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Equity-linked life insurance – a model with stochastic interest rates

J. Aase Nielsen Klaus Sandmann[1]

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Jørgen Aase Nielsen: Department of Operations Research University of Aarhus, Bldg. 530, Ny Munkegade, DK-8000 Aarhus C, Denmark. e-mail: atsjan@mi.aau.dk

Klaus Sandmann: Department of Statistics University of Bonn, Adenauerallee 24-42 ,D-53113 Bonn, Germany. e-mail: sandmann@addi.or.uni-bonn.de

Abstract

In Brennan and Schwartz (1976, 1979), the rational insurance premium on an equity - linked insurance contract was obtained through the application of the theory of contingent claims pricing. The premium was determined in an economy with the equity following a geometric Brownian motion, whereas the interest rate was assumed to be constant. Further considerations with deterministic interest rate have been discussed in Aase and Persson (1992) and in Persson (1993). Bacinello and Ortu (1993) allow for interest rate risk by assuming an Ornstein - Uhlenbeck process implying a closed form solution of the single premium endowment policy.
This paper presents a model for the multi premium case in the context of a stochastic interest rate process. It is shown that the insurance contract includes an Asian - like option contract. No closed form solution will be obtained. We discuss different numerical approaches and apply Monte Carlo simulations with a variance reduction technique.

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Key words : Asian option, forward risk adjusted measure, Monte Carlo simulations

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1 Introduction

In Brennan and Schwartz (1976, 1979) the rational insurance premium on an equity-linked insurance contract was obtained through the application of the theory of contingent claims pricing. The premium was determined in an economy with the equity price following a geometric Brownian motion, whereas the interest rate was assumed to be known and constant throughout the entire life of the insurance contract considered. Further considerations on equity-linked contracts have been discussed in Aase and Persson (1992) and in Persson (1993), but also in these papers the interest rate is assumed to be deterministic. Bacinello and Ortu (1993) allow for interest rate risk as they model the development in the short term interest rate and the underlying fund with an Ornstein-Uhlenbeck process and a closed form solution of the single premium endowment policy.
The purpose of this paper is to present a model for the multi-premium case in the context of a stochastic interest rate process. It is shown that the insurance contract includes an Asian-like option contract. No closed form solution will be obtained, but different numerical procedures will be discussed and results with respect to Monte Carlo simulation will be obtained.
The schedule of the paper is as follows. In section 2, the notation and the definition of the contract as well as a description of the economy is presented. Excluding mortality section 3 is devoted to the pricing of a call option imbedded in the life insurance contract. It is shown that the call option is similar to an Asian option. In section 4, the mortality case is investigated. A discussion of different numerical approaches is given in section 5. Section 6 contains the simulation result. Finally, section 7 concludes.

2 Notation and definition of the contract

An equity-linked contract is an agreement between a buyer and a seller, where the buyer is committed to pay, typically at yearly intervals and until the maturity of the contract or the death of the buyer whichever comes first, a predetermined premium to the seller. At maturity or death of the buyer, the seller is committed to deliver a payment in accordance to the agreement settled when the contract was written. This payment, the benefit, is the max. of 1) a function depending on the periodic premium and on the history of the spot price of the underlying equity from the date of settlement to the expiration date of the contract and 2) a non-random guaranteed amount also depending on the periodic premium.
We will defer the problem of mortality and for now simply assume that the insured person survives the maturity date of the contract. Then the model will structurally be less complicated, and further on due to the assumption that the mortality process is independent of the process describing the development in the financial market, no point of interest will be missed when in the end we regard the possibility of an early death.

The following notation will be applied:

K the periodic premium paid by the insured,

k a share of the periodic premium, $k = a \cdot K$ where $0 \leq a \leq 1$.

t_i a premium payment date, $i = 0, 1, 2, ..., n-1. t_o = 0$.

t_n the maturity date, $t_n = T$.

$S(t)$ the price of an index or a mutual fund at time t.

$D(t, t')$ the price at date t of a zero coupon bond with maturity date $t', t \leq t'$.

The reference portfolio

is defined as the portfolio obtained by investing an amount $k = a \cdot K$ at each of the dates $t_i, i = 0, 1, 2, ..., n-1$, in the fund with price process $S(t)$.

$g(K)$ the guaranteed amount. A deterministic function of the periodic premium.

