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Continuous–Time Limits in the Generalized Ho–Lee Framework under the Forward Measure

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Abstract

The forward measure in the discrete time Ho/Lee model is derived and passages to the continuous time limit are carried out under this measure. In particular the continuous time valuation formula for call options on zero coupon bonds is obtained as a limit of its discrete time equivalent as well as the continuous time distribution of the continuously compounded short rate.
Finally it is shown that the trinomial and quattronomial generalizations of the Ho/Lee model by Bühler and Schulze are essentially equivalent to the Ho/Lee model as concernes their discrete time properties and their continuous time limits.

Key words: Ho/Lee model, forward measure, continuous time limit, trinomial and quattronomial models.

JEL Classification Number: G12, G13

Contents

1 Introduction

The prominence of the Ho/Lee–model [HL86] stems from the fact that it represented the first purely arbitrage–based approach to pricing interest rate derivative securities.

While Ho and Lee have not considered continuous time limits of their model Heath, Jarrow and Morton (HJM)[HJM90] provide limit results for the distribution of instantaneous forward rates under the risk–neutral probability measure.

In this paper we derive the continuous time limit of the discrete time call–option formula for zero coupon bonds. It turns out that this limit is naturally calculated under the forward measure. Due to the fact that interest rates are stochastic this measure differs from the risk neutral measure thus causing considerable differences between the limit arguments presented in this paper and the one by HJM or Cox–Rubinstein [CR85] in the Black–Scholes case. We also derive the continuous time distribution of the short rate under the forward measure.

Finally we consider generalizations of the Ho/Lee model proposed by Bühler/Schulze [BS92],[BS93],[BS95] and show that in every interesting respect these alleged generalizations do not go beyond the original Ho/Lee model. In particular all the models proposed by Bühler/Schulze are one–factor models driven by the stochastic process of the short rate. Moreover, all the limit results obtained in the Ho/Lee framework carry over to the Bühler/Schulze models.

In the second section of the paper, we review the Ho/Lee model and establish necessary notation. The third section contains our limit results for the Ho/Lee model, the fourth contains our analysis of the Bühler/Schulze models while the fifth concludes.

2 The Structure of the Ho/Lee Model

Consider a fixed time interval $I := [0, T]$. Assume that this time interval is divided into $M-1$ sub–intervals of length $\Delta := T/M$ and that trading in zero coupon bonds takes place at times $\{0, \Delta, 2\Delta, \ldots, m\Delta, \ldots, T-\Delta\}$. At time 0 one observes the prices of a given set of zero coupon bonds maturing at $\{\Delta, \ldots, m\Delta, \ldots, T\}$. The central assumption in the Ho/Lee model [HL86] is that the prices of these securities can undergo upward and downward shocks such that the stochastic processes of zero coupon bond prices can be represented on a binomial lattice with step size Δ. For a lattice with this step size we denote by $\tilde{D}_\Delta(m, n, j)$ the price in vertex (m, j) of a zero coupon bond with time to maturity $n\Delta$ where $n \in \{0, 1, \ldots, M-m\}$. Here $j \in \{0, 1, \ldots, m\}$ denotes the number of upward shocks that have occured up to time $m\Delta$. Now take a family $(z_i)_{i \in \mathbb{N}}$ of independent random variables that are 1 with probability p and 0 with probability $1-p$ and define random variables $J_m := \sum_{i=1}^m z_i$. With this notation we can denote the family of processes of prices of zero coupon bonds with times to maturity $n\Delta$, $n \in \{0, 1, \ldots, M\}$ on a binomial lattice with step size Δ by:

$$\left(\tilde{D}_\Delta(m, n, J_m)\right)_{\substack{m \in \{0,1,\ldots,M-n\} \\ n \in \{0,1,\ldots,M\}}}$$

In order to model upward and downward shocks to zero coupon bond prices Ho and Lee introduce functions $h_\Delta(\cdot)$ and $h^*_\Delta(\cdot)$ of time to maturity. For $m > 0$ they then specify the evolution of these prices recursively by

$$\begin{aligned} &\text{upward shock:} \\ \tilde{D}_\Delta(m,n,j+1) &:= \frac{\tilde{D}_\Delta(m-1,n+1,j)}{\tilde{D}_\Delta(m-1,1,j)}\, h_\Delta(n) \\ &\text{downward shock:} \\ \tilde{D}_\Delta(m,n,j) &:= \frac{\tilde{D}_\Delta(m-1,n+1,j)}{\tilde{D}_\Delta(m-1,1,j)}\, h^*_\Delta(n) \end{aligned} \tag{1}$$

For $\Delta = 1$ and $T \in \mathbb{N}$ Ho and Lee show that under the assumption of absence of arbitrage opportunities and given the requirement that it must be possible to represent the price processes of zero coupon bonds on a binomial lattice the functions h_1 and h_1^* are uniquely determined and given by

$$\begin{aligned} h_1(\xi) &= \frac{1}{p + (1-p)\,\delta_1^\xi} \\ h_1^*(\xi) &= \frac{\delta_1^\xi}{p + (1-p)\,\delta_1^\xi} \end{aligned} \qquad \xi \in \{0, 1, \ldots, T\}\,, \tag{2}$$

where $\delta_1 \in (0,1)$ is an exogenous constant that expresses the variability of interest rates and bond prices.

To obtain a generalization of these perturbation functions for an arbitrary step size Δ replace δ_1 by $\delta_\Delta \in (0,1)$ in equation (2). The way in which δ_Δ depends on Δ can be determined as follows. Denote by $\sigma^2\Delta$ the variance of the continuously compounded yield of a zero coupon bond with time to maturity Δ conditional on J_m over a time interval of length Δ . Then using equation (1) and the modified equation (2) we have

$$\sigma^2\Delta := V^p\left[-\frac{1}{\Delta}\ln\,\tilde{D}_\Delta(m+1,1,J_m+z_{m+1})\,\middle|\,J_m\right] = \frac{1}{\Delta^2}\,p(1-p)\,(\ln\,\delta_\Delta)^2.$$

Remembering that $\delta_\Delta \in (0,1)$ this yields

$$\delta_\Delta = \exp\left\{-\frac{\sigma\Delta^{\frac{3}{2}}}{\sqrt{p(1-p)}}\right\}.$$

This generalization of the Ho/Lee perturbation functions has previously been obtained in a different way by Heath, Jarrow and Morton [HJM90].

Given these specifications we obtain the following closed form solution for the family of processes of prices of zero coupon bonds $\tilde{D}_\Delta$.

$$\left(\tilde{D}_\Delta(m,n,J_m)\right)_{\substack{m\,\in\,\{0,1,\ldots,M-n\}\\ n\,\in\,\{0,1,\ldots,M\}}} =$$

$$\left(\frac{\tilde{D}_\Delta(0,m+n,0)}{\tilde{D}_\Delta(0,m,0)}\left(\prod_{i=1}^{m}\frac{h_\Delta(m+n-i)}{h_\Delta(m-i)}\right)\delta_\Delta^{n(m-J_m)}\right)_{\substack{m\,\in\,\{0,1,\ldots,M-n\}\\ n\,\in\,\{0,1,\ldots,M\}}} \tag{3}$$

This formula can easily be proved by induction. It almost immediately gives rise to

Proposition 2.1 *In the Ho/Lee model processes of logarithms of prices of zero coupon bonds for all times to maturity are affine transformations of the short rate process.*

PROOF:
The proof is contained in the proof of Proposition 4.2. □

Finally it is important to notice that the way we have presented the Ho/Lee model the probability measure p is the martingale measure in the sense that for every $m \in \{1,\ldots,M\}$ and $n \in \{0,\ldots,M-m\}$ we have

$$\tilde{D}_\Delta(m-1,n+1,J_{m-1}) = E^p\left[\tilde{D}_\Delta(m-1,1,J_{m-1})\tilde{D}_\Delta(m,n,J_m)\,|\,J_{m-1}\right].$$

More generally, given a binomial lattice with step size Δ on which the price processes of zero coupon bonds are as specified above, denoting by $V_\Delta(\Gamma(J_{\tilde{m}}),m,J_m)$ the price on this lattice in vertex (m,J_m) of a contingent claim with a single payoff $\Gamma(J_{\tilde{m}})$ at time $\tilde{m}\Delta := [t/\Delta]\Delta$, one necessarily has by absence of arbitrage opportunities

$$V_\Delta(\Gamma(J_{\tilde{m}}),m-1,J_{m-1}) = E^p\left[\tilde{D}_\Delta(m-1,1,J_{m-1})V_\Delta(\Gamma(J_{\tilde{m}}),m,J_m)\,|\,J_{m-1}\right] \tag{4}$$

for $m \in \{1,\ldots,\tilde{m}\}$, as shown for this particular case in [HL86].

