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Closed Form Term Structure Derivatives in a Heath-Jarrow-Morton Model with Log-Normal Annually Compounded Interest Rates[1]

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Abstract

Starting with observable annually compounded forward rates we derive a term structure model of interest rates. The model relies upon the assumption that a specific set of annually compounded forward rates is log-normally distributed. We derive solutions for interest rate caps and floors as well as puts and calls written on zero-coupon bonds. In particular, for caplets with payment periods of same length as the compounding period (in our paper we have chosen one year, but it could be as well three or six months with quarterly or biannual compounding) we obtain the same Black formula as often used by market practioners, however, without making the unrealistic assumption that forward rates are independent of the accummulation process. Moreover, the log-normal assumption is shown to be consistent with the Heath-Jarrow-Morton model for a specific choice of volatility.

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1 Introduction

Closed-form solutions for interest rate derivatives, in particular caplets, bond options and swaptions, have been obtained by a number of authors for Markovian term structure models with normally distributed interest rates resp. log-normal bond prices, see e.g. El Karoui–Rochet (1989), Jamshidian (1990), Heath–Jarrow–Morton (1992) and Brace–Musiela (1993). These models support Black–Scholes type formulae most frequently used by practioners for pricing caplets, bond options and swaptions. Unfortunately these models imply negative interest rates with positive probability, and hence are not arbitrage free in an economy where money exists. Log-normal volatility structures avoid these problem of negative interest rates. However as shown by Morton (1988) and Hogan–Weintraub (1993) these rates explode with positive probability implying zero prices for bonds and hence also arbitrage opportunities. Furthermore so far no closed form solutions are known for these models.

As has been observed by Sandmann–Sondermann (1993) the problems of rate explosion result from modelling the "wrong" rate, namely the continuously compounded rate. Assuming that the continuous rate is log-normal results in "double exponential" expressions, i.e. where the exponential function is itself an argument of an exponential, thus giving rise to infinite expectations under the martingale measure. The problem disappears if, instead of the continuous rate, one assumes that the effective annual rates are log-normal. In practice interest rates, both spot and forward are quoted as rates per annum, even if the compounding period is different, e.g. three months. Effective annual rates are then used as benchmark for comparing nominal rates with different compounding periods. Hence these rates and their volatilities are directly observable in the market and form a natural starting point for modelling the term structure[2]. Assuming that the effective annual rates are log-normal, it follows that the continuously compounded short rate $r_c(t)$ follows a diffusion which is neither normal nor log-normal, but a dynamic combination of both with the following properties: for small values of $r_c(t)$ the diffusion process approaches a log-normal diffusion[3], thus generating positive rates, whereas for large values of $r_c(t)$ the diffusion approaches a normal diffusion process, generating stable finite expected returns and futures values. It thus combines – in a very simple and straightforward manner – the strengths of the normal and log-normal model and avoids their shortcomings. This type of dynamics of $r_c(t)$ has been supported by an independent empirical study by Miltersen (1993a).

The main result of this paper is the derivation of solutions for interest rate caps and floors as well as puts and calls written on zero-coupon bonds within the context of a log-normal interest rate model. In some specific cases these solutions are closed

[2]See also Goldys, Musiela and Sondermann (1994).

[3]Indeed, if the continuous rate $r_c(t)$ becomes infinitesimally small, i.e. $r_c(t) = 0(dt)$, then the two dynamics coincide, for further details see Sandmann-Sondermann (1993).

forms and coincide with modifications of the Black-Scholes formula. In particular, for caplets with payment periods of same length as the compounding period[4] we obtain the usual Black formula often used by market practioners, however, without making the unrealistic assumption that forward rates are independent of the accummulation process. Thus in this case our model supports market practice. Moreover, the log-normal assumption is shown to be consistent with the Heath-Jarrow-Morton model for a specific choice of volatility. Since the model imply non-negative interest rates with probability one, the model is, in addition, arbitrage free in an economy where money exists.

The paper is organized as follows. Section 2 contains the model and the relationship to the Heath-Jarrow-Morton framework. A discussion about the induced forward yield dynamics is given in Section 3. The solutions for interest rate derivatives are derived in Section 4. Appart from the proof of Proposition 1 the Appendix contains a further discussion about the situation within a log-normal multi-factor interest rate model.

2 A Model of Annually Compounded Forward Rates

By $r(t,x,\alpha)$ we will denote the annually compounded interest rate prevailing at time t for the future time interval $[t+x, t+x+\alpha]$. With α we fix the length of the time period and by x the forward starting point of the contract is specified. Note that $r(t,x,\alpha)$ are observable market rates. I.e. for $\alpha = 0.25$ and $x_i = i \cdot \alpha$ the sequence $\{r(t,x_i,\alpha)\}_{i=1}^{n}$ is a sequence of three month forward rate agreements valid at time t quoted on annually compounding basis. Due to the definition of $r(t,x,\alpha)$, as t moves, we observe this sequence of forward rate agreements as a time series as t moves. Furthermore, we can now define the relationships between on the one side zero-coupon bond prices $B(t,x)$, the continuously compounded forward rates $f(t,x)$, and the continuously compounded (forward) yields $y(t,x)$ or $y(t,x,\alpha)$ and on the other side to the annually compounded forward rate $r(t,x,\alpha)$.

a) Let $B(t,x+\alpha)$ be the price of a default free zero-coupon bond, at time t, with maturity $t+x+\alpha$ and face value 1. Per definition

$$B(t,x+\alpha) = B(t,x) \cdot (1 + r(t,x,\alpha))^{-\alpha}.$$

b) We define the continuously compounded forward rate prevailing at time t for the time $t+x$ as $f(t,x) = \lim_{\alpha\to 0} \ln(1 + r(t,x,\alpha))$. Since

$$\ln(1 + r(t,x,\alpha)) = -\frac{1}{\alpha}[\ln B(t,x+\alpha) - \ln B(t,x)], \quad (1)$$

implying that

$$f(t,x) = -\frac{\partial}{\partial x} \ln B(t,x). \quad (2)$$

[4] In the paper we have chosen one year, but it could be as well three or six months with quarterly or biannual compounding.