$V(T) + g(K) = g(K) + \max\left\{k \cdot \sum_{i=0}^{n-1} \frac{S(T)}{S(t_i)} - g(K), 0\right\}$

the benefit from the insurance contract received at maturity date T .

Fair periodic premium

The periodic premium is fair if the value at date t_0 of the benefit equals the value at the same date of the premium payments, where the latter could also be denoted the cost of the insurance contract.

$r(t)$ the instantaneous risk free rate of interest at time t.

$B(t)$ the bank account. $B(t)$ is an accumulation factor corresponding to the price of a bank account, rolling over at $r(t)$, with the date t_0 investment of one unit of account.

$$B(t) = \exp\left\{\int_0^t r(u)du\right\}, \qquad dB(t) = r(t) \cdot B(t)dt.$$

The benefit at maturity is composed of the guaranteed amount plus a call option with exercise price $g(K)$ and with the reference portfolio as the underlying asset. The benefit is the proceeds from a financial contract and its price at time t_0 will be found in accordance to the absence of arbitrage possibilities in the financial market. As $g(K)$ is a deterministic function its value at time t_0 is equal to $g(K) \cdot D(t_0, T)$, and as the periodic premium is also known at date t_0 the cost of the contract is $K \cdot \sum_{i=0}^{n-1} D(t_0, t_i)$.

Therefore, the fair premium in the absence of mortality risk is the solution to the equation

$$K \cdot \sum_{i=0}^{n-1} D(t_0, t_i) = g(K) \cdot D(t_0, T) + V(t_0) \quad ,$$

so as $V(t_0)$ is the only term missing to be determined, we will in the following concentrate on the call option pricing.

3 Pricing of the call option in the absence of mortality risk

The fund from which the reference portfolio is created, consists of a linear combination of traded stocks and its value $S(t)$ is assumed to satisfy the differential equation[2]

$$dS(t)/S(t) = \mu dt + \sigma_1 dW_1(t) + \sigma_2 dW_2(t) \ ,$$

The development of the bonds is described by

$$dD(t,t')/D(t,t') = \mu(t,t')dt + \sigma(t,t')dW_1(t) \ ,$$

where the time dependence is such that $\sigma(t,t') = 0$ and $D(t,t) = 1$.

In a general setup we could allow for stochastic and time dependent coefficients in the differential equations for the bonds and the fund, but as anyway we will be forced to restrict ourselves to nonstochastic coefficients when looking for a solution the restriction is introduced at once.
The absence of arbitrage in the financial market implies certain restrictions on the μ's. If there is no arbitrage in the economy considered, then there exist functions $\lambda_1(t)$ and $\lambda_2(t)$, which are asset-independent [3]:

$$\begin{aligned}
\lambda_1(t) &= \frac{\mu(t,t') - r(t)}{\sigma(t,t')} \\
\lambda_2(t) &= \frac{\mu - r(t)}{\sigma_2} - \frac{\sigma_1}{\sigma_2} \cdot \frac{\mu(t,t') - r(t)}{\sigma(t,t')} \quad .
\end{aligned}$$

Denoting the objective probability measure by P an equivalent probability measure P^* is given by

$$\frac{dP^*}{dP} = \exp\left\{ -\int_{t_0}^{T} \lambda_1 dW_1 - \int_{t_0}^{T} \lambda_2 dW_2 - \frac{1}{2}\int_{t_0}^{T} (\lambda_1^2 + \lambda_2^2) dt \right\}$$

[2] In other words, we assume a Black-Scholes type behavior of the reference portfolio. This is from the empirical point of view more robust than the assumption of lognormal distributed stocks.

[3] For simplicity, we assume that $\lambda_1(t)$ and $\lambda_2(t)$ are independent of $r(t)$

and using Girsanov's Theorem, the processes

$$(dW_1^*, dW_2^*) = (dW_1 + \lambda_1(t)dt_1 dW_2 + \lambda_2(t)dt)$$

are standard Wiener processes under the P^* - measure.
The change of probability measure has no influence on the volatility coefficients in the differential equations whereas all the μ's are replaced by $r(t)$. In this artificial economy, the expected rate of return over the next time interval of length dt will for any asset be equal to $r(t)$:

$$\begin{aligned} dS(t)/S(t) &= r(t)dt + \sigma_1 dW_1^*(t) + \sigma_2 dW_2^*(t). \\ dD(t,t')/D(t,t') &= r(t)dt + \sigma(t,t')dW_1^*(t) \end{aligned}$$

