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An Evolutionary Approach to Financial Innovation

by

Marc Oliver Bettzüge and Thorsten Hens *)

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SONDERFORSCHUNGSBEREICH 303
Rheinische Friedrich-Wilhelms-Universität Bonn
Lennéstraße 37
D–53113 Bonn
Germany
Telefon (0228) 73 92 42
Telefax (0228) 73 79 40
E–Mail: with2 @ track.econ2.uni-bonn.de

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Abstract

This paper presents an evolutionary approach to the analysis of innovation in financial markets. In contrast to the existing literature on this topic, no innovating entities are considered in the evolutionary model studied. The paper can therefore be regarded as offering an evolutionary foundation for the strong rationality assumptions needed in the existing static models. First, existence of the standard GEI-equilibrium is established in this set-up. Next, an evolutionary process governing the participation structure is superimposed and corresponding notions for evolutionary equilibria are introduced. Finally, these equilibria are analysed for important examples, especially for the case, where the underlying stage economy takes the form of a standard CAPM-economy.

Keywords: Financial innovation, Evolution, CAPM.

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Contents

1 Introduction

The rapid development and unprecedented growth of international financial markets over the last two decades[1] has become the object of intensive economic research. Literally every day an abundance of new financial assets is created on these markets; some of these new securities soon become standard instruments of financial trade, other ones disappear as quickly as they have emerged. This extensive process of financial innovation has led to deep structural changes in the workings and the usage of international financial markets.

This paper sets out from the claim that the development of financial markets must be regarded as a fundamentally dynamic process, connecting a series of basic historical changes. Therefore we postulate that the analysis of the process of financial innovation requires treatment within a truely dynamic model[2]. As the process of financial innovation is stimulated by a variety of different reasons, each leading to a different range of financial products, our concept regards the causes for a specific innovation as a black box, assuming that for whatever reason, new assets can perpetually enter the market. Then we consider an evolutionary selection process which distinguishes stable asset structures from such asset structures which are likely to be modified by the innovation of some new financial product.

Building on the applied literature on this topic we assume that the trading volume generated by a certain financial security is one of the key determinants of its "survival" in the market[3]. Since in our model we are mainly concerned with the stability of asset structures we will qualify an asset as successful in a certain environment if it is able to generate a self-sustaining trading volume. Thus, the focus is not on large trading volume *per se* but on the possibility that trading volumes might be affected by the introduction of new assets. In this view, certain assets might survive even at a trading volume which is low relative to the total trading volume on the financial market, if they happen to successfully occupy some "niche", satisfying certain needs for the exchange of risks which cannot be better taken care of by another set of assets.

[1] For an assessment cf. Tufano (1989) and Miller (1992).

[2] We note that this should be true for any economic analysis of causes and effects of innovations.

[3] Cf. e.g. Tufano (1989). His empirical studies suggest that innovators profit from their first-mover-advantage by selling larger quantity rather than by charging higher prices. This indicates that a new asset will be judged to be successful if it generates a high trading volume. Verbal statements in this direction can be found e.g. in Miller (1986) or in a recent supplement of "The Economist" (1996). The main justification given for the importance of trading volume for the success of a financial instrument are the presence of bid-ask-spreads and network effects.

The essential ingredient of our evolutionary model is a standard (static) general equilibrium model with incomplete financial markets (GEI-model) where investors' market participation is assumed to be asymmetrically restricted[4]. In such a model, investors are characterised not only by their respective endowments and their preferences, but also by the subset of assets which they are able to trade on the market. We then consider an intertemporal sequence of these fundamental stage economies, where we assume stationarity of all the standard GEI-characteristics. Trading volume, however, links consecutive periods by determining the proportion of investors trading in certain assets in the next period. This proportion can be interpreted as a "market participation rate". Using some "fitness functions", which again are stationary, the effect of last period's trading volume on this period's market participation rates is modelled. Thus, starting from arbitrary initial conditions, an iterative dynamical process is defined and stationary equilibria (fixed points) of this process can be analysed.

Plausible qualitative assumptions on the fitness functions imply that both the situations where no asset is traded by any investor (no participation) and where every asset is traded by every investor (complete participation) are stationary equilibria. It also turns out that in general there are additional stationary equilibria. Especially, it is then interesting to analyse which of the stationary equilibria are robust with respect to small perturbations of the market participation rates. We investigate whether some fixed new assets which have not been traded so far ("mutations") can succeed in being established in the market. We call stationary equilibria evolutionarily stable if there is an entry barrier for some possible new asset, below which the asset will be pushed back out of the market. Moreover, a stationary equilibrium is called asymptotically stable, if any small enough perturbation of the corresponding market participation rate will induce convergence of the dynamical process back to the stationary equilibrium for some given asset structure. An asymptotically stable equilibrium thus always is evolutionarily stable. In our model, the complete-participation-case is evolutionarily stable by construction. This evolutionary approach is then applied to the analysis of financial innovations in prominent examples, especially for the well-known CAPM-economies widely used in the finance literature. Several interesting properties of the evolution of such CAPM-economies are derived.

The issue of financial innovation has recently received a lot of attention in economic theory[5]. The main focus is on optimal security design, i.e. on the innovator's decision problem, especially in the presence of asymmetric information.

[4]Such models were studied first by Siconolfi (1986, 1989) and Balasko, Cass and Siconolfi (1990).

[5]Cf. especially the survey article by Duffie and Rahi (1995) in the JET Symposium on Financial Innovation and the book by Allen and Gale (1994).

The models presented so far in a framework of general equilibrium theory are of an inherently static nature. In general only two time periods are considered[6]. In the first period, imperfect competition between financial intermediaries (banks, brokering institutions, exchanges) is modelled, which determines some endogenous financial market structure. In a second step, this structure is used as the exogenous market structure closing the well-known static GEI-financial market model. All of these models have to make rather strong assumptions on innovators' rationality. In particular, in these models financial intermediaries can perfectly anticipate every possible consequence of their alternative financial innovations. Our model is closely related to these approaches. It can be viewed as a limit case analysis where imperfect competition is replaced by an evolutionary process. Thus, it can serve as a dynamic, bounded rationality foundation for the results obtained in the corresponding static models under rather strong rationality assumptions. In particular, we therefore analyse how the so-called "nuts-and-bolts"-example derived in a static setting by Heller (1995) allows for a dynamic interpretation. Also, we show that our results for innovation in CAPM-economies exactly match the static analysis carried out in Duffie and Jackson (1989).

The paper is organised as follows. In section 2, the static GEI-economy with restricted participation is presented and sufficient conditions for the existence of equilibria are derived. Section 3 sets up the evolutionary process and introduces the intertemporal equilibrium and stability notions underlying the analysis of the model. Section 4 then applies the machinery established so far to a discussion of the "nuts-and-bolts"-example, before section 5 analyses the important special case where the stage economies are of the CAPM-type. The outlook on intended further research given in section 6 then concludes the paper.

2 The Stage Economy

The first step in constructing the evolutionary approach to financial innovation consists in the definition of a suitable stationary stage economy. The basis of the financial markets model at each stage of the evolutionary process is given by the standard GEI-model, where for simplicity we assume that there is only one consumption good, which is divisible and perishable, and which is interpreted as a composite commodity. Moreover, there are two periods $t = 0, 1$ with uncertainty in period 1, which is modelled by S possible states $s = 1, \ldots, S$. There are I individuals $i = 1, \ldots, I$ having utility functions $U^i : X^i \longrightarrow \mathbb{R}$, where

[6] Models of the type discussed here are used e.g. in Duffie and Jackson (1989), Heller (1993), Allen and Gale (1994), Bisin (1994) and Pesendorfer (1995).

consumption sets $X^i := \mathbb{R}_+^{S+1}$ are identical for each consumer and initial endowments $\omega^i \in X^i$ are contained in the consumption sets. By abuse of notation, we denote index sets and their cardinalities by the same letter, e.g. $I = \{1, \ldots, I\}$. Since the main point of this paper is not to investigate the most general allocation problem, the following rather strong assumption on the individuals' utility functions will be made, which serves to greatly facilitate the exposition.

Assumption (U): $U^i : X^i \longrightarrow \mathbb{R}$ is continuous, strictly monotone and stricly quasi-concave for $i = 1, \ldots, I$.

We will also need the following interiority assumption on endowments.