3 Limit Results for Continuous Trading in the Ho/Lee Model

3.1 The Derivation of the Call Option Formula in the Ho/Lee Model

3.1.1 The Formula in Discrete Time

Let $C_\Delta(\tilde{m},\tilde{n},K)$ be a European call option with exercise price K and expiration date $\tilde{m}\Delta$ on the zero coupon bond that at time $\tilde{m}\Delta$ has time to maturity $\tilde{n}\Delta$, where $\tilde{n}\Delta := [x/\Delta]\Delta$ for $x \in [0,T-t]$. Its payoff is given by $\left[\tilde{D}_\Delta(\tilde{m},\tilde{n},J_{\tilde{m}})-K\right]^+$. To determine the price of this option we first consider Arrow–Debreu securities whose payoff in some vertex (m,j) in the binomial lattice is defined by:

$$AD_{m,j}(\mu,\iota) = \begin{cases} 1 & \text{if } \mu = m \wedge \iota = j \\ 0 & \text{else} \end{cases} \tag{5}$$

Using equation (4) we can determine the value of such a derivative security in the binomial model of zero coupon bond prices. We have

Proposition 3.1 *Define:*

$$l_\Delta(m,j) \;:=\; \begin{cases} 0 & \text{if } m<j \vee j<0 \\ \delta_\Delta^{m-j} & \text{else} \end{cases}$$

$$w_\Delta(m,j) \;:=\; \begin{cases} 1 & \text{if } m=j=0 \\ 0 & \text{if } m<j \vee j<0 \\ \begin{array}{l} w_\Delta(m-1,j-1)l_\Delta(m-1,j-1) + \\ w_\Delta(m-1,j)l_\Delta(m-1,j) \end{array} & \text{else} \end{cases}$$

Then the time zero value of the Arrow–Debreu security $AD_{m,j}$ with $m \in \{1,\dots,M-1\}$ and $j \in \{0,\dots,m\}$in the binomial lattice with step size Δ is given by

$$V_\Delta(AD_{m,j},0,0) = \tilde{D}_\Delta(0,m,0)\,p^j(1-p)^{m-j}\left(\prod_{i=1}^{m} h_\Delta(i-1)\right) w_\Delta(m,j)\,. \tag{6}$$

Proof:

The proof is given by induction.

First consider $AD_{1,1}$. From equation (4) its value at time 0 is given by $\tilde{D}_\Delta(0,1,0)\,p$. This result is also obtained from equation (6) for $m=j=1$. Similarly it is easy to see that for $m=1$ and $j=0$ equation (6) yields a value $\tilde{D}_\Delta(0,1,0)\,(1-p)$ for $AD_{1,0}$, which is also the value obtained from equation (4).

Now assume that equation (6) holds for all the Arrow–Debreu securities up to some period $\bar{m}$ and consider $AD_{\bar{m}+1,j^*}$ with $\bar{m}+1 > j^* > 0$. Again by equation (4) the values of this security in vertices $(\bar{m},j^*)$ and $(\bar{m},j^*-1)$ are

$$\begin{aligned} V_\Delta(AD_{\bar{m}+1,j^*},\bar{m},j^*) &= \tilde{D}_\Delta(\bar{m},1,j^*)\,(1-p) \\ V_\Delta(AD_{\bar{m}+1,j^*},\bar{m},j^*-1) &= \tilde{D}_\Delta(\bar{m},1,j^*-1)\,p\,. \end{aligned}$$

In every vertex $(\bar{m},j)$, $j \neq j^*$ or j^*-1 the value of $AD_{\bar{m}+1,j^*}$ is zero. Now consider a portfolio consisting of $\tilde{D}_\Delta(\bar{m},1,j^*)\,(1-p)$ units of $AD_{\bar{m},j^*}$ and $\tilde{D}_\Delta(\bar{m},1,j^*-1)\,p$ units of $AD_{\bar{m},j^*-1}$. As this portfolio is worth the same as $AD_{\bar{m}+1,j^*}$ in every vertex $(\bar{m},j)$, $j \in \{0,\dots,\bar{m}\}$ by absence of arbitrage opportunities it must also be worth the same as $AD_{\bar{m},j^*-1}$ in vertex $(0,0)$. Therefore, we have

$$\begin{aligned} &V_\Delta(AD_{\bar{m}+1,j^*},0,0) \\ &= \tilde{D}_\Delta(\bar{m},1,j^*)\,(1-p)\,V_\Delta(AD_{\bar{m},j^*},0,0) + \tilde{D}_\Delta(\bar{m},1,j^*-1)\,p\,V_\Delta(AD_{\bar{m},j^*-1},0,0) \\ &= \tilde{D}_\Delta(0,\bar{m}+1,0)\,p^{j^*}(1-p)^{\bar{m}+1-j^*}\left(\prod_{i=1}^{\bar{m}+1} h_\Delta(i-1)\right) \\ &\quad \Big(l_\Delta(\bar{m},j^*)\,w_\Delta(\bar{m},j^*) \;+\; l_\Delta(\bar{m},j^*-1)\,w_\Delta(\bar{m},j^*-1)\Big) \\ &= \tilde{D}_\Delta(0,\bar{m}+1,0)\,p^{j^*}(1-p)^{\bar{m}+1-j^*}\left(\prod_{i=1}^{\bar{m}+1} h_\Delta(i-1)\right) w_\Delta(\bar{m}+1,j^*) \end{aligned}$$

As it is easy to see that a similar argument can be made if $j^* = \bar{m}+1$ or $j^*=0$ this completes the proof. □

REMARKS:
a) The basic idea behind this proof is, of course, the well known concept of forward induction (see [Jam91], [SS96]).

b) It is interesting to compare equation (6) to the pricing formula for Arrow–Debreu securities as we have defined them in equation (5) that would obtain in the Cox/Rubinstein model for stock options [CR85]. Cox/Rubinstein assume short term interest rates to be constant and equal to some value r_Δ. Hence, $\tilde{D}_\Delta(0,m,0) = e^{-r_\Delta m\Delta}$. More importantly the coefficient $w_\Delta(m,j)$ would simply be the binomial coefficient $\binom{m}{j}$ in Cox/Rubinstein, a feature which facilitates the explicit valuation of path dependent securities in their model and which is destroyed through stochastic interest rates.

c) By absence of arbitrage opportunities, the term multiplying $\tilde{D}_\Delta(0,m,0)$ in (6) can be interpreted as a probability measure different from but equivalent to the original probability measure. We shall refer to this measure as the Arrow–Debreu– or (as the term is more common in the literature) the forward–measure. In continuous time this measure was first introduced by El/Karoui/Rochet [EKR89] and Jamshidian [Jam87]. The example of the discrete time model Ho/Lee model brings out clearly the link between the forward measure and Arrow–Debreu prices.

Given the Arrow–Debreu prices it is easy to state a closed formula for zero coupon bond options in discrete time.

Corollary 3.2 *The time zero value of $C_\Delta(\tilde{m},\tilde{n},K)$ is given by*

$$\begin{aligned} V_\Delta\left(C_\Delta(\tilde{m},\tilde{n},K),0,0\right) = \\ \tilde{D}_\Delta(0,\tilde{m}+\tilde{n},0) \sum_{j>a_\Delta}^{\tilde{m}} \left(\prod_{i=1}^{\tilde{m}} h_\Delta(\tilde{m}+\tilde{n}-i)\right) \delta^{\tilde{n}(\tilde{m}-j)}\, p^j\,(1-p)^{\tilde{m}-j}\, w_\Delta(\tilde{m},j) \\ -K\tilde{D}_\Delta(0,\tilde{m},0) \sum_{j>a_\Delta}^{\tilde{m}} \left(\prod_{i=1}^{\tilde{m}} h_\Delta(i-1)\right) p^j\,(1-p)^{\tilde{m}-j}\, w_\Delta(\tilde{m},j) \end{aligned} \tag{7}$$

where

$$a_\Delta = \frac{\ln\left(\dfrac{\tilde{D}_\Delta(0,\tilde{m}+\tilde{n},0)}{\tilde{D}_\Delta(0,\tilde{m},0)K} \displaystyle\prod_{i=1}^{\tilde{m}} \dfrac{h_\Delta(\tilde{m}+\tilde{n}-i)}{h_\Delta(\tilde{m}-i)}\right)}{\tilde{n}\,\ln\,\delta_\Delta} + \tilde{m}$$

PROOF:
Follows immediately by the requirement of absence of arbitrage opportunities, the definition of a European call option, equation (3), and Proposition 3.1. □

3.1.2 The Continuous–Time Limit of the Discrete Time Formula

We determine the limit of the value of the call option in equation (7) in two steps. First we show how the central limit theorem can be invoked, second we calculate the arguments at which the standard normal distribution must be evaluated.

