On the existence of equivalent τ–measures in finite discrete time

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Abstract

Suppose that $(X(n))$ is a finite adapted sequence of d–dimensional random variables defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n), P)$. We obtain conditions which are necessary and sufficient for the existence of a probability measure Q equivalent to P (which we call an equivalent τ–measure) such that each of the d component sequences of $(X(n))$ has a prescribed martingale property w.r.t. Q (i.e., it is either a Q–martingale, a Q–sub– or a Q–supermartingale). This extends a version of the Fundamental Theorem of Asset Pricing due to Dalang, Morton and Willinger (1990).

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equivalent martingale measure $*$ no–arbitrage $*$ security market

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1. Introduction

In the sequel $X(n) = (X_1(n), ..., X_d(n)), n = 0, 1, ..., T \quad (T \geq 1)$ will denote R^d–valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}_T \subset \mathcal{F}$ be any filtration such that the process $(X(n))$ is adapted to $(\mathcal{F}_n)$, i.e. $X(n)$ is $\mathcal{F}_n$–measurable for all n . Let Q be any probability measure on $\mathcal{F}$ which is *equivalent* to P $(Q \sim P)$, i.e. Q and P have the same null sets. One says that Q is an *equivalent martingale measure* for $(X(n))$ if $(X(n))$ is an Q–martingale w.r.t. $(\mathcal{F}_n)$, i.e. $E_Q[\ \|X(n)\|\ ] < \infty$, $n = 0, ..., T(\ \|\cdot\|$ denoting the Euclidean norm) and $E_Q[X(n) - X(n-1)|\mathcal{F}_{n-1}] = 0$ a.s., $n = 1, ..., T$. It is easy to verify that in this case $(X(n))$ satisfies the following *"no–arbitrage" condition*: For $n = 1, ..., T$ and each bounded R^d–valued random variable h which is $\mathcal{F}_{n-1}$–measurable, $(h, X(n) - X(n-1)) \geq 0$ a.s. implies $(h, X(n) - X(n-1)) = 0$ a.s. (here, (x, y) denotes the scalar product of $x, y \in R^d$).

The no–arbitrage condition has the following economic interpretation: Let $X_i(n)$ denote the price of a certain security i at time n and let h represent the investor's portfolio during the period $]n-1, n]$, where h_i is the quantity of security i (here, h_i may assume negative as well as positive values). The $\mathcal{F}_{n-1}$–measurability of h means that the selection of the portfolio only uses the information available to the investor at time $n-1$. The no–arbitrage condition then says that the total net gain $(h, X(n) - X(n-1))$ at time n is either almost surely equal to zero or negative with probability $0 < p < 1$.

It is remarkable that the no–arbitrage condition is also sufficient for the existence of an equivalent martingale measure. More precisely we have the following beautiful version of the Fundamental Theorem of Asset Pricing:

Theorem 1.1 (Dalang, Morton and Willinger (1990)).

There exists an equivalent martingale measure Q for the process $(X(n))$ iff $(X(n))$ satisfies the no–arbitrage condition. In this case Q may be chosen such that the Radon–Nikodym derivative dQ/dP is $\mathcal{F}_T$–measurable and bounded.

Note that in Theorem 1.1 $(X(n))$ is not assumed to be integrable, and there are no additional assumptions on the filtration $(\mathcal{F}_n)$. (In Theorem 2.6 of Dalang et al. (1990) the probability space $(\Omega, \mathcal{F}, P)$ and the σ–algebras $\mathcal{F}_n$ are assumed to be complete. It is, however, easy to see that these additional hypotheses are unnecessary.) Special cases of Theorem 1.1 were derived e.g. by Harrison and Pliska (1981), Taqqu and Willinger (1987) and Back and Pliska (1991). The original proof of Theorem 1.1 given in Dalang et al.(1990) is based on measurable selection and measure–decomposition theorems. Alternative proofs are due to Schachermayer (1992) (using certain Hilbert space techniques), Kabanov and Kramkov (1994) and Rogers (1994).

Note that Theorem 1.1 holds if we allow for positive *and* negative amounts of any security in the portfolio. One might ask whether Theorem 1.1 remains true for markets in which trading of some securities is restricted to *either* positive *or* negative amounts. To be more specific, let us consider a market not allowing short sales of the d securities involved (i.e., trading of these securities is restricted to positive amounts). One could think for instance of small investors avoiding short sales. It follows from our main result (see Theorem 2.4 below) that in this case the absence of arbitrage opportunities is equivalent to the existence of an *equivalent supermartingale measure* Q for $X = (X(n))$, i.e. we have $Q \sim P$ and, for any $1 \leq i \leq d$, the sequence $(X_i(n))$ is a Q-supermartingale w.r.t. $(\mathcal{F}_n)$.

It turns out that there is an interesting connection between the set $\mathcal{M}$ of all equivalent supermartingale measures for X and the existence of some self-financing hedging strategy for a given *contingent claim* f (i.e., f is a nonnegative real-valued random variable which is $\mathcal{F}_T$-measurable). One could think for instance of an investor who sells at time zero a certain option which obliges him to pay at time T the cash amount f to the option buyer. In order to hedge himself against this situation, the investor might apply a self-financing hedging strategy $H = (H(n)), 0 \leq n \leq T$, for f. Here, $H(n)$ is a d-dimensional random vector which represents the investor's portfolio during the period $]n-1, n]$ where $H_i(n)$ is the amount of security i in the portfolio. Let us assume for the moment that $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Then H is called a *self-financing hedging strategy* for the contingent claim f with (constant) *initial value* $x > 0$, provided H is predictable w.r.t. $(\mathcal{F}_n)$ and has the following properties:

$$H(n) \geq 0 \quad \text{(componentwise)}, \quad 0 \leq n \leq T, \tag{H1}$$

$$(H(0), X(0)) = x, \tag{H2}$$

$$(H(n), X(n)) \geq 0 \quad \text{a.s.}, \ 1 \leq n \leq T-1, \tag{H3}$$

$$(H(T), X(T)) \geq f \quad \text{a.s.} \tag{H4}$$

and

$$(H(n+1) - H(n), X(n)) = 0 \quad \text{a.s.}, 0 \leq n \leq T-1. \tag{H5}$$

(The predictability of H means that, for any $1 \leq n \leq T, H(n)$ is $\mathcal{F}_{n-1}$ - measurable, and $H(0)$ is $\mathcal{F}_0$–measurable (hence constant).) Note that (H1) means that the investor avoids short sales. On the other hand, (H5) expresses the self–financing property of H since the scalar product in (H5) equals the change of the value of the portfolio immediately after time n due to the investor's rearrangement of his portfolio. (H2) expresses the fact that the initial value of the portfolio equals x. According to (H4) the value of the portfolio at time T is at least equal to f, and (H3) guarantees that the investor is never put into a position of debt. The set of all hedging strategies satisfying (H1)–(H5) will be denoted by $\mathcal{H}(x, f)$. For any $H \in \mathcal{H}(x, f)$ let $V_n^H = (H(n), X(n)), 0 \leq n \leq T$, denote the value of the resulting portfolio at time n. By (H5) and (H2), $V^H = (V_n^H)$ is a discrete stochastic integral of the form

$$\begin{aligned} V_n^H &= x + (H \cdot X)_n \\ &= x + \sum_{m=1}^{n} (H(m), X(m) - X(m-1)), 0 \leq n \leq T \end{aligned} \tag{1.1}$$

(note that $(H(m), X(m) - X(m-1))$ equals the change of the value of the portfolio due to the change of the prices of securities at time m). Let us put $\mathcal{H}(f) = \bigcup_{x>0} \mathcal{H}(x, f)$ (note that $\mathcal{H}(x, f) \neq \emptyset$ implies $\mathcal{H}(y, f) \neq \emptyset$ for any $y > x$).

Lemma 1.2.

For any $Q \in \mathcal{M}$ and $H \in \mathcal{H}(f), V^H$ is a Q–supermartingale.

