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On the Disaggregation of Excess Demand Functions when Markets are Incomplete: The Case of Nominal Returns.

by

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Abstract

We demonstrate that locally, Walras Law and Homogeneity characterize the structure of market excess demand functions when financial markets are incomplete and assets' returns are nominal. As an application it is shown that – in contrast to the complete markets case – when markets are incomplete there is no general rule describing the effects which monetary policy has on asset prices.

Keywords: excess demand functions, incomplete markets, monetary policy
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1 Introduction

On compact sets of prices Walras' Law and Homogeneity characterize the structure of market excess demand functions in the Arrow-Debreu model. This result, that goes back to Sonnenschein (1973), Mantel (1974) and Debreu (1974), has sometimes been attributed to the fact that in the Arrow-Debreu model the trading mechanism is overly simplified. In this model a complete set of contingent contracts is traded at the beginning of time, and there is no further trade in later periods.

A possibly more realistic setting is considered in the General Equilibrium Incomplete Markets' (abbreviated GEI) model, where we have a sequence of spot markets, connected via an incomplete system of financial markets. Bottazzi and Hens (1996) have shown that the result of Debreu (1974) can be extended to the GEI-model with real assets. This result exploits the fact that with real assets individual demand exhibits similar properties as when markets are complete. Hence the structure of the argument used in Debreu's proof of the excess demand decomposition can be extended to this case.

This is no longer true when assets are nominal. In this case individual demand has some distinctive features which do not allow an immediate parallel with the complete market case. In particular, the dimension of the price domain over which demand is defined is larger, with nominal returns, than the dimension of the space where demand lies. Since by a revealed preference argument the space where demand moves as prices vary has to be of a lower dimension than the space where it lies, the larger dimension of the price domain imposes some restriction on the way demand varies with prices.[1] Furthermore, unlike for the real asset case, a decomposition of the Jacobian of agents' demand into two terms describing an income and a substitution effect is no longer possible. Also the conjecture was advanced (see Mas-Colell (1986)) that with nominal returns Walras' Law and Homogenity do not suffice to characterize the structure of market excess demand when financial markets are incomplete.

The purpose of this paper is to show that the Sonnenschein-Mantel-Debreu result nevertheless extends to the case of incomplete financial markets with nominal returns. The claim will be proved by considering the Jacobian of the aggregate demand function. By the above considerations a simple extension of the argument provided for the complete market case (see in particular Geanakoplos-Polemarchakis (1980)) is not possible with incomplete markets and nominal re-

[1]Building on this, John Geanakoplos (1994) gave an example of a function, satisfying continuity and the budget constraint with nominal incomplete asset markets which cannot be rationalized as an individual agent's demand.

turns. A different argument is then presented. The result will be established through a series of intermediate results which are of independent interest.

As an interesting application we consider the effects of monetary policy on equilibrium asset prices. In doing this we follow Magill and Quinzii (1992) and close the model by adding to the market clearing conditions a simple system of equations implying the validity of the quantity theory of money. In this setting an increase in the money supply in some state of nature proportionally increases the spot price level in that state. When financial markets are complete, one gets then a simple rule describing the transmission of monetary policy to equilibrium asset prices: prices of assets with positive pay-off in that state will decrease while those with negative pay-offs increase. Our result shows that with incomplete markets equilibrium asset prices can move in any direction, following a given change in monetary policy. Hence no statement on a money transmission rule can be made without the knowledge of the characteristics of the agents' utility functions and endowments.

The rest of this paper is organized as follows. In the next section we describe the GEI-model with nominal returns. Thereafter we characterize the local structure of an individual's excess demand. In the main part the disaggregation result is established. Finally, as an application, we demonstrate that with incomplete markets the effects of monetary policy on asset prices (and welfare) can go in any direction.

2 The Model

We consider here a standard two period general equilibrium model with an incomplete set of financial assets whose payoffs are denominated in abstract units of account. We will assume that in every spot market a single commodity is available. This assumption is mainly motivated by the need to make the argument, and the notation, clearer and should not be viewed as being restrictive. Our main focus is in fact on the consequences of the nominal structure of returns and, in any case, our results extend to economies with several goods.[2] The details of the model are as follows.

There is a finite set of agents $h = 1, \ldots, H$ symmetrically informed about the uncertainty in the economy. At date $t = 0$ the agents take portfolio decisions and make consumption plans, knowing that at $t = 1$, when assets pay off and

[2]This follows from an appropriate combination of the results in this paper with the disaggregation result for economies with complete markets (as in each state there is a complete set of spot markets).

consumption plans are carried through, one of S states of the world occurs. There is no first period consumption; the consumption set is then $\mathbb{R}^S_+$, the non-negative orthant of the S-dimensional Euclidean space. It is common to all agents. Agent's endowments w^h are vectors in $\mathbb{R}^S_+$ and their utility functions $U^h(\cdot)$ evaluate consumption plans $x \in \mathbb{R}^S_+$.

In each state of the world, $s = 1, \ldots, S$, a spot market opens, where the consumption good is traded. Let p_s denote the price of this good in state s, $s = 1, \ldots, S$. It is expressed in terms of abstract units of account in which assets' returns are denominated; $p \in \mathbb{R}^S$ is the vector of these prices.

To transfer income across the various spot markets, the agents can buy and sell – without any short-sale restriction – a finite set of assets, $j = 1, \ldots, J$. Let $\theta^i \in \mathbb{R}^J$ denote the portfolio of assets that agent i purchases at $t = 0$. The agents' portfolios have to be self-financing, i.e. the budget restriction $q \cdot \theta^i \leq 0$ has to be satisfied for all vectors of asset prices $q \in \mathbb{R}^J$. Asset j delivers r^j_s units of account if state s occurs. Let $R = (r^j_s)_{\substack{s=1,\ldots,S \\ j=1,\ldots,J}}$ denote the $(S \times J)$ payoff matrix describing the structure of the financial markets. Asset markets are said to be complete if the asset structure is sufficiently rich to allow agents to transfer income freely across all states, i.e. if rank $R = S$. They are incomplete if rank $R < S$.

Throughout this paper we will assume that there are no redundant assets. Furthermore, we assume that there are portfolios which are strictly desirable. More precisely:

Assumption 1

rank $R = J$ *and* $R\theta > 0$ *for some* $\theta \in \mathbb{R}^J$.

This assumption guarantees that agents with strictly monotonic utility functions satisfy the budget restriction with equality.

Let $r^j \in \mathbb{R}^S$ denote the j-th column of R and $r_s \in \mathbb{R}^J$ denote the s-th row of R. Using this notation, the agent's decision problem, (P^h), can be written as

$$(P^h) \qquad \max_{\substack{x \in \mathbb{R}^S_+ \\ \theta \in \mathbb{R}^J}} \quad U^h(x)$$

$$q \cdot \theta \leq 0$$

$$p_s(x_s - w^h_s) \leq r_s \theta \ , \quad s = 1, \ldots, S \ .$$

Let $\theta^h(q,p)$ and $x^h(q,p)$ be the optimal solution of (P^h) at prices (q,p). Also we will denote by $\Lambda(\cdot)$ an operator that transforms a vector into a diagonal matrix with the vector on its main diagonal. Thus $\Lambda(p) = \begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_s \end{bmatrix} \in \mathbb{R}^{S\times S}$.

Furthermore, let $v_s = \frac{1}{p_s}$, $s = 1, \ldots, S$, so that $\Lambda(v) = \Lambda^{-1}(p)$.

Agents are characterized as follows:

Assumption 2

For all $h = 1, \ldots, H$, $w^h \in \mathbb{R}^S_{++}$ and $U^h : \mathbb{R}^S_+ \longrightarrow \mathbb{R}$ is differentiably strictly monotonic and quasi-concave, i.e.

(i) U^h is twice continuously differentiable,

(ii) $DU^h(x) >> 0$ for all $x \in \mathbb{R}^S_{++}$,

(iii) $y^T D^2U^h(x)y \leq 0\ \forall y$; in addition, for any vector $v \in \mathbb{R}^S_{++}$, $y^T(R^T\Lambda(v)D^2U^h(x)\Lambda(v)R)y < 0$ for all y such that $DU^h(x)^T\Lambda(v)Ry = 0$.

(iv) closure $\{y \in \mathbb{R}^S_+ \mid U^h(y) \geq U^h(x)\} \subset \mathbb{R}^S_{++}$

Assumption 2 is a direct translation of Debreu's (1970) assumption of smooth preferences to our framework[3]. It ensures that the agents decision problem has a unique solution which can be characterized by its first order conditions.

Under assumption 2, a necessary (and sufficient) condition for the existence of a solution to the agents' problem (P^h) is that asset prices are arbitrage-free:

$$\nexists\ \theta \in \mathbb{R}^J :\ q \cdot \theta \leq 0 \ \text{ and } \ R\theta > 0\ . \tag{1}$$

It is well known that this condition restricts asset prices to lie in the interior of the positive cone spanned by the rows of the matrix R, i.e. $q^T = \pi^T R$ for some $\pi >> 0$.

A competitive equilibrium is a vector of asset prices, a vector of spot prices and an allocation of assets and consumption plans such that agents optimize and markets clear:

Definition 1

An asset-spot-market equilibrium is an array $(\overset{*}{q}, \overset{*}{p}, \overset{*}{\theta}, \overset{*}{x}) \in \mathbb{R}^J \times \mathbb{R}^S_+ \times \mathbb{R}^{JH} \times \mathbb{R}^{SH}_+$ such that

(i) $(\overset{*}{\theta}, \overset{*}{x})^h$ solves (P^h) at the prices $(\overset{*}{q}, \overset{*}{p})$, $h = 1, \ldots, H$

[3]Condition (iii) is a relaxation of strict quasi-concavity, required to hold only on the marketed subspace.

(ii) $\sum_{h=1}^{H} \overset{*}{\theta}^{h} = 0$

(iii) $\sum_{h=1}^{H} \overset{*}{x}^{h} = \sum_{h=1}^{H} w^h$.

It will be useful to compare asset-spot-market equilibria with Arrow-Debreu equilibria.

Definition 2

An Arrow-Debreu equilibrium is an array $(\overset{*}{p}, \overset{*}{x}) \in \mathrm{I\!R}^S_+ \times \mathrm{I\!R}^{SH}_+$ *such that*

(i) $\overset{*}{x}^{h} \in \arg\max_{x \in \mathrm{I\!R}^S_+} U^h(x)$ *s.t.* $\overset{*}{p} \cdot x \leq \overset{*}{p} \cdot w^h$, $h = 1, \ldots, H$

(ii) $\sum_{h=1}^{H} \overset{*}{x}^{h} = \sum_{h=1}^{H} w^h$.

Recall that if markets are complete then the allocations $\overset{*}{x}$ of the asset-spot-markets equilibria coincide with those of the Arrow-Debreu equilibria, (Arrow (1953)).