Proposition 3.3 **(a)** *Let*

$$y_k^{(1)} = \begin{cases} 1 \sim p\, h_\Delta(k) \\ 0 \sim (1-p)\, h_\Delta(k)\, \delta_\Delta^k \end{cases} \qquad k \in N_0$$

be independent random variables and set $Y_m^{(1)} := \sum_{k=0}^{m-1} y_k^{(1)}$ *then*

$$P^p\left[Y_m^{(1)} = j\right] = \left(\prod_{i=1}^{m} h_\Delta(i-1)\right) p^j\,(1-p)^{m-j}\, w_\Delta(m,j)\,.$$

(b) *Let*

$$y_k^{(2)} = \begin{cases} 1 \sim p\, h_\Delta(m+n-(k+1)) \\ 0 \sim (1-p)\, h_\Delta(m+n-(k+1))\, \delta_\Delta^{(m+n-(k+1))} \end{cases} \qquad k \in \{0,\ldots,m+n-1\}$$

be independent random variables and for $0 < m^* \leq m$ *set* $Y_{m^*}^{(2)} := \sum_{k=0}^{m^*-1} y_k^{(2)}$ *then*

$$P^p\left[Y_{m^*}^{(2)} = j\right] = \left(\prod_{i=1}^{m^*} h_\Delta(m+n-i)\right) \delta_\Delta^{(m+n-m^*)(m^*-j)}\, p^j\,(1-p)^{m^*-j}\, w_\Delta(m^*,j)\,.$$

PROOF:
See Appendix A. □

Note that if $m^* = m = \tilde{m}$ and $n = \tilde{n}$ in part (b) of the proposition we obtain exactly the type of expression multiplying $D_\Delta(0, \tilde{m}+\tilde{n}, 0)$ in equation (7). This means that by Proposition 3.3 we can now interpret the sums in equation (7) as complementary distribution functions of the random variables $Y_m^{(2)}$ and $Y_m^{(1)}$ respectively, which in turn are both sums of independent but no longer identically distributed random variables. Using Proposition 3.3 we have

Corollary 3.4

$$\begin{aligned}
E^p[Y_m^{(1)}] &= \sum_{k=0}^{m-1} p\, h_\Delta(k) \\
E^p[Y_m^{(2)}] &= \sum_{k=0}^{m-1} p\, h_\Delta(m+n-(k+1)) \\
V^p[Y_m^{(1)}] &= \sum_{k=0}^{m-1} (p\, h_\Delta(k) \;-\; p^2\, h_\Delta^2(k)) \\
V^p[Y_m^{(2)}] &= \sum_{k=0}^{m-1} (p\, h_\Delta(m+n-(k+1)) \;-\; p^2\, h_\Delta^2(m+n-(k+1)))
\end{aligned}$$

where V^p *is the variance operator under* p.

PROOF:
Immediate from Proposition 3.3. □

This result enables us to state

Proposition 3.5 *As* $\Delta \to 0$

$$\frac{Y_{\tilde{m}}^{(i)} - E^p[Y_{\tilde{m}}^{(i)}]}{\sqrt{V^p[Y_{\tilde{m}}^{(i)}]}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \quad i = 1, 2.$$

PROOF:
See Appendix B. □

Now we can return to the call option formula in discrete time as given in equation (7). We can restate this formula in the following way:

$$\begin{aligned} & V_\Delta\left(C(\tilde{m},\tilde{n},K),0,0\right) = \\ & \quad \tilde{D}_\Delta(0,\tilde{m}+\tilde{n},0)\left(1 - E^p\left[\frac{Y_{\tilde{m}}^{(2)} - E^p[Y_{\tilde{m}}^{(2)}]}{\sqrt{V^p[Y_{\tilde{m}}^{(2)}]}} \le \frac{a_\Delta - E^p[Y_{\tilde{m}}^{(2)}]}{\sqrt{V^p[Y_{\tilde{m}}^{(2)}]}}\right]\right) \\ & \quad -K\tilde{D}_\Delta(0,\tilde{m},0)\left(1 - E^p\left[\frac{Y_{\tilde{m}}^{(1)} - E^p[Y_{\tilde{m}}^{(1)}]}{\sqrt{V^p[Y_{\tilde{m}}^{(1)}]}} \le \frac{a_\Delta - E^p[Y_{\tilde{m}}^{(1)}]}{\sqrt{V^p[Y_{\tilde{m}}^{(1)}]}}\right]\right) \end{aligned} \tag{8}$$

Having obtained this version of the discrete time valuation formula we are now in a position to give its continuous time limit.

Theorem 3.6

$$\begin{aligned} \lim_{\Delta\to 0} V_\Delta\left(C(\tilde{m},\tilde{n},K),0,0\right) &= D(0,t+x)\mathcal{N}\left(\frac{\ln\left(\frac{D(0,t+x,0)}{KD(0,t,0)}\right) + \frac{1}{2}x^2\sigma^2 t}{x\sqrt{t}\,\sigma}\right) \\ & \quad -D(0,t)K\mathcal{N}\left(\frac{\ln\left(\frac{D(0,t+x,0)}{KD(0,t,0)}\right) - \frac{1}{2}x^2\sigma^2 t}{x\sqrt{t}\,\sigma}\right) \end{aligned}$$

where $\mathcal{N}$ *is the standard normal distribution function and* $D(0,\varphi)$ *is the price at time* 0 *of a zero coupon bond which at time* 0 *has time to maturity* $\varphi \in \mathbb{R}_+$.

PROOF:
See Appendix C. □

This formula is a special case of the well known general zero bond option formula for Gaussian interest rate models (see e.g.[Jam87], [EKR89], [HJM92]). It should be clear that similar limit arguments as the above can also be made for European options on other underlyings that can be considered in the Ho/Lee framework.

3.2 Limit Distributions of the Short Rate in the Ho/Lee Model

It follows from Heath, Jarrow and Morton [HJM90] that under the transition probabilities p the continuously compounded short rate at some time t in the Ho/Lee model converges in distribution to a normally distributed random variable with expectation $r(0,t)+\frac{1}{2}t^2\sigma^2$ and variance $t\sigma^2$. Under the forward measure we have

Proposition 3.7 *Under the forward measure as* $\Delta \to 0$

$$r_\Delta(\tilde{m},1,J_{\tilde{m}}) \xrightarrow{\mathcal{D}} \mathcal{N}\left(r(0,t);\, \sigma^2 t\right) .$$

PROOF:
See Appendix D. □

This agrees with a result obtained by El Karoui/Rochet [EKR89] in a continuous time setting.

4 Why the Bühler/Schulze Models are essentilly equal to the Ho/Lee Model

4.1 The Structure of the Bühler/Schulze Models

Bühler and Schulze propose modifying the Ho/Lee model by assuming that processes of prices of zero coupon bonds can be represented on a trinomial [BS92], [BS95] or quattronomial [BS93] lattice. Since the quattronomial model is the more general we consider this first. As we have done in the case of the Ho/Lee model we shall also present this model in the framework of a lattice with arbitrary step size Δ. Similar to Section 2 we introduce two independent stochastic processes $\left(J_m^{(k)}\right)_{m\in\{0,\ldots,M\}} := \left(\sum_{i=1}^m z_i^{(k)}\right)_{m\in\{0,\ldots,M\}}$ with $z_i^{(k)} \overset{iid}{\sim} \begin{cases} 1 \sim p^{(k)} \\ 0 \sim 1-p^{(k)} \end{cases}$, $k \in \{1,2\}$ which count the number of upward shocks of type (k) to zero coupon bond prices that have occurred up to time $m\Delta$. The processes of prices of zero coupon bonds on a quattronomial lattice with step size Δ can now be denoted by:

$$\left(\tilde{D}_\Delta(m,n,J_m^{(1)},J_m^{(2)})\right)_{\substack{m\,\in\,\{0,1,\ldots,M-n\}\\ n\,\in\,\{0,1,\ldots,M\}}}$$

In order to model upward and downward shocks of the two kinds Bühler and Schulze introduce two pairs of perturbation functions:

$$\begin{aligned} h_\Delta^{(k)}(\xi) &= \frac{1}{p^{(k)}+(1-p^{(k)})\,\delta_\Delta^{(k)\,\xi}} \\ h_\Delta^{(k)*}(\xi) &= \frac{\delta_\Delta^{(k)\,\xi}}{p^{(k)}+(1-p^{(k)})\,\delta_\Delta^{(k)\,\xi}} \end{aligned} \qquad \begin{aligned} &\xi \in \{0,1,\ldots,M\} \\ &k \in \{1,2\} \end{aligned}$$

From every vertex $(m, j^{(1)}, j^{(2)})$, $m \in \{1, \dots, M-n\}$ of the quattronomial lattice prices of zero coupon bonds can move in four directions according to

$$
\begin{aligned}
\tilde{D}_\Delta(m, n, j^{(1)}+1, j^{(2)}+1) &:= \frac{\tilde{D}_\Delta(m-1, n+1, j^{(1)}, j^{(2)})}{\tilde{D}_\Delta(m-1, 1, j^{(1)}, j^{(2)})}\, h_\Delta^{(1)}(n)\, h_\Delta^{(2)}(n) \\
&\;\;\vdots \\
\tilde{D}_\Delta(m, n, j^{(1)}, j^{(2)}) &:= \frac{\tilde{D}_\Delta(m-1, n+1, j^{(1)}, j^{(2)})}{\tilde{D}_\Delta(m-1, 1, j^{(1)}, j^{(2)})}\, h_\Delta^{*(1)}(n)\, h_\Delta^{*(2)}(n)\,.
\end{aligned} \tag{9}
$$

From these recursive formulae we can again obtain a recursion for the continuously compounded yield of a zero coupon bond with time to maturity Δ:

$$
\begin{aligned}
r_\Delta(m, 1, j^{(1)} + z_m^{(1)}, j^{(2)} + z_m^{(2)}) &= r_\Delta(m, 1, j^{(1)}, j^{(2)}) - \frac{1}{\Delta}\left(\ln h_\Delta^{(1)}(1) + (1 - z_m^{(1)}) \ln \delta_\Delta^{(1)}\right) \\
&\quad - \frac{1}{\Delta}\left(\ln h_\Delta^{(2)}(2) + (1 - z_m^{(2)}) \ln \delta_\Delta^{(2)}\right)
\end{aligned}
$$