c) The continuously compounded yield $y(t,x)$, at time t, for the time interval $[t, t+x]$ is thus defined by

$$B(t,x) = \exp\{-x \cdot y(t,x)\} = \exp\left\{-\int_0^x f(t,\theta)d\theta\right\}$$

which imply

$$y(t,x) = \frac{1}{x}\int_0^x f(t,\theta)d\theta$$

and the continuously compounded forward yield $y(t,x,\alpha)$ is determined by

$$\frac{B(t,x+\alpha)}{B(t,x)} = \exp\left\{-\int_x^{x+\alpha} f(t,\theta)d\theta\right\} = \exp\{-\alpha y(t,x,\alpha)\} \qquad (3)$$

implying

$$y(t,x,\alpha) = \frac{1}{\alpha}\int_x^{x+\alpha} f(t,\theta)d\theta.$$

So far these relations have to hold per definition. Any stochastic assumption upon one of these instruments induces the stochastic behaviour of the remaining objects. The starting point of the Heath-Jarrow-Morton model is a stochastic description of the continuously compounded forward rates. The significance of this approach is that the consequences of no arbitrage can be made precise. In other words the Heath-Jarrow-Morton model is a brilliant theoretical framework for the modelling of the term structure of interest rates under no arbitrage. In this paper, we introduce the Heath-Jarrow-Morton model in order to enlighten the question of the existence of an equivalent martingal measure for the description of the interest rate market. In this sense the Heath-Jarrow-Morton model can be interpreted as the term structure of interest rates counter part to the Black-Scholes model in that both models describe the stochastic behaviour of the underlying security. Despite this consistency of the theoretical basis both modelling backgrounds are of different quality from an empirical point of view. Whereas the distributional assumption of the Black-Scholes model is relative to observable prices as stock prices, exchange rates, stock index notations, etc. The continuously compounded forward rates in the Heath-Jarrow-Morton model are not at first glance observable market data. The question remains why one should start with assumptions on the stochastic behaviour of continuously compounded rates since the empirical investigation of the plausability of these assumptions can only be analysed on the basis of observable prices or rates. However, the behaviour of these observable prices or rates is endogenously determined by the model of the continuously compounded forward rate process. That is, there is only an indirect relation between the assumptions and the empirically observed behaviour of the model. The question is, is this relation one-to-one? The idea of the paper is in a sense to turn the Heath-Jarrow-Morton model upside-down. More precisely, starting with exogenously given behaviour of annually compounded forward rates we want to characterize the endogeneous Heath-Jarrow-Morton model supporting this structure. The building block of the model is a family of annually compounded forward rates. Each annually compounded forward rate is determined by the two parameters (x,α). The market-data prevailing at time t is thus defined by the following two dimensional finite set

$$\Gamma = \{(x_1, \alpha_1), \ldots, (x_k, \alpha_k)\}$$

where for each $(x, \alpha) \in \Gamma$ there exists an empirically observable annually compounded forward rate with starting point at time $t + x$ and maturity $t + x + \alpha$.

In general, this set can change over time but for simplicity we assume that it is invariant with respect to time. Furthermore, we assume that for each $(x, \alpha) \in \Gamma$ the annually compounded forward rate process $r(t, x, \alpha)$ follows a log-normal diffusion process, i.e.,

$$dr(t, x, \alpha) = \mu(t, x, \alpha, r)dt + \gamma(t, x, \alpha)r(t, x, \alpha)dW_{x,\alpha}, \tag{4}$$

where the instantaneous drift and volatility are such that there exists a strong unique solution of the stochastic differential equation which is log-normally distributed.[5] By $W_{x,\alpha}(t)$ we denote a standard Wiener process which could be dependent on (x, α). For the moment there are no restrictions on the set Γ, but obviously given the distributional assumptions on the annually compounded forward rates the no arbitrage condition will imply restrictions on the set Γ, i.e. on the number and structure of elements of Γ which can be exogenously considered. First of all we want to consider the consequences of the assumptions implied by Equation (4) on the dynamics of the continuously compounded forward rate process $f(t, x)$, if we assume that $W_{x,\alpha}(t) = W(t)$ is independent of (x, α).

From Equation (3) we know that

$$(1 + r(t, x, \alpha))^{-\alpha} = \exp\left\{-\int_x^{x+\alpha} f(t, \theta)d\theta\right\} = \exp\{-\alpha y(t, x, \alpha)\}.$$

Setting $y(t) = y(t, x, \alpha) = \frac{1}{\alpha}\int_x^{x+\alpha} f(t, \theta)d\theta$, Itô's lemma gives

$$dr(t, x, \alpha) = d\left(e^{y(t)} - 1\right) = e^{y(t)}\left(dy(t) + \frac{1}{2}d\langle y\rangle(t)\right). \tag{5}$$

From Musiela and Sondermann [1993] we know how to specify the drift term of the dynamics of any zero-coupon bond $B(t, x)$ under the risk neutral measure. This is given by

$$dB(t, x) = B(t, x)\left[(f(t, 0) - f(t, x))dt - \tau(t, x)dW(t)\right], \tag{6}$$

where τ is a given return volatility of the zero-coupon bond. Furthermore, Equation (2) and (6) and Itô's lemma imply

$$df(t, x) = \frac{\partial}{\partial x}\left[(f(t, x) + \frac{1}{2}\tau^2(t, x))dt + \tau(t, x)dW(t)\right] \tag{7}$$

where $\frac{\partial \tau(t,x)}{\partial x}$ determines the volatility function of the continuously compounded forward rate $f(t, x)$. We can now derive

$$dy(t) = \frac{1}{\alpha}\int_x^{x+\alpha} df(t, \theta)d\theta$$

[5] This implies that $\gamma(t, x, \alpha)$ is deterministic and it also puts strong conditions on $\mu(t, x, \alpha, r)$. E.g. deterministic or linear in r works.

$$
\begin{aligned}
&= \frac{1}{\alpha}\int_x^{x+\alpha} \frac{\partial}{\partial\theta}\left[(f(t,\theta)+\frac{1}{2}\tau^2(t,\theta))dt + \tau(t,\theta)dW(t)\right]d\theta \\
&= \frac{1}{\alpha}\left[f(t,x+\alpha)-f(t,x)+\frac{1}{2}(\tau^2(t,x+\alpha)-\tau^2(t,x))\right]dt \\
&\quad +\frac{1}{\alpha}\left[\tau(t,x+\alpha)-\tau(t,x)\right]dW(t) \\
d\langle y\rangle(t) &= \frac{1}{\alpha^2}\left[\tau(t,x+\alpha)-\tau(t,x)\right]^2 dt
\end{aligned}
$$

Define $\gamma(t,x,\alpha)$ as the exogeneously given volatility of the annually compounded forward rate process $r(t,x,\alpha)$. We can then conclude by matching drift terms of the quadratic variation processes that the function $\tau(t,x)$ must satisfy the condition:

$$
\begin{aligned}
d\langle r(\cdot,x,\alpha)\rangle(t) &= \gamma^2(t,x,\alpha)\cdot r^2(t,x,\alpha)dt \\
&= \frac{1}{\alpha^2}\left(e^{y(t)}\right)^2(\tau(t,x+\alpha)-\tau(t,x))^2 dt \\
&= \left(e^{y(t)}\right)^2 d\langle y\rangle(t)
\end{aligned}
$$

This yields the following necessary condition for the Heath-Jarrow-Morton term structure model that supports the assumption of a log-normally distributed annually compounded forward rate $r(t,x,\alpha)$:

$$
\begin{aligned}
\frac{1}{\alpha}(\tau(t,x+\alpha)-\tau(t,x)) &= \left(1-\exp\left\{-\frac{1}{\alpha}\int_x^{x+\alpha} f(t,\theta)d\theta\right\}\right)\gamma(t,x,\alpha) \qquad (8) \\
&= (1-\exp\{-y(t,x,\alpha)\})\,\gamma(t,x,\alpha).
\end{aligned}
$$