Assumption (E): $\omega^i \in \mathrm{int}X^i = \mathbb{R}_{++}^{S+1}$ for $i = 1, \ldots, I$.

There is an **exogenously** given set of assets $j = 1, \ldots, J$, a subset of which will be selected by the evolutionary process as the **endogenous** asset structure. However, we already point out at this stage that this does not impose any restrictions on the level of generality of our analysis. This will become clear at the end of the next section. Assets are 'real', i.e. they pay off in units of the single consumption good. They can then be distinguished by their pay-off-vectors, i.e. $a_j \in \mathbb{R}^S$ is the j-th asset. The set of all assets A can then be viewed as a $S \times J$-matrix, i.e. $A = [a_1, \ldots, a_J] \in \mathbb{R}^{S \times J}$. In the sequel, A will always be assumed to have full column rank, i.e. $rank\ A = J$. Thus redundant assets are excluded. This seemingly restrictive assumption is justified by at least the following two reasons. From a theoretical point of view it is not meaningful to specify agents' trading restrictions in some asset if they could attain the same asset pay-off by a portfolio of other assets. That is to say with redundant assets restricted participation should not be modelled with particular assets but with trading subspaces to which agents are restricted (Siconolfi (1986)). From a practical point of view our model is supposed to summarize the fundamental properties of an asset market with transaction costs. The volume driven process of market participation that we study in the main part of our paper is well justified by transaction costs. It is well known that assets that look redundant in a model without transaction costs can serve the important purpose of minimizing cost in a model with transaction costs. We claim that assets that are certainly not redundant in the presence of transaction costs should be modeled in the model without transaction cost as non-redundant assets as well - this can always be done by an appropriate definition of the state space.

In the typical GEI-model, every agent is allowed to trade each available asset. Here, agents can only use certain subsets of the assets; such a restriction of the assets to be utilized by an individual agent can be motivated by claiming that not all the agents in the economy might be aware of all existing financial instruments. Considering the enormous amount of different assets available on today's financial markets this does not seem to be an implausible assumption. Especially for the discussion of the **emergence** of new assets (innovation) it seems rather reasonable to consider situations where - to start with - only a small proportion of all the traders are aware of this new trading opportunity. Therefore, we denote by $J^i \subseteq \{1, \cdots, J\}$ the set of assets possibly traded by agent i. The entire GEI-economy, augmented by restricted participation, is thus given by the tuple $RPGEI = \{\mathbb{R}_+^{S+1}, A, (U^i, \omega^i, J^i)_{i=1}^I\}$.

The decision problem of an individual agent in this $RPGEI$-economy can then be written as

$$(M^i) \qquad \begin{array}{l} \max\limits_{\substack{x^i \in \mathbb{R}_+^{S+1} \\ \theta^i \in \mathbb{R}^J}} \quad U^i(x^i) \\ \text{s.t. } (x^i - \omega^i) \leq \begin{pmatrix} -q \\ A \end{pmatrix} \theta^i \\ \qquad \theta^i_j = 0 \text{ if } j \notin J^i \end{array}$$

where $\theta^i \in \mathbb{R}^J$ is agent i's asset portfolio and $q \in \mathbb{R}^J$ is the vector of asset prices.

Using this definition of the individual decision problem, one obtains the following version of a Walrasian (general) equilibrium, which is very close to the standard GEI-equilibrium.

Definition 1: A tuple $(\overset{*}{x}, \overset{*}{\theta}, \overset{*}{q}) \in \mathbb{R}_+^{I\times(S+1)} \times \mathbb{R}^{I\times J} \times \mathbb{R}^J$ is called a restricted participation equilibrium with incomplete markets ($RPGEI$-equilibrium) if it satisfies the following conditions:

(1) $(\overset{*}{x}{}^i, \overset{*}{\theta}{}^i)$ solves (M^i) given $\overset{*}{q}$ $\forall i \in I$;

(2) $\sum_{i\in I} \overset{*}{\theta}{}^i = 0$;

(3) $\sum_{i\in I} \overset{*}{x}{}^i = \sum_{i\in I} \omega^i$.

In order to prove existence of such $RPGEI$-equilibria one first has to study the set of no-arbitrage prices. Since the trading of assets by certain agents is now restricted, the classical result - called Fundamental Theorem of Assets Prices' - that this set is an open convex cone yielded by multiplying the transpose of the asset matrix with a strictly positive state price vector[7], can no longer be expected to hold. To see what happens to the set of no arbitrage prices when market participation is restricted, it is convenient to introduce for each agent $i = 1, \ldots, I$ the set Q^i of individual no-arbitrage prices defined by

$$Q^i = \left\{ q \in \mathbb{R}^J | \not\exists \theta \in \mathbb{R}^J : \theta_j = 0, j \notin J^i; \begin{pmatrix} -q \\ A \end{pmatrix} \theta > 0 \right\}.$$

It has to be observed that under assumption (U) $\overset{*}{q}$ can be a Restricted Participation equilibrium price-vector only if $\overset{*}{q} \in \cap_{i=1}^I Q^i$, because otherwise some agent i, $i = 1, \ldots, I$ would face an individual arbitrage opportunity. Therefore, a necessary condition for proving the existence of such a $RPGEI$-price-vector is that $\cap_{i=1}^I Q^i \neq \emptyset$.

Before proceding to stating and proving our existence theorem, we have to briefly list some consequences of the individualization of the no-arbitrage cones. It turns out that the transition from individual participation sets J^i to individual no-arbitrage cones Q^i is inclusion-reversing.

Proposition 1: If $J^i \subseteq J^k$ then $Q^k \subseteq Q^i$.

Proof: Suppose $J^i \subseteq J^k$ but there exists a $q \in Q^k \backslash Q^i$. Then there is a trading strategy $\tilde{\theta}$ offering an arbitrage opportunity such that $\tilde{\theta}_j = 0$ for $j \notin J_k$. But since $J^i \subseteq J^k$ it follows that $\tilde{\theta}_j = 0$ for $j \notin J^i$.

Hence $\tilde{\theta}$ is a trading strategy offering an arbitrage opportunity to agent k. Contradiction. □

The following corollary states a simple condition under which one can restrict one's attention to the "standard" cone Q of individualised no-arbitrage prices even with restricted participation. It suffices that there be (at least) one agent whose participation is not restricted.

Corollary 1: If $J^k = J$ for some $k \in \{1, \ldots, I\}$ then $\cap_{i=1}^I Q^i = Q^k = Q := \{q \in \mathbb{R}^J | q = A^T \pi \quad \text{for some } \pi >> 0\}$. □

We are now in a position to state and to prove the existence theorem for the $RPGEI$-economy. The additional assumption needed on top of the standard assumptions[8] required for the existence of GEI-equilibria can be given by

[7] cf. Magill and Quinzii (1995), Theorem 9.3.
[8] cf. Magill and Quinzii (1995), Theorem 10.5.

requiring one of the agents to be able to trade in all the assets in which some other agent is not restricted.

Proposition 2:[9] Suppose that utility functions U^i satisfy assumption (U) and that individual endowments ω^i satisfy assumption (E) for $i = 1, \ldots, I$. If there exists some $k \in \{1, \ldots, I\}$ such that $\cup_{i \in I \setminus \{k\}} J^i \subseteq J^k$ then there are *RPGEI*-equilibria. □

The proof of Proposition 2 follows from straightforward extensions of the proof of existence of *GEI*-equilibria. For the sake of completeness, it is included in the appendix.

3 The Evolutionary Process

Having set-up the one-shot stage economy and consequently having shown that equilibria for this stage economy do in fact exist, we can now turn our attention to the evolutionary process which will generate a sequence of repeated such stage economies. The main driving force of this sequence consists in endogenising the individual agents' market participation by an iterative, volume driven process.

Therefore, consider the sequence of stage economies $RPGEI_{(t)}$ for $t = 1, 2, \ldots$. In order to simplify matters as much as possible, all but one relevant variable will be assumed to be stationary. Thus, we assume the invariance of

- X^i - the commodity space;
- U^i - the utility functions;
- ω^i - the endowments and
- A - the exogenous asset structure

over the time path of the *RPGEI*-economies. Only individual market participation (J^i in the notation of section 2) will be updated in each period.