Now setting $\left(\sigma^{(k)}\right)^2 \Delta := V^{p^{(k)}}\left[\frac{1}{\Delta}\ z_m^{(k)}\ \ln \delta_\Delta^{(k)}\right]$ we obtain similar to the Ho/Lee model

$$
\delta_\Delta^{(k)} = \exp\left\{-\frac{\sigma^{(k)} \Delta^{\frac{3}{2}}}{\sqrt{p^{(k)}(1 - p^{(k)})}}\right\}.
$$

Given these specifications we obtain the following closed form solution for the family of processes of prices of zero coupon bonds $\tilde{D}_\Delta$, which is a generalization of equation (3).

$$
\left(\tilde{D}_\Delta(m, n, J_m^{(1)}, J_m^{(2)})\right)_{\substack{m \in \{0,1,\dots,M-n\} \\ n \in \{0,1,\dots,M\}}} =
$$

$$
\left(\frac{\tilde{D}_\Delta(0, m+n, 0)}{\tilde{D}_\Delta(0, m, 0)} \prod_{k=1}^{2} \left(\prod_{i=1}^{m} \frac{h_\Delta^{(k)}(m+n-i)}{h_\Delta^{(k)}(m-i)}\right) \delta_\Delta^{(k)\, n(m - J_m^{(k)})}\right)_{\substack{m \in \{0,1,\dots,M-n\} \\ n \in \{0,1,\dots,M\}}} \tag{10}
$$

This completes our exposition of the quattronomial model by Bühler and Schulze.
To introduce briefly the trinomial model by Bühler and Schulze assume we are given random variables $z_m^{(3)} \overset{iid}{\sim} \begin{cases} 0 & \sim & 1 - p_1^{(3)} - p_2^{(3)} \\ 1 & \sim & p_2^{(3)} \\ 2 & \sim & p_1^{(3)} \end{cases}$. Then the recursive formula for the development of prices of zero coupon bonds proposed by Bühler and Schulze in their trinomial model can be denoted by

$$
\tilde{D}_\Delta(m, n, j + z_m^{(3)}) := \frac{\tilde{D}_\Delta(m-1, n+1, j)}{\tilde{D}_\Delta(m-1, 1, j)}\, h_\Delta^{(3)}(n)\, \delta_\Delta^{(3)\, 2 - z_m^{(3)}}
$$

where

$$
h_\Delta^{(3)}(n) = \frac{1}{p_1^{(3)} + p_2^{(3)}\, \delta_\Delta^{(3)\, n} + (1 - p_1^{(3)} - p_2^{(3)})\, \delta_\Delta^{(3)\, 2n}}\,. \tag{11}
$$

We almost immediately have

Proposition 4.1 *The trinomial model by Bühler and Schulze is a special case of their quattronomial model.*

Proof:
Define $z_i^{(3a)} \overset{iid}{\sim} \begin{cases} 1 \sim p^{(3a)} \\ 0 \sim 1-p^{(3a)} \end{cases}$ and $z_i^{(3b)} \overset{iid}{\sim} \begin{cases} 1 \sim p^{(3b)} \\ 0 \sim 1-p^{(3b)} \end{cases}$ and set $z_i^{(3)} := z_i^{(3a)} + z_i^{(3b)}$. Hence

$$\begin{aligned} p_1^{(3)} &= p^{(3a)} p^{(3b)} , \\ p_2^{(3)} &= p^{(3a)}(1-p^{(3b)}) + (1-p^{(3a)})p^{(3b)} , \\ 1 - p_1^{(3)} - p_2^{(3)} &= (1-p^{(3a)})(1-p^{(3b)}) . \end{aligned}$$

Further set $\delta_\Delta^{(1)\xi} = \delta_\Delta^{(2)\xi} = \delta_\Delta^{(3)\xi}$ and put

$$\begin{aligned} h_\Delta^{(1)}(\xi) &= h_\Delta^{(3a)}(\xi) := \frac{1}{p^{(3a)} + (1-p^{(3a)})\left(\delta_\Delta^{(3)}\right)^\xi} \\ h_\Delta^{(2)}(\xi) &= h_\Delta^{(3b)}(\xi) := \frac{1}{p^{(3b)} + (1-p^{(3b)})\left(\delta_\Delta^{(3)}\right)^\xi} . \end{aligned}$$

It is easy to see that with these definitions equations (9) and (11) are equal. □

Consequently we shall from now on confine our interest to the quattronomial model. We have

Proposition 4.2 *In the quattronomial model by Bühler and Schulze processes of logarithms of prices of zero coupon bonds for all times to maturity are affine transformations of the short rate process.*

Proof:
The result immediately follows from equation (10). For the process of the short rate we have

$$\begin{aligned} &\left(r_\Delta(m, 1, J_m^{(1)}, J_m^{(2)})\right)_{m\in\{0,\ldots,M-1\}} = \\ &\quad \left(\frac{1}{\Delta}\left(\ln\frac{\tilde{D}_\Delta(0,m,0)}{\tilde{D}_\Delta(0,m+1,0)} - \sum_{k=1}^{2} \ln h_\Delta^{(k)} + (m - J_m^{(k)}) \ln \delta_\Delta^{(k)}\right)\right)_{m\in\{0,\ldots,M-1\}} \end{aligned} \tag{12}$$

Using this we can replace the stochastic process

$$\left(\sum_{k=1}^{2} (m - J_m^{(k)}) \ln \delta_\Delta^{(k)}\right)_{m\in\{0,\ldots,M-1\}}$$

which drives the processes of the logarithms of prices of zero coupon bonds by an affine transformation of the short rate process to obtain

$$\left(\ln \tilde{D}_\Delta(m,n,J_m^{(1)},J_m^{(2)})\right)_{\substack{m\in\{0,1,\ldots,M-n\}\\ n\in\{0,1,\ldots,M\}}} =$$
$$\left(n\left(-\Delta r_\Delta(m,1,J_m^{(1)},J_m^{(2)}) - \sum_{k=1}^{2}\ln h_\Delta^{(k)}(m) + \ln\frac{\tilde{D}_\Delta(0,m,0)}{\tilde{D}_\Delta(0,m+1,0)}\right)\right.$$
$$\left. + \ln\frac{\tilde{D}_\Delta(0,m+n,0)}{\tilde{D}_\Delta(0,m,0)} + \sum_{k=1}^{2}\ln\left(\prod_{i=1}^{m}\frac{h_\Delta^{(k)}(m+n-i)}{h_\Delta^{(k)}(m-i)}\right)\right)_{\substack{m\in\{0,1,\ldots,M-n\}\\ n\in\{0,1,\ldots,M\}}} \tag{13}$$

This proves the claim. □

Remark:
a) This result shows that contrary to what has been claimed by Bühler and Schulze their quattronomial model is not a two–factor but in fact a one–factor model the driving factor being as in the Ho/Lee model the continuously compounded short rate.

4.2 Limit Results for Continuous Trading in the Bühler/Schulze Models

As may be suspected all the limit results that hold in the simple Ho/Lee model also hold in the versions of this model proposed by Bühler and Schulze. It is actually sufficient to give a generalized version of Proposition 3.1 to see that all the arguments in Section 3 can be applied analogously in the Bühler/Schulze framework:

Proposition 4.3 *Define $l_\Delta^{(k)}(m,j^{(k)})$ and $w_\Delta^{(k)}(m,j^{(k)})$, $k\in\{1,2\}$ analogously to the definitions in Proposition 3.1. Then the time zero value of the Arrow–Debreu security $AD_{m,j^{(1)},j^{(2)}}$ with $m\in\{1,\ldots,M-1\}$ and $j^{(k)}\in\{0,\ldots,m\}$, $k\in\{1,2\}$ in the quattronomial lattice with step size Δ is given by*

$$V_\Delta(AD_{m,j^{(1)},j^{(2)}},0,0) =$$
$$\tilde{D}_\Delta(0,m,0,0)\prod_{k=1}^{2} p^{(k)\,j}(1-p^{(k)})^{m-j^{(k)}}\left(\prod_{i=1}^{m} h_\Delta^{(k)}(i-1)\right) w_\Delta^{(k)}(m,j^{(k)}) \tag{14}$$

Proof:
The proof is parallel to the proof of Proposition 3.1 the only difference being that in general the verteces in the Bühler/Schulze lattice have four predecessors instead of two in the Ho/Lee model. Therefore, in the induction step of the proof we need to consider the following portfolio of Arrow– Debreu securities:

$$\begin{array}{rll}
\tilde{D}_\Delta(m,1,j^{(1)},j^{(2)})\,(1-p^{(1)})\,(1-p^{(2)}) & \text{units of} & AD_{m,j^{(1)},j^{(2)}}\\
\tilde{D}_\Delta(m,1,j^{(1)}-1,j^{(2)})\,p^{(1)}\,(1-p^{(2)}) & \text{units of} & AD_{m,j^{(1)}-1,j^{(2)}}\\
\tilde{D}_\Delta(m,1,j^{(1)},j^{(2)}-1)\,(1-p^{(1)})\,p^{(2)} & \text{units of} & AD_{m,j^{(1)},j^{(2)}-1}\\
\tilde{D}_\Delta(m,1,j^{(1)}-1,j^{(2)}-1)\,p^{(1)}\,p^{(2)} & \text{units of} & AD_{m,j^{(1)}-1,j^{(2)}-1}
\end{array}$$

The claim then immediately follows. □

From this proposition it is immediately clear that all the arguments from Section 3 can be applied analogously if one keeps in mind that for every sum of independent random variables in the Ho/Lee model one now has two independent sums of independent random variables. Defining $\sigma^2 := \left(\sigma^{(1)}\right)^2 + \left(\sigma^{(2)}\right)^2$ one obtains the same continuous time call option formula in the Bühler/Schulze models as in the Ho/Lee model.