For $\alpha = 0$ the case is elementary because we have the following simple relation between $r(t,x,0)$ and $f(t,x)$:

$$1 + r(t,x,0) = e^{f(t,x)},$$

implying

$$
\begin{aligned}
\sigma(t,x) &= \frac{\partial\tau(t,x)}{\partial x} = \lim_{\alpha\to 0}\frac{1}{\alpha}\left[\tau(t,x+\alpha)-\tau(t,x)\right] \qquad (9) \\
&= (1-\exp\{-f(t,x)\})\cdot\lim_{\alpha\to 0}\gamma(t,x,\alpha)
\end{aligned}
$$

where $\sigma(t,x)$ is the volatility function of the corresponding Heath-Jarrow-Morton model. From Morton (1988) we know that the stochastic differential equation describing the continuously compounded forward rate process of the Heath-Jarrow-Morton model, $f(t,x)$, has a unique strong solution with this volatility function. Moreover this volatility function fulfills the conditions in Miltersen (1993b) which implies that the continuously compounded forward rates are positive. This case of $\alpha = 0$ is also the subject in Goldys, Musiela, and Sondermann (1994)[6].

[6] For $\alpha > 0$ the case is not so simple. And we have not so far been able to find the volatility of the underlying supporting Heath-Jarrow-Morton model. We will hopefully be able to solve this problem soon.

3 Dynamics of the forward price process

Let us denote by $F(t,T,\alpha)$ the forward price, at time $t \le T$, of a zero coupon bond for delivery at time T which pays 1 \$ at time $T+\alpha$. No arbitrage implies

$$F(t,T,\alpha) := \frac{B(t,T-t+\alpha)}{B(t,T-t)} = \exp\left\{ -\int_{T-t}^{T-t+\alpha} f(t,\theta)d\theta \right\}. \qquad (10)$$

In contrast to Section 2, we are now considering the situation where the time to maturity is not fixed, i.e. if time t moves the time to maturity $T-t$ gets smaller. The reason is when looking at interest rate contingent claims as e.g. caps and floors we have to consider hedging strategies. As usual these hedging strategies consist of a self-financing portfolio which dublicates a given contingent claim. Therefore the dynamics of the portfolio value is only dependent on the dynamics of the involved interest rate securities. Under no-arbitrage the initial value of the portfolio, at time t_0, coincides with the arbitrage price of the contingent claim which is duplicated by this portfolio strategy. Thus the stochastic processes of the underlying interest rate securities determines the arbitrage price of any redundant contingent claim.

In line with the assumptions in Section 2, we assume that there exists an effective rate $r(t,T-t,\alpha)$ which satisfies the log-normal assumption (3) such that

$$F(t,T,\alpha) = (1+r(t,T-t,\alpha))^{-\alpha}.$$

In other words, we assume that $(T-t,\alpha)\in\Gamma \quad \forall t \le T$. As a consequence the set of observable effective rates is now time dependent on the whole interval. Setting

$$y(t) = y(t,T-t,\alpha) := \frac{1}{\alpha}\int_{T-t}^{T-t+\alpha} f(t,\theta)d\theta = \frac{1}{\alpha}\int_0^{\alpha} f(t,T-t+\theta)d\theta$$

we can compute the dynamic behaviour of the forward price process in analogy to Section 2 by:

$$\begin{aligned} dF(t,T,\alpha) &= d\,(1+r(t,T-t,\alpha))^{-\alpha} = d\left[\ \exp\{-\alpha y(t)\}\ \right] \qquad (11)\\ &= -\alpha \exp\{-\alpha y(t)\}\left[\ dy(t) - \frac{\alpha}{2} d\langle y\rangle(t)\ \right] \\ &= -\alpha\ F(t,T,\alpha)\left[\ dy(t) - \frac{\alpha}{2}d\langle y\rangle(t)\ \right]. \end{aligned}$$

Furthermore, we know that

$$\begin{aligned} & dy(t) = \frac{1}{\alpha}\int_0^{\alpha} df(t,T-t+\theta)d\theta \\ =\ & \frac{1}{\alpha}\int_0^{\alpha}\frac{\partial}{\partial\theta}\left[(f(t,T-t+\theta)+\frac{1}{2}\tau^2(t,T-t+\theta))dt + \tau(t,T-t+\theta)dW(t)\right]d\theta \\ =\ & \frac{1}{\alpha}\left[f(t,T-t+\alpha) - f(t,T-t) + \frac{1}{2}(\tau^2(t,T-t+\alpha) - \tau^2(t,T-t))\right]dt \\ & + \frac{1}{\alpha}\left[\tau(t,T-t+\alpha) - \tau(t,T-t)\right]dW(t) \end{aligned}$$

$$\begin{aligned} d\langle y\rangle(t) &= \frac{1}{\alpha^2}\left[\tau(t,T-t+\alpha)-\tau(t,T-t)\right]^2 dt \\ &= \left(1-\exp\{-y(t,T-t,\alpha)\}\right)^2 \gamma^2(t,T-t,\alpha) \end{aligned}$$

Given the log-normal assumption for the effective rate per annum $r(t, T-t, \alpha)$, the stochastic process of the forward price is determined by the above equations. In particular the process of the quadratic variation of the forward price is given by:

$$d\langle F\rangle(t,T,\alpha) = \alpha^2 F^2 \left(1-F^{\frac{1}{\alpha}}\right)^2 \gamma^2(t,T-t,\alpha) \quad dt \quad . \tag{12}$$

We can conclude that the resulting volatility $\alpha\left(1-F^{\frac{1}{\alpha}}\right)\gamma(t,T-t,\alpha)$ of the forward price process is state and time dependent.

4 Pricing of interest rate derivatives

In this section we focus on the arbitrage price of interest rate derivatives. More precisely we consider two special interest rate derivatives: interest rate caps and floors and European type debt options where the underlying security is a zero coupon bond. Since the construction of the underlying term structure model is very closely related to the Black-Scholes model, we should expect similar pricing formulae for these derivatives within our model. Nevertheless we show that only in very special situations modifications of known closed form solutions for the pricing of interest rate derivatives are supported by this model. In most circumstances numeric methods are needed to compute the arbitrage price for these interest rate derivatives.

