Comments welcome!

Projectbereich B

Discussion Paper No. B-298

Evolution in Partnership Games, an Equivalence Result*

Karl H. Schlag

December, 1994

Wirtschaftstheoretische Abteilung III
Friedrich-Wilhelms-Universität Bonn
Adenauerallee 24-26
D-53113 Bonn
Germany
e-mail: schlag@glider.econ3.uni-bonn.de

---

* Partial financial support by Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 is gratefully acknowledged.

Abstract

A partnership game is a two person game in which both players necessarily receive the same payoff. For symmetric partnership games it is shown that asymptotic stability with respect to the replicator dynamics, evolutionary stability (Maynard Smith and Price [1973], Thomas [1985]) and equilibrium evolutionary stability (Swinkels [1992]) are equivalent concepts. This equivalence result is also derived for asymmetric partnership games, both in the asymmetric contest (Selten [1980]) and in the two population setting (Balkenborg and Schlag [1994]). A side result shows for general games that equilibrium evolutionary stability is weaker than evolutionary stability.

Keywords: evolutionarily stable set, strict equilibrium set, equilibrium evolutionarily stable, local efficiency, replicator dynamics, asymptotic stability, minimal attracting set.

JEL classification number: C79.

0. Introduction

Evolutionary game theory has become a popular tool to select among outcomes of a game. However there are various theoretical approaches that lead to different solution concepts. An issue that previously arose in the refinement literature reappears: which solution concept should be used? We show that such considerations do not arise in partnership games because the major concepts are equivalent in these games. Partnership games are two person games in which both players always receive the same payoff, especially they include coordination games. A side result shows for general games that equilibrium evolutionary stability is weaker than evolutionary stability.

In the following we give a short review of the solution concepts of evolutionary game theory we are referring to.

Maynard Smith and Price [1973] introduced the concept of an evolutionarily stable strategy (ESS) in order to capture intuitively the properties a strategy might need to survive in an evolutionary process. Stability means that a monomorphic population can resist any one time entry of a small frequency of mutants.

Later this concept was related to the replicator dynamics, a dynamic system common in population genetics and biology in general. The relevant solution concept in this context is asymptotic stability. A strategy or set of strategies is called asymptotically stable if trajectories starting close stay close and eventually converge to an element in the set. Taylor and Jonker [1978] and Zeeman [1980] showed that each ESS is asymptotically stable in the replicator dynamics. These replicator dynamics have appeared in other fields too, lately they have especially started emerging from models of human learning (e.g. Binmore, Gale and Samuelson [1993], Börgers and Sarin [1993], Cabrales [1993] and Schlag [1994]).

Thomas [1985] extended the concept of an ESS to sets of strategies (called evolutionarily stable set) and showed that this concept is sufficient for asymptotic stability of sets in the replicator dynamics. Balkenborg and Schlag [1994] reinterpreted the ES set condition as close to the original ESS setting as possible.

Swinkels [1992] weakened the definition of an ESS in a different manner and introduced the concept of an equilibrium evolutionarily stable set. Here mutants may only enter if they get the best payoff once they are in the population. Although elements of singleton EES sets are evolutionarily stable strategies the concept of EES remains an intuitive one because a general connection to some explicit dynamic process has not been found.

In this context we extend the understanding of equilibrium evolutionary stability by showing that connected evolutionarily stable sets are equilibrium evolutionarily stable. The previous proof of a weaker version by Blume, Kim and Sobel [1993] is shown to be incomplete.

The object of this paper is to convey some findings as to the equivalence of various evolutionary concepts in partnership games. It namely turns out that the structure of the payoffs in partnership games is such that the differences apparent in the various concepts are eliminated. One lemma that gives some initial idea of the specific payoff structure in partnership games states that payoffs in connected sets of Nash equilibria are constant.

Concerning the evolutionary analysis we must distinguish two matching situations. In the (classical) symmetric setting the agents can not be distinguished, hence the game is symmetric. In the biological terms this coincides to the case in which all agents belong to the same specie. In the asymmetric setting, there are two kinds of agents, one coinciding to each player in the game. Here, the biological story tells about two species where one specie is matched against the other and then breeds among its own specie.

At first we consider the specific properties of partnership games in the symmetric setting. Early work on partnership games in the context of mendelian dynamics has shown that the average payoff in the population increases over time in the replicator dynamics, a result known as the fundamental theorem of natural selection (see Hofbauer and Sigmund [1988] for the various contributions to this result). Other work by Hofbauer and Sigmund [1988] has shown that evolutionary stability and asymptotic stability of a strategy are equivalent in this class of games.

Our main theorem for the symmetric case shows that the set valued concepts of asymptotic stability in the replicator dynamics, evolutionary stability and equilibrium evolutionary stability are equivalent in partnership games. Moreover such sets always exist and are precisely the sets with payoff allocations that are locally efficient. A set of strategies is called locally efficient (this term is due to Weibull [1994]) if the payoff to a strategy against itself achieves a strict local maximum and is constant on the set. Due to our result, the calculation of e.g. the evolutionarily stable sets in partnership games is reduced to a maximization problem.

We then replicate the above analysis for asymmetric games. Here there are two alternative approaches in the literature. One approach is to symmetrize the game by a random assignment of roles before the game is played. Players are identical and have a strategy they will play when they are player one and a strategy to play in the position of player two. Depending on whether a player uses a tuple of mixed strategies or mixes over tuples of pure strategies the resulting game is called the asymmetric contest (Selten [1980]) or the symmetrized game (Balkenborg and Schlag [1994]). Given this symmetrization the results we derived in the symmetric setup can be applied in a straightforward manner. The only difference is that in a way the resulting concepts are now more stringent. Evolutionarily stable strategies become equivalent to strict Nash equilibria (Selten [1980]) and evolutionarily stable sets are equivalent to strict equilibrium sets (Balkenborg [1994], Balkenborg and Schlag [1994]). Especially we prove the analog to the symmetric setting that each connected strict equilibrium set (and hence each connected evolutionarily stable set) is equilibrium evolutionarily stable.