In order to see this, it suffices to note that, by (H1), (H2) and (H3), the Q–integrability of $(H(n), X(n) - X(n-1))$ follows from the fact that if

$$(H(n), X(n) - X(n-1)) \geq \xi \quad Q - a.s.$$

for some Q–integrable random variable ξ, then

$$E_Q\left[|(H(n), X(n) - X(n-1))|\right] \leq 2E_Q\left[\xi_-\right]$$

where $a = \max\{-a, 0\},\ a \in R$.

In the case $\mathcal{M} \neq \emptyset$ our next result gives a *necessary* condition for $\mathcal{H}(f)$ to be nonvoid. (Note that, according to Theorem 2.4 below, $\mathcal{M} \neq \emptyset$ holds iff $(X(n))$ satisfies a no–arbitrage condition in the case when short sales are excluded.) Provided $\mathcal{M} \neq \emptyset$ we put, for any contingent claim f,

$$\tilde{V}_n^f = \operatorname*{ess\,sup}_{Q \in \mathcal{M}} E_Q[f|\mathcal{F}_n], 0 \leq n \leq T. \tag{1.2}$$

According to Theorem 1.3(i) below, $\tilde{V}_n^f$ provides a uniform lower bound for the values $V_n^H, H \in \mathcal{H}(f)$. In particular, this implies (since $\mathcal{F}_0 = \{\emptyset, \Omega\}$)

$$\tilde{V}_0^f = \sup_{Q \in \mathcal{M}} E_Q[f] \leq \inf\{x > 0|\mathcal{H}(x, f) \neq \emptyset\}. \tag{1.3}$$

Note that the right–hand side of (1.3) provides an upper bound for the fair price (at time zero) of an option consisting of a payment f to the option buyer at time T. In fact, let $\mathcal{H}(x, f) \neq \emptyset$. An investor who contemplates buying the option at time zero can instead apply some strategy $H \in \mathcal{H}(x, f)$ to a certain portfolio of initial value x which guarantees him the wealth $V_T^H \geq f$ at time T.

According to Theorem 1.3 (ii) $\tilde{V}^f$ has nice martingale properties which were recently used to obtain a condition which is *sufficient* for $\mathcal{H}(f)$ to be nonvoid (see Remark 1.4 below).

Theorem 1.3.

Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and assume $\mathcal{M} \neq \emptyset$.

(i) For any contingent claim f such that $\mathcal{H}(f) \neq \emptyset$ we have

$$\sup_{Q \in \mathcal{M}} E_Q[f] < \infty \tag{1.4}$$

and, for any $H \in \mathcal{H}(f)$,

$$\tilde{V}_n^f \leq V_n^H \ \ a.s., \quad 0 \leq n \leq T. \tag{1.5}$$

(ii) Let $(X(n))$ be (componentwise) nonnegative. Then, for any contingent claim f such that (1.4) holds, we have that

$$(\tilde{V}_n^f) \text{ is a } Q\text{–supermartingale for any } Q \in \mathcal{M}. \tag{1.6}$$

Remark 1.4.

Recently, H. Föllmer and D.O. Kramkov (oral communication) have shown by using a Hahn–Banach type argument that (1.6) implies $\mathcal{H}(f) \neq \emptyset$. Therefore, under the nonnegativity assumption in Theorem 1.3 (ii), (1.4) is equivalent to $\mathcal{H}(f) \neq \emptyset$.

Note that Theorem 1.3 (i) is an easy consequence of Lemma 1.2 and (H4). Let us now outline the proof of part (ii). In the sequel let $Q_* \in \mathcal{M}$ and $0 \leq m < n \leq T$ be fixed. Expectations w.r.t. Q_* will be denoted by E_*. We put

$$z(Q) = E_*[dQ/dQ_* \mid \mathcal{F}_n], \quad Q \in \mathcal{M}.$$

We will need the following properties of $\mathcal{M}$. For any $P_1, P_2 \in \mathcal{M}$ and $A \in \mathcal{F}_n$ we have (1_B denoting the indicator function of a set B)

$$\frac{1}{z(P_1)} \frac{dP_1}{dQ_*} 1_A + \frac{1}{z(P_2)} \frac{dP_2}{dQ_*} 1_{\Omega \setminus A} = \frac{dP_0}{dQ_*} \quad \text{for some } P_0 \in \mathcal{M} \tag{1.7}$$

(note that $z(P_0)$=1 a.s.) which implies that, for any nonnegative random variable ξ,

$$E_{P_0}[\xi \mid \mathcal{F}_n] = 1_A E_{P_1}[\xi \mid \mathcal{F}_n] + 1_{\Omega \setminus A} \ E_{P_2}[\xi \mid \mathcal{F}_n] \quad \text{a.s.} \tag{1.8}$$

and

$$E_{P_0}[\xi] = E_* \left[1_A E_{P_1}[\xi|\mathcal{F}_n] + 1_{\Omega \setminus A} E_{P_2}[\xi|\mathcal{F}_n] \right] . \tag{1.9}$$

Furthermore, for any $Q \in \mathcal{M}$, we have that

$$E_Q[dQ_*/dQ \mid \mathcal{F}_n] = d\tilde{Q}/dQ \quad \text{for some} \quad \tilde{Q} \in \mathcal{M} \tag{1.10}$$

which implies that, for any σ–algebra $\mathcal{G} \subset \mathcal{F}_n$ and any nonnegative random variable ξ,

$$E_*[E_Q[\xi \mid \mathcal{F}_n] \mid \mathcal{G}] \ = \ E_{\tilde{Q}}[\xi \mid \mathcal{G}] \quad \text{a.s.} \tag{1.11}$$

(We omit the proof that P_0 (given by (1.7)) and $\tilde{Q}$ (given by (1.10)) belong to $\mathcal{M}$, which, in particular, uses the Bayes' rule for conditional expectations (see for instance

Dalang et al. (1990), p. 188).) Now let $P_1, P_2 \in \mathcal{M}$ and put $A = \{E_{P_1}[f \mid \mathcal{F}_n] \geq E_{P_2}[f \mid \mathcal{F}_n]\}$. Then (1.8) implies that, for some $P_0 \in \mathcal{M}$,

$$E_{P_0}[f \mid \mathcal{F}_n] = \max\{E_{P_1}[f \mid \mathcal{F}_n],\ E_{P_2}[f \mid \mathcal{F}_n]\}. \tag{1.12}$$

It follows from (1.12) that there exists a sequence $(Q_k) \subset \mathcal{M}$ such that the sequence $(E_{Q_k}[f \mid \mathcal{F}_n])$, $k \geq 1$, is a.s. *increasing* and

$$\tilde{V}_n^f = \lim_{k\to\infty} E_{Q_k}[f \mid \mathcal{F}_n] \quad a.s. \tag{1.13}$$

Hence, by (1.9), (1.4) and the monotone convergence theorem, $\tilde{V}_n^f$ is Q_*–integrable. It remains to show that

$$E_*[\tilde{V}_n^f \mid \mathcal{F}_m] \leq \tilde{V}_m^f \quad \text{a.s.} \tag{1.14}$$

In order to see this, note that, by (1.13),

$$E_*\left[\tilde{V}_n^f \mid \mathcal{F}_m\right] = \operatorname*{ess\,sup}_{Q\in\mathcal{M}} E_*\left[E_Q[f \mid \mathcal{F}_n] \mid \mathcal{F}_m\right] \quad \text{a.s.}$$

which, by (1.11), entails (1.14). This completes the proof of Theorem 1.3.

The nonnegativity assumption in Theorem 1.3 (ii) is clearly satisfied if we assume that the random vectors $X(n)$ represent *prices* of certain securities. The following example shows however that, in general, (1.7) does not hold if $(X(n))$ is not assumed to be nonnegative.

Example 1.5.