The equations for the relative prices where the numeraire is the bank account are especially interesting as these relative prices are martingales under the P^* - measure. Denoting the bank account at time t by $B(t)$ we have

$$\begin{aligned} d(S(t)/B(t))/(S(t)/B(t)) &= \sigma_1 dW_1^*(t) + \sigma_2 dW_2^*(t) \\ d(D(t,t')/B(t))/(D(t,t')/B(t)) &= \sigma(t,t')dW_1^*(t) \end{aligned}$$

It follows that

$$\frac{S(T)}{S(t)} = \exp\left\{ \int_t^T r(u)du - \frac{1}{2}\int_t^T (\sigma_1^2 + \sigma_2^2)du + \int_t^T \sigma_1 dW_1^*(u) + \int_t^T \sigma_2 dW_2^*(u) \right\},$$

or

$$S(t) = E_t^* \left[\exp\left\{ -\int_t^T r(u)du \right\} \cdot S(T) \right] \tag{1}$$

However, due to the stochastic development of $r(t)$, it is not an easy task to determine the distribution of the ratio $S(T)/S(t)$ or to calculate the expected value in (**??**). For this reason it will be convenient to make another change of the probability measure, and this time to the measure under which the expected spot price is equal to the forward price. This will cause the integral over the short term interest rate to be replaced by the zero coupon bond price, $D(t,T)$. Observe that

$$\begin{aligned} d(D(t,t')/D(t,T))/(D(t,t')/D(t,T)) &= -\sigma(t,T)\cdot(\sigma(t,t') - \sigma(t,T))dt \\ &\quad + (\sigma(t,t') - \sigma(t,T))dW_1^*(t). \end{aligned}$$

A new equivalent P^T - measure given by

$$\frac{dP^T}{dP^*} = \exp\left\{ \int_{t_0}^T \sigma(t,T)dW_1^*(t)) - \frac{1}{2}\int_{t_0}^T \sigma^2(t,T)dt \right\}$$

leads again through Girsanov's Theorem to the standard P^T - Wiener processes

$$(dW_1^T(t), dW_2^T(t)) = (dW_1^*(t) - \sigma(t,T)dt, dW_2^*(t)) \quad ,$$

under which

$$\frac{d(D(t,t')/D(t,T))}{(D(t,t')/D(t,T))} = (\sigma(t,t') - \sigma(t,T))dW_1^T(t), \tag{2}$$

$$\frac{d(S(t)/D(t,T))}{(S(t)/D(t,T))} = (\sigma_1 - \sigma(t,T))dW_1^T(t) + \sigma_2 dW_2^T(t) \tag{3}$$

and

$$\frac{S(t)}{D(t,T)} = E_t^T\left[\frac{S(T)}{D(T,T)}\right] = E_t^T[S(T)] \tag{4}$$

Comparing (**??**) and (**??**), we notice that the stochastic discounting in (**??**) has been replaced by the time-t measurable discounting in (**??**). From (**??**) and (**??**) we derive next that

$$D(t,T) = \frac{D(t_0,T)}{D(t_0,t)} \cdot \exp\left\{-\int_{t_0}^{t}(\sigma(u,t)) - \sigma(u,T))dW_1^T(u) + \frac{1}{2}\int_{t_0}^{t}(\sigma(u,t) - \sigma(u,T))^2 du\right\}$$

and

$$\begin{aligned}\frac{S(T)}{S(t)} &= \frac{1}{D(t,T)} \cdot \exp\left\{-\frac{1}{2}\int_{t}^{T}\left((\sigma_1 - \sigma(u,T))^2 + \sigma_2^2\right)du\right.\\ &\quad \left. + \int_{t}^{T}(\sigma_1 - \sigma(u,T))dW_1^T(u) + \int_{t}^{T}\sigma_2 dW_2^T(u)\right\} \quad ,\end{aligned}$$

and combining these expressions, we obtain

$$\begin{aligned}\frac{S(T)}{S(t)} &= \frac{D(t_0,t)}{D(t_0,T)} \cdot \exp\left\{\int_{t_0}^{t}(\sigma(u,t) - \sigma(u,T))dW_1^T(u)\right.\\ &\quad -\frac{1}{2}\int_{t_0}^{t}(\sigma(u,t) - \sigma(u,T))^2 du - \frac{1}{2}\int_{t}^{T}((\sigma_1 - \sigma(u,T))^2 + \sigma_2^2)du \\ &\quad \left. + \int_{t}^{T}(\sigma_1 - \sigma(u,T))dW_1^T(u) + \int_{t}^{T}\sigma_2 dW_2^T(u)\right\}\end{aligned} \tag{5}$$

