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Liquidity and the Endogenous Number of Financial Assets

by

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Abstract

This note studies a financial markets model with per unit transaction costs which are decreasing with the liquidity of the specific asset. The generic existence of non-trivial Walrasian equilibria is established and the properties of the equilibrium pricing function are discussed. Most importantly, it is shown that the number of assets actually traded is bounded above irrespectively of the number of assets available for trade.

Keywords: Incomplete Markets, Transaction Costs, Liquidity, Endogenous Asset Structure.

J.E.L. Classification: C 62, D 52, G 12

1 Introduction

Due to the rapid growth of international capital markets, the attempt to endogenise the set of securities in models of financial markets has recently become a very active area of research.[1] Contributing to this literature, this note is concerned with the question how many financial securities will effectively be traded on a financial market where a large number of securities is available for trade. A model is suggested where traders choose to concentrate their asset demand on a bounded number of securities even if the number of assets available on the market tends to infinity.

Most of the existing literature on financial innovation introduces one or more innovating institutions (or agents) into rather standard GEI-models. The set of traded assets is then determined endogenously as the solution to some well-defined maximisation problem of the innovators. All of these models have to impose rather strong rationality assumptions on the innovators. In particular, innovators are usually assumed to know the complete equilibrium correspondence for any set of marketed securities.

The model suggested in this paper takes a slightly different point of view. It regards the process of asset creation as a black box and sets out from the assumption that there is a large set of securities which is potentially available for trade; i.e. it is implicitly claimed that if an asset were to be demanded by the traders then some (unmodelled) financial innovator would be ready to supply this asset. In our model, transaction costs for trading the assets become the main driving force for the concentration of asset demand on a particular and endogenous set of assets. These transaction costs are assumed to be proportional to individual portfolio holdings, but they are supposed to be decreasing in the liquidity of the assets.[2] Investors face some kind of proportional transaction costs in alsmost all the models featuring financial innovators.[3] The specific choice of a liquidity-dependent transaction cost function reflects the casual observation that fixed costs are of particular relevance in the context of financial innovation - especially when the issue of setting-up new exchanges is addressed.[4]

[1] For an account of the theoretical efforts see e.g. Duffie and Rahi (1995) and Allen and Gale (1994); for an empirical assessment cf. Miller (1992).

[2] Liquidity here is defined as the total trading volume generated by an asset. For a model specifying the cost reducing role of the liquidity in an asset see Lippmann and McCall (1986). In their model, liquidity is measured as the expected waiting time until trade in an asset can actually be effectuated.

[3] Most notably in the models by Allen and Gale (1988, 1991), Bisin (1993), Che and Rajan (1994), Duffie and Jackson (1989), and Pesendorfer (1995).

[4] See for example the models on financial innovation proposed by Allen and Gale (1990), Che and Rajan (1994) and Heller (1993). A model incorporating the liquidity effect on transaction fees charged by the innovators has been presented by Cuny (1993).

First, generic existence of a non-trivial Walrasian equilibrium is shown for the situation modelled in this paper. Then we derive specific assumptions on the function mapping liquidity into transaction costs which guarantee that the number of assets actually demanded by the investors is bounded above, irrespectively of the number of assets potentially available for trade.[5] The intuition behind this result is simple: the liquidity effects on transaction costs provide an incentive for the economy to concentrate asset trading on a certain limited number of financial securities in order to save on transaction costs.[6]

This paper partly builds on the work of Laitenberger (1996) and Préchac (1996) who have studied GEI-economies with strictly linear transaction costs. Their main focus is on the question of existence of an equilibrium, but they both also provide a characterisation of no-arbitrage prices in their settings. Both Laitenberger (1996) and Préchac (1996) consider the more general situation where there are several physical commodities. The main difference between their approaches consists in the way transaction costs are disposed of. In Laitenberger (1996), as in traditional General Equilibrium Theory with transaction costs (cf. e.g. Hahn (1971, 1973) and Kurz (1974)), transaction costs are 'real' in the sense that they disappear from the economy after having been met by the agents. This point of view will also be taken within this paper. In the context of the one good-GEI model, this assumption can be interpreted as stating that the transaction costs leave the financial sector of the economy. Préchac (1996), in contrast, regards transaction costs as pure intermediation costs charged by financial institutions which enable asset trade. He assumes that these costs are redistributed among the agents, and, hence, that they do not leave the financial sector. It will be discussed in the course of the paper how approaches with strictly linear transaction costs, such as the ones by Laitenberger (1996) and Préchac (1996), fit into the more general framework considered here. In particular, it will be demonstrated that the characterisation for the set of no-arbitrage prices in the presence of transaction costs obtained both by Laitenberger (1996) and Préchac (1996) carries over to our setting.

This paper is organised as follows. Section 2 introduces the model and the necessary notation. Section 3 then studies the problem of the existence of a non-trivial equilibrium. Section 4 derives the upper bound on the number of assets traded in equilibrium. Section 5 discusses the set of no-arbitrage prices, while section 6 concludes.

[5]This reflects the Lancastrian analysis of increasing returns to scale in the context of optimal product differentiation. In our framework, financial assets can be interpreted as differentiated products with the state contingent pay-offs as their characteristics. Cf. Lancaster (1975, Theorem 2), and Lancaster (1971).

[6]This result should be compared with the discussion of standardised assets in the presence of information asymmetries contained in Gale (1992).

2 The Setup

The basis of the model is given by the standard two period-GEI-economy with a single consumption good, where consumption takes place in both periods. Thus, there are $s = 1, \ldots, S$ possible states of the world at $t = 1$. There are $i = 1, \ldots, I$ consumers who are characterised by their utility functions U^i defined on the consumption set $\mathbb{R}_+^{S+1}$ and by their endowments $\omega^i \in \mathbb{R}_+^{S+1}$ for $i = 1, \ldots, I$. Since this paper is not concerned with the most general conditions guaranteeing the existence of an equilibrium, and since we intend to study differentiable GEI-economies, the following assumption on utility functions will be maintained throughout.