The alternative approach is to consider the asymmetric game as a contest between two separate populations. Balkenborg and Schlag [1994] apply the notion of evolutionary stability to this two population setting (for earlier related work restricted to extending ESS to this setting see Swinkels [1992] and Cressman [1992]). They show that (analog to the symmetrization approach) ESS in the two population setting is equivalent to the notion of a strict Nash equilibrium and ES set is equivalent to the concept of a strict equilibrium set. Therefore with our analysis of the symmetrized game we again obtain that a connected ES set is equilibrium evolutionarily stable. For our analysis of partnership games in this two population setting we allow for two different versions of the replicator dynamics, one due to Maynard Smith [1982] and one to Taylor [1979]. We derive the fundamental theorem of natural selection for both versions of the dynamics. The subsequent equivalence theorem for the two population setup is then quite analogous to the one that is derived from symmetrizing the asymmetric game.

In the last section we utilize the invariance of the payoff functions to translations and show how our equivalence theorems can be applied to a wider class of games.

1. Preliminaries

We will start out by introducing some notation and by reviewing some equilibrium concepts. For a finite set $A=\{a^1,..,a^N\}$ let ΔA be the set of probability distributions on A, i.e., $\Delta A=\{x\in\Re^N \text{ s.t. } x_i\geq 0 \text{ and } \sum_{i=1}^{N} x_i=1\}$. Consider a two person game in normal form $\Gamma(S_1,S_2,E_1,E_2)$

with the pure strategies $S_1=\{e^i, i=1,..,N_1\}$, $S_2=\{a^j, j=1,..,N_2\}$ and the payoff functions $E_i:\Delta S_1\times\Delta S_2\rightarrow\Re$, i=1,2. The game Γ is called symmetric if $S_1=S_2$ and $E_2(x,y)=E_1(y,x)$ for all $x,y\in\Delta S_1$. In symmetric games we will simplify notation by dropping the indices (e.g.$S=S_1$) and setting $\Gamma(S,E)=\Gamma$.

For $z\in\Delta S_1\cup\Delta S_2$ let C(z) be the support of x, i.e., $C(z)=\{e\in S_1\cup S_2 \text{ s.t. } z(e)>0\}$. For $j\in\{1,2\}$, $i\in\{1,2\}\backslash\{j\}$ and $z\in\Delta S_j$ let $BR_i(z)$ be the set of best replies of player i to the strategy z of player j, i.e., given $(x,y)\in\Delta S_1\times\Delta S_2$, $BR_1(y)=\text{argmax}\{E_1(x',y), x'\in\Delta S_1\}$ and $BR_2(x)=\text{argmax}\{E_2(x,y'), y'\in\Delta S_2\}$. The pair of strategies $(x,y)\in\Delta S_1\times\Delta S_2$ is called a Nash equilibrium if $x\in BR_1(y)$ and $y\in BR_2(x)$.

Next we will review some dynamic stability concepts. Let X be either ΔS_1 or $\Delta S_1\times\Delta S_2$ and consider a dynamic process on X given by the solutions to the differential equation $\dot{x}=f(x)$ where $f:X\rightarrow X$ is Lipschitz continuous. A closed and non empty set $G\subseteq X$ is called attracting if there exists an open neighborhood U of G such that each trajectory starting in U converges to G $(U\subseteq X)$. G is called a minimal attracting set if there is no set G' that is attracting such that $G'\subseteq G$ and $G\neq G'$. Following Zorn's lemma a minimal attracting set always exists. Notice that minimal attracting sets are candidates for the dynamics to get "caught" if mutations are very rare. A strategy $p\in\Delta S$ is called stable if for every open neighborhood U of p there exists an open neighborhood V of p such that the trajectories starting in V do not leave U $(U,V\subseteq X)$. A set $G\subseteq X$ is called an asymptotically stable set (AS set) if it is attracting and each $x\in G$ is stable. The element of a singleton AS set is called an asymptotically stable strategy.

In the following we add some notes on the above definitions. A trajectory starting in W converges to L $(L,W\subseteq X)$ if for any $\bar{x}\in W$ and $(t_k)_{k\in\mathbb{N}}$ such that $t_k\rightarrow\infty$ when $k\rightarrow\infty$ $(t_k\in\Re)$ it follows that inf{dist(x^{t_k} ,z), z∈L}→0 as k→∞ where x^t solves $\dot{x}=f(x)$ starting at $x^0=\bar{x}$ The above

definition of asymptotic stability is slightly stronger than the classical one (see e.g. Bhatia and Szegö [1970]): in the standard definition additional to attracting the set as a whole must be stable, not necessarily each point. Finally, w.l.o.g. we also require additionally to the standard definition for an attracting set to be closed. We find it intuitive to include rest points on the border of an attracting set into the set.

Notice that a consequence of our definition of asymptotic stability is that trajectories starting sufficiently close to such a set will converge to an element of the set (this follows easily from the pointwise stability condition).

2. Evolutionary Solution Concepts (Symmetric Case):

In this section we will review the major concepts of evolutionary game theory together their interdependence relationships. Some original work will be provided regarding equilibrium evolutionary stability. In this section we will restrict attention to the setup in which the agents participating are identical, in the biological setup referred to as the one specie case. Since there is no role identification, the players (or agents) can only be distinguished according to the strategy they play when matched, especially the game that is associated with the matching process must be symmetric. The asymmetric case can be found in sections 5 and 6.

An evolutionarily stable strategy (short, ESS) is a strategy that as a monomorphic population can drive out any one time mutation of a sufficiently small frequency of mutants playing some strategy q. Thomas [1985] extends this notion to sets without giving much intuition.

DEFINITION 2.1: (Thomas [1985])

Let $\Gamma(S,E)$ be a symmetric game. Then

$G \subseteq \Delta S$ is called an evolutionarily stable set (ES set) if

i) G is non empty and closed,

ii) (p,p) is a Nash equilibrium and

iii) for any $p \in G$ there exists an open neighborhood U(p) such that $E(p,x) \geq E(x,x)$ for all $x \in U(p) \cap BR(p)$ and where $E(p,x)=E(x,x)$ implies $x \in G$.