Counting equations and unknowns we can see that the asset-spot-market equilibrium system $\sum_h \theta^h(q,p) = 0$, $\sum_h (x^h(q,p) - w^h) = 0$ is indeterminate. The system has $J + S$ unknowns and $J + S$ equations. However, due to the $S + 1$ budget equations, there are $S + 1$ Walras Laws restricting the equilibrium equations so that if all but $S + 1$ markets are in equilibrium then all markets clear. On the other hand, there are only 2 degrees of homogeneity. If in the decision problem (P^h) q is replaced by λq for some $\lambda > 0$, then the agent's choice is unaffected. Furthermore, if p changes to λp for $\lambda > 0$, the agents can still afford the same consumption plan as at p by scaling up the portfolio θ by the factor λ. These two are the only homogeneity properties.

Hence there are $S - 1$ Walras Laws more than homogeneity properties and consequently the system of equilibrium equations is indeterminate.

With complete markets, given the allocational equivalence with the Arrow-Debreu model this indeterminacy will only be nominal, whereas with incomplete markets the indeterminacy is real, as shown by Balasko and Cass (1989) and by Geanakoplos and Mas-Colell (1989)). Therefore with incomplete markets, the dimension of the set of the equilibrium consumption allocations is $S - 1$ whereas with complete markets it is zero. To get an intuitive understanding of this result one could think of the equilibrium equations as being solved in terms of the asset prices q for a given p. Thus the spot prices can be taken as parameters of the system of equilibrium equations. As p changes the matrix of 'real asset payoffs' $\Lambda(v)R$ changes too, and this will affect the risk sharing opportunities of the agents, if markets are incomplete.

Evidently, this indeterminacy will disappear if $S-1$ extra independent equations are introduced, determining the price level in the various states of the world. A simple argument allowing this is based on an application of the quantity theory of money. This theory postulates that in each state s the volume of transactions $p_s \sum_{h=1}^{H} x_s^h$, is proportional to the money supply in that state, m_s , $s = 1, \ldots, S$.[4] For our purpose here it will be sufficient to introduce the following additional equations:

$$p_s \sum_{h=1}^{H} w_s^h = \gamma_s m_s \quad s = 1, \ldots, S \ ,$$

where $\gamma_s > 0$ is the velocity of circulation of money in state s. Monetary policy, by fixing $m = (m_1, \ldots, m_s)$, fixes the values of the S spot prices. Hence, the set of equilibrium allocations is now parameterized by the value m of monetary policy.[5]

2.1 Individual Comparative Statics

In this section the properties of the individual Jacobian matrix, $\partial_{(q,p)}(\theta, x)^h(q, p)$, are analyzed. This will lay the foundations for the disaggregation result in the main part of the paper.

The budget restrictions and the homogeneity properties of demand imply a set of general restrictions on the Jacobian. From $q \cdot \theta^h(q,p) = 0 \ \ \forall q, p$, the date 0 budget equation, we get:

$$q^T \partial_q \theta^h(q,p) = -{\theta^h}^T \ , \quad q^T \partial_p \theta^h(q,p) = 0 \ , \tag{2}$$

The remaining budget equations, $\left(x^h(q,p) - w^h\right) = \Lambda(v) R \theta^h(q,p)$, then im-

[4]See Magill and Quinzii (1992) for an explicit modeling of 'cash in advance' constraints leading to the quantity theory of money in this framework; also Gottardi (1994).

[5]In effect we have introduced S additional equations. However, these are not homogeneous with respect to p, so that there is now only one homogeneity property of the equilibrium system, and the number of independent equations is then equal to the number of independent variables.

ply:

$$\begin{aligned} \partial_q x^h(q,p) &= \Lambda(v) R \partial_q \theta^h(q,p) \;, \\ \partial_p x^h(q,p) &= -\Lambda^2(v)\Lambda(x^h(q,p)-w^h) + \Lambda(v) R \partial_p \theta^h(q,p) \end{aligned} \tag{3}$$

The homogeneity properties are

$$\theta^h(\lambda q, p) = \theta^h(q,p) \;, \;\; x^h(\lambda q, p) = x^h(q,p) \;,$$

from which we obtain

$$\partial_q \theta^h(q,p) \cdot q = 0 \;, \;\; \partial_q x^h(q,p) \cdot q = 0 \;. \tag{4}$$

Further homogeneity properties are

$$\theta^h(q, \lambda p) = \lambda \theta^h(q,p) \;, \;\; x^h(q, \lambda p) = x^h(q,p)$$

which implies

$$\partial_p \theta^h(q,p) \cdot p = \theta^h \;, \;\; \partial_p x^h(q,p) \cdot p = 0 \tag{5}$$

Remark 3.1

When markets are complete we get some additional homogeneity properties for the agents' demand function. To derive them, note that the no arbitrage condition implies that the set of attainable consumption $\{x \in \mathbb{R}^S_+ \mid \exists \theta : \Lambda(p)(x - w) = R\theta \;, \;\; q \cdot \theta \leq 0\}$, is equivalent to the set $\{x \in \mathbb{R}^S_+ \mid \;\; \pi^T \Lambda(p)(x-w) \leq 0 \;\; \text{and} \;\; \Lambda(p)(x-w) \in < R >\}$.[6] With complete markets the latter reduces to $\{x \in \mathbb{R}^S_+ \mid \;\; \pi^T \Lambda(p)(x-w) \leq 0\}$. Let $\bar{x}^h(\Lambda(p)\pi) := \arg\max U^h(x)$ s.t. $x \in \{x \in \mathbb{R}^S_+ : \pi^T\Lambda(p)(x-w) \leq 0\}$ and let $\Lambda(p)\pi = \bar{p}$. With complete markets $\bar{x}^h(\Lambda(p)\pi) = x^h(q,p) = x^h(R^T\pi, p)$. Differentiating this identity with respect to π yields $\partial_q x^h R^T = \partial_{\bar{p}} \bar{x}^h \Lambda(p)$ and differentiating it with respect to p gives $\partial_p x^h = \partial_{\bar{p}} \bar{x}^h \Lambda(\pi)$. Combining these two relations we get the following additional property of the Jacobian, $\partial_{(q,p)}(\theta, x)^h(q,p)$, when markets are complete:

$$\partial_p x^h(q,p) = \left(\partial_q x^h(q,p)\right) R^T \Lambda(\pi)\Lambda(v) \;. \tag{6}$$

The following proposition characterizes the structure of an individual's Jacobian matrix.

[6] We denote by $< R >$ the linear subspace generated by the columns of R.

Proposition 1

Suppose assumptions 1, 2 hold, $p \in \mathbb{R}^S_{++}$ and q is arbitrage free. Then

$$\partial_{(q,p)}(\theta, x)^h(q,p) =$$

$$\begin{bmatrix} K_0^h & K_0^h R^T \Lambda(v) \\ \Lambda(v) R K_0^h & \Lambda(v) R K_0^h R^T \Lambda(v) \end{bmatrix} \Lambda \begin{pmatrix} 1\!\!\mathrm{I}_J \lambda_0^h \\ \lambda_1^h \end{pmatrix} - \begin{bmatrix} v_0^h \theta^{hT} & 0 \\ 0 & \Lambda(v)\Lambda(x^h - w^h) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & K_0^h R^T \Lambda(v) D^2 U^h \Lambda(v) \Lambda(x^h - w^h) \\ -\Lambda(v) R v_0^h \theta^{hT} & \Lambda(v) R K_0^h R^T \Lambda(v) D^2 U^h \Lambda(v) \Lambda(x^h - w^h) \end{bmatrix} \tag{7}$$

where $K_0^h = M^h + \frac{v_0^h v_0^{hT}}{\alpha_0^h}$, $M^h = [R^T \Lambda(v) D^2 U^h \Lambda(v) R]^{-1}$, $v_0^h = -\alpha_0^h M^h q$, $\alpha_0^h = -(q^T M^h q)^{-1}$, $1\!\!\mathrm{I}_J = (1, \ldots, 1)^T \in \mathbb{R}^J$, *and* λ_0^h, $\lambda_1^h = \Lambda(v) D U^h$ *are the Lagrange multipliers associated with date* 0 *and date* 1 *budget constraints.*

Proof

From the budget identities it follows that $\partial_{q,p} x^h$ can be derived from $\partial_{q,p}\theta^h$ (see equation (3)). Hence it suffices to consider $\partial_{q,p}\theta^h$; this can be obtained from the solution of the problem

$$(P_\theta)^h \qquad \max_{\theta \in \mathbb{R}^J} \quad U^h(w^h + \Lambda(v) R\theta)$$
$$q \cdot \theta \le 0$$
$$w^h + \Lambda(v) R \theta \ge 0 \ .$$

Since U^h satisfies assumption 2, the following first order conditions characterize the solutions of the problem (P_θ^h).

$$(FOC) \qquad DU^h(w^h + \Lambda(v)R\theta)^T \Lambda(v) R = \lambda_0^h q^T \ \text{ and } \ q \cdot \theta = 0 \ .$$

Totally differentiating (FOC) yields

$$\begin{bmatrix} R^T \Lambda(v) D^2 U^h \Lambda(v) R & -q \\ -q^T & 0 \end{bmatrix} \begin{pmatrix} d\theta^h \\ d\lambda_0^h \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_0^h I \\ \theta^{hT} \end{pmatrix} dq + R^T \begin{pmatrix} \Lambda(v) D^2 U^h \Lambda(R\theta^h)\Lambda(v) + \Lambda(\lambda_1^h) \\ 0 \end{pmatrix} \Lambda(v) dp \tag{8}$$

From the strict quasi-concavity of $U^h(\cdot)$, the coefficient matrix on the left-hand side of (8) is invertible. Let

$$\begin{bmatrix} R^T \Lambda(v) D^2 U^h \Lambda(v) R & -q \\ -q^T & 0 \end{bmatrix}^{-1} =: \begin{bmatrix} K_0^h & -v_0^h \\ -v_0^{hT} & \alpha_0^h \end{bmatrix} .$$

It is easy to see that K_0^h, v_0^h and α_0^h are as in the statement of the proposition. Substituting the above into (8) and using (3) yields (7).

□

Note that $v_0^h \cdot q = 1$ and hence $K_0^h q = 0$, $q^T K_0^h = 0$. Furthermore, it can be verified that K_0^h is a $(J \times J)$ negative semi-definite matrix, negative definite on $q^\perp$. The matrix K_0^h exhibits then the properties of a substitution matrix. It can be shown that it is in fact the Slutsky substitution matrix arising from the solution of problem (P_θ^h), for a given p. The matrix appearing in the first term of (7) can be written as $[I, R^T\Lambda(v)]^T K_0^h [I, R^T\Lambda(v)]$. Thus it also is negative semi-definite and symmetric. However, its interpretation as a substitution matrix, and an interpretation of the matrices appearing in the second and third terms of (7) as income effects terms is not warranted.