Assumption (U):[7] Utility functions $U^i : \mathbb{R}_+^{S+1} \longrightarrow \mathbb{R}$, $i = 1, \ldots, I$, are continuous on $\mathbb{R}_+^{S+1}$ and infinitely often differentiable on $\mathbb{R}_{++}^{S+1}$. Furthermore, the closure of the set $\{x \in \mathbb{R}_+^{S+1} | \ U^i(x) \geq U^i(y)\}$ is contained in $\mathbb{R}_{++}^{S+1}$ for every $y \in \mathbb{R}_{++}^{S+1}$, and, for each $x \in \mathbb{R}_{++}^{S+1}$,

$$DU^i(x) := \Big(\partial_{x_0} U^i(x)\Big)^{-1} \Big(\partial_{x_1} U^i(x), \ldots, \partial_{x_S} U^i(x)\Big) \gg 0$$

as well as $h^T D^2 U^i(x) h < 0$ for all $h \in \mathbb{R}^{S+1}$, $h \neq 0$, with $DU^i(x) \cdot h = 0$.

We also make the following assumption on the interiority of individual endowments which simplifies the exposition without reducing the relevance of our results.[8]

Assumption (E): $\omega^i \gg 0$ for all $i = 1, \ldots, I$.

In order to effectuate transactions between uncertain states, we introduce a set of assets $A \subseteq \mathbb{R}^S$ which can be traded without any short selling restrictions. Throughout this paper we will assume that A is an arbitrary large but finite set, where we let $|A| := \text{card}\,(A)$ denote the cardinality of A. Assets A^j are normalised such as to have unit Euclidean norm, i.e. $||A^j||_2 = 1$ for $j = 1, ..., J$. Let $\sum^{S-1} = \{a \in \mathbb{R}^S : \ \|a\|_2 = 1\}$ denote the corresponding unit sphere, and let $\sum_{(+)}^{S-1} = \{a = (a_1, a_2, \ldots, a_S) \in \sum^{S-1} : \ a_1 \geq 0\}$ be the intersection of $\sum^{S-1}$ with the halfspace of asset vectors paying off a non-negative amount in the first state.[9] Thus we have that $A \subseteq \sum_{(+)}^{S-1}$ is the set of tradable assets.

[7]Assumption (U) goes back to Debreu (1972) who uses it in order to obtain differentiable demand functions for goods in the standard GEI-economies.

[8]Gottardi and Hens (1996) give a detailed account how existence of an equilibrium can be derived with a relaxed version of Assumption (E).

[9]Attention is focussed on $\sum_{(+)}^{S-1}$ because a and $(-a)$ should be regarded as the identical

Observe that the assumption that the number of assets is finite does not rule out the existence of linearly dependent payoff vectors within the set of assets. In the typical GEI-model without trading frictions, there is only a "small" number $J \leq S$ of such assets. This is also true for the paper by Bisin (1994) which studies existence and local uniqueness of equilibria in financial markets with a general class of trading frictions which incorporate the case of transaction costs considered here. In the presence of proportional transaction costs, however, assuming linearly independent asset payoff vectors is not very plausible since linear dependent assets may no longer be regarded as redundant due to the market imperfection introduced. In the main sections of this paper, the number of assets will typically be assumed to be much larger than the number of states.

Next note that the assumption that A is finite implies that agents' portfolios can be written as vectors $\theta = (\theta_+, \theta_-) \in \mathbb{R}^J_+ \times \mathbb{R}^J_+$, where $J = |A|$ and $\theta_+(j)$ ($\theta_-(j)$, resp.) denotes the purchases (sales, resp.) of asset $j \in J$. Thus, a portfolio $\theta \in \mathbb{R}^J_+ \times \mathbb{R}^J_+$ generates time 1-payoffs

$$R(\theta) = \sum_{j \in J} [\theta_+(j) - \theta_-(j)] \cdot A^j \ .$$

Furthermore, let $\Theta^i := \mathbb{R}^J_+ \times \mathbb{R}^J_+$ be the set of portfolios which are attainable for the i'th consumer.[10]

The essential point of departure from the standard GEI-models consists in the introduction of transaction costs. Thus, trading of the assets contained in the exogenously given asset structure A is not frictionless; each individual portfolio will incur transaction costs, which have to be met in period 0 and then disappear from the financial sector of the economy.

In this paper we suggest to study a unit transaction cost function which is decreasing in the total volume of the asset traded, and contains some strictly positive constant part (independent of volume). This function can be interpreted as capturing both the presence of bid-ask-spreads, i.e. transaction costs which are proportional to the individual agents' portfolio positions, and of fixed set-up costs for the trade in a particular asset. By assuming negative dependence on total trading volume, there will be a tendency within the economy to use only a certain number of rather standardised assets. On the other hand, since transaction costs are proportional, agents have an incentive to demand such a portfolio of agent specific assets which generates their desired net-trade with minimal

asset.

[10]By allowing $\Theta^i \neq \mathbb{R}^J_+ \times \mathbb{R}^J_+$, situations of restricted participation would be incorporated. For a discussion of the case of restricted participation without transaction costs, see e.g. Siconolfi (1986, 1989). For simplicity and clarity of the results, we focus our attention on the unrestricted case here.

transaction costs. It is important to note already at this point that transaction costs will not be assumed to be decreasing in each individual agent's trading volume, but to be decreasing in the total trading volume per asset generated in the economy. Thus, this paper focusses on economies of scale obtained not on the individual but on the aggregate level, where they should be supposed to be much more important.