Before tieing down the corresponding updating-rule, we introduce a slight change of notation which will turn out to be rather instructive. Let $i = 1, \ldots, I$ now be **types** of agents specified by the respective (U^i, ω^i)-tuples, and suppose

[9] Following Gottardi and Hens (1996) the assumption $\omega_i >> 0$ for all i can be considerably relaxed by taking A into account.

for convenience that there is a continuum of agents of each type of the same mass. In order to describe the restrictions on market participation for each of the agents (the former J^i), one has to consider all the 2^J possible subsets of J. By p^i_K we now denote the percentage of agents of type i who can trade precisely those assets contained in $K \subseteq J$. Instead of dealing directly with these coefficients p^i_K, we prefer to summarise the corresponding information by coefficients p^i_j for $i = 1, \ldots, I$ and $j = 1, \ldots, J$. Here, p^i_j is the percentage of agents of type i who consider trading in asset j. The following simplifying assumption now relates the different coefficients with each other.

Assumption (P1): $p^i_K = (\Pi_{k \in K}\, p^i_k)(\Pi_{l \notin K}(1 - p^i_l))$ for every $i \in I$ and every $K \subseteq J$, i.e. assets are "independent" with respect to their respective market participation.

"Independence" here is supposed to mean that the knowledge of some particular asset does not favour the knowledge of any other asset. We will also make the following assumption on independence of types and assets. Lacking a clear understanding how various endowment-utility-combinations may affect market participation of an agent, this seems to be the most plausible assumption possible in this context.

Assumption (P2): For every $j \in J$ there is a p_j such that $p_j = p^i_j$ for every $i \in I$.

Remark 1: In order to see the importance of these assumptions, let - for any $\bar{p} \in [0,1]^J$ - $K_+(\bar{p})$ be the set of known assets, i.e. $K_+(\bar{p}) := \{k \in J : \bar{p}_k > 0\}$. Thus $K_+(\bar{p})$ is the set of all the assets possibly traded by a positive mass of agents. By (P1) and (P2) the mass of agents of each type trading in **all** those assets is obtained as $p_{K_+(\bar{p})} := \Pi_{k \in K_+(\bar{p})} p_k > 0$. Therefore, this set of agents (of a certain type) satisfies the additional assumption in Proposition 4, whence existence of one-shot-$RPGEI$-equilibria can be concluded for an arbitrary participation structure $\bar{p}$.

The idea underlying the evolutionary construction which follows is to consider a process of market participation, where the volume of trade effected in the (one-shot-)$RPGEI$-equilibrium by a certain type of agents determines the market participation possibilities for this class of agents in the next period. Using assumptions (P1) and (P2) we can reduce this transition process to considering tuples $p \in [0,1]^J$. Call such tuples the "phase" of the process and their coordinates "market participation rates" of the corresponding assets.

Given such a phase $p \in [0,1]^J$ let $(\overset{*}{x}, \overset{*}{\theta}, \overset{*}{q})_{(p)} \in \mathbb{R}^{I(S+1)\times 2^J} \times \mathbb{R}^{IJ\times 2^J} \times \mathbb{R}^J$ be an associated restricted participation-equilibrium. Note that from the mere knowledge of the phase p in a restricted participation equilibrium the allocation of each type for all possible combinations of asset market participations is well determined.

For a given phase p let now $v_j^i(K,p) := |\overset{*}{\theta}{}_j^i(K,p)|$ be the RPGEI-equilibrium trading volume in asset j as effected by an agent of type i who is restricted to trade only in those assets contained in $K \subseteq J$. Aggregating over types of agents and all the possible restricting subsets of J, one obtains total trading volume in asset j as

$$v_j(p) := \sum_{i=1}^{I} \sum_{K\subseteq J} p_K^i v_j^i(K,p).$$

Because of (P2), the order of summation can be reversed such that one obtains

$$v_j(p) = \sum_{K\subseteq J} p_K \cdot v_j(K,p),$$

where $v_j(K,p) = \sum_{i=1}^{I} v_j^i(K,p)$.

Note that the trading volume defined by v_j^i is not independent of the size of the asset's pay-off. If $\overset{*}{\theta}{}_j^i$ is the equilibrium asset holding of asset j given $a_j \in \mathbb{R}^S$ then $\frac{1}{\lambda}\, \overset{*}{\theta}{}_j^i$ is the equilibrium asset holding when the assets' pay-offs are $\lambda a_j \in \mathbb{R}^S$. Hence, before considering the evolutionary process, we will first normalise the trading volume. Let $p = \mathbb{1} \in \mathbb{R}^J$ denote the phase where every asset is known to every type with probability 1, i.e. $\mathbb{1} = (1,1,\ldots,1)$. This situation of complete participation henceforth serves as the reference situation for evaluating an asset's performance. Normalising trading volume with respect to complete participation $p = \mathbb{1}$, we thus define $vol_j(p) := \frac{v_j(p)}{v_j(\mathbb{1})}$ if $v_j(p) \neq 0$ and $vol_j(p) = 0$ otherwise[10]. Here $v_j(\mathbb{1})$ is the trading volume obtained in some equilibrium with complete participation. We will later study a volume driven dynamical process and ask which of its equilibria are stable. When considering the stability of one of the complete participation equilibria we will choose to normalise with respect to the equilibrium considered. In an economy enjoying the comfort of a unique complete participation equilibrium e.g. in the quadratic CAPM, the matter of taking care of the right normalisation does not raise any

[10] Observe that in general it might well be the case that $vol_j(p) > 1$.

problems. Although other ways of normalising the trading volume might be investigated in the evolutionary set-up chosen, our normalisation seems justified at this starting point of the evolutionary analysis of the process of financial innovation for at least two reasons. Firstly, by taking the phase $p = \mathbb{1}$ as the reference situation, all assets are maximally fit in the ideal situation of complete awareness of all the assets available. This idea follows from the intuitive claim that the situation with complete participation should be a resting point for the economy. Secondly, the normalisation chosen in this paper does not differentiate between the assets with respect to the trading volume which they generate in the $\mathbb{1}$-phase: this appears to be reasonable, since differences in the absolute trading volumes of the assets in the standard GEI-economy should not have any influence on the evolutionary process leading (or not leading) to such an ideal situation. Thus, our normalisation does discriminate between the assets in fact only for phases different from $\mathbb{1}$.

It should be observed at this point, that as a consequence of the normalisation chosen, assets generating a trading volume which is small but constant over all the possible participation phases is maximally fit in every phase. This corresponds to the observation that there are financial securities which do not generate as high as trading volume as, say, IBM-shares or government T-bills, but which nonetheless prevail in the market for a long period of time. Such assets serve agents to hedge against otherwise uninsurable risk and they cannot be replaced by other assets better performing this task. Restricting attention solely on the absolute size of trading volumes would neglect the important rôle played by the hedging demand in such "niches".

Next the relation between consecutive time periods has to be established. This will be done in a Markovian fashion, i.e. market participation in period $t+1$ will depend only on the equilibrium outcome in period t and it will remain unaffected by the history of the process in the periods preceding time t. More precisely, let there be transition functions $f_j^t : \mathbb{R}_+ \to [0,1]$ for each time period $t = 1, 2, \ldots$ and each asset $j = 1, \ldots J$ such that the transition from participation $p(t)$ to $p(t+1)$ is defined as

$$p_j(t+1) := f_j^t(vol_j(p(t))) \quad \text{for } j = 1, \ldots, J.$$

Note that in general this iterative process is defined only up to a selection mechanism selecting - if necessary - one of possibly many $RPGEI$-equilibria corresponding to a phase p. For the purpose of this paper, however, such a mechanism does not have to be explicitly modelled. In the sequel, we will restrict our attention to the local stabilitiy properties of stationary equilibria. Locally, $RPGEI$-equilibria are generically unique when assets are real. Consequently, locally we do not have to consider problems of selecting between multiple equilibria.

As for an interpretation of this kind of a transition function, suppose that individual market participation in a certain asset depends on the individual awareness of the existence of that asset. Obviously, individuals are not able to trade in assets they do not know. Now suppose that the awareness of the existence of some asset depends e.g. on the amount of "advertising" made for this asset and that agents' propensity to learn about the existence of assets is asymmetrically distributed among agents. If the "marketing effort" for an asset was financed by some intermediary out of the transaction cost revenue generated by the asset in the last period and if transaction costs were proportional to the trading volume[11], then awareness in the new period would directly depend on the trading volume in the previous period. This then would give rise to a functional dependence of the kind assumed here[12]. Since it is not our main concern to derive micro-foundations for awareness processes, we leave this issue for further research.