Let $\Omega = \{1, 2, \ldots\}$, let $\mathcal{F}_0$ be the σ–algebra generated by the sets $\{3n-2, 3n-1, 3n\}$, $n \geq 1$, and let $\mathcal{F}_1 = \mathcal{F}$ be the power set of Ω. Let the real–valued random variables $Y_0,\ Y_1$ be defined on Ω by $Y_0 \equiv 0$ and

$$Y_1(3n-2) = 0, \qquad Y_1(3n-1) = 2^n, \qquad Y_1(3n) = -2^n, \qquad n \geq 1.$$

Define probability measures P and Q on $\mathcal{F}$ by

$$\begin{aligned} P\{3n-2\} &= \frac{3}{4\cdot 2^n}, \quad P\{3n-1\} = P\{3n\} = \frac{3}{8\cdot 4^n}, \\ Q\{3n-2\} &= Q\{3n-1\} = Q\{3n\} = \frac{1}{4^n}, \quad n \geq 1. \end{aligned}$$

Then $Y = (Y_n)$, $n = 0, 1$, is a martingale w.r.t. P and Q. Let $P_0 \sim P$ be given by

$$\frac{dP_0}{dP} = \frac{dQ/dP}{E_P[dQ/dP \mid \mathcal{F}_0]}.$$

An easy calculation gives

$$\frac{dP_0}{dP} = \sum_{n=1}^{\infty} \frac{2^n+1}{3\cdot 2^n} 1_{\{3n-2\}} + \sum_{n=1}^{\infty} \frac{2(2^n+1)}{3} 1_{\{3n-1,3n\}}$$

which implies (putting $a_+ = \max\{a, 0\}, a \in R$)

$$E_{P_0}[(Y_1)_+] = E_P\left[E_Q[(Y_1)_+ \mid \mathcal{F}_0]\right] = \infty$$

and

$$E_{P_0}[(Y_1)_-] = E_P\left[E_Q[(Y_1)_- \mid \mathcal{F}_0]\right] = \infty.$$

Hence Y is not a P_0–supermartingale.

It is clear that Q is an *equivalent submartingale measure* for $(X(n))$ (in the obvious sense) iff Q is an equivalent supermartingale measure for $(-X(n))$. Therefore Theorem 2.4 below can be interpreted in this case as saying: "if one can't *lose* betting on a process (playing at nonnegative stakes)then it must be a submartingale under an equivalent change of measure".

Let us now introduce the notion of an equivalent τ–measure which generalizes the notion of an equivalent martingale measure and that of an equivalent super–(sub–) martingale measure (in the sequel $(\mathcal{F}_n)$ denotes any filtration to which $(X(n))$ is adapted).

Definiton 1.6.

Let $\tau \in \{-1, 0, 1\}^d$. Let $Q \sim P$ be a probability measure such that each $X(n)$ is Q–integrable. Then Q is called an *equivalent τ–measure* for $(X(n))$ if, for all $1 \leq i \leq d$, we have that, w.r.t. $(\mathcal{F}_n), (X_i(n))$ is a Q–martingale if $\tau_i = 0$, a Q–submartingale if $\tau_i = -1$, and a Q–supermartingale if $\tau_i = 1$. Clearly, by an *equivalent sub– (super–) martingale measure* we mean an equivalent $(-1, ..., -1)-((1, ..., 1)-)$ measure.

The main result of the present paper (see Theorem 2.4 in Section 2), extending Theorem 1.1, gives a condition which is necessary and sufficient for the existence of an equivalent τ–measure for $(X(n))$. We will also deal with the question how equivalent τ–measures for various τ's are related to each other (see Corollary 2.5). In the one–dimensional case it turns out that there exists an equivalent martingale measure iff there exists an equivalent sub–and supermartingale measure.

Let us outline some ideas used in the proof of Theorem 2.4 which is given in Section 3. Using induction on T, one only needs to prove the desired result for processes $(X(n))$, $n = 0, 1$, such that $X(0) \equiv 0$ and $X(1) = Y$, Y being integrable. Now, the basic strategy is to decompose Ω into suitable sets belonging to $\mathcal{F}_0$, and to prove the desired result for the restrictions of X to these sets (the filtrations being the "traces" of $(\mathcal{F}_n)$ on these sets). "Putting together" the equivalent τ–measures thus obtained yields the desired τ–measure for X (see the proof of Lemma 3.3 which is based on a result due to Yan (1980) (see The orem 3.2 below)). In order to obtain a decomposition of Ω being suitable for our purposes, we apply, in a first step, an elementary result due to Kabanov and Kramkov (1994) (see Lemma 3.4 below) which provides a decomposition of Ω into sets $\Omega(i) \in \mathcal{F}_0$, $i = 1, 2$, with the following properties:

(D1) The components of Y are on $\Omega(1)$ "$\mathcal{F}_0$– linearly independent" in the following sense: If h is any bounded $\mathcal{F}_0$–measurable R^d–valued random variable, then

$$(h1_{\Omega(1)}, Y) = 0 \quad \text{a.s. implies } h1_{\Omega(1)} = 0 \quad \text{a.s.}$$

(D2) The components of Y are on $\Omega(2)$ "$\mathcal{F}_0$–linearly dependent" in the following sense: There exists some bounded $\mathcal{F}_0$–measurable R^d–valued random variable g such that

$$g \neq 0 \text{ on } \Omega(2), \; g = 0 \text{ on } \Omega(1), \text{ and } (g, Y) = 0 \quad \text{a.s.}$$

Now suppose that, for some $\tau \in \{-1, 0, 1\}^d$, X (or, for short, Y) satisfies the "*no–τ–arbitrage*" condition occurring in Lemma 2.1 (here, $\tau_i = +1$ (-1) means that trading of security i is restricted to positive (negative) amounts, whereas in the case $\tau_i = 0$ trading of security i is not subject to any restrictions). Then (D1) implies that, on $\Omega(1)$, Y satisfies the following stronger form of no–τ–arbitrage: If, on Ω, a portfolio has a nonnegative value, then, on $\Omega(1)$, it does not contain any securities. This stronger form of no–τ–arbitrage enables us to derive the existence of an equivalent τ–measure for the restriction of X to $\Omega(1)$ in a straightforward manner by using a deep result due to Komlós (1967) (see Theorem 3.7 below). It thus remains to prove the desired result for the restriction of X to $\Omega(2)$. A simple conditioning argument (which again involves Lemma 3.3) shows that we may additionally assume that the random variable g in (D2) has the property that, for any $1 \leq i \leq d$, one of the sets $\{g_i \geq 1\}$, $\{g_i = 0\}$ and $\{g_i \leq -1\}$ equals $\Omega(2)$. Finally, we may replace Y and τ by Y^* and τ^*, respectively, where, for any $1 \leq i \leq d$,

$$Y_i^* = \begin{cases} g_i Y_i & \text{if} \quad g_i \neq 0 \\ Y_i & \text{otherwise,} \end{cases}$$

and

$$\tau_i^* = \begin{cases} \tau_i & \text{if } g_i \geq 1 \quad \text{or} \quad g_i = 0 \\ -\tau_i & \text{otherwise.} \end{cases}$$

Applying the same permutation to the components of Y^* and τ^* carries Y^* and τ^* over into (say) $\tilde{Y}$ and $\tilde{\tau}$, respectively. Obviously, Y has an equivalent τ–measure iff $\tilde{Y}$ has an equivalent $\tilde{\tau}$–measure. This shows that in order to finish the proof of Theorem 2.4 it suffices to derive the desired result for all Y which are of type τ for some $\tau \in \{-1, 0, 1\}^d$ and, additionally, satisfy the condition

$$Y_1 + \ldots + Y_\pi \equiv 0 \quad \text{on} \quad \Omega \quad \text{for some} \quad 1 \leq \pi \leq d. \tag{1.15}$$