It is also totally obvious that the limiting distributions of the continuously compounded short rate in the Bühler/Schulze models are the same as in the Ho/Lee model.

In sum the results in this section show that the Bühler/Schulze models are but minor generalizations of the Ho/Lee model which in discrete and continuous time preserve all the essential characteristics and features of this model.

5 Conclusion

In this paper we have generalized the limit argument by HJM concerning the Ho/Lee model in two respects.

First, we have shown how to construct limit arguments under the forward measure. In particular we have derived the continuous time distribution of the short rate and the continuous time limit of formula for a European call on a zero coupon bond. Clearly, the ideas developed in this context would carry over to the analysis of a number of other derivatives that can be considered in the Ho/Lee framework.

Second, we have analyzed multinomial generalizations of the Ho/Lee model that have been proposed by Bühler and Schulze. We have shown that they are essentially equivalent to the Ho/Lee model.

A Proof of Proposition 3.3

In order to prove part (a) of Proposition 3.3 we need the following lemma, which we shall prove later.

Lemma A.1

$$w_\Delta(m,j)\delta_\Delta^m + w_\Delta(m,j-1) = w_\Delta(m+1,j)\,,$$

where (m,j) *is a vertex of the binomial lattice.*

Given this Lemma we can prove part (a) of Proposition 3.3. The proof is given by induction.
It is easy to see that the probability distribution for $Y_m^{(1)}$ given in the proposition is correct for $m=1$. Now assume that part (a) of Proposition 3.3 holds for all $m \le \bar{m}$ and consider

$$Y_{\bar{m}+1}^{(1)} = \sum_{k=0}^{\bar{m}} +y_k^{(1)}$$

then for $0 < j < \bar{m}+1$ we have

$$\begin{aligned}
P^p[Y_{\bar{m}+1}^{(1)} = j] &= P^p[Y_{\bar{m}}^{(1)} = j \wedge y_{\bar{m}}^{(1)} = 0] + P^p[Y_{\bar{m}}^{(1)} = j-1 \wedge y_{\bar{m}}^{(1)} = 1] \\
&= p^j(1-p)^{\bar{m}-j}\left(\prod_{i=1}^{\bar{m}} h_\Delta(i-1)\right) w_\Delta(\bar{m},j)\,(1-p)\,h_\Delta(\bar{m})\,\delta_\Delta^{\bar{m}} \\
&\quad + p^{j-1}(1-p)^{\bar{m}-(j-1)}\left(\prod_{i=1}^{\bar{m}} h_\Delta(i-1)\right) w_\Delta(\bar{m},j-1)\,p\,h_\Delta(\bar{m}) \\
&= p^j(1-p)^{\bar{m}+1-j}\left(\prod_{i=1}^{\bar{m}+1} h_\Delta(i-1)\right)[w_\Delta(\bar{m},j-1) \;+\; w_\Delta(\bar{m},j)\,\delta_\Delta^{\bar{m}}] \\
&\quad \text{and by the above lemma} \\
&= p^j(1-p)^{\bar{m}+1-j}\left(\prod_{i=1}^{\bar{m}+1} h_\Delta(i-1)\right) w_\Delta(\bar{m}+1,j)\,.
\end{aligned}$$

As the problem is trivial for $j=0$ or $j=\bar{m}+1$ we only have to prove the lemma to complete the proof of part (a) of Proposition 3.3.

Proof of Lemma A.1:
The problem being trivial if $j-1 \ge m$ or if $j=0$ we shall concentrate on the case where $0 < j \le m$. By definition we have

$$w_\Delta(m+1,j) = w_\Delta(m,j-1)\delta_\Delta^{m-(j-1)} + w_\Delta(m,j)\delta_\Delta^{m-j}\,.$$

So it is sufficient to show that

$$w_\Delta(m,j-1)\delta_\Delta^{m-(j-1)} + w_\Delta(m,j)\delta_\Delta^{m-j} = w_\Delta(m,j-1) + w_\Delta(m,j)\delta_\Delta^m\,.$$

The proof is given by induction. It is easy to see that for $m=0$ we have

$$w_\Delta(1,0) \;=\; w_\Delta(0,0)\delta_\Delta^0 + 0 \;=\; 0 + w_\Delta(0,0) \;=\; w_\Delta(1,1) \;=\; 1\,.$$

Now assume the lemma to hold for $w_\Delta(m,j)$, $m \le \bar{m}$. Then for $\bar{m}+1$ we have

$$w_\Delta(\bar{m},j-1) + w_\Delta(\bar{m},j)\,\delta_\Delta^{\bar{m}}$$

$$= w_\Delta(\bar{m},j-1) + \left(w_\Delta(\bar{m}-1,j-1)\,\delta_\Delta^{(\bar{m}-1)-(j-1)} + w_\Delta(\bar{m}-1,j)\,\delta_\Delta^{(\bar{m}-1)-j}\right)\delta_\Delta^{\bar{m}}$$

by the definition of $w_\Delta(\bar{m},j)$

$$= w_\Delta(\bar{m},j-1) + \Big(w_\Delta(\bar{m}-1,j-1)\,\delta_\Delta^{\bar{m}-j} - w_\Delta(\bar{m}-1,j-1)\,\delta_\Delta^{-j} + w_\Delta(\bar{m}-1,j-1)\,\delta_\Delta^{-j} + w_\Delta(\bar{m}-1,j)\,\delta_\Delta^{(\bar{m}-1)-j}\Big)\,\delta_\Delta^{\bar{m}}$$

$$= w_\Delta(\bar{m},j-1) + \Big(w_\Delta(\bar{m}-1,j-1)\,\delta_\Delta^{\bar{m}-j} - w_\Delta(\bar{m}-1,j-1)\,\delta_\Delta^{-j} + \left[w_\Delta(\bar{m}-1,j-1) + w_\Delta(\bar{m}-1,j)\,\delta_\Delta^{(\bar{m}-1)}\right]\delta_\Delta^{-j}\Big)\,\delta_\Delta^{\bar{m}}$$

$$= w_\Delta(\bar{m},j-1) + w_\Delta(\bar{m}-1,j-1)\,\delta_\Delta^{2\bar{m}-j} - w_\Delta(\bar{m}-1,j-1)\,\delta_\Delta^{\bar{m}-j} + w_\Delta(\bar{m},j)\,\delta_\Delta^{\bar{m}-j}$$

by the induction assumption

$$= w_\Delta(\bar{m}-1,j-2)\,\delta_\Delta^{(\bar{m}-1)-(j-2)} + w_\Delta(\bar{m}-1,j-1)\,\delta_\Delta^{(\bar{m}-1)-(j-1)} + w_\Delta(\bar{m}-1,j-1)\,\delta_\Delta^{2\bar{m}-j} - w_\Delta(\bar{m}-1,j-1)\,\delta_\Delta^{\bar{m}-j} + w_\Delta(\bar{m},j)\,\delta_\Delta^{\bar{m}-j}$$

by the definition of $w_\Delta(\bar{m},j-1)$

$$= \left(w_\Delta(\bar{m}-1,j-2) + w_\Delta(\bar{m}-1,j-1)\,\delta_\Delta^{\bar{m}-1}\right)\delta_\Delta^{\bar{m}-(j-1)} + w_\Delta(\bar{m},j)\,\delta_\Delta^{\bar{m}-j}$$

$$= w_\Delta(\bar{m},j-1)\,\delta_\Delta^{\bar{m}-(j-1)} + w_\Delta(\bar{m},j)\,\delta_\Delta^{\bar{m}-j}$$

by the induction assumption.

Together with the definition of $w_\Delta(\bar{m}+1,j)$ this proves the lemma and hence part (a) of Proposition 3.3.

The proof of part (b) of Proposition 3.3 is similar to the above. However, the induction is made over m^*. Notice also that nothing like the above lemma is needed because the definition of w_Δ can now be immediately applied. This completes the proof of Proposition 3.3.

B Proof of Proposition 3.5

It is sufficient to check the Liapunoff–Condition for $\delta = 1$ (see Bauer [Bau74], p.268).