Regarding the transition functions (or rather, sticking to the terminology used in the context of Evolutionary Game Theory, the fitness functions) the following assumption is made.

Assumption (T): Transition functions are identical across assets and independent of the time period, i.e.

$$p_j(t+1) = f(vol_j(p(t))) \quad \text{for } j = 1, \ldots, J \quad \text{and} t = 1, 2, \ldots,$$

where $f : \mathbb{R}_+ \to [0,1]$ is continuously differentiable and such that $f(0) = 0$ and $f(1) = 1$.

All the parts of assumption (T) may be justified by economic intuition. Firstly, there is no sensible presupposition as to how different transition functions should be related to different assets; thus - for purposes of simplification - one might as well assume that they are identical across agents. Secondly, assuming stationary transition functions is perfectly in line with the other stationarity assumptions made before; here, we wish to consider the simplest possible world by letting only participation rates fluctuate over time. Finally, among the technical assumptions on f note that $vol_j(\mathbb{1})$ serves as a maximum reference level above which the asset will be known to everybody. Hence $f(1) = 1$ whereas when the

[11] For a static innovation model in this spirit, see Duffie and Jackson (1989).

[12] Observe that for this interpretation only relative proportions of agents knowing about a certain asset are relevant.

asset is not known to anybody then nobody will learn about the existence of the asset, hence $f(0) = 0$. This seems to be a plausible assumption.

The evolutionary process governing the sequence of $RPGEI$-economies is now completely specified. We thus can ask for intertemporal equilibria of this process. The equilibrium notion underlying all of the following is that of a "stationary equilibrium" which is nothing else than a fixed point of the transition mapping defined by f.

Definition 2: $\bar{p} \in [0,1]^J$ is called a stationary equilibrium if $\bar{p}_j = f(vol_j(\bar{p}))$ for all $j = 1, \ldots, J$.

We will first devote our attention to some relatively simple but important stationary equilibria. Define $\mathcal{L}$ to be the lattice consisting of the vertices of the cube $[0,1]^J$, i.e. $\mathcal{L} = \{p \in [0,1]^J | \, p_j \in \{0,1\} \quad \text{for every } j \in J\}$. We stress that $p \in \mathcal{L}$ if and only if agents are uniformly restricted , i.e. every agent is considering trading within the same subset of assets. Such phases are called uniform participation phases while the other phases are called mixed participation phases.

At this point, an essential departure of our model from the existing literature on financial innovation has to be observed. In the usual approaches, some innovating intermediaries decide upon the introduction of a possible new financial security according to some objective function, e.g. maximisation of total trading volume. Once the market for some such asset has been opened, however, the asset is available to all the traders in the economy. In our notation this corresponds to a discrete jump from one uniform participation phase, i.e. some $p \in \mathcal{L}$, to another, with one "0"-coordinate turned into a "1". By adding the important and, as we believe, relevant generalisation that agents' information about the existence of the new asset might be asymmetric and by constructing the economy such as to have a continuum of possible participation phases, we are able to conduct the analysis of the inherently discrete phenomenon of innovation within a continous framework. The resulting local perspective onto equilibrium and stability notions we use yields conclusions which are different from the ones obtained in models where innovation is viewed as a clear-cut 0-1-event for the economy. One of these differences concerning the so-called "nuts-and-bolts"-example due to Heller (1995) is discussed at length in the following section.

Note that (T) trivially implies that $p = 0$ and $p = \mathbb{1}$ are stationary equilibria. Hence, uniform participation stationary equilibria do always exist. In general, however, it must not be the case that **all** the lattice points in $\mathcal{L}$ are stationary equilibria. For this to be true one needs e.g. that $p_j = 1$ implies $vol_j(p) = 1$ for each asset $j = 1, \ldots, J$. In such a situation, every asset can achieve its maximal reference level even if no other asset can be traded. This motivates the following definition.

Definition 3: Suppose $(RPGEI, f)$ is such that every $p \in \mathcal{L}$ is a stationary equilibrium. Then the evolutionary economy given by $(RPGEI, f)$ is called volume separating.

We will later show that the widely used $CAPM$-economies with quadratic utilities are volume separating.

Next, two nested concepts for the stability of stationary equilbria are presented. Again, these follow closely the corresponding concepts in evolutionary Game Theory[13].

Definition 4: A stationary equilibrium $\bar{p} \in [0,1]^J$ is called an evolutionarily stable equilibrium if for all $k \in J$ with $\bar{p}_k = 0$ there exists some $\varepsilon_k > 0$ such that for all $\hat{p}$ with $\hat{p}_k \leq \varepsilon_k$ and $\hat{p}_j = \bar{p}_j, j \neq k$, $\hat{p}$ converges to $\bar{p}$, i.e. if it is true that for $p(0) = \hat{p}$ and $p_j(t+1) = f(vol_j(p(t)))$

$$\lim_{t \to \infty} p(t) = \bar{p}.$$

Thus a stationary equilibrium is evolutionarily stable if the traded assets can be protected from mutants by certain entry barriers below which the new assets would not "survive". Note that $\bar{p} = \mathbb{1}$ always is an evolutionarily stable equilibrium, since trivially no more new entrants can appear.

Remark 2: Since J is finite we could as well require ε to be independent of the subscripts of the assets.

Definition 5: A stationary equilibrium $\bar{p} \in [0,1]^J$ is called an asymptotically stable equilibrium if there exists some $\varepsilon > 0$ such that for all $\hat{p}$ with $||\hat{p} - \bar{p}|| \leq \varepsilon$, $\hat{p}$ converges to $\bar{p}$, i.e. if it is true that for $p(0) = \hat{p}$ and $p_j(t+1) = f(vol_j(p(t)))$

$$\lim_{t \to \infty} p(t) = \bar{p}.$$

Note that asymptotically stable equilibria always also are evolutionarily stable while the opposite implication in general fails to hold. The concepts differ,

[13] cf. e.g. Weibull (1996).

since evolutionary stability only requires robustness with respect to a single mutation of a new entrant ("entry barrier") while asymptotic stability considers small deviations from the market participation rate in any possible direction. Thus the latter concept allows for simultaneous mutations and for variations of the incumbent assets' participation rates.

These definitions conclude the set-up of the evolutionary structure of the model. The following questions should first be attacked by a careful analysis of this general model. Such an analysis will be carried out in the next two sections.

Q1: Does a financial market "get started" when following a process of infinitesimal small trials and errors or is the economy without any asset market resistent to any innovation, i.e. is no participation $\bar{p} = 0$ asymptotically or at least evolutionarily stable?

Q2: Is there a tendency to maintain the situation in which every agent considers trading in all assets, i.e. is complete participation $\bar{p} = \mathbb{I}$ asymptotically stable?

Q3: Can the economy get stuck in an equilibrium with incomplete participation, i.e. do there exist stationary equilibria other than no participation and complete participation? Are these equilibria asymptotically or at least evolutionarily stable?

4 The "Nuts-and-Bolts"-Example

Heller (1995) has suggested an example where a coordination failure between innovating intermediaries may lead to an inefficient outcome of the innovation process. In a different framework, Che and Rajan (1994) repeat this example. The idea behind this example, is the following. Suppose there are two market makers who can open a market for certain asset at some fixed cost. Market makers recover these fixed costs through transaction costs they charge for each unit of the asset traded. Suppose furthermore that the two assets possibly traded on the two markets are complements with respect to trading volume. In that case every asset generates a low trading volume if introduced while the market for the other asset remains closed; if both markets are opened simultaneously, however, then trading volume in both assets will be high. Heller (1995) and Che and Rajan (1994) now point out that if fixed costs for the opening of a market are sufficiently high, both market makers may prefer to keep their market closed because each market maker expects the other one to do so. Due to this coordination failure the economy might, therefore, get stuck in the inefficient no trade-equilibrium without any available assets. For obvious reasons, this example is usually referred to as the "nuts-and-bolts"-example.