Now, (1.15) enables us to use induction on d. Assume that Therem 2.4 holds for all k–dimensional Y's $(1 \leq k \leq d-1)$ satisfying (1.15). If one tries to prove Theorem 2.4 for all d–dimensional Y's subject to the constraint (1.15), then the only difficult case is where the type of Y equals some τ which, for some $1 \leq m \leq \pi/2$, satisfies the condition

$$\tau_i = 1, \quad \tau_{m+i} = -1, \quad 1 \leq i \leq m, \quad \text{and} \quad \tau_i \leq 0, \quad 2m+1 \leq i \leq \pi. \tag{1.16}$$

Assume that, at time 1, Y_i is the price of security i and let Z be the value (at time 1) of some portfolio such that the amounts of securities $1, \ldots, d$ in the portfolio are subject to the constraints given by τ. Then (1.15) and (1.16) imply (see relation (3.15) below) that there exist m portfolios having total value Z at time 1 such that, for any $1 \leq j \leq m$, the j–th portfolio does *not* contain security j, satisfies the constraints given by $\tau_1, \ldots, \tau_{j-1}, \tau_{j+1}, \ldots, \tau_d$ and, furthermore, has (at time 1) value $\eta_j Z$ where $0 \leq \eta_j \leq 1$ is $\mathcal{F}_0$–measurable. According to this observation the induction hypothesis applies, and using Komlós's theorem once more proves the desired result for d–dimensional Y's satisfying (1.15). This finishes the proof of Theorem 2.4. Let us note that the proof of Theorem 2.4 in the case $d = 1$ is much shorter than in the case $d \geq 2$ (see the proof of Lemma 3.8 in Section 3).

2. On the existence of equivalent τ–measures

If $\mathcal{G} \subset \mathcal{F}$ is any σ–algebra, we denote by $\mathcal{L}_d^\infty(\mathcal{G})$ $(\mathcal{L}_d^1(\mathcal{G}))$ the family of all R^d–valued random variables defined on $(\Omega, \mathcal{F}, P)$ which are $\mathcal{G}$–measurable and bounded (integrable); we write $\mathcal{L}^\infty(\mathcal{G})$ $(\mathcal{L}^1(\mathcal{G}))$ instead of $\mathcal{L}_1^\infty(\mathcal{G})$ $(\mathcal{L}_1^1(\mathcal{G}))$. If $\mathcal{M}$ is any family of real–valued functions, we put $\mathcal{M}_0 = \mathcal{M}$ and let $\mathcal{M}_{-1}(\mathcal{M}_1)$ denote the family of all $f \in \mathcal{M}$ which are nonpositive (nonnegative). For any $\tau \in \{-1, 0, 1\}^d$ let $\mathcal{L}_d^\infty(\mathcal{G})_\tau$ $(\mathcal{L}_d^1(\mathcal{G})_\tau)$ consist of all h in $\mathcal{L}_d^\infty(\mathcal{G})$ $(\mathcal{L}_d^1(\mathcal{G}))$ such that h_i belongs to $\mathcal{L}^\infty(\mathcal{G})_{\tau_i}$ $(\mathcal{L}^1(\mathcal{G})_{\tau_i})$, $1 \leq i \leq d$.

Lemma 2.1.

If there exists an equivalent τ–measure for $(X(n))$, then, for all $1 \leq n \leq T$ and $h \in \mathcal{L}_d^\infty(\mathcal{F}_{n-1})_\tau$,

$$(h, X(n) - X(n-1)) \geq 0 \quad a.s. \quad \textit{implies} \quad (h, X(n) - X(n-1)) = 0 \quad a.s.$$

Proof.

Let Q be an equivalent τ–measure for $(X(n))$. Then, for each $1 \leq n \leq T$,

$$(g, E_Q[X(n) - X(n-1) | \mathcal{F}_{n-1}]) \leq 0 \quad Q\text{–a.s.}, \ g \in \mathcal{L}_d^\infty(\mathcal{F}_{n-1})_\tau \ ,$$

and hence

$$E_Q[(g, X(n) - X(n-1))] \leq 0 \ , \ g \in \mathcal{L}_d^\infty(\mathcal{F}_{n-1})_\tau \ . \tag{2.1}$$

If, for some $1 \leq n \leq T$ and $h \in \mathcal{L}_d^\infty(\mathcal{F}_{n-1})_\tau$,

$$(h, X(n) - X(n-1)) \geq 0 \quad P\text{–a.s. and therefore} \quad Q\text{–a.s.} \ ,$$

then, by (2.1),

$$(h, X(n) - X(n-1)) = 0 \quad Q\text{–a.s. and therefore} \quad P\text{–a.s.}$$

Remark 2.2.

Using backward induction it is easy to show that the condition of Lemma 2.1 is equivalent to the following condition:

For each random variable Z of the form

$$Z = \sum_{n=1}^{T} (h(n), X(n) - X(n-1)), \ \text{where } h(n) \in \mathcal{L}_d^\infty(\mathcal{F}_{n-1})_\tau, \ 1 \le n \le T,$$

we have that $Z \ge 0$ a.s. implies $Z = 0$ a.s..

(Note that Z is a discrete stochastic integral.) In fact, let us prove this claim using backward induction on

$$N = \min\{1 \le n \le T | P(h(n) \ne 0) > 0\}$$

(we put $\min \emptyset = T$). The claim clearly holds in the case $N = T$. Suppose it holds for all Z for which $2 \le m \le N \le T$. Let Z be a random variable of the above form for which $N = m - 1$ and let $Z \ge 0$ a.s. Then

$$\sum_{n=m}^{T} (h(n), X(n) - X(n-1)) \ge 0$$

a.s. on the set $A = \{(h(m-1), X(m-1) - X(m-2)) \le 0\}$ (note that $A \in \mathcal{F}_{m-1}$). The induction hypothesis implies

$$\sum_{n=m}^{T} (h(n)1_A, X(n) - X(n-1)) = 0 \quad \text{a.s.}$$

Since $Z \ge 0$ a.s., this yields $1_A(h(m-1), X(m-1) - X(m-2)) \ge 0$ a.s. Therefore $(h(m-1), X(m-1) - X(m-2)) \ge 0$ a.s. which, by the condition of Lemma 2.1, implies $(h(m-1), X(m-1) - X(m-2)) = 0$ a.s. Applying the induction hypothesis once more gives $Z = 0$ a.s.

Definition 2.3.

Let $\tau \in \{-1, 0, 1\}^d$. We say that the process $(X(n))$ is of *type* τ if $(X(n))$ satisfies the condition of Lemma 2.1 (or, equivalently, the condition in Remark 2.2).

The following theorem generalizing Theorem 1.1 is our main result (its proof will be given in Section 3).

Theorem 2.4.

There exists an equivalent τ–measure Q for the process $(X(n))$ iff $(X(n))$ is of type τ. In this case Q may be chosen such that dQ/dP is $\mathcal{F}_T$–measurable and bounded.

The following easy consequence of Theorem 2.4 shows how equivalent τ–measures for various τ's are related to each other (note that part (ii) is an easy consequence of Lemma 2.1 and Theorem 1.1).

Corollary 2.5.

(i) *Let $\tau \in \{-1,0,1\}^d$ be such that $\tau_i = 0$ for at least one $1 \le i \le d$. Suppose that $(X(n))$ has an equivalent ρ–measure for all ρ such that $\rho_j = \tau_j$ whenever $\tau_j \ne 0$, and $\rho_j \ne 0$ whenever $\tau_j = 0$. Then $(X(n))$ has an equivalent τ–measure.*

(ii) *In particular, $(X(n))$ has an equivalent martingale measure iff $(X(n))$ has an equivalent τ–measure for all $\tau \in \{-1,1\}^d$.*

Remark 2.6.

The following example shows that the condition in Corollary 2.5 (ii) cannot be relaxed. In fact, fix any $\tau^* \in \{-1,1\}^d$. Then there exists a process $(X(n))$ such that $(X(n))$ is of type τ for all $\tau \in \{-1,1\}^d \setminus \{\tau^*\}$ but does not have an equivalent martingale measure.

Example 2.7.