A simple but important observation is that there is an immediate converse to Proposition 1.

Proposition 2

Let R, q, v be, respectively, the asset payoff matrix, satisfying assumption 1, and the vectors of asset and (the inverse of) commodity prices. Let A be a $(S+J)\times(S+J)$ matrix. If there is a $S \times S$ matrix $\bar{D}$, and vectors $(\lambda_0, \lambda_1^T)^T, z, \theta \in \mathbb{R}_{++}^{S+1} \times \mathbb{R}^S \times \mathbb{R}^J$ such that:

(i) $\bar{D}$ is symmetric and satisfies assumption 2(iii),

(ii) $q \cdot \theta = 0$,

(iii) $z = \Lambda(v)R\theta$,

(iv) $\lambda_1^T R = \lambda_0 \cdot q^T$

and

$$\begin{aligned} A \;=\; & [I_J, R^T\Lambda(v)]^T \left[M - \frac{Mqq^TM}{q^TMq}\right] \left[I_J, R^T\Lambda(v)\right] \Lambda \begin{pmatrix} \lambda_0 \mathbb{1} \\ \lambda_1 \end{pmatrix} \\ & + [I_J, R^T\Lambda(v)]^T \left(0, \left[M - \frac{Mqq^TM}{q^TMq}\right] R^T\Lambda(v)\bar{D}\Lambda(v)\Lambda(z)\right) \\ & - [I_J, R^T\Lambda(v)]^T \left(\frac{Mq\theta^T}{q^TMq}, 0\right) - \begin{pmatrix} 0 & 0 \\ 0 & \Lambda(v)\Lambda(z) \end{pmatrix} \end{aligned}$$

where $M = (R^T\Lambda(v)\bar{D}\Lambda(v)R)^{-1}$; then there exists an agent with endowments w^h and preferences $U^h : \mathbb{R}_+^S \longrightarrow \mathbb{R}$ satisfying assumption 2 whose demand function $\theta^h(q,p)$, $x^h(q,p)$ satisfies:
$\theta^h(q,p) = \theta$, $x^h(q,p) = z + w^h$, and $\partial_{q,p}(\theta, x)^h(q,p) = A$.

Proof

Choose $w^h \in \mathbb{R}^S_{++}$ such that $z + w^h >> 0$, and let
$U^h(x) = \lambda_1^T \Lambda(v)^{-1} x + \frac{1}{2}(x - z + w^h)^T \bar{D}(x - z + w^h)$.
Then $DU^h(z + w^h) = \Lambda(v)^{-1}\lambda_1$, $D^2U^h(z + w^h) = \bar{D}$ and, in a neighborhood of $x = z + w^h$, $U^h(\cdot)$ satisfies assumption 2. Therefore (θ, z) is indeed the agent's excess demand at the prices (q, p): $(\theta, z + w^h) = (\theta, x)^h(q, p)$. It is then immediate to verify, given (6), that $\partial_{q,p}(\theta, x)^h(q, p) = A$.

□

3 Disaggregation of market excess demand

In this section we demonstrate that any matrix $A \in \mathbb{R}^{(S+J)^2}$ satisfying the homogeneity and Walras Law restrictions can be the Jacobian matrix of the market excess demand function of some GEI-economy satisfying assumptions 1 and 2. In reaching the conclusion we will get some intermediate results which are of independent interest.

We will show first how any matrix $A \in \mathbb{R}^{J^2}$ satisfying the homogeneity and Walras Law restrictions can be the matrix of the derivative of the market excess demand for assets with respect to asset prices. The agents in the economy can be chosen to have additively separable preferences. Our result is then an extension of Polemarchakis (1983) to the case of incomplete markets. Also it is the one commodity version under the possible restriction of additively separable preferences, and in terms of properties of the Jacobian, of Bottazzi - Hens (1996). As a corollary of this result we get a characterization of market excess demand functions (both for assets *and* consumption) in the case of complete markets. A second corollary shows that there are no restrictions on the effects that changes of the parameters of agent's utility functions have on equilibrium asset prices.

In the main result of the paper we turn to the whole Jacobian matrix, to show that again it has no stucture. The argument here has to depart from known techniques, as a consequence of the peculiar structure of the Jacobian of individual demand, and in particular – as we argued – of the impossibility of a clear decomposition of the overall Jacobian into an income and substitution effect. A corollary then demonstrates that there are no comparative statics restrictions for the effects of monetary policy on asset prices.

The Walras Law and homogeneity restrictions (2) - (5) extend, evidently, to aggregate demand. We have then:

$$q^T \cdot \sum_h \partial_q \theta^h(q, p) = -\sum_h \theta^h(q, p)^T, \quad \begin{array}{l} \sum_h \partial_q \theta^h(q, p) \cdot q = 0, \\ \sum_h \partial_q x^h(q, p) \cdot q = 0 \end{array} \tag{9}$$

$$q^T \cdot \sum_h \partial_p \theta^h(q,p) = 0, \quad \begin{array}{l} \sum_h \partial_p \theta^h(q,p) \cdot p = \sum_h \theta^h(q,p), \\ \sum_h \partial_p x^h(q,p) \cdot p = 0 \end{array} \tag{10}$$

$$\begin{array}{l} \sum_h \partial_q x^h(q,p) = \Lambda(v) R(\sum_h \partial_q \theta^h(q,p)), \\ \sum_h \partial_p x^h(q,p) = -\Lambda(v)\Lambda(\sum_h x^h(q,p) - \sum_h w^h) + \Lambda(v) R \sum_h \partial_p \theta^h(q,p) \end{array} \tag{11}$$

Theorem 1

Let R satisfy assumption 1, $p \in \mathbb{R}^S_{++}$ and $q \in \mathbb{R}^J$ be arbitrage free. Given any vector $\bar{\theta} \in \mathbb{R}^J$ satisfying $q \cdot \bar{\theta} = 0$ and any matrix $A \in \mathbb{R}^{J\times J}$ such that $q^T A = -\bar{\theta}^T$ and $Aq = 0$, we can find J agents with additively separable utility functions satisfying assumption 2 such that their aggregate demand for assets, $\sum_h \theta^h(q,p)$, satisfies $\partial_q \sum_{h=1}^J \theta^h(q,p) = A$ and $\sum_{h=1}^J \theta^h(q,p) = \bar{\theta}$.

Proof

It will suffice to generate the first $(J-1)$ rows and columns of A. The last row and column will then be determined so as to satisfy the conditions $q^T A = -\bar{\theta}^T$ and $Aq = 0$.

Let $\hat{}$ denote the truncation operator that eliminates the last component of a vector and the last row and column of a matrix.[7] With this notation our problem is to find $J-1$ agents such that

$$\hat{A} = \sum_{h=1}^{J} \left(\lambda_0^h \hat{K}_0^h - \hat{v}_0^h \hat{\theta^h}^T \right) \tag{12}$$

where K_0^h and v_0^h are as defined in proposition 1. Note that $\sum_{h=1}^{J-1} \hat{v}_0^h \hat{\theta^h}^T = \hat{V}_0 \hat{\Theta}^T$, where $\hat{V}_0 = [\hat{v}_0^1, \ldots, \hat{v}_0^{J-1}]$ and $\hat{\Theta} = [\hat{\theta}^1, \ldots, \hat{\theta}^{J-1}]$. Equation (12) can then be rewritten as follows:

$$\hat{A} - \sum_{h=1}^{J} \lambda_0^h \hat{K}_0^h + \hat{v}_0^J \hat{\bar{\theta}}^T = -[\hat{V}_0 - \hat{v}_0^J \mathbb{1}^T]\hat{\Theta}^T, \quad \text{where } \mathbb{1}_{J-1} = (1,\ldots,1)^T \in \mathbb{R}^{J-1} . \tag{13}$$

Let $D^2 U^h = -\Lambda(\alpha_1^h)$ for some vector $\alpha_1^h \in \mathbb{R}^S_{++}$, $h = 1, \ldots, J$. We have then:

$$v_0^h = -\alpha_0^h [R^T \Lambda(v) \Lambda(\alpha_1^h) \Lambda(v) R]^{-1} q \;, \tag{14}$$

where α_0^h is as defined in Proposition 1.

[7] Thus, if A is a $J \times J$ matrix, $\hat{A} = [I_{J-1}, 0] A [\begin{smallmatrix} I_{J-1} \\ 0^T \end{smallmatrix}]$.

Claim 1.1: $\alpha_1^h,\ h=1,\ldots,J$ *can always be chosen so that the matrix* $[\hat{V}_0 - \hat{v}_0^J \mathbb{1}_{J-1}^T]$ *has full rank.*

Proof: Inverting the matrix in (14) which premultiplies q and using the fact $\Lambda(\alpha_1^h)Rv_0^h = \Lambda(Rv_0^h)\alpha_1^h$ it is obtained that:

$$R^T \Lambda^2(v)\Lambda(Rv_0^h)\alpha_1^h = -\alpha_0^h q\ . \tag{15}$$

Without loss of generality R^T can be partitioned as $[R_1\ R_2]^T$, where R_1 is $J \times J$ and invertible. Partition the vectors $v = (v_1, v_2)$, $y^h = (y_1^h, y_2^h)$ and $\alpha_1^h = (\alpha_{11}^h, \alpha_{12}^h)$ accordingly. Using this notation we can solve (15) for α_{11}^h.

$$\alpha_{11}^h = -[\Lambda^2(v_1)\Lambda(R_1 v_0^h)]^{-1}[R_1^T]^{-1}[R_2^T \Lambda^2(v_2)\Lambda(R_2 v_0^h)\alpha_{12}^h + \alpha_0^h q]\ . \tag{16}$$

Consider a vector $\bar{\alpha}_1 >> 0$ and let $\bar{v}_0$ be the corresponding value of v_0 given by (14). Then for all values v_0^h in some open neighborhood of $\bar{v}_0$ the corresponding value of α_1^h, obtained from (16), will still be strictly positive. Hence by letting v_0^h vary, we can find $\hat{v}_0^h,\ h=1,\ldots,J$, in a neighborhood of $\hat{\bar{v}}_0$ so that the matrix $[\hat{V}_0 - \hat{v}_0^J \mathbb{1}^T]$ is invertible and the corresponding $\alpha_1^h,\ h=1,\ldots,J$, are strictly positive. This completes the proof of the claim.

Therefore, we can always solve (13) with respect to the matrix $\hat{\Theta}^T$. Choosing $\theta_J^h = -\hat{q}\cdot\hat{\theta}^h(\frac{1}{q_J})$, $h=1,\ldots,J$, ensures that the budget restriction holds.

Finally, let w^h be such that $x^h = \Lambda(v)R\theta^h + w^h \geq 0$, and $\lambda_1^h >> 0$ such that ${\lambda_1^h}^T\Lambda(v)R = \lambda_0^h q^T$ for some $\lambda_0^h > 0$. Applying then proposition 2 to the collection $(-\Lambda(\alpha^h), \lambda^h, z^h, \theta^h)$, $\forall h=1,\ldots,J$ completes the proof.