Especially, $p \in \Delta S$ is called an evolutionarily stable strategy (ESS) (Maynard Smith and Price [1973]) if $\{p\}$ is an evolutionarily stable set (ES set).

Balkenborg and Schlag [1994] reinterpret the condition of an ES set along the lines of the original ESS interpretation. A set $G \subseteq \Delta S$ is evolutionarily stable if for sufficiently small mutations the following holds: given $q \in \Delta S$ and $p \in G$ the mutant strategy q can not spread in a population playing p and is driven out if $q \notin G$. This leads to the following intuitive characterization of an ES set.

THEOREM 2.1: (Balkenborg and Schlag [1994])

$G \subseteq \Delta S$ is an evolutionarily stable set if and only if there exists $\epsilon^\circ > 0$ such that for all $p \in G$, $q \in \Delta S$ and $0 < \epsilon < \epsilon^\circ$, $E(p,(1-\epsilon)p+\epsilon q) \geq E(q,(1-\epsilon)p+\epsilon q)$ where the inequality holds strict if $q \notin G$.

The main innovation in the above theorem is that ES sets have a uniform invasion barrier, i.e., ϵ° is independent of p and q.

The concept of an ES set has been shown to be closely related to the stability properties of the so called replicator dynamics.

DEFINITION 2.2: (see Taylor and Jonker [1978]):

The <u>replicator dynamics</u> of $\Gamma(S,E)$ on ΔS for continuous time and pure strategy types is defined as follows:

$x^0 = \bar{x}$ and $\dot{x}^t_i = [E(e^i,x^t) - E(x^t,x^t)]x^t_i$, $i=1,..,N$; $t \geq 0$, (RD)

where $\bar{x} \in \Delta S$ is the initial state and x^t_i is the frequency of the type using strategy e^i ($e^i \in S$) at time t. To simplify notation we will drop the parameter t from the expressions (e.g., $x = x^t$).

The following theorem states that ES sets are asymptotically stable with respect to the trajectories of the continuous replicator dynamics (RD).

THEOREM 2.2 (Thomas [1985]):

If $G \subseteq \Delta S$ is an evolutionarily stable set (ES set) then G is an asymptotically stable set w.r.t. the replicator dynamics (RD). The converse is not true.

We refer to Thomas [1985] for the proof of this theorem. It should be mentioned that both Taylor and Jonker [1978] and Zeeman [1980] previously proved this theorem for the case of singleton sets.

An alternative evolutionary solution concept that is not related to an explicit dynamic process was introduced by Swinkels [1992], here given in its symmetric version.

DEFINITION 2.3: (Swinkels [1992])

A set $G \subseteq \Delta S$ is called (symmetric) equilibrium evolutionarily stable (short, EES) if it is minimal with respect to the following properties:

i) G is closed and non empty,

ii) if $p \in G$ then (p,p) is a Nash equilibrium and

iii) there exists $\epsilon^\circ \in (0,1)$ such that for all $\epsilon \in (0,\epsilon^\circ)$, $p \in G$ and $q \in \Delta S$, if $q \in BR((1-\epsilon)p+\epsilon q)$ then $(1-\epsilon)p+\epsilon q \in G$.

It follows easily that an equilibrium evolutionarily stable set (EES set) is a connected component of $\Delta^{NE}=\{x \in \Delta S \text{ s.t. } x \in BR(x)\}$ (see Swinkels [1992]).

The following theorem shows that equilibrium evolutionary stability is weaker than evolutionary stability.

THEOREM 2.3:

Let $\Gamma(S,E)$ be a symmetric game. If $G \subseteq \Delta S$ is a connected evolutionarily stable set then G is a symmetric equilibrium evolutionarily stable set. In general the converse is not true.

We will first present an example to demonstrate that in general EES sets must not be ES sets. Consider the symmetric game $\Gamma(S,E)$ with strategy set $S=\{T,M,B\}$ and payoffs given in table I.

Table I: A symmetric game with an EES set that is not an ES set.

	T	M	B
T	1,1	1,1	1,1
M	1,1	2,2	0,3
B	1,1	3,0	0,0

It follows that {T} is EES but {T} is clearly not an ES set.

The proof of theorem 2.3 is simple when theorem 2.1 is available since equilibrium evolutionary stability requires a uniform invasion barrier (i.e., ϵ° does not depend on $p\in G$ or q in definition 2.3). In fact, previously a weaker version of theorem 2.3 was claimed by Blume, Kim and Sobel [1993]. However their proof remains incomplete because prior to the result of Balkenborg and Schlag [1994] it was conceivable that an ES set does not have a uniform invasion barrier and consequently must not contain an EES set.

PROOF of theorem 2.3:

Let $G\in\Delta S$ be a connected ES set and let ϵ° be defined as in theorem 2.1. We will first show that G satisfies conditions i) - iii) in the definition of EES.

Parts i) and ii) follow immediately from the definition of an ES set.

We will now show part iii). Let $p\in G$, $\epsilon\in(0,\epsilon^\circ)$, $z\in\Delta S$ and $x=(1-\epsilon)p+\epsilon z$. Assume $z\in BR(x)$. Then $E(z,x)\geq E(p,x)$. Since G is an ES set it follows that $E(p,x)\geq E(x,x)$ (if and only if $E(p,x)\geq E(z,x)$) where equality implies $x\in G$. It follows $E(p,x)=E(x,x)$ and hence $x\in G$.

Finally, G is minimal because an ES set is a connected subset of Δ^{NE} and each EES set is a connected component of Δ^{NE} (see note made after definition 2.3). □

3. Partnership Games

In this section introduce the class of games in which both players have the same payoff function, called partnership games.

DEFINITION 3.1: (Hofbauer and Sigmund [1988])

The two person game $\Gamma(S_1,S_2,E_1,E_2)$ is called a partnership game if $E_1(e,e')=E_2(e,e')$ for all $e\in S_1$ and $e'\in S_2$.

A special property of partnership games is that in any connected set of Nash equilibria the associated payoffs are constant.