To make the concept precise, we now turn to the construction of the transaction cost function. We first have to define a notion of the trading volume generated by a financial asset.

Definition 1: For a tuple of asset trades $\bar{\theta} := \left(\theta^1, \ldots, \theta^I\right) \in \Theta^1 \times \ldots \times \Theta^I$ the trading volume of asset $j \in J$ is given by $v_j := v_j(\bar{\theta}) := \sum_{i \in I} \left[\theta^i_+(j) + \theta^i_-(j)\right]$.

Now let $f : \mathrm{I\!R}_+ \to \mathrm{I\!R}_+$ be a continuous cost function which is strictly decreasing and which satisfies

(i) $\lim_{x \to 0} f(x) = \lim_{x \to 0} x \cdot f(x) = \infty$, aswellas

(ii) $\lim_{x \to \infty} f(x) = \alpha$,

where $\alpha > 0$ is some constant[11]. Then the unit transaction cost to be met by every agent for the trade of one unit of asset j, $j \in J$, is given by $f(v_j)$. Thus, the higher the trading volume in an asset, the lower are the transaction costs per unit of this asset. The first one of the two assumptions imposed on f reflects the idea that the technology for the "production" of assets contains a fixed cost component. In fact it has often been observed that for many types of new financial products new exchanges had to be created on which these (standardised) instruments could then be traded.[12] The second assumption stated is of a rather more technical nature. It defines a minimal transaction fee which has to be met independently of the size of the asset specific trading volume. For simplicity, transaction costs are assumed to be symmetric with respect to long and short positions. Also note, that due to prohibitive costs agents can only demand assets already traded by other agents.

Summarising this discussion the cost functional is defined as follows.

Definition 2: The transaction costs incurred by a vector of asset trades $\theta^i := (\theta^i_1, \ldots, \theta^i_J) \in \Theta^i$ for some agent and at total trading volume $v := (v_1, \ldots, v_J)$

[11]E.g. the function $f(x) = \frac{1}{x^2} + \alpha$ has these properties.

[12]This has e.g. been the case for commodity and later stock futures, where the existence of a corresponding central market place (such as e.g. the Chicago Board of Exchange) allowed for the successful introduction of new assets.

are given by

$$C(v;\theta^i) := \sum_{j\in J} f(v_j)\cdot\left[\theta^i_+(j)+\theta^i_-(j)\right] .$$

With this definition of transaction costs, the introduction of the individual budget set is straightforward. Note that as in Laitenberger (1996) transaction costs are assumed to disappear from the financial side of the economy after the first period[13].

Definition 3: For $i\in I$, $v := (v_1,\ldots,v_J)$, $j\in I$ and prices $p_0\in\mathbb{R}$ and $q : A\to\mathbb{R}$, the individual budget set correspondence is given by

$$\mathbb{B}^i(p_0,q;v) := \left\{(x,\theta)\in\mathbb{R}^{S+1}_+\times\Theta^i \,\middle|\, \begin{array}{ll} (i) & p_0(x_0+C^i(v;\theta))+Q(\theta)\le p_0\omega^i_0 \\ (ii) & x_{\mathrm{I}}\le\omega^i_{\mathrm{I}}+R(\theta)\end{array}\right\},$$

where $\Theta^i = \mathbb{R}^{|A|}_+\times\mathbb{R}^{|A|}_+$.

This leads to the following notion of a Walrasian financial markets equilibrium with transaction costs.

Definition 4: Given $A\subseteq\sum^{S-1}_{(+)}$, a tuple $(\overset{*}{p}_0,\overset{*}{q},(\overset{*}{x}{}^i,\overset{*}{\theta}{}^i)^I_{i=1})\in R^{|A|+1}\times\prod^I_{i=1}(X^i\times\Theta^i)$ is called a financial markets liquidity equilibrium (A-liq-equilibrium) if it satisfies the following conditions.

(*i*) $\sum^I_{i=1}\overset{*}{\theta}{}^i_+(j)=\sum^I_{i=1}\overset{*}{\theta}{}^i_-(j)$, for every $j\in J$,

(*ii*) $\sum^I_{i=1}\overset{*}{x}{}^i_0+\sum^J_{j=1}f(v_j)\cdot v_j=\sum^I_{i=1}\omega^i_0$, where $v_j := \sum^I_{i=1}\left[\theta^i_+(j)+\theta^i_-(j)\right]$,

(*iii*) $\sum^I_{i=1}\overset{*}{x}{}^i_{\mathrm{I}}=\sum^I_{i=1}\omega^i_{\mathrm{I}}$, and

(*iv*) for every $i\in I$, $(\overset{*}{x}{}^i,\overset{*}{\theta}{}^i)$ solves

$$(P^i_{\mathrm{liq}}) \qquad \max U^i(x^i) \text{ s.t. } (x^i,\theta^i)\in\mathbb{B}^i(\overset{*}{p}_0,\overset{*}{q};v)$$

with $v=(v_1,\ldots,v_J)$ defined as in (*ii*).

Furthermore let $\mathcal{W}_{liq}(A)$ denote the set of A-liq-equilibria corresponding to the given structure A.

[13] This is in contrast to Préchac (1996) where the individual agents hold shares of the financial intermediaries and transaction costs are thus redistributed among the agents.

Note that in this definition of financial markets equilibrium all agents take prices and transaction costs as given. In particular, they are assumed to ignore the impact of their own asset trades on overall transaction costs. This seems to be a reasonable extension of the price-taking paradigm to a situation with liquidity-dependent transaction costs.