PROOF OF PART (a) OF PROPOSITION 3.5:
The Liapunoff–Condition takes the following form:

$$\lim_{\Delta\to 0} L(\Delta) \;=\; \lim_{\Delta\to 0} \frac{\sum_{k=0}^{\tilde m-1} E^p[\,|y_k^{(1)} - p\,h_\Delta(k)|^3\,]}{\left(\sum_{k=0}^{\tilde m-1} V^p[\,y_k^{(1)}\,]\right)^{\frac{3}{2}}} \;=\; \lim_{\Delta\to 0} \frac{N(\Delta)}{D(\Delta)} \;\overset{!}{=}\; 0$$

$$\begin{aligned}
N(\Delta) \;&=\; \sum_{k=0}^{\tilde m-1} h_\Delta(k)\,[p\,(1-p\,h_\Delta(k))^3 + (1-p)\,\delta_1^{k\Delta^{\frac{3}{2}}}\,p^3\,h_\Delta^3(k)] \\
&\qquad \text{using that} \quad 1 \le h_\Delta(k) \le \frac{1}{p} \quad \text{and} \quad 0 \le \delta_1^{k\Delta^{\frac{3}{2}}} \le 1 \\
&\le\; \sum_{k=0}^{\tilde m-1} \frac{1}{p}\left[p\,(1-p)^3 \;+\; (1-p)\,p^3\,p^{-3}\right] \;\le\; \frac{\tilde m}{p}
\end{aligned}$$

$$\begin{aligned}
D(\Delta) \;&=\; \left(\sum_{k=0}^{\tilde m-1} p\,h_\Delta(k) \;-\; p^2 h_\Delta^2(k)\right)^{\frac{3}{2}} \\
&\qquad \text{using Taylor expansions of } h_\Delta \text{ and } h_\Delta^2 \text{ around } \bar\sigma = 0 \\
&=\; \left(\sum_{k=0}^{\tilde m-1}(p-p^2) \;+\; \underbrace{\sqrt{\frac{1-p}{p}}\sigma(p-2p^2)}_{=:\,a}\,\Delta^{\frac{3}{2}}\sum_{k=0}^{\tilde m-1} k \;+\; o(\Delta^{-\frac{1}{2}})\right)^{\frac{3}{2}} \\
&=\; \left(\tilde m(p-p^2) \;+\; a\Delta^{\frac{3}{2}}\frac{\tilde m(\tilde m-1)}{2} \;+\; o(\Delta^{-\frac{1}{2}})\right)^{\frac{3}{2}} \\
&=\; \tilde m^{\frac{3}{2}}\left((p-p^2) \;+\; o(1)\right)^{\frac{3}{2}}
\end{aligned}$$

Hence

$$\lim_{\Delta\to 0} L(\Delta) \;\le\; \lim_{\Delta\to 0} \frac{1/p}{(\tilde m)^{\frac{1}{2}}\,((p-p^2) \;+\; o(1))^{\frac{3}{2}}} \;=\; 0$$

PROOF OF PART (b) OF PROPOSITION 3.5:
The Liapunoff–Condition takes the following form:

$$\lim_{\Delta\to 0} L(\Delta) \;=\; \lim_{\Delta\to 0} \frac{\sum_{k=0}^{\tilde m-1} E^p[\,|\,y_k^{(2)} \;-\; p\,h_\Delta(\tilde m \;+\; \tilde n - (k+1))\,|^3\,]}{\left(\sum_{k=0}^{\tilde m-1} p\,h_\Delta(\tilde m \;+\; \tilde n \;-\; (k+1)) \;-\; p^2 h_\Delta^2(\tilde m \;+\; \tilde n \;-\; (k+1))\right)^{\frac{3}{2}}} \;\overset{!}{=}\; 0$$

By the same procedures as employed in the proof of part (a) of Proposition 3.5 it can be shown that this expression also converges to zero as Δ goes to zero.

This completes the proof of Proposition 3.5.

C Proof of Theorem 3.6

In view of equation (8) and Proposition 3.5 we need to calculate

$$\lim_{\Delta \to 0} \frac{a_\Delta \, - \, E^p\left[Y_{\tilde{m}}^{(k)}\right]}{\sqrt{V^p\left[Y_{\tilde{m}}^{(k)}\right]}} .$$

Let us first consider the case $k = 1$.

(a) We first deal with (-1) times the numerator of the above expression:

(i)

$$\begin{aligned}
&\ln\left\{\prod_{k=1}^{\tilde{m}} \frac{h_\Delta(\tilde{m}-k)}{h_\Delta(\tilde{m}+\tilde{n}-k)}\right\} \\
&= \left[\sum_{k=1}^{\tilde{m}} \left(\sqrt{\frac{1-p}{p}}\Delta^{\frac{3}{2}}(\tilde{m}-k)\sigma - \frac{1}{2}\Delta^3\left(\tilde{m}-k\right)^2\sigma^2\right)\right] - \\
&\quad \left[\sum_{k=1}^{\tilde{m}} \left(\sqrt{\frac{1-p}{p}}\Delta^{\frac{3}{2}}(\tilde{m}+\tilde{n}-k)\sigma - \frac{1}{2}\Delta^3(\tilde{m}+\tilde{n}-k)^2\sigma^2\right)\right] + o(1) \\
&= \Delta^{\frac{3}{2}}(1-p)\,\tilde{m}\,\tilde{n}\,(\ln\delta_1) + \frac{1}{2}((t+x)^2 t - (t+x)\,t^2)\,p\,(1-p)\,(\ln\delta_1)^2 + o(1)
\end{aligned}$$

(ii)

$$\begin{aligned}
&\Delta^{\frac{1}{2}} \sum_{k=0}^{\tilde{m}-1} p\, h_\Delta(k) \\
&= \Delta^{\frac{1}{2}} p \left[\sum_{k=0}^{\tilde{m}-1} \left(1 \, + \, \Delta^{\frac{3}{2}} k \sqrt{\frac{1-p}{p}}\,\sigma \, + \, \frac{1}{2}\Delta^3\, k^2\, \frac{1-2p}{p}\,\sigma^2\right) \, + \, o(1)\right] \\
&= \Delta^{\frac{3}{2}} p\, \tilde{m} \, - \, \frac{p\,(1-p)\,t^2(\ln\delta_1)}{2} \, + \, o(1)
\end{aligned}$$

(iii)

$$\begin{aligned}
&(i) \, - \, \Delta^{\frac{3}{2}}\tilde{m}\,\tilde{n}\,\ln\delta_1 \, + \, (ii)\cdot(\tilde{n}\,\ln\delta_1) \\
&= \frac{1}{2}\left((t+x)^2 t - (t+x)t^2\right) p(1-p)(\ln\delta_1)^2 \, - \, \frac{1}{2}\, x\, p\,(1-p)\,t^2\,(\ln\delta_1)^2 \, + \, o(1) \\
&= \frac{1}{2}(\ln\delta_1)^2 p\,(1-p)\left[(t+x)^2\, t \, - \, (t+x)\,t^2 \, - \, x\,t^2\right] \, + \, o(1) \\
&= \frac{1}{2}(\ln\delta_1)^2 p\,(1-p)\left[(t^2 \, + \, 2t\,x \, + \, x^2)\,t \, - \, t^3 \, - \, t^2 x \, - \, x\,t^2\right] \, + \, o(1) \\
&= \frac{1}{2}(\ln\delta_1)^2 p\,(1-p)\left[(2t^2\,x \, - \, 2t^2\,x \, + \, t\,x^2\right] \, + \, o(1) \\
&= \frac{1}{2}(\ln\delta_1)^2 p\,(1-p)\,t\,x^2 \, + \, o(1) \\
&= \frac{1}{2}\sigma^2 t\,x^2 \, + \, o(1)
\end{aligned}$$

Thus for (-1) times the complete numerator we have

$$\ln\left\{\frac{K\tilde{D}(0,\tilde{m},0)}{\tilde{D}(0,\tilde{m}+\tilde{n},0)}\right\} + \frac{1}{2}\,\sigma^2\,t\,x^2 + o(1)\,.$$

(b) The denominator:

$$\Delta^{\frac{3}{2}}\tilde{n}\ln\delta_1\left(\sum_{k=0}^{\tilde{m}-1}(p\,h_\Delta(k) - p^2 h_\Delta^2(k))\right)^{\frac{1}{2}}$$
$$= \Delta\tilde{n}\ln\delta_1\left(\Delta\sum_{k=0}^{\tilde{m}-1}(p-p^2) + o(1)\right)^{\frac{1}{2}} = \Delta\tilde{n}\ln\delta_1\Big(t\,p\,(1-p) + o(1)\Big)^{\frac{1}{2}}$$

(c) Hence we have for the case $k = 1$:

$$\lim_{\Delta\to 0}\frac{a_\Delta - E^p\left[Y_{\tilde{m}}^{(k)}\right]}{\sqrt{V^p\left[Y_{\tilde{m}}^{(k)}\right]}}$$
$$= \lim_{\Delta\to 0} -\frac{\ln\left\{\frac{K\tilde{D}(0,\tilde{m},0)}{\tilde{D}(0,\tilde{m}+\tilde{n},0)}\right\} + \frac{1}{2}\,\sigma^2\,t\,x^2 + o(1)}{\Delta\tilde{n}\ln\delta_1\Big(t\,p\,(1-p) + o(1)\Big)^{\frac{1}{2}}} = \frac{\ln\left\{\frac{KD(0,t)}{D(0,t+x)}\right\} + \frac{1}{2}\sigma^2 t\,x^2}{x\,t^{\frac{1}{2}}\,\sigma}$$