□

As we showed in Remark 3.1, with complete markets we have an additional homogeneity restriction, so that the matrix $\partial_q(\sum_h \theta^h(q,p))$ determines uniquely the remaining terms of the matrix $\partial_{q,p}(\sum_h(\theta,x)^h(q,p))$, via (11) and (6), applied to aggregate demand. Therefore, from Theorem 1 we get:

Corollary 1

Let rank $R = S$, $v \in \mathbb{R}_{++}^S$ *and* q *be arbitrage free. Given any vector* $\bar{\theta}$, *such that* $q\cdot\bar{\theta} = 0$, *and any matrix* $A \in \mathbb{R}^{(S+J)^2}$, *partitioned as* $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, *where* $A_1 \in \mathbb{R}^{J\times J}$, $A_2 \in \mathbb{R}^{J\times S}$, $A_3 \in \mathbb{R}^{S\times J}$, $A_4 \in \mathbb{R}^{S\times S}$ *are such that*

$q^T A_1 = -\bar{\theta}^T$, *and* $A_1 q = 0$,

$A_2 = A_1 R^T \Lambda(\pi)\Lambda(v) + R^{-1}\Lambda(v)\Lambda(R\bar{\theta})$

$A_3 = \Lambda(v) R A_1$

$A_4 = \Lambda(v) R A_1 R^T \Lambda(\pi)\Lambda(v)$

for π a vector such that $\pi^T R = q^T$,

then there exist J agents with additively separable utility functions satisfying assumption 2 such that $\sum_{h=1}^{J} \theta^h(q,p) = \bar{\theta}$ and $\partial_{(q,p)} \sum_{h=1}^{J} (\theta, x)^h(q,p) = A$.

Proof

By Theorem 1 we can find J agents such that $\sum_h \partial_q \theta^h(q,p) = A_1,\ \sum_h \theta^h(q,p) = \bar{\theta}$. It is enough then to verify that the matrices $A_2,\ A_3,\ A_4$ satisfy the Walras Law and homogeneity restrictions for aggregate demand with complete markets: (6) and (11) as we saw uniquely determine the remaining terms of $\partial_{q,p}(\theta, x)$ once $\partial_q \theta$ is given.

It is immediate to see that $A_1,\ A_2,\ A_3,\ A_4$ as above satisfy (6) and (11). It remains to check that (9), (10) also hold:

$$A_3 q = \Lambda(v) R A_1 q = 0$$

$$\begin{aligned} A_4 p &= \Lambda(v) R A_1 R^T \Lambda(\pi)\Lambda(v) p \\ &= \Lambda(v) R A_1 R^T \pi \\ &= \Lambda(v) R A_1 R^T (R^T)^{-1} q \\ &= \Lambda(v) R A_1 q = 0 \end{aligned}$$

$$\begin{aligned} A_2 p &= R^{-1}\left[R A_1 R^T \Lambda(\pi)\Lambda(v) + \Lambda(v)\Lambda(R\bar{\theta})\right] \cdot p \\ &= R^{-1}\Lambda(v)\Lambda(R\bar{\theta}) \cdot p \\ &= R^{-1} R\bar{\theta} = \bar{\theta} \end{aligned}$$

$$\begin{aligned} q^T A_2 &= q^T R^{-1}\left[R A_1 R^T \Lambda(\pi)\Lambda(v) + \Lambda(v)\Lambda(R\bar{\theta})\right] \\ &= q^T A_1 R^T \Lambda(\pi)\Lambda(v) + q^T R^{-1}\Lambda(v)\Lambda(R\bar{\theta}) \\ &= -\theta^T R^T \Lambda(\pi)\Lambda(v) + \theta^T R^T \Lambda(v)\Lambda(R^{-1} q) = 0 \ \text{ since } \ q^T R^{-1} = \pi\ . \end{aligned}$$

□

As a second consequence of Theorem 1 we demonstrate that, if there are ‘enough’ agents in the economy (i.e. $H \geq J$), there are in general no restrictions on the direction in which equilibrium asset prices move when the parameters determining the agents’ demand functions (e.g. their preferences) change.

Let $\beta_h \in B_h$ be a finite smooth parameterisation of agent h's choice problem (P^h) (and hence of his demand $(\theta, x)^h(q, p, \beta_h)$). For instance β_h could describe a finite-dimensional parameterisation of his utility function, e.g. $\sum_{s=1}^{S} \beta_s^h u^h(x_s)$. In this case one may expect that an increase in β_s^h increases the equilibrium prices of those assets that have a relatively high payoff in state s. Alternatively, $U^h(x)$ might be of the form $\left(\sum_{s=1}^{S}(u_s^h(x_s))^{\beta^h}\right)^{\frac{1}{\beta^h}}$, for $\beta_h < 1$; an increase of β_h now amounts to a decrease of the agent's risk aversion and again we may think that as a result the equilibrium prices of relatively riskier assets tend to decrease.

As the following corollary shows, in general, none of these conjectures need to be true. Even more, without any condition on the agents' preferences and endowments, a change in the parameters of the agents' demand can have any effect on equilibrium asset prices. More precisely, we show that whatever parametrisation we choose of the agents' choice problem, the effect of a pertubation of K agents is arbitrary if in the economy there are at least $K + J$ agents.[8]

Corollary 2

Let R satisfy assumption 1, and K agents be given, with preferences and endowments satisfying assumption 2. Let $\beta_h \in \mathbb{B}^h$ describe a finite smooth parameterisation of agent h's choice problem (P^h), $h = 1, \ldots, K$; $\bar{K} = \sum_{h=1}^{K} dim(\mathbb{B}^h) \geq J - 1$.

Then, generically[9], any matrix $B \in \mathbb{R}^{(J-1)\times\bar{K}}$ of full row rank can be obtained as the matrix $\partial_\beta \hat{q}$, describing the effects of changes in $\beta = (\beta_1, \ldots, \beta_K)$ on the normalized equilibrium asset prices $q = (\hat{q}, 1)$ for an economy with $J + K$ agents (K being the number of the given agents), for a given vector of prices p.

Proof

Let

$$\theta(q, \beta) = \sum_{h=1}^{K} \theta^h(q, \beta^h) + \sum_{h=K+1}^{K+J} \theta^h(q)$$

be the market excess demand for assets for an economy with $K + J$ agents, the first being the given K agents.

Due to the homogeneity property, $\theta(\lambda q; \beta) = \theta(q; \beta)$, and Walras' Law, $q \cdot \theta(q; \beta) = 0$, we can ignore the demand and the price of the last asset and consider

[8] If we are willing to be specific about the nature of the parametrisation β, so that $\partial_{\beta_h}(\theta, x)^h$ can be derived, it can be shown that we only need J agents in the whole economy for the result to hold; the argument is similar, though the proof is lenghtier and more elaborate.

[9] Here we mean generically with respect to the given parameterization $(\ldots, \beta_h, \ldots)$, though any other parameterization of demand will do.

$\hat{\theta}(\hat{q}, \beta)$. Differentiating the equilibrium system $\hat{\theta}(\hat{q}, \beta) = 0$, if, at equilibrium, $\partial_{\hat{q}}\hat{\theta}$ is invertible, we get

$$\partial_\beta \hat{q} = -\left(\partial_{\hat{q}}\hat{\theta}(\hat{q}; \beta)\right)^{-1} \partial_\beta \hat{\theta}(\hat{q}; \beta) \tag{17}$$

To show that we can find an economy such that

$$\partial_\beta \hat{q} = B \tag{18}$$

note that (18) can be rewritten as

$$\sum_{h=K+1}^{K+J} \partial_{\hat{q}}\hat{\theta}^h(\hat{q}) = -\sum_{h=1}^{K} \partial_\beta \hat{\theta}^h(\hat{q}, \beta) B^T (BB^T)^{-1} - \sum_{h=1}^{K} \partial_q \hat{\theta}^h(\hat{q}, \beta) \tag{19}$$

By Theorem 1 the left-hand side term can be an arbitrary matrix for a suitable choice of J agents satsfying assumption 2. These agents can then be chosen so that (19) holds, for the given B.

Finally, by a standard argument we can verify that, generically, $\partial_{\hat{q}}\hat{\theta}$ is indeed invertible at equilibrium.

□

So far we have considered, with the exception of the complete market case, only one of the terms of $\partial_{q,p}(\theta, x)$. We turn now our attention to the whole Jacobian. This is the subject of the main result of this chapter.

Theorem 2

Let R be in general position and satisfy assumption 1, $S > J$, $v \in \mathrm{I\!R}^S_{++}$ and $q \in \mathrm{I\!R}^J$ be arbitrage free. Given any vector $\bar{\theta}$ such that $q \cdot \bar{\theta} = 0$, and any matrix $A \in \mathrm{I\!R}^{(S+J)^2}$, partitioned as
$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, where $A_1 \in \mathrm{I\!R}^{J\times J}$, $A_2 \in \mathrm{I\!R}^{J\times S}$, $A_3 \in \mathrm{I\!R}^{S\times J}$, $A_4 \in \mathrm{I\!R}^{S\times S}$ are such that $q^T A_1 = -\bar{\theta}^T$, $A_1 q = 0$, $A_2 p = \bar{\theta}$, $q^T A_2 = 0$, $A_3 = \Lambda(v) R A_1$, $A_4 = -\Lambda^2(v)\Lambda(R\bar{\theta}) + \Lambda(v) R A_2$
we can find $2J-1$ agents satisfying assumption 2 such that their demand satisfies:

$$\sum_{h=1}^{2J-1} \theta^h(q, p) = \bar{\theta} \ \text{ and } \ \partial_{(q,p)} \sum_{h=1}^{2J-1} (\theta, x)^h(q, p) = A\ .$$

Proof

The strategy of the proof has to depart from the standard argument used in the decomposition of the Jacobian of aggregate demand. The problem is that

$\partial_p \theta^h(\cdot)$, as we remarked, does not have a clear decomposition into a substitution effect, with well-defined properties, and an income effect, with few properties and a low rank, on which the argument can then be built. Here the symmetric negative semi-definite matrix K_0^h, of rank $J-1$, appears in all terms of $\partial_p \theta^h$. The idea of our proof is then to consider a particular specification of the agents' Hessian matrix $D^2 U^h$ which allows a decomposition of K_0^h into a full rank matrix and a rank one matrix on which the decomposition is then based.

To prove the result we have to show that the equations $\sum_h \partial_{(q,p)}(\theta, x)^h(q,p) = A$, $\sum_h \theta^h(q,p) = \bar{\theta}$ have a solution, with respect to the parameters describing preferences, endowments and demands of the agents.