LEMMA 3.1:

Let $\Gamma(S_1,S_2,E_1,E_2)$ be a partnership game and $G\subseteq\Delta S_1\times\Delta S_2$ be a connected set of Nash equilibria. Then $(x,y),(x',y')\in G$ implies $E_i(x,y)=E_i(x',y')$, $i=1,2$.

PROOF:

Let $(x,y),(x',y')\in G$. Since G is connected there exists a continuous function $f:[0,1]\rightarrow G$ such that $f(0)=(x,y)$ and $f(1)=(x',y')$.

With $f=(f_1,f_2)$ such that $f_i\in\Delta S_i$, let $a=\max\{t \text{ s.t. } t\in[0,1], E_1(f_1(t),f_2(t))=E_1(x,y)\}$. Assume that $a<1$. Since $f_2(t)\rightarrow f_2(a)$ as $t\rightarrow a$ it follows that there exists $b_1\in(a,1]$ such that $C(f_2(a))\subseteq C(f_2(t))$ and hence $f_2(a)\in BR_2(f_1(t))$ for $t\in(a,b_1)$. From the upperhemicontinuity of the best response correspondence it follows that there exists $b_2\in(a,b_1]$ such that $f_1(t)\in BR_1(f_2(a))$ for all $t\in(a,b_2)$. We therefore obtain for $t\in(a,b_2)$ that

$E_1(f_1(a),f_2(a))=E_1(f_1(t),f_2(a))=E_2(f_1(t),f_2(a))=E_2(f_1(t),f_2(t))=E_1(f_1(t),f_2(t))$ which contradicts the fact that $a<1$ and thus proves the lemma. ☐

4. Evolution in Symmetric Partnership Games

In this section we will analyze the relationship of the various solution concepts in the one population setting (see section two) in partnership games. Since we are now only concerned with the symmetric case (one specie hypothesis) we must restrict our attention to symmetric partnership games.

Previous work on evolution in partnership games relevant to our analysis was undergone by Hofbauer and Sigmund [1988] and is summarized in the following theorem. The first part states that the ESS condition is not only a sufficient (see Taylor and Jonker [1978], Zeeman [1980]) but also a necessary condition for the asymptotic stability of a strategy in partnership games. Part ii) is self explanatory and is referred to as the fundamental theorem of natural selection.

THEOREM 4.1: (Hofbauer and Sigmund [1988])

Let $\Gamma(S,E)$ be a symmetric partnership game. Then

i) A strategy $p \in \Delta S$ is an asymptotically stable strategy of (RD) if and only if p is an evolutionarily stable strategy.

ii) Given $(x^t)_{t\geq 0}$ solves (RD), the average payoff in the population $E(x^t,x^t)$ increases strictly over time if x^0 is not a rest point of (RD).

In our main theorem that is coming up we will need the concept of local efficiency (see Weibull [1994]).

DEFINITION 4.1:

A set $G \subseteq \Delta S$ is called <u>locally efficient</u> if there exists an open neighborhood $U \subseteq \Delta S$ of G such that for $y,z \in G$ and $x \in U \backslash G$, $E(x,x)<E(y,y)=E(z,z)$.

The following characterization theorem shows that in the class of partnership games various evolutionary solution concepts are equivalent and essentially select locally efficient sets.

THEOREM 4.2:

Let $\Gamma(S,E)$ be a symmetric partnership game and let $G \subseteq \Delta S$ be non empty. Then the following statements are equivalent:

i) G is a connected evolutionarily stable set (ES set).

ii) G is a connected asymptotically stable set of (RD).

iii) G is a minimal attracting set of (RD).

iv) G is an equilibrium evolutionarily stable set (EES set).

v) G is connected and locally efficient.

In particular $\mathrm{argmax}\{E(x,x),\ x \in \Delta S\}$ is an evolutionarily stable set (ES set).

PROOF:

"v) $\Rightarrow$ i)":

Let $G \subseteq \Delta S$ be a connected locally efficient set and let U be the corresponding neighborhood of G from definition 4.1. From the continuity of E() it follows that G is closed.

Let $p \in G$ and $y \in \Delta S$. For $\lambda \in [0,1)$ let $p^{\lambda}=(1-\lambda)p+\lambda y$. If λ is sufficiently small then $p^{\lambda} \in U$ and $E(p,p) \geq E(p^{\lambda},p^{\lambda})$ which implies $(2-\lambda)E(p,p) \geq 2(1-\lambda)E(y,p)+\lambda E(y,y)$. It follows that $E(p,p) \geq E(y,p)$ and since y was arbitrary we obtain that (p,p) is a Nash equilibrium.

Let $y \in BR(p) \cap U$. Then $E(p,y)=E(y,p)=E(p,p) \geq E(y,y)$ and $E(p,y)=E(y,y)$ implies $y \in G$. Hence G is an ES set.

"i) $\Rightarrow$ ii)" is stated in theorem 2.2 and "ii) $\Rightarrow$ iii)" follows directly from the definitions.

"iii) $\Rightarrow$ v)":

Let G be a minimal attracting set and let $U \subseteq \Delta S$ be the corresponding open neighborhood of G such that trajectories starting in U will converge to G. Using the fact that G is closed, let $x^{\circ} \in \mathrm{argmax}\{E(x,x),\ x \in G\}$. From part ii) of theorem 4.1 it follows that $E(x^{\circ},x^{\circ})>E(x,x)$ for all $x \in U \backslash G$. Let G° be a (non empty) connected component of $\{x \in G \text{ s.t. } E(x,x)=E(x^{\circ},x^{\circ})\}$. It follows that G° together with U satisfy iv). From the proof of "v) $\Rightarrow$ i)" it follows that G° is an ES set. Since $G^{\circ} \subseteq G$ and G is minimal it follows that $G=G^{\circ}$.

"i) $\Rightarrow$ iv)" is stated in theorem 2.3.

"iv) $\Rightarrow$ v)":

Let $G \subseteq \Delta S$ be a symmetric EES and Γ be a partnership game. Since G is EES it is connected. Let $p \in G$. Lemma 3.1 implies that $G \subseteq \{z \in \Delta S \text{ s.t. } E(z,z)=E(p,p)\}$. Therefore all that is left to show is that there exists an open neighborhood U of p such that $E(x,x)<E(p,p)$ when $x \in U \backslash G$.