Remark 1: *It is interesting to note that Definition 4 can be brought in line with the equilibrium concept with linear transaction costs used by Laitenberger (1996) (and, similarly, by Préchac (1996)). To this end one only has to modify the transaction cost function by assuming that $f(v)$ is a constant for every $v \geq 0$. In particular, this implies that individual transaction costs no longer are prohibitive if none of the other agents does not trade the asset. With this definition of the transaction cost function Laitenberger (1996)'s proof of the existence of a Walrasian equilibrium as in Definition 4 can be carried over with no difficulty.*[14] *Also, Laitenberger (1996)'s characterisation of no-arbitrage prices can be seen to be a special instance of the characterisation obtained below in section 5. This will be discussed in more detail at that point.*

This completes the description of the set-up of the model considered in this paper.

3 Existence

With the definition of A-liq-equilibria as given in Definition 4, a trivial A-liq-equilibrium always exists: since asset trading costs are prohibitive if no agent engages in asset trading, the situation where every agent just consumes his initial endowments is, in fact, an A-liq-equilibrium for any exogenous asset structure A. The result of this section states that under assumptions standard in the literature the Walrasian solution concept defined in the previous section carries meaning, i.e. that for transaction costs which are sufficiently small for low levels of trading volume, non-trivial A-liq-equilibria do in fact exist.

The intuition behind this result is rather simple. Generically in endowments, standard GEI-equilibria (without transaction costs) are non-trivial in the sense that every agent is actively trading in every asset. If sufficiently small transaction costs are introduced in such a generic situation, then - for continuity of the demand functions involved - it must still be the case that every agent will trade every asset (although maybe at a different level). As an additional piece of

[14] It should be observed that both Laitenberger (1996) and Préchac (1996) need to assume asset structures with non-negative pay-offs in order to derive existence.

notation denote by A^K the submatrix $A^K := (A^j_s)^{j \in K}_{s \in S}$ for any $K \subseteq J$.

Theorem 1: Let $A \in \mathrm{IR}^{S \times J}$ and assume (U) and (E). For any $K \subseteq J$ with $\operatorname{rank} A^K = \operatorname{rank} A$ there is a full measure set $\Omega(K) \subseteq \Omega$ with the following property: For any $\omega \in \Omega(K)$ there are constants $\beta_K, \gamma_K > 0$ such that if $f(x) < \gamma_K$ for every $x > \beta_K$, there exists an A-liq-equilibrium $\left(\overset{*}{p}_0, \overset{*}{q}, (\overset{*}{x}^i, \overset{*}{\theta}^i)_{i=1}^I\right) \in \mathcal{W}_{\text{liq}}(A)$ with $\overset{*}{\theta}^i_j \neq 0$ for every $i \in I$ and $j \in K$.

Proof: see appendix.

4 Endogenous Number of Assets

This section contains the main point of this paper. It shows that independently of the number $J = |A|$ of available assets, the number of assets actually traded will remain "small", i.e. bounded above, as J tends to infinity. In our setup, asset trading has to be concentrated on a bounded number of financial securities for two reasons. On the one hand, transaction costs per traded unit of an asset never are zero; hence, total trading volume must be bounded. On the other hand, transaction costs tend to infinity if trading volume per asset tends to zero; hence, the bounded feasible amount of total trading volume must be divided among a limited number of assets.

Theorem 2: Let $\bar{\omega}_0 := \sum_{i=1}^I \omega_0^i$. For any finite asset structure $A \subseteq \sum_{(+)}^{S-1}$ and for any tuple $\overset{*}{\Gamma} \in \mathcal{W}_{liq}(A)$ there exists a constant m depending only on $\bar{\omega}_0$ and f such that the traded asset structure $\overset{*}{A} = \overset{*}{A}(\overset{*}{\Gamma}) := \{j \in J | \, \exists i \in I : \overset{*}{\theta}^i(j) \neq 0\}$ satisfies $|\overset{*}{A}| < \frac{\bar{\omega}_0}{\alpha \cdot m}$.

Proof: see appendix.

Hence, the traded asset structure substantially differs from the structure which is exogenously given. This important conclusion, noted as a corollary, shows that it is possible to built a meaningful theory of traded asset structures on the transaction cost approach presented here.

Corollary 2: For any $A \subseteq \sum_{(+)}^{S-1}$ satisfying $J := |A| < \infty$ such that $|A| \geq \frac{\omega_0}{\alpha \cdot m}$, and for any $\overset{*}{\Gamma} \in \mathcal{W}_{liq}(A)$ one has that $\overset{*}{A}(\overset{*}{\Gamma}) \neq A$.

It is interesting to note the comparative statics of the model which follow economic intuitition. Higher variable cost components α and lower first period endowments imply a decreasing upper bound on the endogenous number of assets. On the other hand, reducing the steepness of f at $x = 0$ (i.e. increasing the constant m) will allow for more assets to be traded. This latter property should indeed hold since the slope of f in a neighbourhood of zero measures in some sense the extent of the fixed costs effects.

5 Characterisations of Walrasian Equilibria

Two intimately related properties characterise equilibrium prices in the standard GEI-economies: they are arbitrage-free, and they satisfy the linear pricing rule. Both of them are subject to qualifications in the presence of transaction costs. One is first led to ask for a characterisation of arbitrage free prices in this set-up. Obviously, since there are transaction costs depending on asset liquidity, the definition of arbitrage free prices (cf. e.g. Definition 9.1 in Magill and Quinzii (1996)) has to be modified accordingly.

Definition 5: A tuple $(q, \theta) \in \mathbb{R}^J \times (\mathbb{R}^J_+ \times \mathbb{R}^J_+)^I$, where $J = |A|$ is called a no arbitrage price portfolio situation (PPS) if

$$\nexists \mu \in \mathbb{R}^J_+ \times \mathbb{R}^J_+ : \begin{pmatrix} -Q(\mu) - C^i(v; \mu) \\ R(\mu) \end{pmatrix} > 0,$$

where $v = (v_1, \ldots, v_J)$ and $v_j := \sum_{i=1}^I \left[\theta^i_+(j) + \theta^i_-(j)\right]$.