Caps and floors are special types of options where a nominal interest rate is the underlying security. The underlying interest rate could be for example the 3- or 6-month LIBOR. A cap is an insurance against upward moving interest rate and a floor against downward moving interest rate. Let L be a nominal interest rate with compounding period α, i.e. for $\alpha = 0.25$ the 3-month LIBOR. Let $\underline{\underline{T}} = \{0 = t_0 < t_1 < \cdots < t_N\}$ be a set of times such that $\alpha = t_{i+1} - t_i \quad \forall i = 0, \cdots, N-1$. Fix now some date $t_i \in \underline{\underline{T}}$, then a cap contract with level L, face value V, underlying nominal interest rate $\tilde{r}$ and payment dates $\underline{\underline{T}} \setminus \{t_0\}$ is defined by the payoff at all times $t_i \in \underline{\underline{T}} \setminus \{t_0\}$:

$$V\alpha\left[\tilde{r}(t_{i-1}) - L\right]^+ = V\alpha \max\left\{\tilde{r}(t_{i-1}) - L, 0\right\}.$$

Here we denote with $\tilde{r}(t)$ the nominal interest rate valid at t with compounding period of length α. Clearly the relation between $\tilde{r}(t)$ and the effective rate $r(t, 0, \alpha)$ is given by

$$(1 + \alpha\tilde{r}(t))^{\frac{1}{\alpha}} = 1 + r(t, 0, \alpha).$$

Since the nominal rate of the underlying interest rate $\tilde{r}(t_{i-1})$ is known at time t_{i-1} the present value of this payoff at time t_{i-1} is equal to:

$$B(t_{i-1}, \alpha) V\alpha\left[\tilde{r}(t_{i-1}) - L\right]^+$$

$$
\begin{aligned}
&= B(t_{i-1},\alpha)V\left[1+\alpha\tilde{r}(t_{i-1})-(1+\alpha L)\right]^{+} \\
&= V\left[\quad 1-\frac{1+\alpha L}{(1+r(t_{i-1},0,\alpha))^{\alpha}}\quad\right]^{+} \qquad (13)\\
&= V(1+\alpha L)\left[\quad\frac{1}{1+\alpha L}-\frac{1}{(1+r(t_{i-1},0,\alpha))^{\alpha}}\quad\right]^{+}
\end{aligned}
$$

where under the no-arbitrage assumption and the absence of transaction costs we have

$$B(t_{i-1},\alpha)\left(1+\alpha\tilde{r}(t_{i-1})\right)=1 \quad .$$

Since a cap pays at each time $t_i \in \underline{\underline{T}}\backslash\{t_0\}$, the total payment is a linear combination of these so-called caplets with payoff given by (13). The floor is just the opposite contract. At each time $t_i \in \underline{\underline{T}} \setminus \{t_0\}$ the payoff is defined by:

$$V\alpha\left[L-\tilde{r}(t_{i-1})\right]^{+}$$

and the present value at time t_{i-1} is determined by:

$$B(t_{i-1},\alpha)V\alpha\left[L-\tilde{r}(t_{i-1})\right]^{+}=V(1+\alpha L)\left[\frac{1}{(1+r(t_{i-1},0,\alpha))^{\alpha}}-\frac{1}{1+\alpha L}\quad\right]^{+} \quad . \qquad (14)$$

Therefore the payoff of a cap resp. a floor at each time t_{i-1} is equivalent to $V(1+\alpha L)$ times that of a European type put option resp. a call option with exercise date t_{i-1}, exercise price $K=\frac{1}{1+\alpha L}$ where the underlying security is a zero coupon bond with maturity $t_{i-1}+\alpha$. Thus the arbitrage price of a cap or a floor is equal to the arbitrage price of a portfolio formed by european type put resp. call options. Furthermore focusing on one payment date the time t_{i-1} the forward arbitrage price of the payoff (13) resp. (14) is completely determined by the stochastic process of the forward price

$$\frac{B(t,t_{i-1}-t+\alpha)}{B(t,t_{i-1}-t)}=\frac{1}{(1+r(t,t_{i-1}-t,\alpha))^{\alpha}}=F(t,t_{i-1},\alpha) \quad .$$

The same argument is also valid for european type debt options on zero coupon or coupon bonds[7], i.e. the arbitrage price of these contracts is determinded by the forward price process. This has now a very useful consequence with respect to those portfolio strategies which are self-financing and duplicate a given contingent claim. As already shown by Müller (1985) a portfolio strategy is self-financing and duplicates a given contingent claim on the spot market if and only if it is self-financing and duplicates this contingent claim on the forward market. In other words we are free to choose whether we consider the spot or forward market. From these observations we can now compute the arbitrage price of any contingent claim which depends only on some forward price processes $F(t,T,\alpha)$. As already proved by Rady and Sandmann in the context of european type debt options the arbitrage price on the forward market is characterized by the following theorem:

[7] see e.g. Rady, Sandmann (1994)

Theorem 1:
Suppose the process of the quadratic variation of the forward price process $F(t,T,\alpha)$ is given by a function $\nu(t,z)$ on $[0,T]\times[0,1]$, i.e.

$$d\langle F\rangle(t,T,\alpha)=\nu^2(t,F(t,T,\alpha))dt \quad .$$

Let $u(t,z)$ be a continuous function on $[0,T]\times[0,1]$ and a solution of the partial differential equation

$$0=u_t(t,z)+\frac{1}{2}\nu^2(t,z)u_{z,z}(t,z) \qquad (15)$$

on $[0,T]\times]0,1[$. Then, the portfolio strategy $\phi=(\phi^1,\phi^2)$ on the forward market defined by

$$\phi^1(t)=u_z(t,F(t,T,\alpha)) \quad \text{and} \quad \phi^2(t)=u(t,F(t,T,\alpha))-u_z(t,F(t,T,\alpha))F(t,T,\alpha)$$

is self-financing.

a) *Moreover, suppose u has a terminal value $u(T,x)=A[K-x]^+$ for some $A>0$ and satisfies $0\le u(t,x)\le AK$, then for $A=1$ the portfolio strategy ϕ duplicates a European put option with exercise price K, exercise date T and underlying zero coupon bond $B(t,T-t+\alpha)$, resp. for $A=V(1+\alpha L)$ the time T present value of the cap payment at time $T+\alpha$ with face value V, cap level $L=\frac{1-K}{\alpha K}$ and underlying nominal interest rate $\tilde{r}(t,T,\alpha)$.*

b) *If u has a terminal value $u(T,x)=A[x-K]^+$ for some $A>0$ and satisfies $A[x-K]^+\le u(t,x)\le A\min\{x,1-K\}$, then for $A=1$ the portfolio strategy ϕ duplicates a European call option with exercise price K, exercise date T and underlying zero coupon bond $B(t,T-t+\alpha)$, resp. for $A=V(1+\alpha L)$ the time T present value of the floor payment at time $T+\alpha$ with face value V, cap level $L=\frac{1-K}{\alpha K}$ and underlying nominal interest rate $\tilde{r}(t,T,\alpha)$.*

Proof of theorem 1:
The definition of the portfolio strategy $\phi=\phi^1,\phi^2$ implies that the value process of the portfolio on the forward market satisfies

$$\begin{aligned} V(\phi(t)) &= \phi^1(t)\frac{B(t,T-t+\alpha)}{B(t,T-t)}+\phi^2(t) \\ &= \phi^1(t)F(t,T,\alpha)+\phi^2(t)=u(t,F(t,T,\alpha)) \quad . \end{aligned}$$