In our framework, the "nuts-and-bolts"-situation can be seen to depend crucially on the discrete nature of the innovation models considered by Heller (1995) and Che and Rajan (1994). In fact, the following proposition shows that locally assets can never be sufficiently complementary to give rise to the type of coordination failure described in the previous paragraph. Observe that in order to decide whether a given stationary equilibrium also is evolutionarily or even asymptotically stable, it suffices to consider the Jacobian matrix $Jac(p)$ of $(f(vol_1(p)), \ldots, f(vol_J(p)))$ evaluated at the phase $\bar{p}$. It follows from fundamental results of the Theory of Dynamical Systems that a stationary equilibrium is asymptotically stable if all the eigenvalues of the Jacobian at the fixed point have absolute value less than 1 (cf. e.g. Hirsch and Smale, (1974)). Similarly, a stationary phase is evolutionarily stable with respect to the mutation of a new asset, if the eigenvalue corresponding to this mutation has absolute value less than 1. Thus the derivative of the transition function is one of the two key determinants of the stability of stationary equilibria. Note that this derivative has a clear economic interpretation. It measures the speed with which innovations spread in the economy which we suggest to call the 'inertia to innovation'. This entails the following result on the stability properties of the origin, i.e. of the no-participation-situation.

Proposition 3: Suppose the economy is such that $vol(p)$ is continuously differentiable at $p = 0$. Then the stationary phase $\bar{p} = 0$ is evolutionarily stable with respect to a mutation in asset $j \in J$ if $f'(0) \cdot vol_j(\{j\}) < 1$. Moreover, if $f'(0) \cdot vol_j(\{j\}) < 1$ holds for every $j \in J$ then $\bar{p} = 0$ also is asymptotically stable.

Thus the financial market never gets started (Q1) if every asset in isolation does not generate a sufficiently high trading volume. Therefore assets are seen to be independent in the evolutionary set-up: the "nuts-and-bolts"-example cannot occur.

Remark 3:

1) Note as a corollary of Proposition 3 that in differentiable economies evolutionary stability of the origin implies its asymptotic stability.

2) The seminal paper on financial innovation by Duffie and Jackson (1989) considers the innovation problem of a monopolistic exchange choosing to create the asset which maximises trading volume[14]. They give an endogenous characteri-

[14]This approach is motivated by the introduction of positive transaction costs which are proportional to volume.

sation of such an optimal asset. Proposition 3 can thus be read as stating that Duffie and Jackson's optimal security will also be the "fittest" one in our evolutionary set-up. "Fittest" is here taken to mean that on increasing the slope of f at the origin, i.e. on decreasing the degree of inertia to innovation, Duffie and Jackson's optimal security will be the first asset to be able to survive in the market. Therefore, the evolutionary argument given here nicely reinforces the result obtained by considering rational innovators in a static setting.

Proof of Proposition 3: Since $\bar{p} = 0$, the derivative (from the right) of $f(vol_j(p))$ with respect to p_k at $p = \bar{p}$ is given by

$$\partial_{p_k} f(vol_j(p)) = f'(vol_j(0)) \cdot \partial_{p_k} vol_j(\bar{p}) = f'(0) \cdot \partial_{p_k} vol_j(0),$$

for every $j \in J$.

But the remaining derivative reduces to

$$\begin{aligned} \partial_{p_k} vol_j(0) &= \partial_{p_k} \sum_{K \subset J} p_K vol_j(p, K)|_{p=0} \\ &= \sum_{K \subset J} (\partial_{p_k} p_K) \cdot vol_j(p, K)|_{p=0} + \sum_{K \subset J} p_K \cdot (\partial_{p_k} vol_j(p, K))\,|_{p=0} \\ &= vol_j(0, \{k\}) + \partial_{p_k} vol_j(0, \emptyset) = vol_j(0, \{k\}), \end{aligned}$$

where one has to recall that $p_K = (\Pi_{k \in K}\, p_k)\,(\Pi_{l \notin K}(1 - p_l))$ by assumption (P1). Since $vol_j(\{k\}) = 0$ if $k \neq j$ it follows that $Jac(0)$ is a diagonal matrix with diagonal entries $Jac_{jj}(0) = vol_j(\{j\})$, $j = 1, \ldots, J$. Thus, the eigenvalues of $Jac(0)$ are given by $\lambda_j(0) = Jac_{jj}(0)$ with the j-th unit vector as the corresponding eigenvectors. This implies that 0 is asymptotically stable if $\lambda_j(0) < 1$ or equivalently if $\frac{1}{f'(0)} > vol_j(\{j\})$ for every $j \in J$.

Finally, since the eigenvectors precisely correspond to the directions of mutation with respect to a single asset, we can similarly deduce that 0 is evolutionarily stable with respect to a mutation in asset j if $\frac{1}{f'(0)} > vol_j(\{j\})$.

$\square$

The following example shows that the situation becomes more complicated if one moves away from the origin. A one-sided complementarity can then arise where the successful introduction of one asset can promote the participation in another asset which when introduced in isolation could not be sustained by the economy. This can be interpreted as half the "nuts-and-bolts"-example.

Example 1:

Suppose that there are two states ($S = 2$) with first period-consumption and that the asset structure is given by two securities with linearly independent pay-offs. Furthermore, assume that the characteristics of the economy are such that for the normalised trading volumes associated with the different subeconomies without restricted participation, the following relations hold:

$$v_1(\{1\}) = 1,\ v_2(\{2\}) = 3,\ v_1(\{1,2\}) = v_2(\{1,2\}) = 4.$$

Also suppose that equilibrium prices for these subeconomies are identical, i.e. that

$$\overset{*}{q}_1(\{1\}) = \overset{*}{q}_1(\{1,2\}) \text{ and } \overset{*}{q}_2(\{2\}) = \overset{*}{q}_2(\{1,2\}).$$[15]

Then normalised trading volume is given by

$$\begin{aligned} vol_1(p) &= p_1(1-p_2)\cdot\frac{1}{4} + p_1p_2, \text{ and} \\ vol_2(p) &= (1-p_1)p_2\cdot\frac{3}{4} + p_1p_2. \end{aligned}$$

Now suppose that the transition function f satisfies

$$f'(0) = 2,\ f(\frac{3}{8}) = \frac{1}{2} \text{ and } f'(\frac{3}{8}) = 1.$$

Then the following facts can be observed.

- The Jacobian matrix of $f(vol(p))$ at $p=0$ is given by

$$Jac(0) = \begin{pmatrix} f'(0) & 0 \\ 0 & f'(0) \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}.$$

  From the preceding discussion we can thus conclude that the origin is evolutionarily stable with respect to the introduction of the first asset, but that it is not evolutionarily stable with respect to the introduction for the second asset.

- The phase $\bar{p} = (0, \frac{1}{2})$ is a stationary equilibrium since $f(vol(\bar{p})) = f(\frac{1}{2}\cdot\frac{3}{4}) = f(\frac{3}{8}) = \frac{1}{2}$.

- The Jacobian of $f(vol(p))$ at the phase $\bar{p}$ is given by

$$\begin{aligned} Jac(\bar{p}) &= \begin{pmatrix} f'(0) & 0 \\ 0 & f'(\frac{3}{8}) \end{pmatrix} \begin{pmatrix} (1-\bar{p}_2)\frac{1}{4} + \bar{p}_2 & 0 \\ -\bar{p}_2\frac{3}{4} + \bar{p}_2 & \frac{3}{4} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{8} & 0 \\ \frac{1}{8} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & 0 \\ \frac{3}{16} & \frac{3}{4} \end{pmatrix} \end{aligned}$$

[15] An economy as the one described can be constructed with quadratic CAPM-preferences. Cf. the next section.

which has eigenvalues $\lambda_1 = \frac{5}{4}$ and $\lambda_2 = \frac{3}{4}$ associated with the eigenvectors $v_1 = (1, \frac{3}{8})$ and $v_2 = (0, 1)$. Hence, $\bar{p}$ is not evolutionarily stable with respect to the introduction of asset 1.

□

Thus, while not being able to be sustained by the market in the situation without any asset trade, asset 1 may successfully enter the market at the (stable) stationary equilibrium where a fraction of the agents is already trading in asset 2.

A final example is intended to demonstrate how the occurrence of a symmetric "nuts-and-bolts"-situation as in Heller (1995) depends on taking a global (i.e. discrete) perspective.