Let ϵ_0 be given from the definition of EES. Moreover, let $\epsilon' \in (0,\epsilon_0]$ be such that for any $x \in \Delta S$ and $\epsilon \in (0,\epsilon')$, if $y \in BR((1-\epsilon)p+\epsilon x)$ then $y \in BR(p)$.

Let $K=\{y' \in \Delta S \text{ s.t. } \exists\, y \in \Delta S \text{ and } \epsilon \in [0,\epsilon'] \text{ s.t. } y'=(1-\epsilon)p+\epsilon y\}$. It follows that K is convex. Let $y^\circ \in \operatorname{argmax}\{E(x,x), x \in K\}$. Since $y^\circ \in K$ there exists $y \in \Delta S$ and $\epsilon^\circ \in (0,\epsilon_0)$ such that $y^\circ=(1-\epsilon^\circ)p+\epsilon^\circ y$.

We will now show that $y \in BR(y^\circ)$. Let $z \in BR(y^\circ)$. By definition of K it follows that $z \in BR(p)$. With $z^\circ=(1-\epsilon^\circ)p+\epsilon^\circ z$ it follows that $z^\circ \in K$ and hence $E(y^\circ,y^\circ) \geq E(z^\circ,z^\circ)$. For $\lambda \in [0,1]$ let $y^\lambda=(1-\lambda)y^\circ+\lambda z^\circ$. Then $y^\lambda \in K$ and hence $E(y^\circ,y^\circ) \geq E(y^\lambda,y^\lambda)$ which implies $(2-\lambda)E(y^\circ,y^\circ) \geq 2(1-\lambda)E(z^\circ,y^\circ)+\lambda E(z^\circ,z^\circ)$ and hence $E(y^\circ,y^\circ) \geq E(z^\circ,y^\circ)$. Since $y^\circ,z^\circ \in BR(p)$ it follows that $E(y,y) \geq E(z,y)$. Therefore $E(y,y^\circ) \geq E(z,y^\circ)$ and hence $z \in BR(y^\circ)$ implies $y \in BR(y^\circ)$.

Let $U \subseteq K$ be an open neighborhood of p. Since $y^\circ \in K$, $y \in BR(y^\circ)$ and G is EES it follows that $y^\circ \in G$. Therefore $E(x,x)<E(p,p)$ for all $x \in U \backslash G$ which completes the proof of part ii).

Finally to the "in particular" statement: $G=\{x \in \Delta S \text{ s.t. } E(x,x) \geq E(y,y) \text{ for all } y \in \Delta S\}$ can be split into a union of disjoint connected sets, each trivially satisfying condition iv) of the theorem. Therefore G is an ES set. ☐

One evolutionary concept we omitted from section two is that of weak equilibrium evolutionary stability (see Kim and Sobel [1991]), a weakening of the EES condition. Formally, $G \subseteq \Delta S$ is called a <u>(symmetric) weakly equilibrium evolutionarily stable</u> (symmetric WEES) if it is minimal with respect to i) and ii) from definition 2.3 and the following condition iii'):

iii') There exists $\epsilon_0 \in (0,1)$ such that for all $\epsilon \in (0,\epsilon_0)$, $x \in G$ and $y \in \Delta S$, if $y \in BR((1-\epsilon)x+\epsilon y)$ and $y \in BR(y)$ then $(1-\epsilon)x+\epsilon y \in G$.

Following theorem 4.2 it might be conjectured that WEES and EES coincide in partnership games. The following example shows that this is not the case. Consider the symmetric partnership $\Gamma(S,E)$ with $S=\{T,M,B\}$ and payoffs given in table II.

Table II: A partnership game with a WEES set that is not an ES set.

	T	M	B
T	1,1	1,1	0,0
M	1,1	2,2	3,3
B	0,0	3,3	0,0

$\{T\}$ is not a (symmetric) EES since $M \in BR((1-\epsilon)T+\epsilon M)$ for all $\epsilon>0$. However $\{T\}$ is a (symmetric) WEES since (T,T) is the unique symmetric Nash equilibrium in $BR(T)\times BR(T)$.

5. The Asymmetric Contest and the Symmetrized Game

In this section we will extend the results of the previous sections to asymmetric games by means of symmetrization. Agents are identical as in the one population setting but each agent has a strategy he uses when he is player one and one he uses when he is player two. When two such agents are matched to play the asymmetric game, an independent random outcome assigns one agent to be player one and the other to be player two. If agents choose a mixed strategy for each role then the resulting game is called the asymmetric contest (see van Damme [1991]).

DEFINITION 5.1:

We will call pair $(\Delta S_1\times\Delta S_2, E^c)$ the asymmetric contest of the asymmetric game $\Gamma(S_1,S_2,E_1,E_2)$ if E^c: $(\Delta S_1\times\Delta S_2)\times(\Delta S_1\times\Delta S_2)\to\Re$ is such that $E^c(x,z)=\frac{1}{2}E_1(x_1,z_2)+\frac{1}{2}E_2(z_1,x_2)$ for all $x,z\in\Delta S_1\times\Delta S_2$.

Notice that the asymmetric contest is not a two person (or bimatrix) game in normal form. Never-the-less the concept of an evolutionarily stable set can be defined using the statement in theorem 2.1 (see Balkenborg and Schlag [1994]). Such a set in $\Delta S_1\times\Delta S_2$ will be called a direct evolutionarily stable set. Balkenborg and Schlag [1994] show that a direct evolutionarily stable set of the asymmetric contest is equivalent to a strict equilibrium set (concept due to Balkenborg [1994]) of the original game.

DEFINITION 5.2: (Balkenborg [1994])

Let Γ be an asymmetric game. The non empty set $G\subseteq\Delta S_1\times\Delta S_2$ is called a strict equilibrium set (short, SE set) if for any $(x,y)\in G$ and $(x',y')\in\Delta S_1\times\Delta S_2$,

i) $E_1(x',y)\leq E_1(x,y)$ where equality implies $(x',y)\in G$ and

ii) $E_2(x,y')\leq E_2(x,y)$ where equality implies $(x,y')\in G$.