By Ito's lemma and (15) this implies:

$$\begin{aligned} dV(\phi(t)) &= \left[u_t(t,F(t,T,\alpha))+\frac{1}{2}\nu^2(t,F(t,T,\alpha))u_{x,x}(t,F(t,T,\alpha))\right]dt \\ &\quad +u_x(t,F(t,T,\alpha))dF(t,T,\alpha) \\ &= u_x(t,F(t,T,\alpha))dF(t,T,\alpha) \\ &= \phi^1(t)dF(t,T,\alpha)=\phi^1(t)\left(d\frac{B(t,T-t+\alpha)}{B(t,T-t)}\right) \end{aligned}$$

Thus the portfolio strategy is self-financing on the forward market. By the usual no-arbitrage argument the forward price of a contingent claim with terminal payoff $u(T, F(T, T, \alpha))$ at time T is therefore equal to $u(t, F(t, T, \alpha))$ whereas the spot arbitrage price of this contingent claim is given by $B(t, T-t)u(t, F(t, T, \alpha))$. The restriction on the function $u(t, x)$ given in a) resp. b) is due to the fact that the state space of the forward price process is equal to $[0, 1]$ given the log-normal distribution assumption on $r(t, x, \alpha) \forall (x, \alpha) \in \Gamma$.

□

To compute the arbitrage price of interest rate derivatives we usually have to compute in the first place the entire term structure curve. This determines then the state space, i.e. the drift process of the underlying factors is endogeneously determined with respect to the initial term structure of interest rates and the model assumptions. This procedure involves in most cases numeric techniques which can be very time consuming. With thereom 1 the problem to price a interest rate contingent claim within the framework of a term structure model is reduced to the solution of the partial differential equation (15) with the appropiate endvalue condition. More precisely the question whether we have to use the entire term structure model to derive the arbitrage price of a specific contingent claim or not depends on two circumstances:

- First, whether the payoff of the contract depends on one interest rate instrument or on some set of the entire term structure. In the case of caps, floors or European type call and put options only a specific forward price process determines the arbitrage price. Independent of the precise log-normal distribution assumption we therefore cannot expect that the arbitrage price of European type debt options is dependent on the entire term structure model as long as the volatility process of each of the continuously compounded forward rate is independent of the other forward rates. Beside the stochastic process of the forward price, the remaining information contained in the term structure is without any effect for the pricing of a debt option under this structural form of the volatlity. The difference between the so-called indirect approach via a model of the term structure of interest rates and the direct approach, i.e. a model of the price processes of zero coupon bonds is thus without consequence for the pricing of European type debt options. If we consider e.g. a term structure model with normal distributed continuously compounded forward rates as generated in the limit by the Ho-Lee model (1986), then the arbitrage price of a European type call or put option where the underlying security is a zero coupon bond is exactly equal to the duration model of Kemna, de Munik and Vorst (1989).

- Secondly, whether the involved forward rate is a factor of the model or not, i.e. an element of the exogeneously given set Γ. The relevance of this restriction is not independent of the distribution assumption. In the case of normal distributed continuously compounded forward rates this has no consequences. If instead the distributional assumption is imposed on effective forward rates then the set Γ is of relevance. As shown in the appendix the log-normal distribution assumption is not satisfied for those processes which can be duplicated by elements in Γ. The process of such rates has to be determined within the term structure model and is endogeneously determined. The arbitrage price of contingent claims depending on such forward rates is influenced by at least a part of the term stucture. This is also valid if we assume normal distributed effective forward rates instead of the log-normal distribution.

Given these remarks we can now consider the arbitrage price of caps, floors and European call and put options under the assumption of log-normal distributed effective forward rates. If the underlying forward price process is not associated with one element in the set of observable effective forward rate processes per annum, Γ, the volatility function $\nu(.,.)$ is a function of some elements of the set Γ. In general, we cannot expect to go far beyond the statement of theorem 1. If instead the effective forward rate process is associated with one effective rate within Γ, the log-normal distribution assumption implies that the arbitrage price of the contingent claim is a solution of the partial differential equation (15), i.e.

$$0 = u_t(t,F) + \frac{1}{2}\alpha^2 F^2 \left(1 - F^{\frac{1}{\alpha}}\right)^2 u_{F,F}(t,F)\gamma^2(t,T-t,\alpha) \tag{16}$$

subject to some terminal value condition $u(T,F)$. In the special situation where $\alpha = 1$ we can now derive closed form solutions for some interest rate contingent claims.

Proposition 1:
Suppose that for the forward price process given by $\frac{B(t,T-t+1)}{B(t,T-t)} = \frac{1}{(1+r(t,T-t,1)}$ *the effective forward rate per annum* $r(t,T-t,1)$ *is an element of* Γ *then the arbitrage price of a European call option with exercise price* K*, exercise time* T *and underlying zero coupon bond* $B(t,T-t+1)$ *is equal to:*

$$\begin{aligned} & \mathit{Call}[t, B(t,T-t+1), B(t,T-t), T, K] \qquad (17) \\ = \; & B(t,T-t+1)\cdot N(e_1) - K\cdot B(t,T-t)\cdot N(e_2) \\ & -K\cdot B(t,T-t+1)\cdot (N(e_1)-N(e_2)) \end{aligned}$$

with:

$$\begin{aligned} e_{1\backslash 2} &= \frac{1}{s(t,T,1)}\left(\ln\left(\frac{B(t,T-t+1)(1-K)}{(B(t,T-t)-B(t,T-t+1))K}\right) \pm \frac{s^2(t,T,1)}{2}\right) \\ s^2(t,T,1) &= \int_t^T \gamma^2(\theta,T-\theta,1)d\theta \quad , \end{aligned}$$

where $N(.)$ *denotes the standard normal distribution.*

Within the context of a bond price based model the closed form solution was first derived by Käsler (1991). A discussion of this model relative to other bond price based models can be found in Rady and Sandmann (1994). The proof of proposition 1 follows the presentation in Rady and Sandmann[8]. We can now apply proposition 1 to the pricing of interest rate caps and floors.

Proposition 2:
Let $\underline{\underline{T}} = \{0 \leq t_0 < t_1 < \cdots < t_N\}$ be a set of times with $t_{i+1} - t_i = \alpha = 1 \quad \forall i = 0, \ldots, N-1$. Suppose that $(t_i - t, 1) \in \Gamma \quad \forall t \leq t_i \in \underline{\underline{T}} \setminus \{t_0\}$, i.e. the effective forward rates $r(t, t_i - t, 1)$ are log-normal distributed, then the arbitrage price of an interest rate cap at time $t \leq t_0$ with yearly payment dates $t_i \in \underline{\underline{T}} \setminus \{t_0\}$, face value V and interest rate level L is given by:

$$\begin{aligned} & Cap[t, L, V, \underline{\underline{T}} \setminus \{t_0\}] \\ = \;& V \cdot \sum_{i=0}^{N-1} B(t, t_{i+1} - t) \left[r(t, t_i - t, 1) \cdot N(d_1(t, t_i, 1)) - L \cdot N(d_2(t, t_i, 1)) \right] \end{aligned} \tag{18}$$

with:

$$\begin{aligned} d_{1\backslash 2}(t, t_i, 1) &= \frac{1}{s(t, t_i, 1)} \left(\ln \left(\frac{r(t, t_i - t, 1)}{L} \right) \pm \frac{s^2(t, t_i, 1)}{2} \right) \\ s^2(t, t_i, 1) &= \int_t^{t_i} \gamma^2(\theta, t_i - \theta, 1) d\theta \quad , \end{aligned}$$

where $N(.)$ denotes the standard normal distribution.