Let $\Omega = [0,1[$ be equipped with the σ–algebra $\mathcal{F}$ of Borel sets and Lebesgue measure. Let $d \ge 2,\ T = 1,\ \mathcal{F}_1 = \mathcal{F}$ and let $\mathcal{F}_0$ be trivial. Let $X(0) \equiv 0,\ X(1) = Y$ where

$$Y_i \;=\; 2d \sum_{k=0}^{2^{i-1}-1} 1_{\left[\frac{2k}{2^i},\frac{2k+1}{2^i}\right[} - \sum_{k=1}^{2^{i-1}} 1_{\left[\frac{2k-1}{2^i},\frac{2k}{2^i}\right[} \qquad ,\ 1 \le i \le d-1,$$

$$Y_d \;=\; \sum_{k=0}^{2^{d-1}-2} 1_{\left[\frac{2k}{2^d},\frac{2k+1}{2^d}\right[} - \sum_{k=1}^{2^{d-1}} 1_{\left[\frac{2k-1}{2^d},\frac{2k}{2^d}\right[} - 1_{\left[\frac{2^d-2}{2^d},\frac{2^d-1}{2^d}\right[}$$

Since $Y_1 + \ldots + Y_{d-1} - dY_d \ge 1, (X(n))$ is not of type τ^*, where $\tau^* = (1,\ldots,1,-1)$. On the other hand, $(X(n))$ is easily shown to be of type τ for all $\tau \in \{-1,1\}^d \setminus \{\tau^*\}$. In fact, let sgn a, $a \in R$, denote the sign of a which equals 1 if $a > 0$, -1 if $a < 0$, 0 if $a = 0$. Fix any $\tau \in \{-1,1\}^d \setminus \{\tau^*\}$. Then $P(\text{sgn } X_i = -\tau_i, 1 \le i \le d) > 0$. Let $(a_1,\ldots,a_d) \in R^d$ be such that, for all $1 \le i \le d$, sgn $a_i \ge 0$ if $\tau_i = 1$, sgn $a_i \le 0$ if $\tau_i = -1$, and let $a_1X_1 + \ldots + a_dX_d \ge 0$ a.s.. Then $a_1X_1 + \ldots + a_dX_d \le 0$ a.s. on $\{\text{sgn } X_i = -\tau_i,\ 1 \le i \le d\}$ which shows that $P(a_1X_1 + \ldots + a_dX_d < 0) > 0$ if $a_i \ne 0$ for some $1 \le i \le d$. Therefore $a_1 = \ldots = a_d = 0$, and $(X(n))$ is of type τ. Replacing Y by $(s_1Y_1,\ldots,s_dY_d)$ where $s_i \in \{-1,1\},\ 1 \le i \le d$, we obtain a process $(X(n))$ which is of type τ for all $\tau \in \{-1,1\}^d \setminus \{(s_1\tau_1^*,...,s_d\tau_d^*)\}$ but does not have an equivalent martingale measure.

Example 2.8.

Let $Y(0),\ldots,Y(T)$ be i.i.d. real valued random variables defined on $(\Omega,\mathcal{F},P)$. For $n = 0,\ldots,T$ put $S(n) = Y(0) + \ldots + Y(n),\ X(n) = c_n^{-1}S(n)$ and $\mathcal{F}_n = \sigma\{Y(0),...,Y(n)\}$. Here, $c_0,...,c_T$ are denoting strictly positive real numbers. Put $a = \text{ess sup } Y(0),\ b = -\text{ess inf } Y(0)$. We shall assume $a > 0$ and $b > 0$. It turns out that the conditions on the constants c_n under which $(X(n))$ has an equivalent τ–measure only depend on a and b and (possibly) on whether the distribution of $Y(0)$ has positive mass at a or b.

Case 1. $0 < a < \infty$ *and* $0 < b < \infty$.

Let

$$P(Y(0) = -b) > 0 \quad , \; P(Y(0) = a) > 0.$$

Then an equivalent submartingale measure exists iff

$$1 - \frac{a}{nb} < \frac{c_n}{c_{n-1}} < 1 + \frac{1}{n} \quad , \; 1 \leq n \leq T.$$

(Note that this excludes the choice $c_n = n + 1 \quad , \; 0 \leq n \leq T$!) Furthermore, an equivalent martingale measure exists iff

$$1 - \frac{1}{n} \min\left(\frac{a}{b}, \frac{b}{a}\right) < \frac{c_n}{c_{n-1}} < 1 + \frac{1}{n} \quad , \; 1 \leq n \leq T.$$

Applying this to the process $(-X(n))$ gives conditions under which an equivalent supermartingale measure exists. Now assume

$$P(Y(0) = -b) = 0 \quad , \; P(Y(0) = a) > 0.$$

Then an equivalent submartingale measure exists iff

$$1 - \frac{a}{nb} \leq \frac{c_n}{c_{n-1}} < 1 + \frac{1}{n} \quad , \; 1 \leq n \leq T.$$

An equivalent supermartingale measure exists iff

$$1 - \frac{b}{na} < \frac{c_n}{c_{n-1}} \leq 1 + \frac{1}{n} \quad , \; 1 \leq n \leq T.$$

This implies (using Corollary 2.5 (ii)) that an equivalent martingale measure exists in the case $a \geq b$ iff

$$1 - \frac{b}{na} < \frac{c_n}{c_{n-1}} < 1 + \frac{1}{n} \quad , \; 1 \leq n \leq T;$$

it exists in the case $a < b$ iff

$$1 - \frac{a}{nb} \leq \frac{c_n}{c_{n-1}} < 1 + \frac{1}{n} \quad , \; 1 \leq n \leq T.$$

Finally assume

$$P(Y(0) = -b) \; = \; P(Y(0) = a) = 0.$$

Then an equivalent submartingale measure exists iff

$$1 - \frac{a}{nb} \leq \frac{c_n}{c_{n-1}} \leq 1 + \frac{1}{n} \quad , \; 1 \leq n \leq T.$$

An equivalent martingale measure exists iff

$$1 - \frac{1}{n} \min\left(\frac{a}{b}, \frac{b}{a}\right) \leq \frac{c_n}{c_{n-1}} \leq 1 + \frac{1}{n} \quad , \; 1 \leq n \leq T.$$

Case 2. $a = \infty$ *and* $0 < b < \infty$.

Then for any choice of the constants c_n, an equivalent submartingale measure exists. On the other hand, an equivalent supermartingale measure exists in the case $P(Y(0) = -b) > 0$ iff

$$1 \leq \frac{c_n}{c_{n-1}} < 1 + \frac{1}{n} \quad , \ 1 \leq n \leq T;$$

it exists in the case $P(Y(0) = -b) = 0$ iff

$$1 \leq \frac{c_n}{c_{n-1}} \leq 1 + \frac{1}{n} \quad , \ 1 \leq n \leq T.$$

Note that, in view of Corollary 2.5 (ii), the last two claims remain true if "supermartingale" is replaced by "martingale".

Case 3. $a = \infty$ *and* $b = \infty$.

Then, for any choice of the constants c_n, an equivalent martingale measure exists!