Example 2:

Again suppose that $S = J = 2$ and that there is first period consumption. Let

$$v_1(\{1\}) = v_2(\{2\}) = 1 \;\text{ and }\; v_1(\{1,2\}) = v_2(\{1,2\}) = 2.$$

Maintaining the assumption on identical asset prices in the subeconomies one obtains

$$\begin{aligned} vol_1(p) &= p_1(1-p_2) \cdot \frac{1}{2} + p_1 p_2, \;\text{ and} \\ vol_2(p) &= (1-p_1)p_2 \cdot \frac{1}{2} + p_1 p_2. \end{aligned}$$

Now assume that f is such that $f'(0) < 2$, $f(\frac{1}{4}x) < x$ for every $x \in (0.1]$ and $f'(1) < \frac{2}{3}$. Then the eigenvalues on the diagonal of $Jac(0)$ satisfy $\lambda_1 = \lambda_2 < 1$ whence one concludes that the origin is evolutionarily and, consequently (Proposition 3) also asymptotically stable. Also, since $f(\frac{1}{2}x) < x$ for every $x \in (0, 1]$, there is no stationary equilibrium of the form $(\bar{p}_1, 0)$ or $(0, \bar{p}_2)$ other than $(0, 0)$. On the other hand, the Jacobian at $\bar{p} = \mathbb{1}$ is given by

$$Jac(\mathbb{1}) = \begin{pmatrix} f'(1) & 0 \\ 0 & f'(1) \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix},$$

whose eigenvalues satisfy $0 < \lambda = f'(1) \cdot (\pm\frac{1}{2} + 1) < 1$. Hence, $\bar{p} = \mathbb{1}$ is asymptotically stable[16]. Therefore, if one starts from the presumption that both

[16] Observe that one actually only needs to consider one of the eigenvalues (the larger one, in this case) because the other one corresponds to an eigenvector whose direction ($\nu = (1, -1)$) points outward of $[0, 1]^2$.

assets get introduced simultaneously and immediately known to (almost) the entire population of traders, then the efficient outcome might be sustained by the economy.

5 Evolution in the CAPM

In this section we will apply the equilibrium concepts introduced so far to the analysis of the evolutionary process emerging from CAPM-economies as underlying stage economies. For a variety of reasons such economies are widely used in the Theory of Finance[17] which renders them quite a natural starting point for our analysis. We will consider one of the simplest versions of the CAPM, where the standard mean-variance preferences are of the quadratic type. This approach leads to explicit and unique solutions for the stage economy which in turn allows to focus on the evolutionary process emerging. Thus, the following assumptions will be in place for the remainder of this section.

Assumption (CAPM-1) Agents' utilities are given by $U^i(x^i) = x_0^i + \sum_{s=1}^S \pi_s (x_s^i - \frac{1}{2}\alpha^i (x_s^i)^2)$ for $i = 1, \ldots, I$, where $\alpha^i > 0$ is a coefficient of individual risk aversion.

In order to guarantee that individual consumption will always be below the satiation point of the quadratic utilities (which in turn implies existence of a unique equilibrium), the following assumption on the relation between aggregate endowments and individual risk aversion is made. This assumption is standard in this context[18].

Assumption (CAPM-2) For $\omega := \sum_{i=1}^I \omega^i$, let $(\omega_s - \frac{1}{2}\alpha^i(\omega_s)^2) > 0$ for every $i = 1, \ldots, I$.

In the case of quadratic utilities, $RPGEI$-equilibria can easily be solved for. In particular, this is a consequence of a result by Oh (1994) which states that innovation of a non-redundant asset in the quadratic CAPM does not change the equilibrium price of the existing assets.

Lemma 1: (Market Partition Lemma): For some $K \subseteq J$ consider the standard GEI-economies given by $\{\mathbb{R}_+^{S+1}, A, (U^i, \omega^i, K)_{i=1}^I\}$ where only those subsets of

[17] e.g. Duffie writes that the CAPM is "a rich source of intuition and the basis for many practical financial decisions.", Duffie (1988) p. 93.

[18] cf. e.g. Duffie and Jackson (1989), Geanakoplos and Shubik (1990).

agents of each type are taken into account who are restricted to trade precisely in those assets contained in K. Under assumptions (CAPM-1) and (CAPM-2) it then follows that the equilibrium asset prices $\bar{q}(K)$ of these economies satisfy $\bar{q}_j(K) = \bar{q}_j(K')$ for each $j = 1, \ldots, J$ with $j \in K \cap K'$.

Proof: This is Corollary 1 in Oh (1994).

□

The Market Partition Lemma demonstrates that when the economy is partitioned into the 2^J subeconomies consisting of all the agents being restricted in the same way, then asset prices are identical in all these subeconomies.

By standard results using the first-order-conditions and the fact that assumption (CAPM-2) guarantees an interior solution it is well-known that equilibria for these subeconmomies exist and that they are unique. As a consequence of the Market Partition Lemma one can therefore deduce the existence of a unique $RPGEI$-equilibrium.

Proposition 4: Add (CAPM-1) and (CAPM-2) to the assumptions in Proposition 2. Then there is a unique $RPGEI$-equilibrium for this economy.

Proof: If $\overset{*}{q}$ is the unique equilibrium price for the subeconomy trading all the assets $\{1, \ldots, J\}$, then $\overset{*}{q}$ is the unique $RPGEI$-equilibrium price for the entire economy by the Market Partition Lemma. □

The Market Partition Lemma also allows to considerably simplify the structure of the transition process. In particular, it leads to a very useful property of the volume function: The aggregate trading volume of asset j in the population of agents considering trade in the subset K of assets $v_j(K,p)$, is independent of the market participation rate, p.

Proposition 5: For each $K \subseteq J$ there exists a positive constant α_j^K such that $v_j(K,p) = \alpha_j^K$ for any $p \in [0,1]^J$.

Proof: Let $v_j(K,p)$ be the trading volume in the GEI-economy $\{\mathbb{R}_+^{S+1}, (A_j)_{j \in K}, (U^i, \omega^i, K)_{i=1}^I\}$; it only depends on the equilibrium prices $\bar{q}(K)$. But prices $\bar{q}(K)$ are independent of the participation structure by the previous Market Participation Lemma. Hence independently of p (i.e. independently of the relative sizes of the 2^J subeconomies restricted to trade assets in K)

equilibrium prices for the $RPGEI$-economy are given as $q \equiv q(J)$. Therefore, $\alpha_j^K := v_j(K,p)$ is constant over p.

$\square$

Corollary 2: For every asset $j \in J$ and every phase $p \in [0,1]^J$ normalised trading volume is given by $vol_j(p) = \sum_{\substack{K \subset J \\ j \in K}} p_K \frac{\alpha_j^K}{\alpha_j^J}$.

Proof: This follows readily from the observation that $v_j(\mathbb{1}) = \mathrm{p_J} \cdot \alpha_\mathrm{j}^\mathrm{J}|_{\mathrm{p=I}} = 1 \cdot \alpha_\mathrm{j}^\mathrm{J}$.

$\square$

This corollary yields a sufficient condition for the stationarity of phases in the lattice $\mathcal{L}$.

Proposition 6: Let $p \in \mathcal{L}$ and let $K_+(p) = \{j \in J | p_j = 1\}$. If $\alpha_j^{K(p)} \geq \alpha_j^J$ for all $j \in K(p)$ then p is a stationary equilibrium.

Proof: If $p \in \mathcal{L}$ then $vol_j(p) = p_{K(p)} \frac{\alpha_j^{K(p)}}{\alpha_j^J}$ and therefore p is a stationary equilibrium since $f(0) = 0$ and $f(1) = 1$.

$\square$

Whether a given phase p now has some or all of the evolutionary equilibrium properties defined in the previous section will of course depend on the specific transition function chosen. Proposition 5 implies that the evolutionary process emerging from such quadratic CAPM-stage economies is completely determined by the coefficients α_j^K, which in turn can be derived from the consideration of the different stage economies for varying $K \subseteq J$. This observation greatly facilitates the analysis of the stability properties of the process, since the Jacobian of $f(vol)$ can be explicitly calculated using the previous corollary. The stability properties of the origin have been completely characterised in Proposition 3. To check for stability of phases other than the origin, the following lemma turns out to be very useful with respect to the special case of phases on the unit lattice $\mathcal{L}$.