THEOREM 5.1: (Balkenborg and Schlag [1994])

Let $\Gamma(S_1,S_2,E_1,E_2)$ be an asymmetric game. Then $G \subseteq \Delta S_1 \times \Delta S_2$ is a strict equilibrium set if and only if G is a direct evolutionarily stable set.

An alternative way to encompass mixed strategies in the above symmetrization is to let agents randomize over tuples of pure strategies. This leads to a symmetric game in normal form with the set of pure strategies $S_1 \times S_2$ and the payoff function that is identical to E^c when restricting play to pure strategies.

DEFINITION 5.3: (Balkenborg and Schlag [1994])

Let $\Gamma(S_1,S_2,E_1,E_2)$ be an asymmetric game. Let $E^s:\Delta(S_1 \times S_2) \times \Delta(S_1 \times S_2) \to \Re$ be bilinear such that $E^s(e,e')=E^c(e,e')$ for all $e,e' \in S_1 \times S_2$. Then the symmetric game $\Gamma(S_1 \times S_2, E^s)$ will be called the <u>symmetrized game</u> of $\Gamma(S_1,S_2,E_1,E_2)$.

Notice that if $\Gamma(S_1,S_2,E_1,E_2)$ is a partnership game then so is the symmetrized game. Since the symmetrized game is a symmetric bimatrix game the concept of an evolutionarily stable set is well defined. Balkenborg and Schlag [1994] show that evolutionarily stable sets of the symmetrized game are essentially equivalent to strict equilibrium sets. This equivalence is to be considered with respect to the projection r() on the marginal distributions. Let $r:\Delta\{S_1 \times S_2\} \to \Delta S_1 \times \Delta S_2$ be such that $r_i(p)$ is the marginal distribution of p in ΔS_i. Formally, r() is the unique linear function such that r() is the identity on $S_1 \times S_2$.

THEOREM 5.2: (Balkenborg [1994])

Let $\Gamma(S_1,S_2,E_1,E_2)$ be an asymmetric game. If $G \subseteq \Delta S_1 \times \Delta S_2$ is a strict equilibrium set then $r^{-1}(G)$ is an evolutionarily stable set of the symmetrized game $\Gamma(S_1 \times S_2, E^s)$. Conversely, if $G' \subseteq \Delta(S_1 \times S_2)$ is an ES set of $\Gamma(S_1 \times S_2, E^s)$ then r(G') is a strict equilibrium set.

For asymmetric games there is a separate definition of EES. The relationship to a symmetric EES set of the asymmetric contest will become apparent later.

DEFINITION 5.4: (Swinkels [1992])

A set $G \subseteq \Delta S_1 \times \Delta S_2$ is called <u>equilibrium evolutionarily stable</u> if it is minimal with respect to the following properties:

i) G is closed and non empty.

ii) $(x,y) \in G$ implies (x,y) is a Nash equilibrium.

iii) There exists $\epsilon^\circ > 0$ such that for all $0 < \epsilon < \epsilon^\circ$, $(x,y) \in G$ and $(u,v) \in \Delta S_1 \times \Delta S_2$, if $u \in BR_1((1-\epsilon)y + \epsilon v)$ and $v \in BR_2((1-\epsilon)x + \epsilon u)$ then $(1-\epsilon)(x,y) + \epsilon(u,v) \in G$.

It is easy to show that an EES set is a connected component of the set of Nash equilibria (see Swinkels [1992]). Moreover together with theorems 5.1 and 5.2 we obtain the pendant to theorem 2.3.

THEOREM 5.3:

Let $\Gamma(S_1,S_2,E_1,E_2)$ be an asymmetric game. If $G \subseteq \Delta S_1 \times \Delta S_2$ is a connected strict equilibrium set then G is equilibrium evolutionarily stable. In general the converse is not true.

We will first present an example to show that EES sets must not be strict equilibrium sets. Consider the game Γ with $S_1=\{T,B\}$, $S_2=\{L,R\}$ and payoffs given in table III.

Table III: An asymmetric game with an EES set that is not a strict equilibrium set.

	L	R
T	1,1	0,1
B	1,0	−1,1

It follows that this game has no strict equilibrium set although $\{(1-\lambda)(T,L)+\lambda(T,R), \lambda\in[0,1]\}$ is an equilibrium evolutionarily stable set.

PROOF of theorem 5.3:

Let G be a connected strict equilibrium set. From theorem 5.2 it follows that $G'=r^{-1}(G)$ is an ES set of the symmetrized game $\Gamma(S_1\times S_2,E^s)$. Therefore using theorem 2.3, G' is a symmetric EES of $\Gamma(S_1\times S_2,E^s)$, i.e., there exists $\epsilon°$ such that $p\in G'$, $\epsilon<\epsilon°$ and $q\in BR^s((1-\epsilon)p+\epsilon q)$ implies $q\in G'$ (the superscript s refers to the fact that we are considering the symmetrized game).

We will now show that G is EES given $\epsilon°$ defined above. Conditions i) and ii) in the definition of EES follow from the properties of strict equilibrium sets. Let $\epsilon<\epsilon°$, $(x,y)\in G$ and consider $(u,v)\in BR_1((1-\epsilon)y+\epsilon v)\times BR_2((1-\epsilon)x+\epsilon u)$. With $p,q\in\Delta\{S_1\times S_2\}$ such that $r(p)=(x,y)$ and $r(q)=(u,v)$ it follows that $q\in BR^s((1-\epsilon)p+\epsilon q)$. Since $p\in G'$ it follows that $q\in G'$ and hence $(u,v)\in G$. This shows that condition iii) holds too.

Finally, the minimal property of G follows as in the proof of theorem 2.3 from the fact that an EES set is a connected component of the set of Nash equilibria. ☐

Especially we showed in the above proof that a symmetric EES set of the symmetrized game corresponds via the projection r() to an EES set of the original asymmetric game. Similarly a converse statement can be derived, i.e., if $G\subseteq\Delta S_1\times\Delta s_2$ is an EES set then $r^{-1}(G)$ is a symmetric EES set of the symmetrized game.