Thus, a tuple of prices and portfolio does not offer arbitrage opportunities, if none of the agents perceives to be able to buy a free lunch given the other agents' portfolio holdings. This definition leads to a straightforward adaptation of the fundamental characterisation of no arbitrage prices.

Proposition 1: $(q,\theta) \in \mathbb{R}^J \times (\mathbb{R}^J_+ \times \mathbb{R}^J_+)^I$ is a no-arbitrage PPS if and only if there exists some $\pi \in \mathbb{R}^{S+1}_+$ such that

$$\pi_{\mathrm{I}} \cdot A^j - \pi_0 \cdot f(v_j(\theta)) \leq \pi_0 \cdot q(j) \leq \pi_{\mathrm{I}} \cdot A^j + \pi_0 \cdot f(v_j(\theta))$$

holds for every $j \in J$.

Proof: The proof can be carried over without essential alterations from Préchac (1996).

Remark 2: *Proposition 1 is a straightforward extension of the corresponding result for the case of strictly linear transaction costs (i.e. $f(v) \equiv \alpha$ for every $v \geq 0$) derived by Laitenberger (1996) and Préchac (1996): (q,θ) is a no-arbitrage PPS if and only if there exists a $\pi \in \mathbb{R}^{S+1}_{++}$ such that $\pi_{\mathrm{I}} \cdot A^j - \pi_0 \cdot \alpha \leq \pi_0 \cdot q(j) \leq \pi_{\mathrm{I}} \cdot A^j + \pi_0 \cdot \alpha$ holds for every $j \in J$.*

Obviously, Proposition 1 implies that the famous linear pricing rule (LPR), which plays a major rôle in the theory of financial markets, is weakend to a chain of inequalities. As transaction costs tend to zero, however, these inequalities can be seen to converge back to the LPR.[15]

Corollary 1: Suppose there is some $k \in J$, and some $\lambda \in \mathbb{R}^J_+ \times \mathbb{R}^J_+$ with $\lambda_k = (0,0)$, such that $A^k = \sum_{j\in J} (\lambda_+(j) - \lambda_-(j)) \cdot A^j$. If (q,θ) is an equilibrium PPS corresponding to an A-liq-equilibrium then

$$Q(\lambda) - \hat{\alpha} \cdot (1 + \|\lambda\|_1) \leq q(k) \leq Q(\lambda) + \hat{\alpha}(1 + \|\lambda\|_1),$$

where $\hat{\alpha} := \max_{j\in J} \hat{\alpha}_j := \max_{j \in J} f(v_j(\theta))$.

6 Conclusion

This paper has studied the standard one good-GEI-model in the presence of of transaction costs. A generalised cost function was introduced which incorporates both proportional and fixed parts. The existence of non-trivial Walrasian equilibria was shown under fairly general assumptions. Then, conditions were derived which imply that the endogenous set of assets used in equilibrium will never comprise more than a certain constant number of assets - independent of the number of assets available for trade. Finally, it was pointed out that the

[15] Note that the inequalities stated certainly are not the best ones which could be obtained. For simplicity of the exposition, however, we leave the issue at this level of generality.

well-known characterisation of the set of no-arbitrage prices in the presence of strictly linear transaction costs carries over to the situation considered here.

The model presented in this paper sheds new light on financial markets with transaction costs. Such markets are especially important in the context of financial innovation which has recently become an active area of research. Besides being of interest in its own right, the result obtained should therefore be useful for further research into more complicated models where the decision problems of financial innovators are explicitly taken into account.

Appendix

Proof (of Theorem 1):
For the course of this proof, let $\theta^i_k := \theta^i_+(k), \theta^i_-(k)$ in order to shorten notation. Choose a set $K \subseteq J$ with $\operatorname{rank} A^K = \operatorname{rank} A$, and consider the set $\mathcal{W}(A^K)$ of (standard) GEI-equilibria for the asset matrix A^K. From Exercise 8 in chapter 2 of Magill and Quinzii (1996, p.132) and generic local uniqueness of GEI-equilibria, it can be concluded that there is a full measure set $\Omega(K) \subseteq \Omega$ such that in every GEI-equilibrium contained in $\mathcal{W}(A^K)$ one has that (1) $\overset{*}{\theta}{}^i_j \neq 0$ for every $i \in I$ and $j \in K$, and (2) the equilibrium is regular. Pick such an equilibrium $(\overset{*}{p}_0, \overset{*}{q}, (\overset{*}{x}, \overset{*}{\theta}))$.

Now consider the budget set of agent $i \in I$ with linear transaction costs $\alpha_k \geq 0$, $k \in K$. It is given by

$$I\!\!B^i_K(p_0, q, \alpha) := \left\{ (x, \theta) \in X^i \times \Theta^i_K \;\middle|\; \begin{array}{l} p_0(x_0 + \alpha(\theta_+ + \theta_-)) + q(\theta_+ - \theta_-) \leq p_0 \cdot \omega^i \\ x_1 \leq \omega^i_1 + A^K \cdot (\theta_+ - \theta_-) \end{array} \right\},$$

where Θ^i_K denotes the restriction of agent i's portfolio set to the assets contained in K. One can rewrite this budget set as

$$I\!\!B^i_K(p_0, q, \alpha) := \left\{ (x, \theta) \in X^i \times \Theta^i_K \;\middle|\; \begin{array}{l} p_0 x_0 + (p_0\alpha + q)\theta_+ + (p_0\alpha - q)\theta_- \leq p_0 \cdot \omega^i \\ x_1 \leq \omega^i_1 + A^K \cdot (\theta_+ - \theta_-) \end{array} \right\}.$$