Proof of proposition 2:
Under the assumptions of proposition 2 we can consider each payment of the cap contract separately. With (13) each cap payment is equivalent to $V(1+L)$ times the payoff of a European put option where the underlying security is a special zero coupon bond. Consider e.g. a caplet with payment at time t_{i+1}. The time t_i present value of this payoff can be written as:

$$V(1+L) = \left[\frac{1}{1+L} - \frac{1}{1 + r(t, 0, 1)} \right]^+$$

which is equal to $V(1+L)$ times the payoff of a European put option with exercise price $K = \frac{1}{1+L}$, exercise date t_i and underlying zero coupon bond $B(t, t_{i+1} - t) = B(t, t_i - t + 1) = B(t, t_i - t)\left(1 + r(t, t_i - t, 1)\right)^{-1}$. By proposition 1 the arbitrage price of this put option is equal to:

$$\begin{aligned} & V(1+L) \left[K B(t, t_i - t) N(-e_2) - B(t, t_{i+1} - t) N(-e_1) \right. \\ & \left. - K B(t, t_{i+1} - t) \left(N(e_1) - N(e_2) \right) \right] \\ = \;& V B(t, t_{i+1} - t) \left[\left(1 + r(t, t_i - t, 1) \right) N(-e_2) - (1+L) N(-e_1) - \left(N(e_1) - N(e_2) \right) \right] \\ = \;& V B(t, t_{i+1} - t) \left[r(t, t_i - t, 1) N(-e_2) - L N(-e_1) \right] \end{aligned}$$

where

$$-e_{1\backslash 2} = \frac{-1}{s(t, t_i, 1)} \left(\ln \left(\frac{B(t, t_{i+1} - t)(1-K)}{(B(t, t_i - t) - B(t, t_{i+1} - t))K} \right) \pm \frac{s^2(t, t_i, 1)}{2} \right)$$

[8] For completeness we give the outline of the proof and the corresponding formula for the European put option in the appendix.

$$
\begin{aligned}
&= \frac{-1}{s(t,t_i,1)}\left(\ln\left(\frac{1-\frac{1}{1+L}}{r(t,t_i-t,1)\frac{1}{1+L}}\right) \pm \frac{s^2(t,t_i,1)}{2}\right) \\
&= \frac{1}{s(t,t_i,1)}\left(\ln\left(\frac{r(t,t_i-t,1)}{L}\right) \mp \frac{s^2(t,t_i,1)}{2}\right) \\
&= d_{2\setminus 1} \quad .
\end{aligned}
$$

By summing up this yields the pricing formular for a cap.

□

The pricing formula (18) for a cap[9] is a modifications of the Black-Scholes formula for a call option. The reason for this is the assumption of log-normal distributed effective interest rates. As already mentioned this formula is only valid for $\alpha = 1$, i.e. for a time grid of one year. If $\alpha < 1$ as it is usually the case for caps and floors the arbitrage price of the caplets resp. floor payments is a solution of the partial differential equation (15). In the situation of proposition 1 and 2 this can be reduced to the partial differential equation (16) . But still for $\alpha < 1$ this solution has to be computed by numeric techniques as e.g. finit differences. In other words, modifications of the Black-Scholes formulae to price interest rate derivatives can only be supported by a log-normal model of the term structure of interest rates in a very specific case.

[9] The pricing formula for a floor is given in the appendix

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Appendix

Proof of proposition 1:
The proof follows exactely the arguments given by Rady, Sandmann (1994). Given the assumptions of proposition 1 we have to solve terminal value problem (22) on $[0,T]\times]0,1[$ for $\alpha = 1$, i.e.

$$\begin{aligned} 0 &= u_t(t,F) + \frac{1}{2}F^2(1-F)^2 u_{FF}(t,F)\gamma^2(t,T-t,1) \\ u(T,F) &= [F-K]^+ \text{ resp. } [K-F]^+ \quad =: g(F) \end{aligned}$$

where $u(t,F)$ is the time T forward price of the option contract. This problem is transformed by introducing the new time variable

$$s = s(t,T,1) = \int_t^T \gamma(\theta, T-\theta, 1)^2 d\theta \ ,$$

and the new space variable

$$z = \ln\frac{F}{1-F} \qquad \text{or} \quad F = \frac{1}{1+e^{-z}}$$

and finally setting

$$u(t,F) = a(z)b(s)h(s,z) \ .$$

The idea is now to choose differentiable functions a and b in such a way that any solution h of the heat equation yields a solution u of the original partial differential equation. As shown by Rady, Sandmann this can be done by setting

$$\begin{aligned} a(z) &= \frac{1}{\exp\{\frac{z}{2}\} + \exp\{-\frac{z}{2}\}} \\ b(s) &= \exp\left\{-\frac{s}{8}\right\} \quad . \end{aligned}$$

By setting

$$u(t,F) = \frac{1}{e^{\frac{z}{2}} + e^{-\frac{z}{2}}} e^{-\frac{s}{8}} h(s,z) \ ,$$

we obtain the transformed problem on $[0,T]\times \mathbb{R}$

$$\begin{aligned} 0 &= \frac{1}{2}h_{zz} - h_s \\ h(0,z) &= \left(e^{\frac{z}{2}} + e^{-\frac{z}{2}}\right) g\left(\frac{1}{1+e^{-z}}\right) \\ \text{with the solution:}\quad h(s,z) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(0, z+\rho\sqrt{s}) e^{-\frac{\rho^2}{2}} d\rho \ . \end{aligned}$$

Consider e.g. the initial value condition for a European call option, i.e.