In order to make the model computationally feasible, $\sigma(t,t')$ should be parametrised in a suitable manner. The specific and convenient form chosen is

$$\sigma(t,t') = (t' - t) \cdot \sigma \tag{6}$$

where σ is constant. This parametrisation, which is the continuous time analogue of Ho and Lee (1986) specification, allows us to reduce (**??**) to

$$\frac{S(T)}{S(t)} = \frac{D(t_0,t)}{D(t_0,T)} \cdot \exp\left\{-\frac{1}{2}(T-t)^2\sigma^2 t - \frac{1}{2}\int_t^T ((\sigma_1-(T-u)\sigma)^2+\sigma_2^2)du\right\}$$
$$\exp\left\{-\sigma(T-t)W_1^T(t) + \int_t^T (\sigma_1-(T-u)\sigma)dW_1^T(u) + \int_t^T \sigma_2 dW_2^T(u)\right\} \quad . \tag{7}$$

4 Mortality case

The *fair premium* determination will involve the modelling of an early death possibility, but as already stated, the death process is assumed to be independent of the processes ruling in the financial market. Furthermore, the insurance company is assumed to behave as risk neutral concerning the mortality risk. With $\pi(t)dt$ denoting the probability that the contract terminates in the time interval $[t, t+dt]$, the value at date t_0 of the benefit is

$$\int_{t_0}^T \pi(t)\cdot D(t_0,t)\cdot E^t\left[g(K)+\max\left\{k\cdot\sum_{i=0}^{n^*-1}\frac{S(t)}{S(t_i)}-g(K),0\right\}\right]dt \tag{8}$$
$$+(1-\int_{t_0}^T \pi(t)dt)\cdot D(t_0,T)\cdot E^T\left[g(K)+\max\left\{k\cdot\sum_{i=0}^{n-1}\frac{S(T)}{S(t_i)}-g(K),0\right\}\right]$$

where $n^* = min(i|t_i > t)$.
$V(t_0)$ is determined through the application of the numerical procedure specified in section 5. In finance, a good procedure should give the price within seconds to keep up with the volatile market, but in the case considered here, the computer time needed is not the limiting factor. The calculations should only be performed once when the contract is entered.
The cost of the contract consists of

$$K\sum_{i=0}^{n-i} D(t_0,t_i)\cdot(1-\int_{t_0}^{t_i}\pi(t)dt) \quad .$$

The fair premium can now be found by an iterative approach. The value (**??**) of the benefit denoted by $V(t_0;a,K)$ depends on a, the fraction of the premium invested, and on the premium K. For a given a, the *fair premium* is the K satisfying

$$V(t_0;a,K) + g(K)\cdot\int_{t_0}^T D(t_0,t)\pi(t)dt + g(K)\cdot(1-\int_{t_0}^T D(t_0,t)\pi(t)dt)$$
$$= K\sum_{i=0}^{n-1} D(t_0,t_i)\cdot(1-\int_{t_0}^{t_i}\pi(t)dt). \tag{9}$$

5 Numerical method

In order to calculate the fair premium K of the insurance contract, we first have to calculate the arbitrage price of the average option [4] and secondly apply an iterative procedure to compute the fair premium defined by (??). Under the specification of the index process $\{S_t\}_t$ and the interest rate of dynamics, we know that (??) is bivariate lognormal distributed. Thus the option pricing problem is very similar to the one of Asian options under the assumption of a geometric Brownian motion. In difference, the insurance contract depends on the sum of the index returns and not on the average index realisation and more important the discounting is stochastic. So far, there exists no closed form solution for the distribution of a sum of correlated lognormal distributed random variables. Therefore, numeric techniques have to be applied to approximate the option value.
Let $0 \leq t_{i-1} < t_i < T = t_n$ be two premium dates, then we can rewrite (??) into:

$$\begin{aligned}
\frac{S(T)}{S(t_i)} &= \frac{S(T)}{S(t_{i-1})}\left[\frac{D(t_0,t_i)}{D(t_0,t_{i-1})}\exp\left\{\frac{1}{2}[(T-t_{i-1})^2 t_{i-1} - (T-t_i)^2 t_i]\sigma^2\right\}\right. \\
&\quad \exp\left\{\frac{1}{2}\int_{t_{i-1}}^{t_i}(\sigma_1 - (T-u)\sigma)^2 + \sigma_2^2 du\right\} \qquad (10) \\
&\quad \exp\left\{[(T-t_{i-1})W_1^T(t_{i-1}) - (T-t_i)W_1^T(t_i)]\sigma - \int_{t_{i-1}}^{t_i}(\sigma_1 - (T-u)\sigma)dW_1^T(u)\right\} \\
&\quad \left.\exp\left\{-\int_{t_{i-1}}^{t_i}\sigma_2 dW_2^T(u)\right\}\right] \\
&=: \frac{S(T)}{S(t_{i-1})}A^T(t_{i-1},t_i)
\end{aligned}$$

Inserting (??) in $\sum_{i=0}^{n-1}\frac{S(T)}{S(t_i)}$, we obtain after a slight reshuffling that

$$\sum_{i=0}^{n-1}\frac{S(T)}{S_i} = \frac{S(T)}{S(t_0)}[1+A^T(t_0,t_1)[1+A^T(t_1,t_2)[1+\ldots A^T(t_{n-3},t_{n-2})[1+A^T(t_{n-2},t_{n-1}),\ldots]]]] \qquad (11)$$

The structure of this equation is similar to the one that Turnbull - Wakeman (1991) apply to calculate the first four central moments of the unknown distribution. Furthermore, Caverhill and Clewlow (1990) suggest to apply interactively the Fast Fourier transformation on an expression which in structure is similar to (??). Both methods explicitly use the fact that for deterministic interest rates the elements of the

[4] In the case of mortality, we have to calculate a series of option prices. Remark that the forward risk adjusted measure P^t depends on n^*. This implies that we have to consider the change of measure dependent on the death distribution.

sequence $A^T(t_{i-1}, t_i), i = 1, 2, \ldots$ are stochastic independent. This is not the case in our situation since for

$$\begin{aligned} X &:= (T - t_{i-1})W_1^T(t_{i-1}) - (T - t_i)W_1^T(t_i) \\ Y &:= (T - t_i)W_1^T(t_1) - (T - t_{i+1})W_1^T(t_{i+1}) \end{aligned}$$

we have

a) $E[X] = E[Y] = 0$

b) $E[X \cdot Y] = [t_{i+1} - t_i] \cdot [(T - t_{i-1})t_{i-1} - (T - t_i)t_i] \neq 0$

which implies that the $A^T(t_{i-1}, t_i)$ are stochastic dependent random variables. Thus we cannot apply the Fast Fourier transformation suggested by Caverhill and Clewlow to our problem.
Turnbull and Wakeman (1991) suggest to approximate the unknown density ρ^T of the sum of lognormal distributed variables by the following Edgeworth expansion:

$$\rho^T(x) \approx f(x) + \frac{c_2}{2!}\frac{\partial^2 f(x)}{\partial x^2} - \frac{c_3}{3!}\frac{\partial^3 f(x)}{\partial x^3} + \frac{c_4}{4!}\frac{\partial^4 f(x)}{\partial x^4} \tag{12}$$

where $f(x)$ is given by a lognormal density function, i.e.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_f}\frac{1}{x}\exp\left\{-\frac{(\ln x - \mu_f)^2}{2\sigma_f^2}\right\}$$

and

$$\begin{aligned} c_2 &= \mathcal{K}(2, \rho^T) - \mathcal{K}(2, f) \\ c_3 &= \mathcal{K}(3, \rho^T) - \mathcal{K}(3, f) \\ c_4 &= \mathcal{K}(4, \rho^T) - \mathcal{K}(4, f) + 3c_3^2 \end{aligned}$$

where $\mathcal{K}(i, f) = E_f[(X - E_f[X])^i]$ equals the i-th central moment with respect to the lognormal distribution given by f, resp. $\mathcal{K}(i, \rho^T)$ with respect to the unknown distribution given by ρ^T. To calculate these moments, the first four non - central moments of (**??**) must be computed. The paramenters μ_f and σ_f are chosen such that the first two non-central moments under both measures are identical. Given the moments and a vanishing error term, the value of the insurance bonus at time $t_n = T$ is approximated by:

$$\begin{aligned} & D(t_0, t_n) \cdot E^T[max\{k \cdot \sum_{i=0}^{n-1} \frac{S(T)}{S(t_i)} - g(K), 0\}] \\ \approx k \cdot nD(t_0, t_n) & \left\{ e^{\mu_f + \frac{\sigma_f^2}{2}} N(x) - \frac{g(K)}{k \cdot n} N(x - \sigma_f) + \frac{c_2}{2!} f\left(\frac{g(K)}{k \cdot n}\right) \right. \\ & \left. - \frac{c_3}{3!}\frac{\partial f}{\partial x}\left(\frac{g(K)}{k \cdot n}\right) + \frac{c_4}{4!}\frac{\partial^2 f}{\partial x^2}\left(\frac{g(K)}{k \cdot n}\right) \right\} \end{aligned}$$

with $x = \frac{\mu_f + \sigma_f^2 - \ln(\frac{g(K)}{k \cdot n})}{\sigma_f}$ and $N(.)$ denoting the standard normal distribution.
Since the $A^T(t_{i-1}, t_i)$ in (**??**) are stochastic dependent variables, it is not possible to calculate the moments of (**??**) as in Turnbull - Wakeman. An alternative but much slower algorithm is given in the Appendix. Apart from this numerical difficulty, the applicability of this approximation to the insurance problem appears not advisable. The usual maturity of Asian options is less than one year whereas the equity linked insurance contract has a maturity between 10 and 35 years. Secondly, in the insurance case, the premium dates are discrete which implies that the contract is based on a discrete average in difference to the continuous average in the Asian option case.
In table 1 we present the four non-central moments and c-coefficients $c_2, c_3,$ and c_4 for the Turnbull - Wakeman approximation. The data used for these calculations are: $\sigma = 8\%, \sigma_1 = 10\%, \sigma_2 = 15\%$ and a flat initial interest rate curve with $D(t_0, t_i) = (1.06)^{-t_i}$. As expected, the moments grow extremely with time to maturity which leads to extreme c - coefficients. As a consequence, the correction of the lognormal distribution suggested by Turnbull-Wakeman is without any control and leads to unreasonable option values.

Table 1: Moments of $X(t_i) = \sum_{j=0}^{i-1} \frac{S(t_i)}{S(t_j)}$ and c-coefficients

t_i-years	$E[X^1(t_i)]$	$E[X^2(t_i)]$	$E[X^3(t_i)]$	$E[X^4(t_i)]$	c_2	c_3	c_4
2	2.1836	4.938873203	1.157153044E1	2.808594171E1	-3.608224830E-16	7.041681995E-4	7.306091592E-4
3	3.374616	12.03111244	4.533130605E1	1.805753740E2	-2.775557562E-15	1.606725317E-2	4.018295328E-2
4	4.63709296	23.53757207	1.309741467E2	8.001377772E2	3.996802889E-15	1.922651545E-1	1.138657036E0
5	5.975318538	41.39886631	3.342386526E2	3.160524105E3	-4.440892099E-15	1.669350636E0	2.290425470E1
6	7.39383765	69.16034229	8.303573898E2	1.298256225E4	-1.243449788E-14	1.196483272E1	3.773601312E2
7	8.897467909	113.5793366	2.157719679E3	6.318076241E4	-1.421085472E-14	7.755033394E1	5.761398349E3
8	10.49131598	188.2398003	6.267727071E3	4.162394608E5	-2.842170943E-14	4.914973363E2	9.248360267E4
9	12.18079494	322.3108235	2.180177327E4	4.263979812E6	2.273736754E-13	3.275085893E3	1.791041567E6
10	13.97164264	583.5278803	9.748415571E4	7.730078475E7	-5.684341886E-14	2.463199682E4	4.873539479E7
11	15.8699412	1144.028252	5.991464196E5	2.758617255E9	3.410605132E-13	2.245321673E5	2.187155218E9
12	17.88213767	2488.291088	5.357771496E6	2.111394207E11	-2.728484105E-12	2.663475845E6	1.882475589E11
13	20.01506593	6145.112321	7.299377020E7	3.740094687E13	9.094947018E-13	4.405244580E7	3.530657062E13
14	22.27596988	17592.90185	1.576026608E9	1.651369077E16	3.637978807E-12	1.083417346E9	1.602456828E16
15	24.67252808	59424.64672	5.596824986E10	1.958121843E19	-2.182787284E-11	4.199624043E10	1.926052058E19