Thus it becomes clear that linear transaction costs affect the budget set in the same way prices do. From standard arguments in demand analysis (cf. e.g. Mas-Colell, Whinston and Green (1996, Appendix A to Chapter 3); or, for the GEI-case, Hens (1991, Chapter II.1.3, Proposition 2(6))), and making use of assumption (U) (especially strict quasi-concavity), one can conclude that the asset demand $\theta^i(p_0, q, \alpha)$ resulting from maximising U^i over $I\!\!B^i_K(p_0, q, \alpha)$ is continuous in the arguments (p_0, q, α). Hence, aggregate demand $\theta(p_0, q, \alpha) := \sum_{i \in I} \theta^i(p_0, q, \alpha)$ is continuous. From the choice of $\Omega(K)$ one infers furthermore that there is an $\varepsilon_1 > 0$ such that $\|(p_0, q, \alpha) - (\overset{*}{p}_0, \overset{*}{q}, 0)\| < \varepsilon_1$ implies that θ^i is differentiable at (p_0, q, α), for $i = 1, \ldots, I$. Such differentiability can be shown along standard lines (cf. e.g. Hens (1991, Chapter II.1.3, Proposition 2(8))), using either asset prices $p_0\alpha_j + q_j$ (if $\overset{*}{\theta}{}^i_+(j) > 0$) or $p_0\alpha_j - q_j$ (if $\overset{*}{\theta}{}^i_-(j) > 0$). (Note that θ^i will fail to be globally differentiable, if transaction costs are positive; cf. Bisin (1994) for a general analysis of this problem.)

Normalising $p_0 \equiv 1$, one can infer from the regularity of the equilibrium at the price $(1, \overset{*}{q}, 0)$ that there is some $\varepsilon_2 > 0$ such that

$$\|\tilde{\alpha}\| < \varepsilon_2 < \varepsilon_1 \implies \operatorname{rank} D_q\theta(1, q, \tilde{\alpha}) = J.$$

(Recalling that there is first period consumption.) The implicit function theorem then implies that there is some $\varepsilon_3 > 0, \varepsilon_3 < \varepsilon_2$ such that for every $\tilde{\alpha} > 0$ with $\|\tilde{\alpha}\| < \varepsilon_3$ there is a (unique) $q(\tilde{\alpha})$ with $\theta(1, q(\tilde{\alpha}), \tilde{\alpha}) = 0$, where the function $q(\cdot)$ such defined is continuous.

Continuity of $q(\cdot)$ and $\theta^i(\cdot)$ now yields the existence of some $\varepsilon_4 > 0$ such that

$$\|\tilde{\alpha}\| < \varepsilon_4 < \varepsilon_3 \implies \forall\, i \in I\; \forall\, k \in K\; \theta^i_k(1, q(\tilde{\alpha}), \tilde{\alpha}) \neq 0.$$

Now choose $\beta_K > 0$ such that $\|\tilde{\alpha}\| < \varepsilon_4$ implies

$$\min_{k \in K} v_k(1, q(\tilde{\alpha}), \tilde{\alpha}) > \beta_K,$$

and suppose that the transaction cost function f satisfies the hypothesis stated in the theorem, i.e. that

$$x > \beta_K \implies f(x) < \gamma_K := \frac{1}{2}\varepsilon_4.$$

Define the compact set $B_K := \{\tilde{\alpha} \in \mathrm{IR}^K |\ \|\tilde{\alpha}\| \leq \gamma_K\}$ and a map $\varphi :\ B_K \longrightarrow \mathrm{IR}^K$

$$\varphi(\tilde{\alpha}) := \left(f(v_k(1, q(\tilde{\alpha}), \tilde{\alpha})\right)_{k \in K}.$$

By construction, φ is continuous and satisfies $\varphi(B_K) \subseteq B_K$. By Brouwer's Fixed Point Theorem (which is a special case of Kakutani's Fixed Point Theorem), φ then has a fixed point $\hat{\alpha}$.

Since transaction costs are prohibitive for every asset which is not traded, it then follows that prices $q(\hat{\alpha})$ and portfolios $\hat{\theta}^i$ defined by $\hat{\theta}^i_k := \theta^i_k(1, q(\hat{\alpha}), \hat{\alpha}) \neq 0$ for $k \in K$, and $\hat{\theta}^i_k = 0$ for $k \notin K$ constitute a non-trivial A-liq-equilibrium.

□

Proof (of Proposition 1): To show the if-part of the proposition let $(q, \theta) \in \mathrm{IR}^J \times (\mathrm{IR}^J_+ \times \mathrm{IR}^J_+)^I$ be an arbitrary PPS, and suppose that $\pi \in \mathrm{IR}^{S+1}_{++}$ has the desired property with respect to (q, θ). Now one has to demonstrate that (q, θ) does not offer an arbitrage opportunity. To see this, let $\mu \in \mathrm{IR}^J_+ \times \mathrm{IR}^J_+$ be an arbitrary portfolio. Letting $\hat{\alpha}_j := f(v_j(\theta))$ for $j \in J$, one then obtains

$$\begin{aligned}
\pi_{\mathrm{I}} \cdot R(\mu) > 0 \ &\Rightarrow\ \sum_{j \in J} \pi_{\mathrm{I}} \cdot A^j \cdot (\mu_+(j) - \mu_-(j)) > 0 \\
&\Rightarrow\ \sum_{j \in J} \pi_0(q(j) + \hat{\alpha}_j) \cdot \mu_+(j) - \sum_{j \in J} \pi_0(q(j) - \hat{\alpha}_j) \cdot \mu_-(j) > 0 \\
&\Rightarrow\ Q(\mu) + C(v(\theta); \mu) > 0\ ,
\end{aligned}$$

for every $i \in I$. Analogously one can show that $Q(\mu) + C(v(\theta);\mu) < 0$ for every $i \in I$ implies $\pi_{\mathrm{I}} \cdot R(\mu) < 0$, and hence (since $\pi_{\mathrm{I}} \gg 0$) that $R(\mu) \neq 0$. Therefore, (q,θ) cannot be a PPS offering an arbitrage opportunity.