Since, given (11), the submatrices $\partial_q x$, $\partial_p x$ are uniquely determined from $\partial_q \theta$, $\partial_p \theta$, and since the matrices A_1, A_2, A_3, A_4 also satisfy (11), we can only consider the equations $\sum_h \partial_q \theta^h = A_1$, $\sum_h \partial_p \theta^h = A_2$.

Agents are partitioned into three groups: agents in a first group are labeled by h_1, $h = 1, \ldots, J-1$, while agents in the second set are labeled by h_2, $h = 1, \ldots, J-1$. In addition to these agents, which we will use to generate the given matrices A_1 and A_2, there is a last agent, labeled J, who will ensure that the adding up conditions, $\sum_h \theta^h = \bar{\theta}$, hold.[10]

We will suppose that the agents' preferences are characterized by the following matrix of the second derivatives, of their utility function:

$$\begin{aligned} D^2 U^{h_1} &= -(\delta D + d_{h_1} d_{h_1}^T)\ , \quad \text{for} \quad h = 1, \ldots, J-1 \\ D^2 U^{h_2} &= -(\delta D - d_{h_2} d_{h_2}^T)\ , \quad \text{for} \quad h = 1, \ldots, J-1 \\ D^2 U^J &= -\delta D \end{aligned}$$

for

$$\begin{aligned} D &= \Lambda^{-1}(v) R (R^T R)^{-1} \Lambda(\xi) (R^T R)^{-1} R^T \Lambda^{-1}(v) \\ d_{h_1} &= T_1 e_h{}^{11}\ , \quad d_{h_2} = T_2 e_h, \ h = 1, \ldots, J-1 \\ T_1 &= \delta_T \Lambda^{-1}(v) R (R^T R)^{-1} \Lambda(\xi) + \delta_\pi \pi k^T \\ T_2 &= \delta_T \Lambda^{-1}(v) R (R^T R)^{-1} \Lambda(\xi) \end{aligned}$$

where δ, δ_T, $\delta_\pi \in \mathbb{R}_+$, $\xi \in \mathbb{R}^J_{++}$, $k \in \mathbb{R}^S$ and $\pi \in \mathbb{R}^S$ such that $\pi_s \neq 0$, $s = 1, \ldots, S$, and $R^T \Lambda(v) \pi = 0$.

Claim 2.1: *Let R be in general position. There exists $\pi \in \mathbb{R}^S$ such that $\pi_s \neq$*

[10] With some abuse of notation we denote by $\sum_h$ the sum across all $(2J-1)$ agents.

[11] e_h denotes the h-th unit vector in $\mathbb{R}^J$.

$0,\ s = 1,\ldots,S$ *and* $R^T\Lambda(v)\pi = 0$.[12]

The proof of the claim is in the appendix.

With the above specification $D^2U^{h_1}$ is negative semi-definite, and the matrix

$$M^{h_1} = [R^T\Lambda(v)D^2U^{h_1}\Lambda(v)R]^{-1} = -\frac{1}{\delta}\left[\Lambda^{-1}(\xi) - \frac{\delta_T^2 e_h e_h^T}{\delta\mu^{h_1}}\right] ,$$

where $\mu^{h_1} = 1 + \frac{\delta_T^2\xi_h}{\delta} > 0$, is negative definite. Analogously, if $\delta > \delta_T^2\xi_h$, $D^2U^{h_2}$ is negative semi-definite and

$$M^{h_2} = -\frac{1}{\delta}\left[\Lambda^{-1}(\xi) + \frac{\delta_T^2 e_h e_h^T}{\delta\mu^{h_2}}\right] ,$$

where $\mu^{h_2} = 1 - \frac{\delta_T^2\xi_h}{\delta} > 0$, is negative definite. Finally, D^2U^J is negative definite and $M^J = -\frac{1}{\delta}\Lambda^{-1}(\xi)$ is negative definite. Hence the above specification is consistent with assumption 2 (iii).

We also get

$$\begin{aligned}
\alpha_0^{h_1} &= \delta(q^T\Lambda^{-1}(\xi)q - \frac{\delta_T^2 q_h^2}{\delta\mu^{h_1}})^{-1} , \\
K_0^{h_1} &= -\frac{1}{\delta}\left[\Lambda^{-1}(\xi) - \frac{\delta_T^2 e_h e_h^T}{\delta\mu^{h_1}}\right]\left[I - \frac{\alpha_0^{h_1}}{\delta}q(q^T\Lambda^{-1}(\xi) - (\frac{\delta_T^2 q_h}{\delta\mu^{h_1}})e_h^T)\right]
\end{aligned}$$

and

$$v_0^{h_1} = \frac{\alpha_0^{h_1}}{\delta}\left[\Lambda^{-1}(\xi)q - (\frac{\delta_T^2 q_h}{\delta\mu^{h_1}})e_h\right] .$$

Similarly,

$$\begin{aligned}
\alpha_0^{h_2} &= \delta(q^T\Lambda^{-1}(\xi)q + \frac{\delta_T^2 q_h^2}{\delta\mu^{h_2}})^{-1} \\
K_0^{h_2} &= -\frac{1}{\delta}\left[\Lambda^{-1}(\xi) + \frac{\delta_T^2 e_h e_h^T}{\delta\mu^{h_2}}\right]\left[I - \frac{\alpha_0^{h_2}}{\delta}q(q^T\Lambda^{-1}(\xi) + (\frac{\delta_T^2 q_h}{\delta\mu^{h_2}})e_h^T)\right] \\
v_0^{h_2} &= \frac{\alpha_0^{h_2}}{\delta}\left[\Lambda^{-1}(\xi)q + (\frac{\delta_T^2 q_h}{\delta\mu^{h_2}})e_h\right] .
\end{aligned}$$

[12]The general position of R is only a sufficient condition for the validity of the claim; it is however also a necessary condition when $J = S - 1$.

Finally,

$$\begin{aligned}
\alpha_0^J &= \delta(q^T\Lambda^{-1}(\xi)q)^{-1} \\
K_0^J &= -\frac{1}{\delta}\Lambda^{-1}(\xi)\left[I - \frac{\alpha_0^J}{\delta}qq^T\Lambda^{-1}(\xi)\right] \\
v_0^J &= \frac{\alpha_0^J}{\delta}\left[\Lambda^{-1}(\xi)q\right]
\end{aligned}$$

The problem can be simplified by an appropriate transformation of basis. If $S_L^1,\ S_R^1,\ S_L^2 \in \mathbb{R}^{J\times J},\ S_R^2 \in \mathbb{R}^{S\times S}$ are invertible matrices to prove the result we can equivalently show that the following system of equations has a solution:

$$S_L^1 A_1 S_R^1 = S_L^1\left(\sum_{h=1}^{H}\partial_q\theta^h(q,p)\right)S_R^1 \tag{20}$$

and

$$S_L^2 A_2 S_R^2 = S_L^2\left(\sum_{h=1}^{H}\partial_p\theta^h(q,p)\right)S_R^2 \tag{21}$$

In particular, it is convenient to consider the following matrices:

$$\begin{aligned}
S_L^1 &= S_L^2 = \left[S^\perp, q\right]^T \\
S_R^1 &= \left[I^{\setminus J}, q\right]^T \\
S_R^2 &= \left[I^{\setminus S}, p\right]^T
\end{aligned}$$

where the superscript $^{\setminus n}$ denotes the fact that the n-th column of a matrix has been deleted, i.e. $I^{\setminus n} = \binom{I_{n-1}}{0}$, and $S^\perp$ denotes a $J\times(J-1)$ basis of the space orthogonal to the vector $\Lambda^{-1}(\xi)q$. It is immediate to verify, given assumption 1 and the fact that $\xi >> 0$, that the above matrices are all of full rank.

Due to properties (9), (10) of the Jacobian of aggregate demand, arising from homogeneity and Walras Law, and to the corresponding properties of the matrix A, with the above specification of $S_L^1,\ S_L^2,\ S_R^1,\ S_R^2$, equations (20) and (21) become:

$$\begin{bmatrix} S^\perp A_1^{\setminus J} & 0 \\ -\hat{\theta}^T & 0 \end{bmatrix} = \begin{bmatrix} S^\perp(\sum_h \partial_q\theta^h)^{\setminus J} & 0 \\ -\hat{\theta}^T & 0 \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} S^\perp A_2^{\setminus S} & S^\perp\bar{\theta} \\ \hat{0}^T & 0 \end{bmatrix} = \begin{bmatrix} S^\perp(\sum_h \partial_p\theta^h)^{\setminus S} & S^\perp\bar{\theta} \\ \hat{0}^T & 0 \end{bmatrix} \tag{23}$$

We can thus limit our attention to showing the existence of a solution for the equations

$$S^{\perp} A_1^{\backslash J} = S^{\perp} (\sum_h \partial_q \theta^h)^{\backslash J} \tag{24}$$

$$S^{\perp} A_2^{\backslash S} = S^{\perp} (\sum_h \partial_p \theta^h)^{\backslash S} \, . \tag{25}$$

The effect of premultiplying a matrix by $S^{\perp}$ is that the terms collinear to $\Lambda^{-1}(\xi)q$ no longer appear. Consider the first system of equations:

$$S^{\perp} A_1^{\backslash J} = S^{\perp} (\sum_h \partial_q \theta^h)^{\backslash J} = S^{\perp} (\sum_h \lambda_0^h K_0^h - v_0^h \theta^{h^T})^{\backslash J} \tag{26}$$

Given the above specification of preferences, (26) can be rewritten as:

$$\begin{array}{rcl} S^{\perp}(A_1 - \sum_h \lambda_0^h K_0^h)^{\backslash J} & = & S^{\perp} \left(\sum_{h=1}^{J-1} \left(\frac{\delta_T^2}{\delta^2} \frac{\alpha_0^{h_1} q_h}{\mu^{h_1}} \right) e_h \hat{\theta}_{h_1}^T - \sum_{h=1}^{J-1} \left(\frac{\delta_T^2}{\delta^2} \frac{\alpha_0^{h_2} q_h}{\mu^{h_2}} \right) e_h \hat{\theta}_{h_2}^T \right) \\ & = & S^{\perp} I^{\backslash J} \left[\Lambda(\beta_1) \hat{\Theta}_1^T - \Lambda(\beta_2) \hat{\Theta}_2^T \right] \end{array} \tag{27}$$

where

$$\begin{array}{rcl} \beta_i & = & (\ldots, \beta_{h_i}, \ldots, \beta_{J-1_i}), \;\; \text{for} \;\; \beta_{h_i} = \dfrac{\delta_T^2}{\delta^2} \dfrac{\alpha_0^{h_i} q_h}{\mu_{h_i}} \\ \Theta_i & = & [\theta_{1_i}, \ldots, \theta_{J-1_i}], \end{array}$$

for $i = 1, 2$. Solving (27) for $\hat{\Theta}_1$ as a function of $\hat{\Theta}_2$, we get[13]

$$\hat{\Theta}_1^T = \Lambda^{-1}(\beta_1) \left[(S^{\perp})^{\backslash J} \right]^{-1} S^{\perp} \left[A_1 - \sum_h \lambda_0^h K_0^h \right]^{\backslash J} + \Lambda^{-1}(\beta_1) \Lambda(\beta_2) \hat{\Theta}_2^T \tag{28}$$