We shall verify the above claims only in two typical cases. Assume $0 < a < \infty$, $0 < b < \infty$ and let

$$1 - \frac{a}{mb} = \frac{c_m}{c_{m-1}} \quad \text{for some } \ 1 \leq m \leq T. \tag{2.2}$$

If $P(Y(0) = -b) > 0$, then no equivalent submartingale measure exists. In fact, consideration of

$$h(m) = -1_{\{S(m-1) = -bm\}}$$

shows that $(X(n))$ is not of type (-1). Now assume $P(Y(0) = -b) = 0$ and (2.2). Let $h(m)$ be any nonpositive $\mathcal{F}_{m-1}$–measurable random variable for which

$$\begin{aligned} h(m)(X(m) - X(m-1)) \ &\geq \ 0 \text{ a.s. , i.e.} \\ h(m)\left(Y(m) + \frac{a}{mb}S(m-1)\right) \ &\geq \ 0 \text{ a.s.} \end{aligned} \tag{2.3}$$

Let us verify that this implies $h(m) = 0$ a.s. . In fact, since $Y(m)$ is independent of $\mathcal{F}_{m-1}$ we obtain

$$\begin{aligned} P(h(m) < 0) \ = \ & P\left(h(m) < 0 \quad , -\frac{a}{mb}S(m-1) \leq \frac{a}{2} \ \middle| \ Y(m) > \frac{a}{2}\right) \\ & + \sum_{k=1}^{\infty} P\left(h(m) < 0 \ , \ a\left(1 - \frac{1}{2^k}\right) < -\frac{a}{mb}S(m-1) \ \leq a\left(1 - \frac{1}{2^{k+1}}\right) \ \middle| \ Y(m) > a\left(1 - \frac{1}{2^{k+1}}\right)\right) \\ & + P\left(h(m) < 0 \ , -\frac{a}{mb}S(m-1) = a\right). \end{aligned}$$

This implies $h(m) = 0$ a.s. since, by (2.3),

$$Y(m) \ \leq \ -\frac{a}{mb}S(m-1) \text{ a.s. on } \{h(m) < 0\}.$$

3. Proof of Theorem 2.4

Using an induction argument (see Dalang et al.(1990)) it is not difficult to verify that it suffices to prove Theorem 2.4 in the case $T = 1$. For the rest of the proof we will therefore assume $T = 1$. Suppose that the desired result holds for all processes $(Y(n))$, $n = 0, 1$, of type τ such that $Y(0) \equiv 0$ and $E[\parallel Y(1) \parallel] < \infty$. Then, if $(X(n))$, $n = 0, 1$, is any process of type τ, the process $(Y(n))$ given by $Y(0) \equiv 0$ and $Y(1) = (\max(1, \parallel X(0) \parallel, \parallel X(1) \parallel))^{-1}(X(1) - X(0))$ is of type τ. By assumption, $(Y(n))$ has an equivalent τ–measure Q such that $\xi = dQ/dP$ is $\mathcal{F}_1$–measurable and bounded. Then the probability measure Q^* on $\mathcal{F}$, given by

$$dQ^*/dP = c^*(\max(1, \parallel X(0) \parallel, \ \parallel X(1) \parallel))^{-1}\xi$$

($c^* > 0$ denoting a normalizing constant) is easily checked to be an equivalent τ–measure for $(X(n))$ with the desired properties.

In the sequel we shall therefore consider only processes $(X(n)), n = 0, 1$, such that $X(0) \equiv 0$ and $X(1) = Y \in \mathcal{L}^1_d(\mathcal{F}_1)$. Following the usual notation, we denote e.g. by $L^\infty_d(\mathcal{G})_\tau$ the family of equivalence classes of random variables in $\mathcal{L}^\infty_d(\mathcal{G})_\tau$ (in order to simplify notation, a random variable and the equivalence class it represents will be denoted by the same symbol). For any $W \in \mathcal{L}^1_d$ and $\tau \in \{-1, 0, 1\}^d$ put

$$K_W(\tau) \quad = \quad \{(h, W) \mid h \in L^\infty_d(\mathcal{F}_0)_\tau\}$$

(note that $K_W(\tau)\imath L^1(\mathcal{F}_1)$ if $W \in \mathcal{L}^1_d(\mathcal{F}_1)$). Then $(X(n))$ (or, for short, Y) is of type τ iff

$$K_Y(\tau) \cap L^1_+(\mathcal{F}_1) = \{0\}.$$

The desired result is a consequence of

Theorem 3.1.

For any τ and any $Y \in \mathcal{L}^1_d(\mathcal{F}_1)$, the following properties are equivalent:

(i) $K_Y(\tau) \cap L^1_+(\mathcal{F}_1) = \{0\}$;

(ii) $\overline{K_Y(\tau) - L^1_+(\mathcal{F}_1)} \ \cap \ L^1_+(\mathcal{F}_1) = \{0\}$

(the bar denoting closure w.r.t. the $L^1(\mathcal{F}_1)$–norm);

(iii) *The process $(X(n))$ given by $X(0) \equiv 0$, $X(1) = Y$ has an equivalent τ–measure Q such that dQ/dP is $\mathcal{F}_1$–measurable and bounded.*

(Here, $K_Y(\tau) - L^1_+(\mathcal{F}_1)$ means $\{\eta - \xi \mid \eta \in K_Y(\tau), \xi \in L^1_+(\mathcal{F}_1).\}$)

The proof of Theorem 3.1 uses the following result which is due to Yan (1980) (see also Ansel and Stricker (1990)):

Theorem 3.2.

For any convex set $K \subset L^1(\mathcal{F}_1)$ such that $0 \in K$, the following conditions are equivalent:

(a) For each $\eta \in L^1_+(\mathcal{F}_1), \eta \neq 0$, there exists a constant $c > 0$ such that $c\eta \notin \overline{K - L^\infty_+(\mathcal{F}_1)}$ (the bar denoting closure w.r.t. the $L^1(\mathcal{F}_1)$–norm);

(b) There exists a random variable $\xi \in \mathcal{L}^\infty(\mathcal{F}_1)$ such that $\xi > 0$ a.s. and

$$\sup_{Z \in K} E[Z\xi] < \infty.$$

(Note that the sets $K - L^\infty_+(\mathcal{F}_1)$ and $K - L^1_+(\mathcal{F}_1)$ have the same closure w.r.t. the $L^1(\mathcal{F}_1)$–norm.)

Proof of Theorem 3.1.

It follows from Lemma 2.1 that (iii) implies (i). On the other hand, (ii) implies that Condition (a) in Theorem 3.2 holds for $K = K_Y(\tau)$. Therefore, by Condition (b) in Theorem 3.2, there exists a random variable $\xi \in \mathcal{L}^\infty(\mathcal{F}_1)$ such that $\xi > 0$ a.s., $E[\xi] = 1$ and (since $K_Y(\tau)$ is a cone)

$$E[Z\xi] \leq 0 \text{ for all } Z \in K_Y(\tau) . \tag{3.1}$$

Let Q be the probability measure on $\mathcal{F}$ with density $dQ/dP = \xi$. Using (3.1) one easily verifies that Q is an equivalent τ–measure of $(X(n))$ having the desired properties. This shows that (ii) implies (iii).

In the sequel we shall show that (i) implies (ii) (this will also complete the proof of Theorem 2.4). Since we might replace P by its restriction to $\mathcal{F}_1$, we may (and will) from now on assume $\mathcal{F}_1 = \mathcal{F}$, and we shall denote $\mathcal{F}_0$ by $\mathcal{G}$.

Let $\Omega(i) \in \mathcal{G}$ be disjoint sets such that

$$P(\Omega(i)) > 0 \ , \ i = 1, 2..., \text{ and } P(\Omega(1)) + P(\Omega(2)) + ... = 1.$$

Put

$$\begin{aligned} \mathcal{F}(i) &= \{A \cap \Omega(i) \mid A \in \mathcal{F}\} , \\ \mathcal{G}(i) &= \{A \cap \Omega(i) \mid A \in \mathcal{G}\} , \\ Y(i) &= Y|\Omega(i) \ (Y \text{ restricted to } \Omega(i)) , \\ P_i &= P(\cdot \mid \Omega(i)) \ (P_i \text{ defined on } \mathcal{F}(i)). \end{aligned}$$

Lemma 3.3.

Fix any $\tau \in \{-1, 0, 1\}^d$ and let $Y \in \mathcal{L}^1_d$. If the desired implication (i) $\Rightarrow$ (ii) holds for each $Y(i)$ (the probability space being $(\Omega(i), \mathcal{F}(i), P_i)$ equipped with the filtration $(\mathcal{G}(i), \mathcal{F}(i))$) , then it also holds for Y.

Proof.