Let us now consider the case $k = 2$.
(a) Again we deal with (-1) times the numerator:
(i)

$$\Delta^{\frac{3}{2}}\,\tilde{n}\ln\delta_1\sum_{k=0}^{\tilde{m}-1}p\,h_\Delta(\tilde{m}+\tilde{n}-(k+1))$$
$$= \Delta\tilde{n}\ln\delta_1\left(p\,\tilde{m}\,\Delta^{\frac{1}{2}} - p(1-p)\left((t+x)\,t - \frac{1}{2}t^2\right)\ln\delta_1 + o(1)\right)$$
$$= \Delta^{\frac{3}{2}}p\,\tilde{m}\,\tilde{n}\ln\delta_1 - p\,(1-p)\,\Delta\tilde{n}\left((t+x)\,t - \frac{1}{2}t^2\right)(\ln\delta_1)^2 + o(1)$$

(ii)

$$\ln\left\{\prod_{k=1}^{\tilde{m}}\frac{h_\Delta(\tilde{m}-k)}{h_\Delta(\tilde{m}+\tilde{n}-k)}\right\} - \Delta^{\frac{3}{2}}\,\tilde{m}\,\tilde{n}\ln\delta_1 + (i)$$
$$= p\,(1-p)\,(\ln\delta_1)^2\left(\frac{1}{2}\left((t+x)^2\,t - (t+x)\,t^2\right) - x\,(t+x)\,t + \frac{1}{2}\,x\,t^2\right) + o(1)$$
$$= p\,(1-p)\,(\ln\delta_1)^2\left(\frac{1}{2}\,x\,t^2 + \frac{1}{2}\,x^2\,t - x\,t^2 - t\,x^2 + \frac{1}{2}\,x\,t^2\right) + o(1)$$
$$= p\,(1-p)\,(\ln\delta_1)^2\left(-\frac{1}{2}\,t\,x^2\right) + o(1)$$
$$= -\frac{1}{2}\sigma^2 t\,x^2 + o(1)$$

(b) The denominator:
Similar to the considerations for $k = 1$ we obtain:

$$\Delta^{\frac{3}{2}}\tilde{n}\ \ln\delta_1 \left(\sum_{k=0}^{\tilde{m}-1} (p\, h_\Delta(\tilde{m}+\tilde{n}-(k+1)) \ -\ p^2\, h_\Delta^2(\tilde{m}+\tilde{n}-(k+1))) \right)^{\frac{1}{2}}$$
$$= \ \Delta\tilde{n}(\ln\delta_1)\,(t\,p\,(1-p)\ +\ o(1))^{\frac{1}{2}}$$

(c) Hence we have for the case $k = 2$

$$\lim_{\Delta\to 0} \frac{a_\Delta \ -\ E^p\left[Y_{\tilde{m}}^{(k)}\right]}{\sqrt{V^p\left[Y_{\tilde{m}}^{(k)}\right]}}$$
$$= \ \lim_{\Delta\to 0}\ -\frac{\ln\left\{\frac{K\tilde{D}(0,\tilde{m},0)}{\tilde{D}(0,\tilde{m}+\tilde{n},0)}\right\}\ -\ \frac{1}{2}\,\sigma^2\,t\,x^2\ +\ o(1)}{\Delta\tilde{n}\Big(\ln\delta_1\Big)\Big(t\,p\,(1-p)\ -\ o(1)\Big)^{\frac{1}{2}}} \ = \ \frac{\ln\left\{\frac{KD(0,t)}{D(0,t+x)}\right\}\ -\ \frac{1}{2}\sigma^2 t\,x^2}{x\,t^{\frac{1}{2}}\sigma}$$

The theorem then follows from equation (8) and Proposition 3.5.

D Proof of Proposition 3.7

The continously compounded short rate of interest at time $\tilde{m}\Delta$ and state j is given by

$$r_\Delta(\tilde{m},1,j)$$
$$= \ -\frac{\ln\tilde{D}_\Delta(\tilde{m},1,j)}{\Delta}$$
$$= \ -\frac{1}{\Delta}\left[\ln\left\{\frac{\tilde{D}_\Delta(0,\tilde{m}+1,0)}{\tilde{D}_\Delta(0,\tilde{m},0)}\right\}\ +\ \ln h_\Delta(\tilde{m})\ +\ \ln\left\{\exp\left\{-\frac{\sigma\Delta^{\frac{3}{2}}(\tilde{m}-Y_{\tilde{m}}^{(1)})}{\sqrt{p(1-p)}}\right\}\right\}\right]$$
$$= \ -\frac{1}{\Delta}\left[\ln\left\{\frac{\tilde{D}_\Delta(0,\tilde{m}+1,0)}{\tilde{D}_\Delta(0,\tilde{m},0)}\right\}\ +\ \ln h_\Delta(\tilde{m})\ +\ \ln\left\{\exp\left\{-\frac{\sigma\Delta^{\frac{3}{2}}(\tilde{m}-E^p[Y_{\tilde{m}}^{(1)}])}{\sqrt{p(1-p)}}\right\}\right\}\right]$$
$$-\ln\left\{\exp\left\{-\frac{\sigma\Delta^{\frac{3}{2}}}{p(1-p)}\right\}\right\}\Big(Y_{\tilde{m}}^{(1)} - E^p[Y_{\tilde{m}}^{(1)}]\Big)$$
$$= \ -\frac{\ln\left\{\frac{\tilde{D}_\Delta(0,\tilde{m}+1,0)}{\tilde{D}_\Delta(0,\tilde{m},0)}\right\}}{\Delta}\ +\ \frac{1}{\Delta}\ln\left\{p\,\exp\left\{-\frac{\sigma\Delta^{\frac{3}{2}}(\tilde{m}-E^p[Y_{\tilde{m}}^{(1)}])}{\sqrt{p(1-p)}}\right\}\right.$$
$$\left.+\,(1-p)\exp\left\{-\frac{\sigma\Delta^{\frac{3}{2}}E^p[Y_{\tilde{m}}^{(1)}]}{\sqrt{p(1-p)}}\right\}\right\}$$
$$+\ \ln\left\{\exp\left\{-\frac{\sigma}{\sqrt{p(1-p)}}\right\}\right\}\Delta^{\frac{1}{2}}(Y_{\tilde{m}}^{(1)} - E^p[Y_{\tilde{m}}^{(1)}])$$

(i)

$$\lim_{\Delta\to 0} -\frac{\ln\left\{\frac{D_\Delta(0,\tilde m+1,0)}{D_\Delta(0,\tilde m,0)}\right\}}{\Delta} = r(0,t)$$

(ii)
For $\Delta^{\frac{1}{2}}\left(Y^{(1)}_{\tilde m} - E^p[Y^{(1)}_{\tilde m}]\right)$ the Liapunoff–condition takes the form

$$L(\Delta) \;=\; \frac{\sum_{k=0}^{\tilde m-1} E^p\left[|\Delta^{\frac{1}{2}}(y_k^{(1)} \,-\, p\,h_\Delta(k)|^3\right]}{\left(\sum_{k=0}^{\tilde m-1}\Delta V^p[y_k^{(1)}]\right)^{\frac{3}{2}}} = \frac{\sum_{k=0}^{\tilde m-1} E^p\left[|y_k^{(1)} - p\,h_\Delta(k)|^3\right]}{\left(\sum_{k=0}^{\tilde m-1} V^p[y_k^{(1)}]\right)^{\frac{3}{2}}}$$

which is equal to the Liapunoff–condition in Appendix B. So we can conclude that for $\Delta\to 0$

$$\frac{\Delta^{\frac{1}{2}}(Y^{(1)}_\Delta - E^p[Y^{(1)}_\Delta])}{V^p[\Delta^{\frac{1}{2}}Y^{(1)}_\Delta]}$$

will have a standard normal distribution.
However,

$$V^p\left[\Delta^{\frac{1}{2}}Y^{(1)}_{\tilde m}\right] = \Delta\left(\tilde m(p-p^2) \;+\; a\,\Delta^{\frac{3}{2}}\frac{\tilde m(\tilde m-1)}{2} + o(\Delta^{\frac{1}{2}})\right)\,,$$

which yields

$$\lim_{\Delta\to 0}\Delta\, V^p[Y^{(1)}_{\tilde m}] = t\,p\,(1-p).$$

So we can conclude that as $\Delta\to 0$

$$\Delta^{\frac{1}{2}}\left(Y^{(1)}_{\tilde m} \;-\; E^p[Y^{(1)}_{\tilde m}]\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,t\,p\,(1-p))$$

Since

$$\ln\left\{\exp\left\{-\frac{\sigma}{\sqrt{p(1-p)}}\right\}\right\} = \ln\,\delta$$

we can conclude that

$$\lim_{\Delta\to 0} V^p[r_\Delta(\tilde m,1,.)] = t(\ln\delta)^2 p(1-p) = t\sigma^2.$$

(iii)
Finally we have to show that

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\ln\left\{p\,\exp\left\{\frac{\sigma\Delta^{\frac{3}{2}}\left(\tilde m \;-\; E^p[Y^{(1)}_{\tilde m}]\right)}{\sqrt{p(1-p)}}\right\} \;+\; (1-p)\exp\left\{-\frac{\sigma\Delta^{\frac{3}{2}}E^p[Y^{(1)}_{\tilde m}]}{\sqrt{p(1-p)}}\right\}\right\} = 0$$