With these remarks, it is not surprising that the comparison with Monte Carlo simulations in section 6 will strongly reject the Turnbull - Wakeman approach to this problem.
Based on the strong relationship between the arithmetic and the geometric average, Vorst (1992) suggests an alternative approximation of the arbitrage price for an Asian option and furthermore derives upper and lower bounds for these prices. With the following notation

$$A(t_n) = \frac{1}{n}\sum_{i=0}^{n-1}\frac{S(t_n)}{S(t_i)}, \qquad G(t_n) = \sqrt[n]{\prod_{i=0}^{n-1}\frac{S(t_n)}{S(t_i)}}$$

the Vorst approximation and bounds on the price of the Asian option are given by

$$\begin{aligned}
& D(t_0,T)\left(e^{m_G+\frac{1}{2}\sigma_G^2}N(d_1) - YN(d_1-\sigma_G)\right) \\
\leq\ & D(t_0,T)E^T\left[\max\{A(t_n)-Y,0\}\right] \\
\approx\ & D(t_0,T)\left(e^{m_G+\frac{1}{2}\sigma_G^2}N(d_2) - Y'N(d_2-\sigma_G)\right) \\
\leq\ & D(t_0,T)\left(e^{m_G+\frac{1}{2}\sigma_G^2}N(d_1) - YN(d_1-\sigma_G) + E^T[A(t_n)] - E^T[G(t_n)]\right)
\end{aligned} \tag{13}$$

where

$$d_1 = \frac{m_G - \ln(Y) + \sigma_G^2}{\sigma_G}, \quad d_2 = \frac{m_G - \ln(Y') + \sigma_G^2}{\sigma_G}$$

$$Y' = Y - \left(E^T[A(t_n)] - E^T[G(t_n)]\right)$$

$$\left.\begin{array}{rcl} m_G & = & E^T[\ln G(t_n)] \\ \sigma_G^2 & = & V^T[\ln G(t_n)] \end{array}\right\} \Rightarrow E^T[G(t_n)] = \exp\left\{m_G + \tfrac{1}{2}\sigma_G^2\right\}$$

Thus the Vorst approximation only involves the computation of the first moment for the arithmetic mean and the mean and variance of the logarithmic geometric mean. Inserting (**??**) we can directly compute

$$m_G = \frac{1}{n}\sum_{i=0}^{n-1}\left[\ln\left(\frac{D(t_0,t_i)}{D(t_0,T)}\right) - \frac{1}{2}(T-t_i)^2\sigma^2 t_i - \frac{1}{2}\int_{t_i}^{T}\left((\sigma_1-(T-u)\sigma)^2+\sigma_2^2\right)du\right]$$

whereas the computation of the variance is more complicated. A recursive algorithm and a formula is given in the Appendix. Inspecting (**??**), we notice that the approximation is derived by transforming the probability measure of a lognormal distribution with support $\mathbb{R}^+$ to a lognormal distribution with support $\left[E^T[A(t_n)] - E^T[G(t_n)], \infty\right[$. Since the support of the random variable $A(t_n)$ is $\mathbb{R}^+$ the distance $E^T[A(t_n)] - E^T[G(t_n)] > 0$ is important for the approximation error. Again for a flat interest rate curve, figure 1 shows the development of this distance if t_n increases. From this we can expect for our insurance problem that the Vorst approximation leads to an overpricing of in the money Asian call options if the maturity increases.

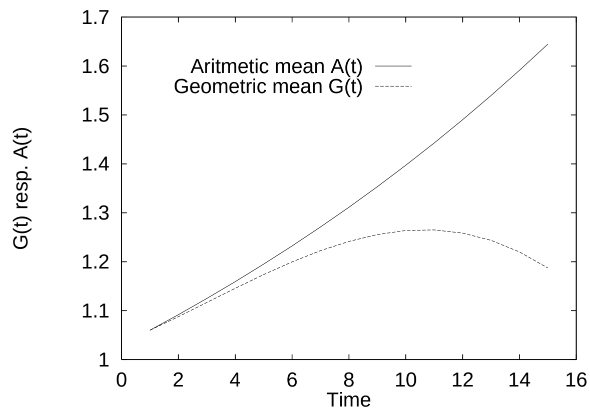

Figure 1: Arithmetic and geometric mean as functions of the time to maturity. Flat initial term structure with constant effective rate per annum of 0.06, $\sigma = 0.08$, $\sigma_1 = 0.10$, and $\sigma_2 = 0.15$.