$$h(0,z) = (e^{\frac{z}{2}} + e^{-\frac{z}{2}})\left[\frac{1}{1+e^{-z}} - K\right]^+$$

then the solution is determined by:

$$\begin{aligned} h(s,z) &= \frac{1}{\sqrt{2\pi}}\int_{\frac{1}{\sqrt{s}}[\ln\frac{K}{1-K} - z]}^{\infty} \left(e^{\frac{1}{2}[z+\rho\sqrt{s}]} + e^{-\frac{1}{2}[z+\rho\sqrt{s}]}\right)\left(\frac{1}{1+e^{-[z+\rho\sqrt{s}]}} - K\right) e^{-\frac{\rho^2}{2}} d\rho \\ &= (1-K)I_1 - KI_2 \end{aligned}$$

with

$$I_1 = \frac{1}{\sqrt{2\pi}}\int_{\frac{1}{\sqrt{s}}[\ln\frac{K}{1-K}-z]}^{\infty} e^{\frac{1}{2}[z+\rho\sqrt{s}]}e^{-\frac{\rho^2}{2}}d\rho = e^{\frac{z}{2}}\cdot e^{\frac{s}{8}}N\left(\frac{1}{\sqrt{s}}\left[z+\ln\frac{1-K}{K}+\frac{s}{2}\right]\right)$$

$$I_2 = \frac{1}{\sqrt{2\pi}}\int_{\frac{1}{\sqrt{s}}[\ln\frac{K}{1-K}-z]}^{\infty} e^{-\frac{1}{2}[z+\rho\sqrt{s}]}e^{-\frac{\rho^2}{2}}d\rho = e^{-\frac{z}{2}}\cdot e^{\frac{s}{8}}N\left(\frac{1}{\sqrt{s}}\left[z+\ln\frac{1-K}{K}-\frac{s}{2}\right]\right)$$

Therefore:

$$u(t,F) = \frac{e^{-\frac{s}{8}}}{e^{\frac{z}{2}}+e^{-\frac{z}{2}}}h(s,z) = (1-K)\underbrace{\frac{e^{\frac{z}{2}}}{e^{\frac{z}{2}}+e^{-\frac{z}{2}}}}_{=F}N(e_1) - K\underbrace{\frac{e^{-\frac{z}{2}}}{e^{\frac{z}{2}}+e^{-\frac{z}{2}}}}_{=1-F}N(e_2)$$

and since $B(t,T-t+1) = B(t,T-t)\cdot F = B(t,T-t)\cdot\frac{1}{1+r(t,T-t,1)}$ the spot arbitrage price of the European call option is equal to

$$\text{Call}[t,B(t,T-t+1),B(t,T-t),T,K] = B(t,T-t)u(t,F) \quad .$$

□

Pricing formula for a European put option
By call-put parity

$$\text{Put} = \text{Call} + KB(t,T-t) - B(t,T-t+1)$$

Hence under the assumptions of proposition 1 the arbitrage price of a European put option with exercise price K, exercise time T and underlying zero coupon bond $B(t,T-t+1)$ is equal to:

$$\begin{aligned}&\text{Put}[t,B(t,T-t+1),B(t,T-t),T,K]\\ =\ & K\cdot B(t,T-t)\cdot N(-e_2) - B(t,T-t+1)\cdot N(-e_1)\\ & -K\cdot B(t,T-t+1)\cdot(N(e_1)-N(e_2))\end{aligned}$$

Pricing formula for a interest rate floor
Under the assumptions of proposition 2 the arbitrage price of an interest rate floor at time $t \le t_0$ with yearly payment dates $t_i \in \underline{\underline{T}} \setminus \{t_0\}$, face value V and interest rate level L is given by:

$$\begin{aligned}&\text{Floor}[t,L,V,\underline{\underline{T}}\setminus\{t_0\}]\\ =\ & V\cdot\sum_{i=0}^{N-1} B(t,t_{i+1}-t)\left[L\cdot N(-d_2(t,t_i,1)) - r(t,t_i-t,1)\cdot N(-d_1(t,t_i,1))\right]\end{aligned}$$

The multifactor situation: Necessary restrictions on the effective forward rate processes

A first restriction on the volatility function of the effective rate process is given by (9). To construct a viable Heath-Jarrow-Morton model we assume that there exists a function

$\gamma(t,x)$ which satisfies the Lipschitz- and growth condition and for each $(x,\alpha)\in\Gamma$ the volatility function of the corresponding effective rate process is given by

$$\gamma(t,x,\alpha)=\frac{1}{\alpha}\int_x^{x+\alpha}\gamma(t,\theta)d\theta \quad . \tag{19}$$

So far the set Γ was characterized by the set of all observable effective forward rates. Together with the log-normal assumption we have to restrict the set Γ to a subset of observable effective rates in such a way that non of the effective rates within the set Γ can be duplicated by using other rates in Γ. Consider for example the following situation: Let $(x,\alpha_1),(x+\alpha_1,\alpha_2)\in\Gamma$ such that $r(t,x,\alpha_1)$ and $r(t,x+\alpha_1,\alpha_2)$ are two observable effective rates satisfying the log-normal assumption (4). Furthermore the involved Wiener processes $W_{x,\alpha_1}(t)$ and $W_{x+\alpha_1,\alpha_2}(t)$ are not necessarily assumed to be identical. Using the definition of the continuously compounded forward rate we can express the effective forward $r(t,x,\alpha_1+\alpha_2)$ by:

$$\begin{aligned}
&1+r(t,x,\alpha_1+\alpha_2)\\
=\;&\exp\left\{\frac{1}{\alpha_1+\alpha_2}\int_x^{x+\alpha_1+\alpha_2}f(t,\theta)d\theta\right\}\\
=\;&\exp\left\{\frac{1}{\alpha_1+\alpha_2}\int_x^{x+\alpha_1}f(t,\theta)d\theta\right\}\exp\left\{\frac{1}{\alpha_1+\alpha_2}\int_{x+\alpha_1}^{x+\alpha_1+\alpha_2}f(t,\theta)d\theta\right\}\\
=\;&\exp\left\{\frac{\alpha_1}{\alpha_1+\alpha_2}y(t,x,\alpha_1)\right\}\exp\left\{\frac{\alpha_2}{\alpha_1+\alpha_2}y(t,x+\alpha_1,\alpha_2)\right\}
\end{aligned}$$

Given the log-normal assumption for the two observable rates the stochastic behaviour for the effective rate $r(t,x,\alpha_1+\alpha_2)$ is endogeneously determined.

$$\begin{aligned}
&dr(t,x,\alpha_1+\alpha_2)\\
=\;&d\left[\exp\left\{\frac{\alpha_1}{\alpha_1+\alpha_2}y(t,x,\alpha_1)\right\}\exp\left\{\frac{\alpha_2}{\alpha_1+\alpha_2}y(t,x+\alpha_1,\alpha_2)\right\}-1\right]\\
=\;&\frac{1}{\alpha_1+\alpha_2}\exp\{y(t,x,\alpha_1+\alpha_2)\}\\
&\times\left[\left(f(t,x+\alpha_1+\alpha_2)-f(t,x)+\frac{1}{2}(\tau^2(t,x+\alpha_1+\alpha_2)-\tau^2(t,x))\right)dt\right.\\
&+(\tau(t,x+\alpha_1)-\tau(t,x))dW_{x,\alpha_1}(t)\\
&\left.+\,(\tau(t,x+\alpha_1+\alpha_2)-\tau(t,x+\alpha_1))dW_{x+\alpha_1,\alpha_2}(t)\quad\right]\\
+\;&\frac{1}{2(\alpha_1+\alpha_2)^2}\exp\{y(t,x,\alpha_1+\alpha_2)\}\times\left[(\tau(t,x+\alpha_1)-\tau(t,x))^2\right.\\
&+2\rho(t,x,\alpha_1,x+\alpha_1,\alpha_2)(\tau(t,x+\alpha_1)-\tau(t,x))(\tau(t,x+\alpha_1+\alpha_2)-\tau(t,x+\alpha_1))\\
&\left.+\tau(t,x+\alpha_1+\alpha_2)-\tau(t,x+\alpha_1))^2dt\quad\right]
\end{aligned}$$

where $\rho(t,x,\alpha_1,x+\alpha_1,\alpha_2)$ is the instantaneous correlation coefficient between the two Wiener processes, i.e

$$dW_{x,\alpha_1}dW_{x+\alpha_1,\alpha_2}(t)=\rho(t,x,\alpha_1,x+\alpha_1,\alpha_2)dt. \tag{20}$$