Consider the remaining system of equations, (25). Aggregating across agents the expression for the individual Jacobian obtained in (7) and using the above specification of preferences to substitute for K_0^h the right-hand side of (25) can be rewritten as:

$$\begin{array}{rcl} S^{\perp}(\sum_h \partial_p \theta^h)^{\backslash S} & = & S^{\perp} \left(\sum_h K_0^h R^T \Lambda(v) \left[\Lambda(\lambda_1^h) + D^2 U^h \Lambda^2(v) \Lambda(R\theta^h) \right] \right)^{\backslash S} \\ & = & -\dfrac{1}{\delta} S^{\perp} \left(\sum_h \left[\Lambda^{-1}(\xi) - e_h z_h^T \right] R^T \Lambda(v) \left[\Lambda(\lambda_1^h) + D^2 U^h \Lambda^2(v) \Lambda(R\theta^h) \right] \right)^{\backslash S} \end{array}$$

[13]Since $S^{\perp}$ has rank $J-1$, up to an appropriate interchange of columns, the matrix $\left[(S^{\perp})^{\backslash J} \right]$ is invertible.

where

$$z_{h_1}^T = \frac{\delta_T^2}{\delta\mu^{h_1}}\left[\left(1+\frac{\delta_T^2\alpha_0^{h_1}q_h^2}{\delta^2\mu^{h_1}}\right)e_h^T - \left(\frac{\alpha_0^{h_1}q_h}{\delta}\right)q^T\Lambda^{-1}(\xi)\right] \ , \quad h=1,\ldots,J-1$$

and

$$z_{h_2}^T = -\frac{\delta_T^2}{\delta\mu^{h_2}}\left[\left(1-\frac{\delta_T^2\alpha_0^{h_2}q_h^2}{\delta^2\mu^{h_2}}\right)e_h^T - \left(\frac{\alpha_0^{h_2}q_h}{\delta}\right)q^T\Lambda^{-1}(\xi)\right] \ , \quad h=1,\ldots,J-1$$

and

$$z_J^T = 0 \ .$$

Substituting then for D^2U^h the specification we adopted, after a series of computations, yields:

$$\begin{array}{rcl} S^\perp(\sum_h \partial_p\theta^h)^{\backslash S} & = & -\frac{1}{\delta}S^\perp\left(\Lambda^{-1}(\xi)R^T\Lambda(v)\Lambda(\bar\lambda_1) - \delta(R^TR)^{-1}R^T\Lambda(v)\Lambda(R\bar\theta)\right)^{\backslash S} \\ & & +\frac{1}{\delta}S^\perp\left(\sum_h e_h z_h^T(R^T\Lambda(v)\Lambda(\lambda_1^h)) + \sum_h e_h a_h^T\Lambda(v)\Lambda(R\theta^h)\right)^{\backslash S} \end{array} \tag{29}$$

where

$$a_{h_1}^T = \left(\frac{\delta_T^2\alpha_0^{h_1}q_h}{\mu^{h_1}\delta}\right)q^T(R^TR)^{-1}R^T + \tilde\nu_{h_1}k_h\pi^T\Lambda(v) \ , \quad h=1,\ldots,J-1 \ ,$$

$$\tilde\nu_{h_1} := \left(\frac{\delta_T}{\delta\mu^{h_1}}(\xi_h\delta_T^2-\delta\mu^{h_1}) + \frac{\delta_T^3q_h^2}{\delta^3(\mu^{h_1})^2}(\xi_h\delta_T^2-\delta\mu^{h_1})\alpha_0^{h_1}\right)\delta_\pi \ ,$$

and

$$a_{h_2}^T = -\left(\frac{\delta_T^2\alpha_0^{h_2}q_h}{\delta\mu^{h_2}}\right)q^T(R^TR)^{-1}R^T \ , \quad h=1,\ldots,J-1 \ ,$$

$$a_J^T = 0$$

$$\bar\lambda_1 = \sum_h \lambda_1^h \ .$$

The agents' budget constraint ($q\cdot\theta^h = 0$) allows us to write θ_J^h in terms of $\hat\theta^h$; therefore given $\hat\theta^h$ the whole vector θ^h is determined: $\theta^h = \tilde I\hat\theta^h$, where $\tilde I = [I_{J-1}, -\hat q(\frac{1}{q_J})]^T$. Using this relation and substituting then the expression obtained in (28) for $\hat\theta^{h_1}$, $h=1,\ldots,J-1$ we get:

$$\begin{array}{rcl} S^\perp(\sum_h \partial_p\theta^h)^{\backslash S} & = & -\frac{1}{\delta}S^\perp(\Lambda^{-1}(\xi)R^T\Lambda(v)\Lambda(\bar\lambda_1) - \delta(R^TR)^{-1}R^T\Lambda(v)\Lambda(R\bar\theta))^{\backslash S} \\ & + & \frac{1}{\delta}S^\perp[\sum_h e_h z_h^T R^T\Lambda(v)\Lambda(\lambda_1^h)^{\backslash S} \\ & + & \sum_{h=1}^{J-1} e_h a_{h_1}^T\Lambda(R\tilde I(\eta_h + \beta_2^h/\beta_1^h\hat\theta^{h_2})) + \sum_{h=1}^{J-1} e_h a_{h_2}^T\Lambda(v)\Lambda(R\theta^{h_2})]^{\backslash S} \ , \end{array} \tag{30}$$

where η_h^T is the h-th row of the matrix $\frac{1}{\delta_T^2}\Lambda^{-1}(\beta_1)[(S^{\perp})^{\setminus J}]^{-1}S^{\perp}[A_1 - \sum_h \lambda_0^h K_0^h]^{\setminus J}$, which is independent of h. Recalling that $\beta_{h_2}/\beta_{h_1} = \frac{\alpha_0^{h_2}}{\alpha_0^{h_1}} \cdot \frac{\mu^{h_1}}{\mu^{h_2}}$, and the above expressions for a_{h_1}, a_{h_2}, (30) can be simplified as follows:

$$\begin{aligned} S^{\perp}(\textstyle\sum_h \partial_p \theta^h)^{\setminus S} &= -\tfrac{1}{\delta}S^{\perp}\Big(\Lambda^{-1}(\xi)R^T\Lambda(v)\Lambda(\bar\lambda_1) - \delta(R^TR)^{-1}R^T\Lambda(v)\Lambda(R\bar\theta) \\ &- \textstyle\sum_{h=1}^{J-1} e_h a_{h_1}^T \Lambda(R\tilde{I}\eta_h)^{\setminus S}\Big)^{\setminus S} \\ &+ \tfrac{1}{\delta}S^{\perp}[\textstyle\sum_h e_h z_h^T R^T \Lambda(v)\Lambda(\lambda_1^h) + \sum_{h=1}^{J-1}\nu_h e_h \pi^T \Lambda(v)\Lambda(R\tilde{I}\hat\theta^{h_2})]^{\setminus S} \end{aligned} \tag{31}$$

where ν_h is the scalar $k_h\left(\frac{\alpha_0^{h_2}\mu^{h_1}}{\alpha_0^{h_1}\mu^{h_2}}\right)\tilde{\nu}_h$, $h = 1, \ldots, J-1$

Substituting (31) into the system (25) and moving the first term appearing in (31) on the other side, we are left, on the right-hand side of such system, with the term:

$$\frac{1}{\delta}S^{\perp}\left(\sum_h e_h z_h^T R^T \Lambda(\lambda_1^h) + \sum_{h_2=1}^{J-1} \nu_h e_h \pi^T \Lambda(R\tilde{I}\hat\theta^{h_2})\right)\Lambda(v)^{\setminus S} \tag{32}$$

The important observation is that this matrix "decomposes agent by agent"; that is to say every h-th row is exclusively generated by the vectors λ^{h_1}, λ^{h_2}, $\hat\theta^{h_2}$. We have then succeded in obtaining a matrix which has a similar structure as the matrix of the income effect terms. This allows us to use each (here pair of) agents to generate independently each row of the matrix $S^{\perp}A_1^{\setminus J}$.

Thus it remains to demonstrate that an appropriate choice of λ^{h_1}, λ^{h_2}, $\hat\theta^{h_2}$, subject to the non-negativity restrictions which have to hold for λ^h ($\lambda^h > 0$, $\sum_{h=1}^{2(J-1)}\lambda^h < \bar\lambda$[14]) allow us to generate an arbitrary $(S-1)$-vector y_h, i.e. that the following system has a solution, for every pair h_1, h_2:

$$\begin{bmatrix} \nu_h[\Lambda^{\setminus S}(v)]^T\Lambda(\pi)R\tilde{I} & [\Lambda^{\setminus S}(v)]^T\Lambda(Rz_{h_1}) & [\Lambda^{\setminus S}(v)]^T\Lambda(Rz_{h_2}) \\ 0 & R^T & 0 \\ 0 & 0 & R^T \end{bmatrix}\begin{pmatrix} \hat\theta^{h_2} \\ \lambda_1^{h_1} \\ \lambda_1^{h_2} \end{pmatrix} = \begin{pmatrix} y_h \\ \lambda_0^{h_1} q \\ \lambda_0^{h_2} q \end{pmatrix} \tag{33}$$

where the last two equations are the no arbitrage restriction which have to hold for λ_{h_1}, λ_{h_2}, and y_h is an arbitrary vector in $\mathbb{R}^{S-1}$.

In the appendix it is shown

Claim 2.2: *We can always find values of $\xi, \delta, \delta_T, \delta_\pi$, (in turn generating α_0^h, μ^h)*

[14] The second inequality comes from the non-negativity condition for λ^J, indeed determined by $\bar\lambda - \sum_h^{2(J-1)}\lambda^h$.

so that the matrix of the coefficients in (33) has full (row) rank and the solution for λ^{h_1}, λ^{h_2} satisfies the non-negativity conditions.

To complete the proof let $DU^h = \Lambda(v)\lambda_1^h,\ h = 1, \ldots 2(J-1)$, $DU^J = \Lambda(v)f$ for some $f \in \mathbb{R}^S_{++}$ such that $f^T R^T = q$, and choose w^h sufficiently large to insure that $w^h + \Lambda(v)R\theta^h \geq 0,\ h = 1, \ldots, 2J-1$ to obtain the existence of utility functions, for every agent, consistent with the above specification.

$\square$

As an application of Theorem 2 we demonstrate that in contrast to the complete markets case, without some knowledge of the agents' characteristics no rule can be given describing the effects of monetary policy on asset prices (and agents' welfare): these effects can then go in any direction.