Assume that (i) holds for Y. Then (i) holds for each $Y(i)$ which, by assumption, implies

$$\overline{K_{Y(i)}(\tau) - L^1_+(\mathcal{F}(i))} \ \cap \ L^1_+(\mathcal{F}(i)) = \{0\} \text{ for each } i.$$

Therefore each process $\left(X^{(i)}(n)\right)$ given by $X^{(i)}(0) \equiv 0,\ X^{(i)}(1) = Y(i)$, has an equivalent τ–measure Q_i defined on $\mathcal{F}(i)$ such that its density $\xi_i := dQ_i/dP_i$ satisfies $\xi_i \leq c_i$ for some constant $c_i > 0$ because (ii) implies (iii) in Theorem 3.1. Put

$$a_i = \max\left(\frac{c_i}{P(\Omega(i))}\ ,\ E_{Q_i}[\ \|\ Y(i)\ \|\ ]\right)$$

and let $\lambda_i > 0$ be real numbers such that $\lambda_1 + \lambda_2 + ... = 1$ and $\lambda_1 a_1 + \lambda_2 a_2 + ... < \infty$. Then the probability measure Q given by

$$Q(A) = \sum_i \lambda_i Q_i\left(A \cap \Omega(i)\right)\ ,\ A \in \mathcal{F},$$

has a density

$$\frac{dQ}{dP} = \sum_i \frac{\lambda_i}{P(\Omega(i))}\ \xi_i\ 1_{\Omega(i)}$$

such that

$$\frac{dQ}{dP} \leq \sum_i \lambda_i a_i < \infty.$$

It is easily verified that Q is an equivalent τ–measure for $(X(n))$. Combining (2.1) and Theorem 3.2 shows that (ii) holds for Y .

The following simple (but crucial) result provides, for any $Y \in \mathcal{L}^1_d$, a decomposition of Ω into sets $\Omega(i) \in \mathcal{G}\ ,\ i = 1, 2,$ to which Lemma 3.3 will be applied.

Lemma 3.4 (Kabanov and Kramkov (1994)).

For any R^d–valued random variable W there exists a decomposition of Ω into sets $\Omega(i) \in \mathcal{G}, i = 1, 2,$ with the following properties:

(a) for each $h \in \mathcal{L}^\infty_d(\mathcal{G})$ we have that $(h1_{\Omega(1)}, W) = 0$ a.s. implies $h1_{\Omega(1)} = 0$ a.s.;

(b) there exists some $g \in \mathcal{L}^\infty_d(\mathcal{G})$ such that $g \neq 0$ on $\Omega(2)$, $g = 0$ on $\Omega(1)$ and $(g, W) = 0$ a.s.

A decomposition with these properties is unique up to null sets.

The following result suggests that Property (a) should become efficient when combining the Lemmas 3.3 and 3.4.

Lemma 3.5.

Let $Y \in \mathcal{L}^1_d$ and $\tau \in \{-1,0,1\}^d$. Assume that, for all $h \in \mathcal{L}^\infty_d(\mathcal{G})_\tau$, we have that

$$(h, Y) \geq 0 \text{ a.s. implies } h = 0 \text{ a.s.} \tag{3.2}$$

Then

$$\overline{K_Y(\tau) - L^1_+} \cap L^1_+ = \{0\} \ . \tag{3.3}$$

Remark 3.6.

In the case where $\tau = (0, \dots, 0)$, Lemma 3.5 was proved by Kabanov and Kramkov (1994) by using the fact that the closed unit ball in $L^\infty_d(\mathcal{G})$ is weak* sequentially compact. Our proof uses instead the following deep result due to Komlós (1967) which makes the proof shorter.

Theorem 3.7.

Let $(Z_n) \subset \mathcal{L}^1$ be any sequence such that, for some constant c, $E[|Z_n|] \leq c$, $n \geq 1$. Then there exists a random variable $Z_\infty \in \mathcal{L}^1$ and a subsequence (n_k) of indices such that, for any further subsequence $(m_k) \subset (n_k)$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} Z_{m_k} \ = \ Z_\infty \quad a.s.$$

Proof of Lemma 3.5.

If (3.3) does not hold, then there exist $\xi \in \mathcal{L}^1_+$ such that $P(\xi > 0) > 0$, and sequences $(h(n)) \subset \mathcal{L}^\infty_d(\mathcal{G})_\tau$, $(f(n)) \subset \mathcal{L}^1_+$ such that, as $n \longrightarrow \infty$,

$$(h(n), Y) - f(n) - \xi \longrightarrow 0 \text{ a.s. and in mean} \tag{3.4}$$

and

$$(h(n), E[Y|\mathcal{G}]) - E[f(n)|\mathcal{G}] - E[\xi|\mathcal{G}] \longrightarrow 0 \text{ a.s. and in mean.} \tag{3.5}$$

Then, by (3.5),

$$\liminf_{n \to \infty} \|h(n)\| > 0 \text{ a.s. on } A := \{E[\xi|\mathcal{G}] > 0\} \ . \tag{3.6}$$

Since ξ is nonnegative and $P(\xi > 0) > 0$, we have $P(A) > 0$. For $r \in R$ put $r^\oplus = \frac{1}{r}$ if $r \neq 0$, and $r^\oplus = 0$ if $r = 0$. Let

$$\tilde{h}(n) = 1_A \|1_A h(n)\|^\oplus h(n) \ , \quad n \geq 1 \ .$$

Note that $\tilde{h}(n)$ is $\mathcal{G}$–measurable and, as $n \to \infty$,

$$\|\tilde{h}(n)\| = 1_{A \cap \{h(n) \neq 0\}} \longrightarrow 1 \text{ a.s. on } A \ . \tag{3.7}$$

By (3.4) and (3.6),

$$(\tilde{h}(n), Y) - 1_A \|1_A h(n)\|^{\oplus}(f(n) + \xi) \longrightarrow 0 \text{ a.s. as } n \to \infty \ . \tag{3.8}$$

For any $\sigma \in \{-1,1\}^d$ put $I_\sigma = I_{\sigma_1} \times \ldots \times I_{\sigma_d}$, where $I_1 := [0, \infty[$ and $I_{-1} :=]-\infty, 0[$. Let

$$g(\sigma, n) = 1_{I_\sigma}(\tilde{h}(n))\tilde{h}(n) \ , \ \sigma \in \{-1,1\}^d \ , \ n \geq 1 \ .$$

An application of Komlós's theorem shows that we may additionally assume that, for all σ,

$$g(\sigma) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g(\sigma, k) \quad \text{exists a.s.} \tag{3.9}$$

where $g(\sigma) \in \mathcal{L}_d^\infty(\mathcal{G})_\tau$. By (3.8), this gives $(g(\sigma), Y) \geq 0$ a.s. and hence, by (3.2),

$$g(\sigma) = 0 \text{ a.s. for all } \sigma \ . \tag{3.10}$$

Since

$$\sum_\sigma \left(\sigma, \frac{1}{n} \sum_{k=1}^{n} g(\sigma, k) \right) \geq \frac{1}{n} \sum_{k=1}^{n} \|\tilde{h}(k)\| = \frac{1}{n} \sum_{k=1}^{n} 1_{\{h(k) \neq 0\}} 1_A \ ,$$

we obtain from (3.9) and (3.7)

$$\sum_\sigma (\sigma, g(\sigma)) \geq 1 \text{ a.s. on } A$$

which contradicts (3.10) since $P(A) > 0$. This proves Lemma 3.5.