Together with theorems 5.1 and 5.2 we are now able to state an equivalence theorem analog to theorem 4.2 for the case where an asymmetric game is symmetrized.

THEOREM 5.4:

Let $\Gamma(S_1,S_2,E_1,E_2)$ be a partnership game and let $G\subseteq\Delta S_1\times\Delta S_2$ be non empty. Then the following statements are equivalent:

i) G is a connected strict equilibrium set (SE set).

ii) $r^{-1}(G)$ is a minimal attracting set of (RD) in the symmetrized game.

iii) G is an equilibrium evolutionarily stable set (EES set).

In particular $\operatorname{argmax}\{E_1(x,y), (x,y)\in\Delta S_1\times\Delta S_2\}$ is a strict equilibrium set.

The proof follows directly from theorems 3.2, 5.2 and 5.3 and from the note made after the proof of theorem 5.3.

6. Evolution in the Asymmetric Two Population Setting

In this section we consider an asymmetric game as a contest between two separate populations. In the biological context, we are considering the case where there are two populations that are matched against each other and breed among themselves. In contrast to the symmetric case games must no longer be symmetric. In this section we will introduce the relevant evolutionary solution concepts.

Evolutionary stability is a very stringent concept in the two population setting. Balkenborg and Schlag [1994] show that the derivation of an evolutionarily stable set for this asymmetric setup leads to the same definition as that of a strict equilibrium set (SE set), due to Balkenborg [1994].

Concerning the replicator dynamics there are two common versions in the asymmetric setup, here distinguished by adding "w/D" and "w/oD".

DEFINITION 6.1:

The replicator dynamics RDw/oD of $\Gamma(S_1,S_2,E_1,E_2)$ on $\Delta S_1\times\Delta S_2$ for continuous time and pure strategy types is defined as follows (see Taylor [1979]):

$x^0=\bar{x},\ y^0=\bar{y},$

$\dot{x}_i=[E_1(e^i,y)-E_1(x,y)]x_i$, $i=1,..,N_1$,

$\dot{y}_j=[E_2(x,a^j)-E_2(x,y)]y_j$, $j=1,..,N_2$; $t\geq 0$. (RDw/oD)

The replicator dynamics RDw/D of $\Gamma(S_1,S_2,E_1,E_2)$ on $\Delta S_1\times\Delta S_2$ for continuous time, pure strategy types and strictly positive payoffs is defined as follows (see Maynard Smith [1982]):

$x^0=\bar{x},\ y^0=\bar{y},$

$$\dot{x}_i=\frac{E_1(e^i,y)-E_1(x,y)}{E_1(x,y)}x_i\ ,\ i=1,..,N_1,$$

$$\dot{y}_j = \frac{E_2(x,a^j)-E_2(x,y)}{E_2(x,y)} y_j \quad , \; i=1,..,N_2; \; t\geq 0. \qquad \text{(RDw/D)}$$

The pendant to theorem 2.2 showing the connection between evolutionary stability and asymptotic stability only exists for RDw/oD.

THEOREM 6.1: (Balkenborg [1994])

If $G\subseteq\Delta S_1\times\Delta S_2$ is a strict equilibrium set then G is an asymptotically stable set w.r.t. the replicator dynamics (RDw/oD).

The pendant to the fundamental theorem of natural selection is easily obtained for two species (compare to part ii) of theorem 4.1).

THEOREM 6.2:

In either replicator dynamics (RDw/D) or (RDw/oD) of a partnership game, the average payoff in the population strictly increases over time if the trajectory is not at a rest point.

PROOF: this is an easy exercise. It uses the following trick, here illustrated for RDw/oD:

$$\frac{d}{dt} E_1(x,y) = \sum_{i=1}^{N_1} E_1(e^i,y) \frac{d}{dt} x_i + \sum_{j=1}^{N_2} E_1(x,e^j) \frac{d}{dt} y_j$$

$$= \sum_{i=1}^{N_1} E_1(e^i,y)[E_1(e^i,y)-E_1(x,y)]x_i + \sum_{j=1}^{N_2} E_1(x,e^j)[E_2(x,e^j)-E_2(x,y)]y_j$$

$$= \sum_{i=1}^{N_1} [E_1(e^i,y)-E_1(x,y)]^2 x_i + \sum_{j=1}^{N_2} [E_1(x,e^j)-E_1(x,y)][E_2(x,e^j)-E_2(x,y)]y_j. \; \square$$

We are now able to state the analog equivalence result to theorems 4.2 and 5.4 for asymmetric partnership games in the two population setting.

DEFINITION 6.4:

A set $G \subseteq \Delta S_1 \times \Delta S_2$ is called <u>locally efficient</u> if there exists an open neighborhood $U \subseteq \Delta S_1 \times \Delta S_2$ of G such that for $(x,y),(w,z) \in G$ and $(u,v) \in U \setminus G$, $E_1(u,v) < E_1(x,y) = E_1(w,z)$.

THEOREM 6.3:

Let $\Gamma(S_1,S_2,E_1,E_2)$ be a partnership game and let $G \subseteq \Delta S_1 \times \Delta S_2$ be non empty. Then the following statements are equivalent:

i) G is a connected strict equilibrium set (SE set).

ii) G is a connected asymptotically stable set of RDw/D1 and RDw/oD.

iii) G is a minimal attracting set of RDw/D[1] and RDw/oD.

iv) G is an equilibrium evolutionarily stable set (EES set).

v) G is connected and locally efficient.

In particular $\operatorname{argmax}\{E_1(x,y), (x,y) \in \Delta S_1 \times \Delta S_2\}$ is a strict equilibrium set.

PROOF:

"i) $\Leftrightarrow$ iv)" is stated in theorem 5.4.

"i) $\Leftrightarrow$ v)": It is easy to verify that G is locally efficient if and only if $r^{-1}(G)$ is locally efficient in the symmetrized game. The rest then follows directly from theorem 5.4 together with theorem 4.2.