Lemma 2: Let $\bar{p} \in \mathcal{L}$ and let $K_+(\bar{p}) = \{j \in J | \bar{p}_j = 1\}$.
(1) Then

$$\partial_{p_j} vol_j(\bar{p}) = \begin{cases} \frac{\alpha^{K(\bar{p})}}{\alpha_j^J} & j \in K_+(\bar{p}) \\ \frac{\alpha_j^{K(\bar{p}) \cup \{j\}}}{\alpha_j^J} & j \notin K_+(\bar{p}) \end{cases}$$

(2) Moreover let $j \neq k$.
(2a) If $j,k \in K(\bar{p})$ then

$$\partial_{p_k} vol_j(\bar{p}) = \frac{\alpha_j^{K_+(\bar{p})} - \alpha_j^{K_+(\bar{p})\setminus\{k\}}}{\alpha_j^J}.$$

(2b) If $j \in K(\bar{p})$ and $k \notin K(\bar{p})$ then

$$\partial_{p_k} vol_j(\bar{p}) = \frac{\alpha_j^{K_+(\bar{p})\cup\{k\}} - \alpha_j^{K_+(\bar{p})}}{\alpha_j^J}.$$

(2c) If $j,k \notin K_+(\bar{p})$ then $\partial_{p_k} vol_j(\bar{p}) = 0$.
(2d) If $k \in K_+(\bar{p})$ and $j \notin K_+(\bar{p})$ then $\partial_{p_k} vol_j(\bar{p}) = 0$.

Proof: Recall that

$$\partial_{\bar{p}_k} vol_j(\bar{p}) = \sum_{\substack{K\subset J\\ j,k\in K}} \left(\Pi_{\substack{l\in K\\ l\neq k}} \bar{p}_l\right)\left(\Pi_{l\notin K}(1-\bar{p}_l)\right)\frac{\alpha_j^K}{\alpha_j^J} - \sum_{\substack{K\subset J\\ j\in K, k\notin K}} \left(\Pi_{l\in K}\bar{p}_l\right)\left(\Pi_{\substack{l\notin K\\ l\neq k}}(1-\bar{p}_l)\right)\frac{\alpha_j^K}{\alpha_j^J}.$$

Evaluating this formula according to the different cases yields the formulas stated.

□

A particularly simple case arises when the asset structure is orthogonal[19]. Before stating the result, we have to introduce some more notation. Let the π-adjusted scalar product in $\mathbb{R}^S$, $< \cdot,\cdot >_\pi$: $\mathbb{R}^S \times \mathbb{R}^S \to \mathbb{R}$ be given by

$< x, y >_\pi := \sum_{s=1}^S \pi_s x_s y_s$

For every matrix $B \in \mathbb{R}^{T\times S}$ this induces a linear mapping, $\cdot_\pi : \mathbb{R}^S \to \mathbb{R}^T$ by

$B\cdot_\pi := (\sum_{s=1}^S \pi_s B_{ts} x_s)_{t=1}^T$ which in turn gives rise to a natural matrix product, $\times_\pi : \mathbb{R}^{U\times S} \times \mathbb{R}^{S\times T} \to \mathbb{R}^{U\times T}$, given by

$B \times_\pi C := (\sum_{s=1}^S \pi_s B_{ts} C_{su})_{t=1,u=1}^{T,U}$.

Observe that these mappings are just the standard linear mappings adjusted by the objective probabilities π[20].

Lemma 3: Suppose $A = \{a_1, \ldots, a_J\}$ is such that $< a_j, a_k >_\pi = 0$ for $j \neq k$ ("π-orthogonal asset structure"). Then under assumptions (CAPM-1) and (CAPM-2) $\alpha_j^K = \alpha_j^J$ if $j \in K$ and $\alpha_j^K = 0$ otherwise.

[19] In Duffie and Jackson (1989) orthogonal asset structures emerge as the equilibrium outcome.

[20] For these definitions and their role in CAPM-economies cf. e.g. Magill and Quinzii (1995), Chapter 3.

Proof: Define the constant $k := \sum_{i=1}^{I} \frac{1}{\alpha^i}$ and let $\omega_{\rm I}$ be the S-dimensional vector of aggregate period 1-endowments. Let A^K be the $S \times K$ submatrix of A formed by the column vectors associated with the assets contained in K. Asset demand of a type i-agent trading only K can then be derived as

$$\theta^i(K) = \frac{1}{\alpha^i}(\tilde{A}^K)^{-1}(A^K)^T \cdot_\pi (\frac{1}{k}\omega_{\rm I} - \alpha^i \omega_1^i),$$

where $\tilde{A}^K = (A^K)^T \times_\pi A^K$. Then π-orthogonality of the assets implies that $\tilde{A}^K$ is a diagonal matrix with the π-adjusted Euclidean norms of the asset-payoff-vectors as diagonal entries. It follows that

$$\theta^i_j(K) = \frac{1}{\alpha^i} \frac{< a_j, y^i >_\pi}{< a_j, a_j >_\pi},$$

where y^i, $i = 1, \ldots, I$, is an S-dimensional vector independent of K.

From this we can conclude that $\theta^i_j(K) = \theta^i_j(K')$ if $j \in K \cap K'$, i.e. every agent being able to trade in asset j will choose to trade the same amount of this asset independent of the other assets she is allowed to trade in. But then, total trading volume in any asset has to be constant over the different separated markets, in particular, hence, it follows that $v_j(K) = v_j(J)$ if $j \in K$.

□

Remark 4: The arguments given in Duffie and Jackson (1989) suggest that with quadratic utilities agents always choose to insure their unhedged endowment risk. Thus, the statement of the proposition is intuitively clear; if assets are π-orthogonal they are used for insuring orthogonal endowment risks such that the trading strategies should indeed be independent of whether there are other (π-orthogonal) insurance opportunities.

For the important special case where the assets are mutually orthogonal this produces the characterisation of CAPM-economies indicated in section 3. This answers our third question (Q3) asking whether the economy might get stuck in some stationary equilibrium with incomplete participation. A preliminary lemma turns out to be useful.

Lemma 4: Suppose A is an π-orthogonal asset structure and let the economy satisfy (CAPM-1) and (CAPM-2). Then $v_j(p) = p_j \cdot \alpha^J_j$ for every asset $j \in J$ and every phase $p \in [0,1]^J$.

Proof: Let $p \in [0,1]^J$. Then it follows that

$$v_j(p) = \sum_{K \subset J \backslash \{j\}} p_{K \cup \{j\}} \alpha_j^{K \cup \{j\}} = p_j \cdot \alpha_j^J \sum_{K \subset J \backslash \{j\}} p_K = p_j \cdot \alpha_j^J,$$

where the penultimate equality follows from the preceding proposition and the ultimate one from the fact that p_K describe a probability measure on $K \backslash \{j\}$ as can be checked by a simple algebraic argument (e.g. using induction on J).

$\square$

Proposition 7: If the asset structure A is π-orthogonal then the evolutionary *RPGEI*-economy specified by $(\mathbb{R}_+^{S+1}, A, \{U^i, \omega^i\}_{i=1}^I, f)$ is volume separating, if it satisfies (CAPM-1) and (CAPM-2). Moreover, if $f(x) \neq x$ for $x \in (0,1)$ then $\mathcal{L}$ are the only stationary equilibria.

Proof: Let $p \in \mathcal{L} \backslash \{0,1\}$. The previous lemma implies that $v_j(p) = 0$ if $p_j = 0$ and $v_j(p) = \alpha_j^J$ otherwise. Assumption (T) then yields stationarity of p.

On the other hand, if $p \in [0,1]^J$ is a stationary equilibrium then the additional assumption implies that $vol_j(\bar{p}) \in \{0,1\}$ for every asset j. But then from Lemma 4 it follows that $p_j \in \{0,1\}$.

$\square$

Proposition 7 demonstrates in particular that if the inertia to innovation is monotonically increasing or monotonically decreasing then any stationary equilibrium is given by a uniform participation phase.

Concluding this discussion of the evolution of CAPM-economies, we can therefore completely characterise the stability structure of the phases on the unit lattice in the case of an orthogonal asset structure, hence providing answers to our fundamental questions Q1-Q3.

Proposition 8: Suppose A is an π-orthogonal asset structure and the economy satisfies (CAPM-1) and (CAPM-2). Then

1. $f'(0) < 1$ implies $\bar{p} = 0$ is asymptotically stable and $\bar{p} \in \mathcal{L}$ is evolutionarily stable.
2. $f'(1) < 1$ implies $\bar{p} = \mathbb{1}$ is asymptotically stable.
3. $f'(0) > 1$ implies $\bar{p} \in \mathcal{L} \backslash \{\mathbb{1}\}$ is not evolutionarily stable.
4. $f'(1) > 1$ implies $\bar{p} \in \mathcal{L}$ is not asymptotically stable.