Combining the Lemmas 3.3, 3.4 and 3.5 shows that in order to finish the proof of Theorem 3.1 it suffices to prove that if $Y \in \mathcal{L}_d^1$ is of type τ and if there exists some $g \in \mathcal{L}_d^\infty(\mathcal{G})$ such that

$$g(\omega) \neq 0 \ , \ (g(\omega), Y(\omega)) = 0 \ , \ \omega \in \Omega \ , \tag{3.11}$$

then

$$\overline{K_Y(\tau) - L_+^1} \cap L_+^1 = \{0\} \ .$$

Let $Y \in \mathcal{L}_d^1$ be of type τ and assume (3.11) for some $g \in \mathcal{L}_d^\infty(\mathcal{G})$. Another application of Lemma 3.3 shows that we may additionally assume that, for each $1 \leq i \leq d$, one of the events $\{g_i \geq 1\}$, $\{g_i = 0\}$ and $\{g_i \leq -1\}$ equals Ω. Applying the same permutation to the components of Y and τ and multiplying the same components of Y and τ by minus one leaves $K_Y(\tau)$ unchanged and shows that we may additionally assume that there exists a number $1 \leq \pi \leq d$ such that

$$g_i(\omega) \geq 1 \ , \ 1 \leq i \leq \pi \ , \ g_i(\omega) = 0 \ , \ \pi + 1 \leq i \leq d \ , \ \omega \in \Omega \ . \tag{3.12}$$

Now consider $\tilde{Y} := (g_1 Y_1, \ldots, g_\pi Y_\pi, Y_{\pi+1}, \ldots, Y_d)$. Clearly $\tilde{Y} \in \mathcal{L}_d^1$ and, by (3.12), $K_Y(\rho) = K_{\tilde{Y}}(\rho)$ for all $\rho \in \{-1,0,1\}^d$. In order to finish the proof of Theorem 3.1 it therefore suffices to derive

Lemma 3.8.

Fix $\tau \in \{-1,0,1\}^d$ *and let* $Y \in \mathcal{L}^1_d$ *be such that, for some number* $1 \le \pi \le d$,

$$Y_1 + \ldots + Y_\pi = 0 \ \ on\ \Omega\ . \tag{3.13}$$

If Y *is of type* τ, *then*

$$\overline{K_Y(\tau) - L^1_+} \cap L^1_+ = \{0\}\ . \tag{3.14}$$

We shall prove Lemma 3.8 by induction on d. In the case $d = 1$ Lemma 3.8 is trivial since (3.13) implies $Y \equiv 0$. Now let $d \ge 2$ and assume that Lemma 3.8 holds in the k–dimensional case for any $1 \le k \le d-1$. For any $x \in R^d$ let $\hat{x}(i) \in R^{d-1}$, $1 \le i \le d$, be defined by $\hat{x}(i) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$. Consider any $Y \in \mathcal{L}^1_d$ which is of type τ and satisfies (3.13) for some $1 \le \pi \le d$.

Case 1. $\quad \tau_i \le 0,\ 1 \le i \le \pi$.

It follows from (3.13) that Y is of type ρ where $\rho_i = 0$, $1 \le i \le \pi$, and $\rho_i = \tau_i$, $\pi + 1 \le i \le d$. By (3.13),

$$K_Y(\rho) = K_{\hat{Y}(\pi)}(\hat{\rho}(\pi))\ .$$

Since $\hat{Y}(\pi)$ is of type $\hat{\rho}(\pi)$, the induction hypothesis implies

$$\overline{K_{\hat{Y}(\pi)}(\hat{\rho}(\pi)) - L^1_+} \cap L^1_+ = \{0\}$$

and hence (3.14).

The case where $\tau_i \ge 0$, $1 \le i \le \pi$, can be reduced to Case 1 by noting that

$$K_Y(\tau) = K_{-Y}(-\tau)\ .$$

In the sequel we will assume $\pi \ge 2$ since otherwise $Y_1 \equiv 0$ and therefore $K_Y(\tau) = K_{\hat{Y}(1)}(\hat{\tau}(1))$. Noting again that $K_Y(\tau)$ is left unchanged if the same permutation is applied to the components of Y and τ, and if the same components of Y and τ are multiplied by minus one, it is clear that all remaining cases can be reduced to

Case 2. $\quad$ *For some* $1 \le m \le \frac{\pi}{2}$, *we have* $\tau_i = 1$, $\tau_{m+i} = -1$, $1 \le i \le m$, *and* $\tau_i \le 0$, $2m + 1 \le i \le \pi$.

Then for each $Z \in K_Y(\tau)$ there exists a partition of Ω into sets $A_j \in \mathcal{G}$, $1 \le j \le m$, such that

$$Z1_{A_j} \in K_{\hat{Y}(j)}(\hat{\tau}(j))\ ,\ 1 \le j \le m\ . \tag{3.15}$$

This can be interpreted as follows. Assume (for the moment) that Y_i is the price of security i at time 1. Then Z is the value (at time 1) of some portfolio such that the amounts of securities $1, \ldots, d$ in the portfolio are subject to the constraints given by τ. Now, (3.15) says that there exist m portfolios having total value Z such that, for any $1 \le j \le m$, the j-th portfolio does *not* contain security j, satisfies the constraints given by $\hat{\tau}(j)$ and, finally, has value $Z1_{A_j}$ (1_{A_j} being $\mathcal{G}$–measurable). In order to prove (3.15) note that there exists some $h \in \mathcal{L}_d^\infty(\mathcal{G})_\tau$ such that, by (3.13),

$$Z = \sum_{i=1}^{\pi}(h_i - h_j)Y_i + \sum_{i=\pi+1}^{d} h_i Y_i \ , \ 1 \le j \le m \ .$$

If we put

$$\mu(\omega) = \min\left\{1 \le j \le m \middle| h_j(\omega) = \min_{1\le i\le m} h_i(\omega)\right\} \ , \ \omega \in \Omega \ ,$$

and

$$A_j = \{\mu = j\} \ , \ 1 \le j \le m \ ,$$

then (3.15) follows. Since Y is of type τ, the induction hypothesis implies

$$\overline{K_{\hat{Y}(j)}(\hat{\tau}(j)) - L^1_+} \cap L^1_+ = \{0\} \ , \ 1 \le j \le m \ . \tag{3.16}$$

Combining (3.15) and (3.16) yields (3.14). In fact, let $\xi \in \overline{K_Y(\tau) - L^1_+} \cap L^1_+$. Then there exist sequences $(h(n)) \subset \mathcal{L}_d^\infty(\mathcal{G})_\tau$ and $(g(n)) \subset \mathcal{L}^1_+$ such that

$$\lim_{n\to\infty} E[|(h(n), Y) - g(n) - \xi|] = 0 \ . \tag{3.17}$$

According to (3.15) it follows that, for each $n \ge 1$, there exists a partition of Ω into sets $A_j(n) \in \mathcal{G}$, $1 \le j \le m$, such that

$$(h(n), Y)1_{A_j(n)} \in K_{\hat{Y}(j)}(\hat{\tau}(j)) \ , \ 1 \le j \le m \ , \ n \ge 1 \ .$$

Put

$$S(j,n) = \frac{1}{n}\sum_{k=1}^{n} 1_{A_j(k)} \ , \ 1 \le j \le m \ , \ n \ge 1 \ .$$

An application of Komlós's theorem shows that we may additionally assume that

$$S(j) := \lim_{n\to\infty} S(j,n) \quad \text{exists a.s.}, 1 \le j \le m \ .$$

Putting

$$H(j,n) = \frac{1}{n}\sum_{k=1}^{n} h(k)1_{A_j(k)} \quad , \quad G(j,n) = \frac{1}{n}\sum_{k=1}^{n} g(k)1_{A_j(k)} \ ,$$

we have

$$(H(j,n), Y) \in K_{\hat{Y}(j)}(\hat{\tau}(j)) \ , \ 1 \le j \le m, \ n \ge 1 \ ,$$

and, by (3.17),

$$\lim_{n\to\infty} E[|(H(j,n), Y) - G(j,n) - \xi S(j,n)|] = 0 \ , \ 1 \le j \le m \ .$$

Hence, by the dominated convergence theorem,

$$\lim_{n\to\infty} E[|(H(j,n), Y) - G(j,n) - \xi S(j)|] \ = \ 0, \ \ 1 \le j \le m,$$

which, by (3.16), implies $\xi S(j) = 0$ a.s., $1 \le j \le m$. Since $S(1) + \ldots + S(m) = 1$ a.s., we obtain $\xi = 0$ a.s. This yields (3.14) and finishes the proof of Lemma 3.8.

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