"i) $\Rightarrow$ ii)": The statement for RDw/oD is stated in theorem 6.1. Especially it follows that G has no arbitrarily close rest points of RDw/oD. Following theorem 6.2 $E_1(x,y)$ increases when (x,y) is not a rest point. With v) we obtain that G maximizes $E_1(x,y)$ in a neighborhood of G. Therefore asymptotically stability of G follows if we show that G has no close rest points with respect to RDw/D. However since RDw/D and RDw/oD have the same rest points the claim follows.

---

[1] Statements concerning RDw/D are only valid for $E_1>0$ and $E_2>0$.

"ii) $\Rightarrow$ iii)" follows from the definitions.

"iii) $\Rightarrow$ v)" follows as in the symmetric case using theorem 6.2. $\square$

7. Generalization of results to other games

In this section we utilize the invariance of the presented evolutionary concepts and of the trajectories of RD and RDw/oD to translations in the payoff functions. This fact implies that our equivalence theorems in sections 4-6 (limited to RDw/oD) apply to a wider class of games, namely to the games that can be translated into partnership games. Of course, local efficiency properties and the fundamental theorems of natural selection must no longer hold.

DEFINITION 7.1:

We will call the asymmetric game $\Gamma(S_1,S_2,E_1,E_2)$ a transformed partnership game if there exist $\{\alpha_i,\ 1\le i\le N_1\}$ and $\{\beta_j,\ 1\le j\le N_2\}$ such that $\Gamma(S_1,S_2,E^\circ_1,E^\circ_2)$ is a partnership game where $E^\circ_1(e^i,y)=E_1(e^i,y)+\alpha_i$ and $E^\circ_2(x,a^j)=E_2(x,a^j)+\beta_j$ for $1\le i\le N_1$, $1\le j\le N_2$, $x\in\Delta S_1$ and $y\in\Delta S_2$.

We will call the symmetric game $\Gamma(S,E)$ a transformed partnership game if $\Gamma(S,S,E,E)$ is a transformed partnership game where $\alpha_i=\beta_i$ for $1\le i\le N$.

Notice that every symmetric game with $|S|=2$ is a transformed partnership game. The following lemma states the invariance of the concepts needed in theorems 4.2, 5.4 and 6.3 to translations. The proof is an easy exercise.

LEMMA 7.1:

Given $\Gamma(S,E)$ the trajectories of RD and the concepts ES set and (symmetric) EES are invariant to a translation of the payoff function E (as given in definition 7.1).

Given $\Gamma(S_1,S_2,E_1,E_2)$, the trajectories of RDw/oD and the concepts SE set and EES are invariant to a translation of the payoff functions E_1 and E_2 (as given in definition 7.1).

We now come to the final result.

COROLLARY 7.1:

Theorem 4.2 without part v) and theorem 5.4 without part holds for symmetric transformed partnership games. Theorem 5.3 applied to RDw/oD omitting part v) holds for transformed partnership games.

Combining the note made after definition 5.1 with the above corollary it follows that in symmetric games with two strategies for each player the presented evolutionary solution concepts are equivalent.

References

Balkenborg, D. (1994). "Strictness and Evolutionary Stability," Mimeo, University of Bonn.

Balkenborg, D., Schlag, K. H. (1994). "Evolutionarily Stable Sets in Symmetric and Asymmetric Population Games," Mimeo, University of Bonn.

Bhatia, N. P., Szegö, G. P. (1970). *Stability Theory of Dynamical Systems*. Berlin: Springer-Verlag.

Binmore, K. G., Gale, J., Samuelson, L. (1993). "Learning to be Imperfect: the Ultimatum Game," forthcoming in *Games Econ. Beh.*.

Blume, A., Kim, Y.-G., Sobel, J. (1993). "Evolutionary Stability in Games of Communication," *Games Econ. Beh.* **5**, 547-575.

Börgers, T., Sarin, R. (1993). "Learning Through Reinforcement and Replicator Dynamics," UCL Disc. Paper No. 93-19.

Cabrales, A. (1993). "Stochastic Replicator Dynamics," Mimeo, University of California, San Diego.

Cressman, R. (1992). *The Stability Concept of Evolutionary Game Theory*, Springer-Verlag, Berlin•Heidelberg.

van Damme, E. (1991). *Stability and Perfection of Nash Equilbria*, Springer Verlag, Berlin.

Hofbauer, J., Sigmund, K. (1988). *The Theory of Evolution and Dynamical Systems. Mathematical Aspects of Selection*. Cambridge: Cambridge University Press.

Kim, Y.-G., Sobel, J. (1991). "An Evolutionary Approach to Pre-Play Communication," Mimeo, Dec. 1991.

Maynard Smith, J. (1982). *Evolution and the theory of games*. Cambridge: Cambridge University Press.

Maynard Smith, J., Price, G. R. (1973). "The Logic of Animal Conflict," *Nature* **246,** 15-18.

Selten, R. (1980). "A Note on Evolutionarily Stable Strategies in Asymmetric Animal Conflicts," *J. Theor. Biol.* **84**, 93-101.

Schlag, K. H. (1994). "Why Imitate, and if so, How? Exploring a Model of Social Evolution," Disc. Paper No. **B-296**, University of Bonn.

Swinkels, J. M. (1992). "Evolutionary Stability with Equilibrium Entrants," *J. Econ. Theory* **57**, 306-332.

Taylor, P. (1979). "Evolutionarily Stable Strategies With Two Types of Players," *J. Applied Prob.* **16**, 76-83.

Taylor, P. D., Jonker, L. B. (1978). "Evolutionarily Stable Strategies and Game Dynamics," *Math. Biosc.* **40**, 145-156.

Thomas, B. (1985). "On Evolutionarily Stable Sets," *J. Math. Biol.* **22**, 105-115.

Weibull, J. W. (1994). *Evolutionary Game Theory*. Forthcoming in MIT Press.

Zeeman. E. C. (1980). "Population Dynamics from Game Theory," in *Global Theory of Dynamical Systems* (Z. Nitecki, C. Robinson, Eds.), pp. 471-497, *Lecture Notes in Mathematics*, **819**, Berlin:Springer-Verlag.