In the case of complete markets, the effects of monetary policy m on asset prices can be explained by a simple rule. Note that among the first order conditions for the agents' choice problem (P^h) under Assumption 2 we have:

$$R^T \Lambda(v) DU^h(x^h) = \lambda_0^h q$$

where λ_0^h is agent h's Lagrange multiplier associated with the constraint $q \cdot \theta \leq 0$. Let $\pi^h(x) := \frac{DU^h(x^h)}{\lambda_0^h} >> 0$. With complete markets at equilibrium π^h is independent of h, say π.

Thus, from (35) we obtain:

$$q = R^T \Lambda(w) \Lambda^{-1}(\gamma) \Lambda^{-1}(m) \pi$$

where $w = (\sum_h w_1^h, \ldots, \sum_h w_s^h)$, $\gamma = (\gamma_1, \ldots, \gamma_s)$; note that $\frac{w_s \pi_s}{\gamma_s m_s}$ is the present value of one unit of money contingent on state s, $s = 1, \ldots, S$.

Since equilibrium allocations are the some as in the Arrow-Debreu model, they are independent of monetary policy. Thus π does not change where m varies. In particular, when m_s increases, $\frac{\omega_s \pi_s}{\gamma_s m_s}$ decreases, hence the price of the assets with positive (negative) payoffs in state s decreases (increases).

On the other hand, with incomplete markets we need to look at the whole system of equilibrium equations (34), (35).

More precisely, let m be a description of a normalized monetary policy, $m = (\hat{m}, 1)$; the associated monetary equilibrium can be defined as a vector of normalized asset prices $(\hat{q}, 1)$ and spot prices p such that the market for the first $J-1$ assets and for money clear:

$$\sum_{h=1}^{H} \theta_j^h(\hat{q}, p) = 0\ , \quad j = 1, \ldots, J-1 \tag{34}$$

$$p_s \sum_h w_s^h - \gamma_s m_s = 0 \ , \quad s = 1, \ldots, S \tag{35}$$

By Walras Law and homogeneity of demand, the market for the last asset and the commodity traded in all states $s = 1, \ldots, S$ will also clear, and asset prices can be normalized in terms of q_J.

Note that this system decomposes in the sense that m does not appear in (34) while $\hat{q}$ does not appear in (35). The latter equation determines then the price levels p, say $p = \psi(m)$ and we are left with $\sum_h \hat{\theta}^h(\hat{q}, \psi(\hat{m})) = 0$. The effects on the equilibrium asset prices of an arbitrary change in monetary policy are then, locally, described by:

$$d_{\hat{m}} \hat{q} = -\left(\partial_{\hat{q}} \hat{\theta}(\hat{q}, \psi(\hat{m}))\right)^{-1} \partial_p \hat{\theta}(\hat{q}, \psi(\hat{m})) \partial_m \psi(\hat{m}) \ . \tag{36}$$

where $\partial_{\hat{m}} \psi(\hat{m})$ is the diagonal matrix with $(\gamma_s / \sum_h w_s^h)$, $s = 1, \ldots, S-1$ on its main diagonal.

Corollary 3

Let R be in general position and satisfy assumption 1 and let B be an arbitrary $(J-1) \times (S-1)$ matrix. Then there exists an economy with $2J-1$ agents, satisfying assumption 2, such that $d_{\hat{m}} \hat{q} = B$.

Proof

In Theorem 2 we have shown that any matrix of the appropriate order can be the Jacobian matrix $\sum_h \partial_{(q,p)} \hat{\theta}^h$ of an economy with $2J-1$ agents. Given the expression of $d_{\hat{m}} \hat{q}$ derived in (36), the result follows.

□

Note that the effects of monetary policy are arbitrary even when the Jacobian of excess demand with respect to asset prices $\sum_h \partial_{\hat{q}} \hat{\theta}^h$ exhibits some of the properties, which yields nice comparative statics results with respect to endowment changes (e.g. negative definiteness or strict gross substitution). This is the case since the Jacobian with respect to asset prices and the Jacobian with respect to commodity prices can be generated independently from each other.

4 Appendix

4.1 Proof of claim 2.1

Let R be a $S \times J$ matrix with rank $R = J$.

Note that if the condition

$$(A1) \qquad \exists \pi \in \mathbb{R}^S,\ \pi_s \neq 0\ \forall s: \quad R^T \Lambda(v) \pi = 0$$

is satisfied for some $v \in \mathbb{R}^S_{++}$ then it is satisfied for all $v \in \mathbb{R}^S_{++}$: if π satisfies $(A1)$ for v, then $\pi' = \Lambda^{-1}(v)\Lambda(v')\pi$ satisfies $(A1)$ for v'. Hence without loss of generality, we can set $v_s = 1,\ s = 1, \ldots, S$.

We need therefore to demonstrate that the general postition of R implies the existence of a vector $\pi \in \mathbb{R}^S,\ \pi_s \neq 0$ all $s = 1, \ldots, S$ such that $R^T \pi = 0$.

Since rank $R = J$, we know that

(1) there exists $\mu \in \mathbb{R}^J$ such that $\sum_{s=1}^J \mu_s r_s = -\sum_{s=J+1}^S r_s$.

Moreover, since R is in general position

(2) there exists $\bar{\pi} \in \mathbb{R}^J,\ \bar{\pi}_s \neq 0,\ s = 1, \ldots, J$ such that $\sum_{s=1}^J \bar{\pi}_s r_s = -r_{J+1}$.

To see the latter, suppose to the contrary that $r_{J+1} = \sum_{j=1}^J \pi_j r_j = 0$ and $\pi_k = 0$ some k. Then $(r_1, \ldots, r_k, r_{J+1}, r_{k+1}, \ldots, r_J)$ are linearly dependent, contradicting general position of R.

From (1) and (2) we see that for all $t \in \mathbb{R}$

$$R^T(t\bar{\pi}_1 + \mu_1, \ldots, t\bar{\pi}_J + \mu_J, t + 1, 1, \ldots, 1) = 0\ .$$

It remains to choose t such that $t \neq -1$ and $t \neq \frac{-\mu_1}{\bar{\pi}_1}, \ldots, t \neq \frac{-\mu_J}{\bar{\pi}_J}$.

$\square$.

4.2 Proof of claim 2.2

The argument that the system of equations (33) has a solution that satisfies the positivity requirement imposed on $\lambda_h,\ h = 1, \ldots, 2J - 1$ is based on two steps. We demonstrate first that for an open set of parameter values $\xi \in \mathbb{R}^J_{++}$ the coefficient matrix of (33) is regular. To satisfy the positivity requirement we take then limits of the solution with respect to the scale factors $\delta,\ \delta_T,\ \delta_\pi$ and δ_J.

4.2.1 Regularity of the coefficient matrix

We demonstrate that for almost all $\xi \in \mathbb{R}^J_{++}$ the systems of equations (33) has a solution $\forall h = 1, \ldots, J - 1$.

Since the coefficient matrix of (33) is block-diagonal it suffices to demonstrate that the upper-left square matrices

(A3)

$$\left[\begin{array}{ccc} [\hat\Lambda(v)]\hat\Lambda(Rz_{h_1}) & 0 & \nu_h[\hat\Lambda(v)\hat\Lambda(\pi) \;\; 0]R\tilde I \\ [R^T]^{\setminus S} & r_s & 0 \end{array}\right], \quad h = 1, \ldots, J-1$$

are all invertible. The lower-right matrix $[R^T]^{\setminus S}$ has full row rank. Following Murata (1977, Theorem 1.3) we have then to demonstrate the invertibility of the matrices

$$\left\{ [r_s, 0] - \nu_n \left[R^{T\setminus S}\hat\Lambda^{-1}(Rz_{h_1})\hat\Lambda(\pi)[R^T]\setminus S \begin{bmatrix} 0 \\ \vdots \; \tilde I \\ 0 \end{bmatrix} \right] \right\} ,$$

and $\hat\Lambda(Rz_{h_1}),\ h = 1, \ldots, J-1$.

Lemma 3

There exists an open set $\overset{*}{Z}_{h_1} \subset \mathbb{R}^J$ *such that* $R^{T\setminus S}\hat\Lambda^{-1}(Rz_{h_1})\hat\Lambda(\pi)[R^T\setminus S]^T$ *is invertible for all* $z \in \overset{*}{Z}_{h_1}$, $\forall h = 1, \ldots, J-1$.

Proof Note that the vectors on the diagonal matrix on the right and the left hand side of (A4) are never colinear, for all appropriate choice of π_2.[15]

By Lemma 4 in Geanakoplos and Mas-Colell (1989), from the general position of R it then follows:

$$< \hat\Lambda \begin{pmatrix} -\hat\Lambda^{-1}(v_1)(R_1^T)^{-1}R_2^T\Lambda(v_2)\pi_2 \\ \pi_2 \end{pmatrix} [R^{T\setminus S}]^T y > \neq < \Lambda \begin{pmatrix} -(R_1^T)^{-1}R_2^{T\setminus S}x_2 \\ x_2 \end{pmatrix} [R^{T\setminus S}]^T z > \quad .$$

Hence there exists an open set $\overset{*}{Z}_{h_1} \subset \mathbb{R}^J$ such that $\forall z \in \overset{*}{Z}_{h_1}$ there is no y satisfying (A3). In particular note that z close to e_J lies in $\overset{*}{Z}_{h_1}$.

□

Recalling the definition of z_{h_1}, it is easy to verify that the map

$$z_{h_1}(\xi) = \frac{\delta_T^2}{\delta\mu^{h_1}(\xi)}\left[\left(1 + \frac{\delta_T^2\alpha_0^{h_1}(\xi)q_h^2}{\delta^2\mu^{h_1}(\xi)}\right)e_h^T - \left(\frac{\alpha_0^{h_1}(\xi)q_h}{\delta}\right)q^T\Lambda^{-1}(\xi)\right]$$

[15] Suppose, on the contrary, that the two diagonal matrices were identical. Then it must be $x_2 = \hat\pi_2$ and

$$-\Lambda^{-1}(v_1)(R_1^T)^{-1}R_2^{T\setminus S}\hat\Lambda(v_2)\hat\pi_2 + \Lambda^{-1}(v_1)R_1^T\Lambda_s v_s\pi_s = (R_1^T)^{-1}R_2^{T\setminus S}\hat\pi_2$$

but we can always choose π_2 so that the above equation does not hold.

is continuous; thus there also exists an open set $X \subset \mathrm{IR}^J_{++}$ such that $z_{h_1}(\xi) \in \overset{*}{Z}_{h_1}$, $\forall h = 1, \ldots, J-1, \ \forall \xi \in X$.[16]