The volatility function $\nu()$ of the effective rate $r(t,x,\alpha_1+\alpha_2)$ is thus endogeneously determined by:

$$
\begin{aligned}
& \nu^2(t;x,\alpha_1+\alpha_2) \quad \left(\frac{r(t,x,\alpha_1+\alpha_2)}{1+r(t,x,\alpha_1+\alpha_2)}\right)^2 \\
= & \left(\frac{\alpha_1}{\alpha_1+\alpha_2} \quad \frac{r(t,x,\alpha_1)}{1+r(t,x,\alpha_1)} \quad \gamma(t,x,\alpha_1)\right)^2 \\
& +2\rho()\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}\frac{r(t,x,\alpha_1)r(t,x+\alpha_1,\alpha_2)}{(1+r(t,x,\alpha_1))(1+r(t,x+\alpha_1,\alpha_2))}\gamma(t,x,\alpha_1)\gamma(t,x+\alpha_1,\alpha_2) \\
& + \left(\frac{\alpha_2}{\alpha_1+\alpha_2} \quad \frac{r(t,x+\alpha_1,\alpha_2)}{1+r(t,x+\alpha_1,\alpha_2)} \quad \gamma(t,x+\alpha_1,\alpha_2)\right)^2 .
\end{aligned}
$$

We conclude that the endogeneously determined effective rate $r(t,x,\alpha_1+\alpha_2)$ does not satisfy the log-normal distribution assumption (4) which is assumed for all rates within the set Γ. In the special case where the instantaneous correlation $\rho()$ is equal to one, i.e. we consider a 1-factor model this can be simplified to

$$
\begin{aligned}
& \nu(t;x,\alpha_1+\alpha_2) \quad r(t,x,\alpha_1+\alpha_2) \\
= & \frac{\alpha_1}{\alpha_1+\alpha_2} \quad r(t,x,\alpha_1) \quad \frac{1+r(t,x,\alpha_1+\alpha_2)}{1+r(t,x,\alpha_1)} \quad \gamma(t,x,\alpha_1) \\
& +\frac{\alpha_2}{\alpha_1+\alpha_2} \quad r(t,x+\alpha_1,\alpha_2) \quad \frac{1+r(t,x,\alpha_1+\alpha_2)}{1+r(t,x+\alpha_1,\alpha_2)} \quad \gamma(t,x+\alpha_1,\alpha_2)
\end{aligned}
$$

This behaviour of the effective volatility is very similar to some estimation rules used in applications of the Black-Scholes model to interest rate derivatives. Nevertheless this shows that even in the 1-factor case the volatility of redundant effective forward rate processes is state dependent. Only in the situation of a flat effective forward rate curve (i.e $r(t,x,\alpha_1) = r(t,x+\alpha_1,\alpha_2)$) the volatility of the redundant rates is given by the time proportional weighted average of the other volatilities. To conclude at this point a necessary condition on the set of exogeneous effective rates Γ to be a viable input under the no-arbitrage restriction is the no-redundency of this set, i.e we furtheron assume that

$$
(x,\alpha)\in\Gamma \qquad \text{iff} \qquad \forall\Delta\subseteq\Gamma\setminus(x,\alpha) \quad [t+x,t+x+\alpha] \neq \bigcup_{(x_i,\alpha_i)\in\Delta} [t+x_i,t+x_i+\alpha_i] \tag{21}
$$

We finally consider the situation when the forward price cannot be expressed in terms of one factor in the set Γ. More precisely let $(T-t,\alpha_1),(T-t+\alpha_1,\alpha_2)\in\Gamma$. By construction we have:

$$
F(t,T,\alpha_1+\alpha_2) = F(t,T,\alpha_1)F(t,T+\alpha_1,\alpha_2)
$$

Following the arguments in section 2 and setting for simplicity

$$
F_1 = F(t,T,\alpha_1) \quad \text{resp.} \quad F_2 = F(t,T+\alpha_1,\alpha_2) \quad ,
$$

$$
\rho(t) = \rho(t,T-t,\alpha_1,T-t+\alpha_1,\alpha_2)
$$

the process of the quadratic variation for this compounded forward price is given by:

$$
\begin{aligned}
& d\langle F\rangle(t,T,\alpha_1+\alpha_2) \\
= & \Big[\quad \alpha_1^2(F_1F_2)^2\left(1-F_1^{\frac{1}{\alpha_1}}\right)^2\gamma^2(t,T-t,\alpha_1) \\
& +2\alpha_1\alpha_2\rho(t)(F_1F_2)^2\left(1-F_1^{\frac{1}{\alpha_1}}\right)\left(1-F_2^{\frac{1}{\alpha_2}}\right)\gamma^2(t,T-t,\alpha_1)\gamma^2(t,T-t+\alpha_1,\alpha_2) \\
& +\alpha_2^2(F_1F_2)^2\left(1-F_2^{\frac{1}{\alpha_2}}\right)^2\gamma^2(t,T-t+\alpha_1,\alpha_2) \quad \Big] \quad dt \quad .
\end{aligned}
$$

In the special case where the instantaneous correlation $\rho()$ is equal to one this relationship between the volatility functions can be simplified to

$$\begin{aligned} & d\left\langle F\right\rangle(t,T,\alpha_1+\alpha_2) \\ = \; & (F_1F_2)^2\left(\alpha_1\left(1-F_1^{\frac{1}{\alpha_1}}\right)\gamma(t,T-t,\alpha_1)-\alpha_2\left(1-F_2^{\frac{1}{\alpha_2}}\right)\gamma(t,T-t+\alpha_1,\alpha_2)\right)^2 dt \quad . \end{aligned}$$

Again this shows that even in the 1-factor case the functional form of the volatility for the redundant forward price is not precisely of the form (11). Only in the situation of a flat effective forward rate curve (i.e $r(t,T-t,\alpha_1)=r(t,T-t+\alpha_1,\alpha_2)$) the volatility of the redundant forward price is given by the time proportional weighted average of the other volatilities.