To do this we define

$$N(k) \;\;:=\;\; p+(1+p)\exp\left\{-\frac{\Delta^{\frac{3}{2}}k\sigma}{\sqrt{p(1-p)}}\right\}$$

$$\begin{aligned}
\sum &:= \sum_{k=0}^{\tilde{m}-1} \frac{p}{N(k)} \\
\exp\{A\} &:= \exp\left\{ \frac{\sigma \tilde{m} \Delta^{\frac{3}{2}} - \sigma \Delta^{\frac{3}{2}} \sum}{\sqrt{p(1-p)}} \right\} \\
\exp\{B\} &:= \exp\left\{ -\frac{\sigma \Delta^{\frac{3}{2}}}{\sqrt{p(1-p)}} \sum \right\} \\
\ln\{C\} &:= \ln\{p \exp\{A\} + (1-p)\exp\{B\}\}
\end{aligned}$$

Obviously we have

$$\begin{aligned}
N(k)|_{\sigma=0} &= 1 \\
\sum|_{\sigma=0} &= \tilde{m}p \\
\exp\{A\}|_{\sigma=0} &= 1 \\
\exp\{B\}|_{\sigma=0} &= 1 \\
\ln\{C\}|_{\sigma=0} &= 0 \\
C|_{\sigma=0} &= 1
\end{aligned}$$

With these definitions our problem can be expressed more succinctly as follows

$$\lim_{\Delta \to 0} \frac{1}{\Delta} \ln\{C\} \stackrel{!}{=} 0.$$

As we shall attack this problem by using a Taylor expansion of $\ln\{C\}$ around $\bar{\sigma} = 0$ we need a number of derivatives.
First order derivatives:

$$\begin{aligned}
\frac{\partial N(k)}{\partial \sigma} &= -\frac{(1-p)\Delta^{\frac{3}{2}} k}{\sqrt{p(1-p)}} \exp\left\{ -\frac{\Delta^{\frac{3}{2}} k \sigma}{\sqrt{p(1-p)}} \right\} \\
\frac{\partial \sum}{\partial \sigma} &= \sum_{k=0}^{\tilde{m}-1} -\frac{p \frac{\partial N(k)}{\partial \sigma}}{(N(k))^2} \\
\frac{\partial \exp\{A\}}{\partial \sigma} &= \frac{\tilde{m}\Delta^{\frac{3}{2}} - \Delta^{\frac{3}{2}} \sum - \sigma \Delta^{\frac{3}{2}} \frac{\partial \sum}{\partial \sigma}}{\sqrt{p(1-p)}} \exp\{A\} = \frac{\partial A}{\partial \sigma} \cdot \exp\{A\} \\
\frac{\partial \exp\{B\}}{\partial \sigma} &= \frac{-\Delta^{\frac{3}{2}} \sum - \sigma \Delta^{\frac{3}{2}} \frac{\partial \sum}{\partial \delta}}{\sqrt{p(1-p)}} \exp\{B\} = \frac{\partial B}{\partial \sigma} \cdot \exp\{B\} \\
\frac{\partial \ln\{C\}}{\partial \sigma} &= \frac{p \frac{\partial \exp\{A\}}{\partial \sigma} + (1-p) \frac{\partial \exp\{B\}}{\partial \sigma}}{C} = \frac{\frac{\partial C}{\partial \sigma}}{C}
\end{aligned}$$

We obtain:

$$\frac{\partial N(\tilde{h})}{\partial \sigma}|_{\sigma=0} = -\frac{(1-p)\Delta^{\frac{3}{2}} k}{\sqrt{p(1-p)}}$$

$$\frac{\partial \sum}{\partial \sigma}|_{\delta=0} = \sum_{k=0}^{\tilde{m}-1} \sqrt{p(1-p)}\Delta^{\frac{3}{2}} k$$

$$\frac{\partial \exp\{A\}}{\partial \sigma}|_{\sigma=0} = \frac{\tilde{m}\Delta^{\frac{3}{2}}(1-p)}{\sqrt{p(1-p)}}$$

$$\frac{\partial \exp\{B\}}{\partial \sigma}|_{\sigma=0} = \frac{-\tilde{m}\Delta^{\frac{3}{2}} p}{\sqrt{p(1-p)}}$$

$$\frac{\partial \ln\{C\}}{\partial \sigma}|_{\sigma=0} = 0\,,\ \frac{\partial C}{\partial \sigma}|_{\sigma=0} = 0$$

Second order derivatives:

$$\frac{\partial^2 N(k)}{\partial \sigma^2} = \frac{\Delta^3 k^2}{p} \exp\left\{-\frac{\Delta^{\frac{3}{2}}\sigma k}{\sqrt{p(1-p)}}\right\}$$

$$\frac{\partial^2 \sum}{\partial \sigma^2} = \sum_{k=0}^{\tilde{m}-1} \frac{-p\frac{\partial^2 N}{\partial \sigma^2} N^2 + 2p\left(\frac{\partial N}{\partial \sigma}\right)^2}{N^4}$$

$$\frac{\partial^2 \exp\{A\}}{\partial \sigma^2} = \frac{-\left[2\Delta^{\frac{3}{2}}\frac{\partial \sum}{\partial \sigma} + \sigma\Delta^{\frac{3}{2}}\frac{\partial^2 \sum}{\partial \sigma^2}\right]}{\sqrt{p(1-p)}} \exp\{A\} + \left(\frac{\partial A}{\partial \sigma}\right)^2 \exp\{A\}$$

$$= \frac{\partial^2 A}{\partial \sigma^2}\exp\{A\} + \left(\frac{\partial A}{\partial \sigma}\right)^2 \exp\{A\}$$

$$\frac{\partial^2 \exp\{B\}}{\partial \sigma^2} = \frac{-\left[2\Delta^{\frac{3}{2}}\frac{\partial \sum}{\partial \sigma} + \sigma\Delta^{\frac{3}{2}}\frac{\partial^2 \sum}{\partial \sigma^2}\right]}{\sqrt{p(1-p)}} \exp\{B\} + \left(\frac{\partial B}{\partial \sigma}\right)^2 \exp\{B\}$$

$$= \frac{\partial^2 B}{\partial \sigma^2}\exp\{B\} + \left(\frac{\partial B}{\partial \sigma}\right)^2 \exp\{B\}$$

$$\frac{\partial^2 \ln(C)}{\partial \sigma^2} = \frac{\frac{\partial^2 C}{\partial \sigma^2} C - \left(\frac{\partial C}{\partial \delta}\right)^2}{C^2} = \frac{\left[p\frac{\partial^2 \exp\{A\}}{\partial \sigma^2} + (1-p)\frac{\partial^2 \exp\{B\}}{\partial \sigma^2}\right] \cdot C - \left(\frac{\partial C}{\partial \sigma}\right)^2}{C^2}$$

We have:

$$\frac{\partial^2 N(k)}{\partial \sigma^2}|_{\sigma=0} = \frac{\Delta^3 k^2}{p} \quad ; \quad \frac{\partial^2 \sum}{\partial \sigma^2}|_{\sigma=0} = (-3+2p)\sum_{k=0}^{\tilde{m}-1} \Delta^3 k^2$$

$$\frac{\partial^2 \exp\{A\}}{\partial \sigma^2}|_{\sigma=0} = -2\Delta^3 \sum_{k=0}^{\tilde{m}-1} k + \frac{\tilde{m}^2\Delta^3(1-p)^2}{p(1-p)};$$

$$\frac{\partial^2 \exp\{B\}}{\partial\sigma^2}|_{\sigma=0} = -2\Delta^3 \sum_{k=0}^{\tilde{m}-1} k + \frac{\tilde{m}^2\Delta^3 p^2}{p(1-p)}$$

$$\begin{aligned}\frac{\partial^2 \ln\{C\}}{\partial\sigma^2}|_{\sigma=0} &= p\left(-2\Delta^3 \sum_{k=0}^{\tilde{m}-1} k + \frac{\tilde{m}^2\Delta^3(1-p)^2}{p(1-p)}\right) \\ &\quad +(1-p)\left(-2\Delta^3 \sum_{k=0}^{\tilde{m}-1} k + \frac{\tilde{m}^2\Delta^3 p^2}{p(1-p)}\right) \\ &= \tilde{m}\Delta^3\end{aligned}$$

Now we can transform our problem to give

$$\begin{aligned}\lim_{\Delta\to 0} \frac{1}{\Delta}\ln\{C\} &= \lim_{\Delta\to 0} \frac{1}{\Delta}\left[(\ln C)|_{\sigma=0} + \left(\frac{\partial \ln C}{\partial\sigma}\right)|_{\sigma=0}\cdot\sigma + \frac{1}{2}\frac{\partial^2 \ln C}{\partial\sigma^2}|_{\sigma=0}\cdot\sigma^2 + o(\Delta)\right] \\ &= \lim_{\Delta\to 0} \frac{1}{\Delta}\left[\frac{1}{2}\tilde{m}\Delta^3\sigma^2 + o(\Delta)\right] = 0\end{aligned}$$

Thus we have proved that as Δ goes to zero $r_\Delta(\tilde{m}, 1, .)$ converges in distribution to a normally distributed random variable with mean $r(0, t)$ and variance $t\sigma^2$.

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