The invertibility of $(R^T)^{\setminus S} \hat{\Lambda}^{-1}(Rz_{h_1})\Lambda(\pi)(R^{T\setminus S})^T$ implies the existence of vectors $b_{h_1} \in \mathrm{IR}^J$ such that $\nu_h(R^T)^{\setminus S}\hat{\Lambda}^{-1}(Rz_{h_1})\Lambda(\pi)(R^{T\setminus S})^T b_{h_1} = r_s$, $h = 1, \ldots, J-1$. Thus, to establish the full rank of (A3) we only have to show the invertibility of the matrices

$$\begin{bmatrix} b_{h_1} & \tilde{I} \end{bmatrix} = \begin{bmatrix} b_{h_1} & \begin{matrix} I_{J-1} \\ -\hat{q}/q_J \end{matrix} \end{bmatrix}, \ h = 1, \ldots, J-1 \ .$$

These matrices are full rank if $q \cdot b_{h_1} \neq 0$, or equivalently:

$$q^T \left[R^{T\setminus S} \hat{\Lambda}^{-1}(Rz_{h_1})\hat{\Lambda}(\pi)[R^{T\setminus S}]^T \right]^{-1} r_s \neq 0, \ h = 1, \ldots, J-1 \ .$$

We have to show that for all z in some open set $\overset{*}{Z}_{h_1} \subset \mathrm{IR}^J$ there does not exist a vector $y \neq 0$ such that

(A4)

$$R^{T\setminus S}\hat{\Lambda}^{-1}(Rz)\hat{\Lambda}(\pi)[R^{T\setminus S}]^T y = 0 \ .$$

Let $R^T = (R_1^T, R_2^T)$ where $R_1^T \in \mathrm{IR}^{J \times J}$ and rank $R_1^T = J$. Using this decomposition, the right nullspace of $R^{T\setminus S}$ can be parameterized by vectors $x_2 \in \mathrm{IR}^{S-J-1}$ using the expression $\binom{-(R_1^T)^{-1}R_2^{T\setminus S}x_2}{x_2}$. Accordingly, the set of π such that $R^T\Lambda(v)\pi = 0$ can be parameterized by $\binom{-\Lambda^{-1}(v_1)(R_1^T)^{-1}R_2^T\Lambda(v_2)\pi_2}{\pi_2}$, where $\pi_2, \ v_2 \in \mathrm{IR}^{S-J}$ and $v_1 \in \mathrm{IR}^J$. Hence (A4) has a solution if and only if there is $y \in \mathrm{IR}^J, \ x_2 \in \mathrm{IR}^{S-J-1}$ such that:

(A5)

$$\hat{\Lambda}\begin{pmatrix} -\Lambda^{-1}(v_1)(R_1^T)^{-1}R_2^T\Lambda(v_2)\pi_2 \\ \pi_2 \end{pmatrix}[R^{T\setminus S}]^T y = \Lambda\begin{pmatrix} -(R_1^T)^{-1}R_2^{T\setminus S}x_2 \\ x_2 \end{pmatrix}[R^{T\setminus S}]z \ .$$

□

4.2.2 Positivity of the solution with respect to λ_1^h

Letting $\Delta\lambda_1^h := \lambda_1^{h_1} - \lambda_1^{h_2}, \ \Delta\lambda_0^h := \lambda_0^{h_1} - \lambda_0^{h_2}$ we can rewrite system (33) as follows:

[16]Let $\overset{*+}{Z}_{h_1}$ be the set of z_{h_1} such that $\Lambda(Rz_{h_1})$ is invertible. $\overset{*+}{Z}_{h_1}$ is open and dense. Thus $\overset{*+}{Z}_{h_1} \cap \overset{*}{Z}_{h_1}$ is an open set.

(A6)

$$\begin{bmatrix} \nu_h[\Lambda^{\backslash S}(v)]^T\Lambda(\pi)R\tilde{I} & \hat{\Lambda}(v)\hat{\Lambda}(Rz_{h_1})\begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \hat{\Lambda}(v)\hat{\Lambda}(R(z_{h_1}+z_{h_2}))\begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\ 0 & R^T & 0 \\ 0 & 0 & R^T \end{bmatrix} \begin{pmatrix} \hat{\theta}^{h_2} \\ \Delta\lambda_1^h \\ \lambda_1^{h_2} \end{pmatrix} = \begin{pmatrix} y_h \\ \Delta\lambda_0^h q \\ \lambda_0^{h_2} q \end{pmatrix}$$

Note that as $\delta \to \infty$, $z_{h_2} \to -\xi$, $z_{h_1} \to -\frac{1}{\delta}\left[e_h - \frac{q_h}{q^T\Lambda^{-1}(\xi)q}\Lambda^{-1}(\xi)q\right]$[17],hence the above systems decomposes.

Let $(\hat{\theta}_h, \Delta\lambda^h)$ be a solution to the upper left subsystem:
(A7)

$$\begin{bmatrix} \nu_h[\Lambda^{\backslash S}(v)]^T\Lambda(\pi)R\tilde{I} & \frac{1}{\delta}\hat{\Lambda}(v)\Lambda\left(R\left(\Lambda^{-1}(\xi)q\left(\frac{q_h}{q^T\Lambda^{-1}(\xi)q}\right)-e_h\right)\right)\begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\ 0 & R^T \end{bmatrix} \begin{pmatrix} \tilde{\theta}^{h_2} \\ \Delta\lambda^h \end{pmatrix} = \begin{pmatrix} y_h \\ \Delta\lambda_0^h q \end{pmatrix}.$$

Set $\lambda_0^{h_1} = \lambda_0^{h_2}$ it is easy to verify that we can always choose a solution $\lambda_1^{h_2}$ of the lower subsystem of (A6), $R^T\lambda_1^{h_2} = \lambda_0^{h_2}q$ sufficiently large so that $\lambda_1^{h_1} = \lambda_1^{h_2} + \Delta\lambda_1^h >> 0$.

Hence, by continuity we can claim that for δ sufficiently big there exist solutions $(\hat{\theta}_h, \lambda^{h_1}, \lambda^{h_2})$, $h = 1, \ldots, J-1$ to (A6) such that $\lambda^{h_1},\ \lambda^{h_2} >> 0\ \forall h$.

4.2.3 Positivity of λ^J

The values for the last agent, θ^J, λ^J, are then determined as follows: $\theta^J = \bar{\theta} - \sum_h \theta^h$, $\lambda^J = \bar{\lambda} - \sum_h \lambda^h$. Thus it remains to check that the solution for λ^J we obtained also satisfies the non-negativity condition. For this we argue that in solving the system (A7) we can keep the solution for $\Delta\lambda^h$ bounded.

Let $\delta_J := \frac{q_J}{\xi_J}$ and take limits $\delta,\ \delta_J,\ \delta_T,\ \delta_\pi \to \infty$ where $\delta_T = \sqrt{\delta_J}$ and $\frac{\delta}{\delta_J} \to 0$.

Then

$$\begin{array}{lllllll} \mu^{h_1} & \approx & \frac{\delta_T^2}{\delta} \to \infty, & \mu^{h_2} & \approx & \frac{-\delta_T^2}{\delta} \to -\infty \\ \alpha_0^{h_1} & \approx & \frac{\delta}{\delta_J} \to 0, & \alpha_0^{h_2} & \approx & \frac{\delta}{\delta_J} \to 0 \\ K_0^{h_1} & \approx & \frac{\delta_J}{\delta} \to \infty, & K_0^{h_2} & \approx & \frac{\delta_j}{\delta} \to \infty \end{array}$$

$$\delta_T^2 R z_{h_1} \approx \frac{\delta_J}{\delta}\left[\frac{\delta_T^3}{\delta_J - \frac{q_h^2\delta_T^2}{\delta+\delta_T^2\xi_h}}\left(\frac{\delta}{\delta+\delta_T^2\xi_h}\right)\right] = \frac{\delta_J}{\delta}\left[\frac{\delta_T^3\delta}{\delta_J(\delta+\delta_T^2\xi_h) - q_h^2\delta_T^2}\right]$$

---

[17]This follows from the facts: $\mu^{h_i} \to 1$, $\frac{\alpha_0^{h_i}q_h^2}{\delta^2\mu^{h_i}} \to 0$, $\left(\frac{\alpha_0^{h_i}q_h}{\delta}\right)q^T\Lambda^{-1}(\xi) \to \frac{q_h q^T\Lambda^{-1}(\xi)}{q^T\Lambda^{-1}(\xi)q}$, $i = 1, 2$, $\nu_h \to k_h$.

and

$$Rz_{h_1} \to r_J$$

note that $\frac{\delta_T^2 \alpha}{\delta^2 \mu^h} \approx \delta_T^2 \frac{1}{\delta_J} \frac{1}{\delta \delta_T^2} \approx 0$ thus indeed $R(z_{h_1} + z_{h_2}) \to 0$ we need to check that the factors on the left hand side obtained moving the first terms in (31) on the other side are of a lower order than the factors of π and Rz_h:

on the left hand side we have:

$$S^\perp A_2^{\setminus J} + \frac{1}{\delta} S^\perp \left(\Lambda^{-1}(\xi) R^T \Lambda(v) \Lambda(\bar{\lambda}_1) - \delta (R^T R)^{-1} R^T \Lambda(R\bar{\theta}) - \sum_h^{J-1} e_h a_{h_1}^T \Lambda(R\tilde{I}^T \eta_h) \right)^{\setminus S}$$

2nd term is of order $\frac{\delta_J}{\delta}$ less than $\delta_T^2 Rz_h$ for $\frac{\delta_T^3 \delta}{\delta_J(\delta + \delta_T^2)} \to \infty$

4th term has various elements:

the ones where π (in a_{h_1}) does not appear are of order

$$\delta_T^2 \frac{\alpha_0^{h_1}}{\mu^{h_1} \delta} \left(\frac{\delta^2 \mu^{h_1}}{\alpha_0^{h_1}} \right) \frac{1}{\delta_T^2} \frac{\delta_J}{\delta} = \delta_J$$

less than Rz_{h_1} for $\frac{\delta_T^3}{\delta_T(\delta + \delta_T^2) - q_h^2 \delta_T^2} \approx \frac{\delta_T^3 \delta_J \delta_T^2}{\to} \infty$ for δ_T of higher order than δ_J.

the ones where π (in a_{h_1}) appears are of order:

$$\zeta_h \delta_\pi \frac{1}{\delta_T^2} \left(\frac{\delta^2 \mu^{h_1}}{\alpha_0^{h_1}} \right) \frac{\delta_J}{\delta}$$

for

$$\zeta_h = \left(\frac{1}{\delta \mu^{h_1}} \delta_T^2 (\xi_h \delta_T - \delta \mu^{h_1}) + \frac{\delta_T^2 (\xi_h \delta_T - \delta \mu^{h_1}) \alpha_0^{h_1} q_h^2 \delta_T^2}{\delta^3 {\mu^{h_1}}^2} \right)$$

while the terms where π appears in system (33) are of order:

$$\zeta_h \delta_\pi$$

so we would like $\frac{1}{\delta_T^2} \delta \frac{\delta_T^2}{\delta} \frac{\delta_J \delta_J}{\delta} \to 0$ which is ok for δ of higher order than δ_J^2.

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