For the only if-part of the proposition let (q,θ) be a no-arbitrage PPS and fix any agent $i \in I$. Define a correspondence $\phi : \mathrm{IR}_+^J \times \mathrm{IR}_+^J \to \mathrm{IR}^{S+1}$ by

$$\phi(\mu) := \phi(\mu_+,\mu_-) := \begin{pmatrix} -Q(\mu) - C(v(\theta);\mu) \\ R(\mu) \end{pmatrix} ,$$

where $Q(\cdot)$ and $C^i(\cdot)$ are computed with respect to the given PPS (q,θ). Since (q,θ) is arbitrage-free, $\phi(\mathrm{IR}_+^J \times \mathrm{IR}_+^J) \cap \mathrm{IR}_+^{S+1} = \{0\}$. Noting that $\phi(\mathrm{IR}_+^J \times \mathrm{IR}_+^J)$ is closed and convex the separating hyperplane theorem can be applied to find a vector $\pi^i \in \mathrm{IR}^{S+1}$, $\pi^i \neq 0$, which strictly separates $\phi(\mathrm{IR}_+^J \times \mathrm{IR}_+^J)$ from the $S+1$-dimensional simplex $\{x \in \mathrm{IR}_+^{S+1} | \sum_{s=0}^S x_s = 1\}$.

Setting $\mu \equiv 0$ one now sees that

$$0 = \pi^i \cdot \phi(\mu) < \pi^i \cdot e_s = \pi_s^i$$

for $s = 0,1,\ldots,S$, where e_s denotes the s-th unit vector. It follows that $\pi^i \gg 0$. Also, since $x \in \phi(\mathrm{IR}_+^J \times \mathrm{IR}_+^J)$ implies $cx \in \phi(\mathrm{IR}_+^J \times \mathrm{IR}_+^J)$ for all $c \in \mathrm{IR}_+$ (i.e. $\phi(\mathrm{IR}_+^J \times \mathrm{IR}_+^J)$ is a cone), one readily obtains

$$\pi^i \cdot \phi(\mu) \leq 0 \quad \forall \mu \in \mathrm{IR}_+^J \times \mathrm{IR}_+^J ,$$

because otherwise letting $c \to \infty$ would yield a contradiction to the separation property of π^i.

Letting $\mu = (\delta^j, 0)$, where δ^j is the portfolio consisting of exactly one unit of the j-th asset and nothing else, now implies

$$\begin{aligned} 0 \geq \pi^i \cdot \phi(\mu) &= \pi_0^i(-Q(\mu) - C(v(\theta);\mu)) + \pi_{\mathrm{I}}^i \cdot R(\mu) \\ &= -\pi_0^i(q(j) + \hat{\alpha}_j) + \pi_{\mathrm{I}}^i \cdot A^j , \end{aligned}$$

where $\hat{\alpha}_j := f(v_j(\theta))$.

Similarily, for $\mu = (0, \delta_j)$, one obtains that

$$0 \geq \pi^i \cdot \phi(\mu) \geq \pi_0^i(q(j) - \hat{\alpha}_j) - \pi_{\mathrm{I}}^i \cdot A^j .$$

Collecting threads then implies

$$\pi_{\mathrm{I}}^i \cdot A^j - \pi_0^i \cdot \hat{\alpha}_j \leq \pi_0^i \cdot q(j) \leq \pi_{\mathrm{I}}^i \cdot A^j + \pi_0^i \cdot \hat{\alpha}_j$$

for every $j \in J$, and hence π^i satisfies the inequalities stated in the proposition.
□

Proof (of Corollary 1):
Let $\lambda \in \mathbb{R}^J_+ \times \mathbb{R}^J_+$ (with $\lambda_k = (\lambda_+(k), \lambda_-(k)) = (0,0)$) be the portfolio generating A^k, and let $\pi \in \mathbb{R}^{S+1}_{++}$ be the state price vector guaranteed by the second part of Proposition 1. Also, let $e_k \in \mathbb{R}^J$ be the k-th unit vector.

Then one obtains by Proposition 1 that

$$\pi_0 \cdot q(k) \leq \pi_1 \cdot A^k + \pi_0 \cdot \hat{\alpha}_k,$$

where $\hat{\alpha}_k := f(v_k(\theta))$. Thus

$$\begin{aligned} \pi_0 \cdot q(k) &\leq \sum_{j \in J} (\lambda_+(j) - \lambda_-(j)) \pi_1 \cdot A^j + \pi_0 \cdot \hat{\alpha}_k \\ &\leq \pi_0 \cdot \sum_{j \in J} (\lambda_+(j) - \lambda_-(j)) \cdot q(j) + \pi_0 \cdot \sum_{j \in J} (\lambda_+(j) + \lambda_-(j)) \cdot \hat{\alpha}_j + \pi_0 \cdot \hat{\alpha}_k \end{aligned}$$

which implies

$$q(k) \leq Q(\lambda) + (1 + \|\lambda\|_1) \cdot \hat{\alpha},$$

where one makes use of the fact that $\lambda_+(\cdot) > 0 \Rightarrow \lambda_-(\cdot) = 0$, and vice versa. The other inequality follows analogously.

