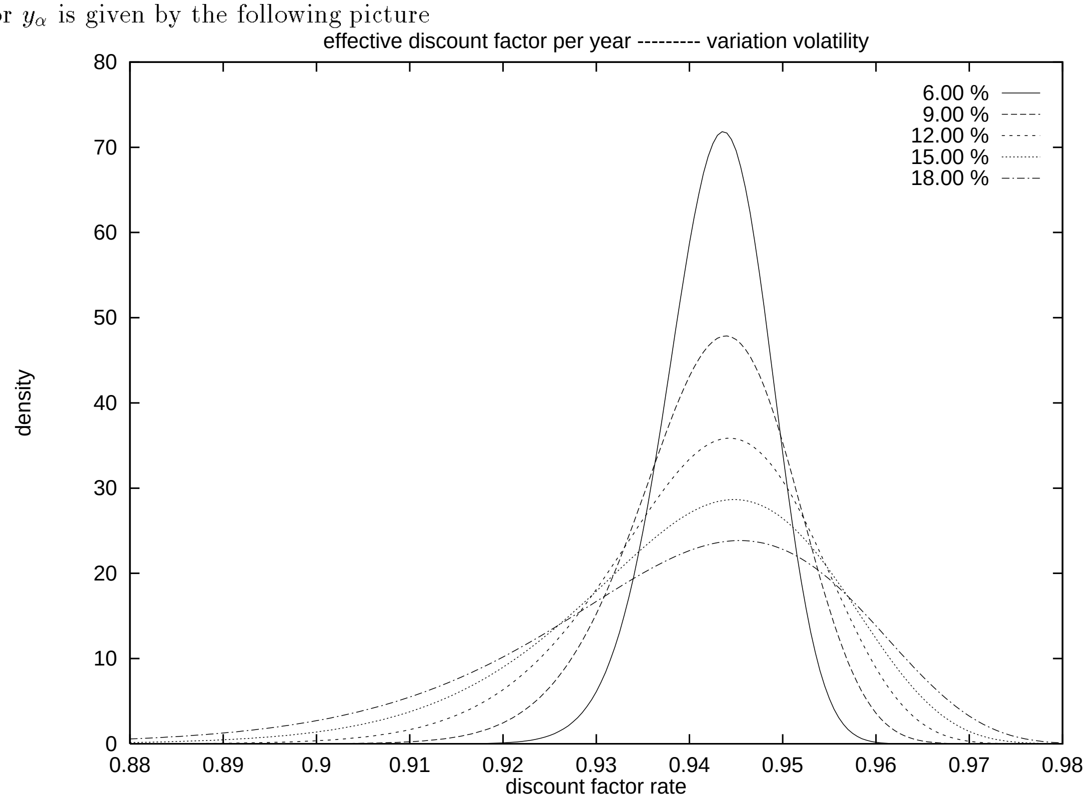

Figure 2 : p.d.f. of y_α for different volatilities of the effective interest rate ($T = 3$, $\alpha = 1, f(0,3) = 6\%$ (forward rate) and $\sigma = 6, 9, 12, 15, 18\%$ resp.).

iii) Due to Johnson (1946) we know that all moments of the discount factor y_α do exist, but there are no closed form expressions for them. For the expected value the following boundary conditions can be shown:

Proposition 2. For $\alpha \in]0,1]$

$$\left(\frac{1}{1+r_0\exp\left\{\int_0^t \mu(s)ds\right\}}\right)^\alpha \le E_g\left[y_\alpha(t)\right] \le \left(\frac{1}{1+r_0\exp\left\{\int_0^t \mu(s)-\sigma^2(t)ds\right\}}\right)^\alpha \tag{20}$$

Proof. Denote by Λ the lognormal distribution associated with the effective interest rate r_e. Since $h(x)=\left(\frac{1}{1+x}\right)^\alpha$ is convex in $x\ge 0$, by Jensens inequality:

$$\begin{aligned} E_g\left[y_\alpha(t)\right] &= E_\Lambda\left[\left(\frac{1}{1+r_e}\right)^\alpha\right] \\ &\ge \left(\frac{1}{1+E_\Lambda[r_e]}\right)^\alpha \\ &= \left(\frac{1}{1+\exp\left\{\bar\mu+\frac{1}{2}\bar\sigma^2\right\}}\right)^\alpha \\ &= \left(\frac{1}{1+r_0\exp\left\{\int_0^t \mu(s)ds\right\}}\right)^\alpha \end{aligned}$$

On the other hand $h(x)=x^\alpha$ is concav in $x>0$. Therefore applying Jensens inequality leads to

$$\begin{aligned} E_g[y_\alpha(t)] &= E_g[(y_1(t))^\alpha] \le (E_g[y_1(t)])^\alpha \\ E_g[y_1(t)] &= \int_0^1 \frac{1}{\sqrt{2\pi}\bar\sigma}\frac{x}{x(1-x)}\exp\left\{-\frac{\left(\ln\left(x^{-1}-1\right)-\bar\mu\right)^2}{2\bar\sigma^2}\right\}dx \\ &= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\bar\sigma}\frac{1}{1+e^x}\exp\left\{-\frac{(x-\bar\mu)^2}{2\bar\sigma^2}\right\}dx \\ &= 1-\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\bar\sigma}\frac{e^x}{1+e^x}\exp\left\{-\frac{(x-\bar\mu)^2}{2\bar\sigma^2}\right\}dx \\ &= 1-e^{\bar\mu+\frac{1}{2}\bar\sigma^2}\cdot\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\bar\sigma}\frac{1}{1+e^x}\exp\left\{-\frac{(x-\bar\mu-\bar\sigma^2)^2}{2\bar\sigma^2}\right\}dx \\ &= 1-e^{\bar\mu-\frac{1}{2}\bar\sigma^2}\cdot\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\bar\sigma}\frac{e^{\bar\sigma^2}}{1+e^{x+\bar\sigma^2}}\exp\left\{-\frac{(x-\bar\mu)^2}{2\bar\sigma^2}\right\}dx \\ &= 1-e^{\bar\mu-\frac{1}{2}\bar\sigma^2}\cdot\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\bar\sigma}\frac{1}{1+e^x}\exp\left\{-\frac{(x-\bar\mu)^2}{2\bar\sigma^2}\right\}dx \\ &= 1-e^{\bar\mu-\frac{1}{2}\bar\sigma^2}\cdot E_g\left[y_1(t)\right] \end{aligned}$$

□

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