Proof: From the π-orthogonality of A one concludes $\alpha_j^K = \alpha_j^J$ if $j \in K$. Hence Lemma 2 implies that $J(p)$ is a diagonal matrix at any phase $p \in \mathcal{L}$. But the diagonal entries are then equal to

$$f'(vol_j(\bar{p})) \cdot \partial_{p_j} vol_j(\bar{p}) = f'(\sum_{K \in J \setminus \{j\}} \bar{p}_{K \cup \{j\}}),$$

whence follow the results stated.

$\square$

For the evolution of financial markets, our model thus predicts the following plausible results. If the inertia to innovation is very high in an economy which is not used to financial markets, then it is most likely to get stuck without any financial markets [Prop. 8 (1)]. Only a "big push" will eventually trigger an evolution of asset markets. If on the other hand the economy is quite ready for financial markets then an evolutionary process driven by arbitrarily small trials and errors will initiate some growing system of financial markets which finds its only natural outcome in the situation of complete market participation [Prop. 7, Prop. 8 (3)]. This market structure in which everybody considers trading in every asset will be stable if close to this phase the economy is rather insensitive to changes in the participation rates [Prop. 8 (2)]. This might naturally be the case since the more agents already consider trading in all assets the less effective will be marketing efforts or network-effects aiming at recruiting additional asset traders.

Remark 5: Finally we would like to highlight the importance of further qualitative properties of the transition function f. The specific functional form to be chosen for f will depend on the particular application. Here, we discuss two examples:

- Suppose that fitness of the assets always is sufficiently strong in the sense that $f(x) > x$ for every $x \in [0, 1]$. Consequently, $f'(0) > 1 > f'(1) > 0$ by assumption (T)[21]. Thus, only the phase $\bar{p} = \mathbb{1}$ would be evolutionarily stable; with a π-orthogonal asset structure, it would in fact be even asymptotically stable. In this case, where information about new assets spreads through process of this kind, the evolutionary approach to financial innovations therefore predicts the complete participation situation as the only stable outcome.

[21] Indeed, this follows from $f(0) = 0$ and $f(1) = 1$.

- One could also study the diffusion of information through a network of interacting agents[22]. When transposed to our setting, such a situation would lead to f being of the logistic family, i.e. f would be S-shaped and monotonically increasing. Again, the derivative of f evaluated at $x = 1$ will always be sufficiently flat as to guarantee asymptotic stability of the complete participation situation. However, if f is sufficiently flat at $x = 0$ as well, then every phase $\bar{p} \in \mathcal{L}$ corresponding to a situation with uniform participation is evolutionarily stable. A big "information push" at the introduction of an asset is needed in such a situation in order to establish an unknown asset. In this fashion our model could serve to explain the empirical fact that many financial innovations disappear from the market rather quickly[23]: the trading volume generated in these assets by those traders initially informed about the innovation simply did not reach the critical size required to get the asset going[24].

6 Conclusion

This paper has presented a novel evolutionary approach to the analysis of innovation in financial markets. We have superimposed a volume-driven evolutionary process onto an extended static GEI-model and have shown that the equilibrium and stability notions known from other branches of evolutionary economics nicely carry over to our setting. As a first application of this approach, we studied the evolution of CAPM economies in the special case where individual preferences are of the quadratic type. We established a sufficient condition for the evolutionary stability of the no-participation-situation and analysed when a complete set of assets is asymptotically stable. In particular, we showed that our results are perfectly in line with the findings of Duffie and Jackson (1989) who study innovation in such special CAPM-economies in a static setting.

This paper could only give an introductory treatment of the evolutionary approach to financial innovation. Further research will have to be conducted along various lines. Firstly, the approach should be applied to examples other than the CAPM. Also, by putting more structure on the fitness functions, the general model should be studied more carefully.

Secondly, possible "spill-over"-effects of the trading volume of some asset on

[22] For a model of this type cf. e.g. Allen (1982).

[23] Cf. e.g. the failure of the inflation-indexed futures market which the Chicago Board of Trade tried to set-up in 1985.

[24] Again we note that "size" here is defined with respect to the trading volume generated in the situation with complete participation.

the assets in a suitably defined neighbourhood of this asset should be taken into account. Finally, it might be considered whether consecutive time periods could be linked by more than just the asset dynamics, e.g. by longer-lived agents and more complicated asset pay-offs.

Given the increasing importance of the evolutionary approach in economics, and given the undeniable and historic fact of rapid financial innovation calling for an economic explanation, we contend that research in these issues will turn out to be a worthwhile project.

7 Appendix

Proof of Proposition 2:

The proof given here follows the excess demand approach[25]. We first define individual excess demand $g^i : Q \to \mathbb{R}^J$ by

$$\begin{aligned} g^i(q) \quad := \quad & \arg\max_{\theta \in \mathbb{R}^J} \quad U^i(\omega^i + \begin{pmatrix} -q \\ A \end{pmatrix} \theta) \\ & \text{s.t. } \omega^i + \begin{pmatrix} -q \\ A \end{pmatrix} \theta \geq 0 \\ & \text{and } \theta_j = 0 \quad \text{if } j \notin J^i. \end{aligned}$$

Note that this formulation uses an individual decision problem which is equivalent to the problem (M^i) stated above since strict monotonicity of the utility functions U^i implies equality in the budget constraint. Also observe that without loss of generality we can assume that $J = \bigcup_{i=1}^{I} J^i$ since assets which cannot be traded by any agent may safely be discarded.

Now compare $g^i(q)$ with the usual excess demand function $\tilde{g}^i(q)$ from standard GEI-models, which is given by

$$\begin{aligned} \tilde{g}^i(q) \quad := \quad & \arg\max_{\theta \in \mathbb{R}^J} U^i(\omega^i + \begin{pmatrix} -q \\ A \end{pmatrix} \theta) \\ & \text{s.t. } \omega^i + \begin{pmatrix} -q \\ A \end{pmatrix} \theta \geq 0. \end{aligned}$$

It is well-known that existence of an equilibrium in standard GEI-economies follows from the following fundamental properties of the aggregate excess demand function $\tilde{g}(q) := \sum_{i=1}^{I} \tilde{g}^i(q)$:

[25] Another way of proving this theorem would consist in letting agent k be the Cass-agent and then to proceed along the lines of Cass (1984).

1. Continuity: $\tilde{g} : Q \to \mathbb{R}^J$ is continuous.

2. Homogeneity: $\tilde{g}(\lambda q) = \tilde{g}(q) \; \forall \lambda > 0, \forall q \in Q$.

3. Walras' Law: $q \cdot \tilde{g}(q) = 0 \; \forall q \in Q$.

4. Boundedness from below: $\exists r > 0 : \tilde{g}_j(q) \geq -r \; \forall q \in Q$, $j = 1, ..., J$.

5. Boundary behaviour: $q^n \in Q, q^n \to q \in \partial Q$ implies $||\tilde{g}(q^n)|| \to \infty$.

Following the proof that $\tilde{g}^i(q)$ does in fact have these properties (as given e.g. in Hens (1991)), one can easily deduce that the properties 1 to 4 are not affected by the additional constraint that $\theta^i_j = 0$ if $j \notin J^i$. Thus, it remains to show that the fifth property, i.e. the boundary behaviour of the excess demand functions, also carries over. In general, this need no longer be true, since if prices for some desired asset tend to zero it might be the case that an agent's demand for this asset does not explode since he might be effectively restricted in trading this asset. For agent k's excess demand function, however, property 5 follows readily from the assumption that this agent is able to trade all the available assets; obviously his excess demand function $g^k(q)$ is not effectively restricted, i.e. $g^k(q) = \tilde{g}^k(q)$, implying that $g^k(q)$ satisfies the boundary behaviour assumption.

We can therefore conclude, that the aggregate excess demand function $\tilde{g}(q)$ satisfies properties 1 through 5. Existence of a fixed point of g and hence existence of a *RPGEI*-equilibrium then follows along standard lines, cf. Hens (1991).

$\square$

8 References

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