□

Proof (of Theorem 2):
The non-negativity constraint on individual consumption, i.e. $x^i \in \mathbb{R}^{S+1}_+$, implies that for period $t = 0$ and for any asset $A^j \in \overset{*}{A} := \overset{*}{A}(\overset{*}{\Gamma})$,

$$\bar{\omega}_0 := \sum_{i=1}^I \omega_0^i \geq \sum_{i=1}^I C(v(\overset{*}{\theta}); \overset{*}{\theta}{}^i) = \sum_{i=1}^I \ \sum_{j=1}^J v_j^i(\overset{*}{\theta}) \cdot f(v_j(\overset{*}{\theta})),$$

where, as usual, one defines $v_j^i(\overset{*}{\theta}{}^i) := \sum_{i \in I} [\overset{*}{\theta}{}^i_+ (j) + \overset{*}{\theta}{}^i_- (j)]$. Then one obtains that

$$\bar{\omega}_0 \geq \max_{j \in J} [v_j(\overset{*}{\theta}) \cdot f(v_j(\overset{*}{\theta}))], \text{ where } v_j(\overset{*}{\theta}) := \sum_{i=1}^I v_j^i(\overset{*}{\theta}{}^i).$$

By assumption on f, $\lim_{x \to 0} x \cdot f(x) = +\infty$; hence there is some $m > 0$ such that

$$0 < x < m \Rightarrow x \cdot f(x) > \bar{\omega}_0.$$

From this, one deduces $v_j(\overset{*}{\theta}) \geq m$ for any traded ass $A^j \in \overset{*}{A}$. Consequently,

$$\begin{aligned} \bar{\omega}_0 &\geq \sum_{i=1}^I C(v(\overset{*}{\theta}); \overset{*}{\theta}{}^i) = \sum_{i=1}^I \sum_{A^j \in \overset{*}{A}} v_j^i(\overset{*}{\theta}) f(v_j(\overset{*}{\theta})) \\ &\geq \sum_{A^j \in \overset{*}{A}} \alpha \cdot v_j(\overset{*}{\theta}) \geq |\overset{*}{A}| \cdot \alpha \cdot m, \end{aligned}$$

which implies the desired inequality.

□

References

Allen, F. and D. Gale (1988) : Optimal Security Design; *Review of Financial Studies 1 (3), 229-263.*

Allen, F. and D. Gale (1991) : Arbitrage, Short Sales and Financial Innovation; *Econometrica 59 (4), 1041-1068.*

Allen, F. and D. Gale (1994) : Financial Innovation and Risk Sharing; *MIT Press, Cambridge MA.*

Bisin, A. (1994) : Walrasian Economies with Non-linear, Individualized Prices; *Working Paper, MIT, Cambridge, MA.*

Che, J. and U. Rajan (1994) : Endogenous Financial Market Formation in a General Equilibrium Model; *Working Paper, Department of Economics, Stanford University.*

Cuny, C. J. (1993) : The Role of Liquidity in Futures Market Innovations; *Review of Financial Studies 6, 57-78.*

Debreu, G. (1972) : Smooth Preferences; *Econometrica 40 (4), 603-615.*

Duffie, D. and M.O. Jackson (1989) : Optimal Security Design; *Review of Financial Studies 2 (3), 275-296.*

Duffie, D. and R. Rahi (1995) : Financial Market Innovation and Security Design; *Journal of Economic Theory Symposium on Financial Innovation and Security Design, vol. 65 (1), 1-42.*

Gale, D. (1992) : Standard Securities; *Review of Economic Studies 59, 731-755.*

Gottardi, P. and Th. Hens (1996) : The Survival Assumption and the Existence of Competitive Equilibria when Markets are Incomplete; *Journal of Economic Theory, forthcoming.*

Hahn, F. (1971) : Equilibrium with Transaction Costs; *Economometrica 39, 417-439.*

Hahn, F. (1973) : On Transaction Costs, Inessential Sequence Economies and Money; *Review of Economic Studies, 40 (4), 449-461.*

Heller, W. (1993) : Equilibrium Market Formation Causes Missing Markets; *Working Paper, Department of Economics, University of California, San Diego.*

Hens, Th. (1991) : Structure of General Equilibrium Models with Incomplete Markets; *Doctoral Dissertation, University of Bonn.*

Kurz, M. (1974) : Arrow-Debreu Equilibrium of an Exchange Economy with Transaction Cost; *International Economic Review 15, 699-717.*

Laitenberger, M. (1996) : Existence of Financial Market Equilbria with Transaction Costs; *Ricerche Economiche, forthcoming.*

Lancaster, K. (1971) : Consumer Demand: A New Approach; *Columbia University Press, New York.*

Lancaster, K. (1975) : Socially Optimal Product Differentiation; *American Economic Review 65, 567-585.*

Lippman, S.A. and J.J. McCall (1986) : An Operational Measure of Liquidity; *American Economic Review 76 (1), 43-55.*

Magill, M. and M. Quinzii (1996) : Theory of Incomplete Markets; *Vol.I, MIT Press, Cambridge MA.*

Mas-Colell, A., M.D. Whinston and J. Green (1995) : Microeconomic Theory; *Oxford University Press.*

Miller, M. (1992) : Financial Innovation: Achievements and Prospects; *Journal of Applied Corporate Finance.*

Pesendorfer, W. (1995) : Financial Innovation in a General Equilibrium Model; *Journal of Economic Theory 65 (1), 79-116.*

Préchac, C. (1996) : Existence of Equilibrium in Incomplete Markets with Intermediation Costs; *Journal of Mathematical Economics.*

Siconolfi, P. (1986) : Equilibrium with Restricted Participation in Incomplete Markets; *Mimeo, University of Pennsylvania.*

Siconolfi, P. (1989) : Equilibrium with Asymmetric Constraints on Portfolio Holdings and Incomplete Financial Markets; *in: M. Galeotti, L. Geronazzo, F. Gori, eds. Non-linear Dynamics in Economics and Social Sciences, Societá Pitagora, December 1989, 271-292.*