Motivated by the complexity of the problem, we apply Monte Carlo simulations to estimate the fair premium K of the equity-linked life insurance contract, and we use antithetic and control variate technique to reduce the variance of the estimation. As control variate we use the corresponding geometric average option. More precisely, the following setup is applied:

Define $\underline{\underline{\Theta}} = \{0 = \tau_0 < \tau_1 < ... < \tau_N = T\}$ with $\Delta\tau = \tau_{i+1} - \tau_i \forall i$ as the finest discretisation of the time axis, where T is the maturity of the insurance contract. The premium K will be paid at each time $t_i \in \underline{\underline{T}}$ with $\underline{\underline{T}} = \{0 = t_0 < ... < t_n = T\} \subset \underline{\underline{\Theta}}$ such that there exists a number $h \in N$ with $\Delta t = t_{i+1} - t_j = h \cdot \Delta\tau$. We assume that if the insured dies at time $\tau_i \in \Theta \backslash \{\tau_0\}$ the insurance company will pay the guaranteed amount $g(K)$ plus the bonus at time τ_{i+1}, which implies that the present value of this payoff is given by

$$D(t_0, t_{i+1}) \left[g(K) + E^{\tau_{i+1}} \left[\max\left\{ a \cdot K \cdot \sum_{j=0}^{v^*(i)} \frac{S(\tau_{i+1})}{S(t_j)} - g(K), 0 \right\} \right] \right] \tag{14}$$

with $v^*(i) = \max\{j \in \{0, ..., n\} | t_j < \tau_i\}$. Note that the expectation has to be formed with respect to the τ_{i+1} forward measure. With M being the number of Monte Carlo simulations ($2M$ antithetic) for each of the two Wiener processes, the value of the bonus is estimated by

$$\begin{aligned} \hat{C}(\tau_{i+1}, K, a) &= \frac{1}{2M} \sum_{m=1}^{2M} \left[\left[a \cdot K \cdot \sum_{j=0}^{v^*(i)} \frac{S_m(\tau_{i+1})}{S_m(t_j)} - g(K) \right]^+ \right. \\ &\quad \left. -(1 + v^*(i)) a \cdot K \left[\sqrt[1+v^*(i)]{\prod_{j=0}^{v^*(i)} \frac{S_m(\tau_{i+1})}{S_m(t_j)}} - \frac{g(K)}{a \cdot K \cdot (1 + v^*(i))} \right]^+ \right] \\ &\quad +(1 + v^*(i)) a \cdot K \cdot G\left(\tau_{i+1}, \frac{g(K)}{a \cdot K \cdot (1 + v^*(i))} \right) \end{aligned} \tag{15}$$

where $\sum\limits_{j=0}^{v^*(i)} \frac{S_m(\tau_{i+1})}{S_m(t_j)}$ is the realisation of the m-th simulation.
The time τ_{i+1} - forward value of the European geometric average option $G(\tau_{i+1}, \frac{g(K)}{a \cdot K v^*(i)})$ with exercise price $\frac{g(K)}{a \cdot K(1+v^*(i))}$ is given by

$$\begin{aligned} G(\tau_{i+1}, Y) &= \exp\{m_G(i) + \frac{1}{2}\sigma_G^2(i)\} N(x) - Y \cdot N(x - \sigma_G(i)) \qquad (16) \\ x &= \frac{-\ln Y + m_G(i) + \sigma_G^2(i)}{\sigma_G(i)} \end{aligned}$$

where $m_G(i)$ and $\sigma_G^2(i)$ are determined as before as the mean resp. variance of the logarithmic geometric average at time τ_{i+1}, i.e.

$$\begin{aligned} m_G(i) &= E^{\tau_{i+1}}\left[\ln(\sqrt[1+v^*(i)]{\prod_{j=0}^{v^*(i)} \frac{S(\tau_{i+1})}{S(t_j)}})\right] = \frac{1}{1+v^*(i)} \sum_{j=0}^{v^*(i)} E^{\tau_{i+1}}\left[\ln\left(\frac{S(\tau_{i+1})}{S(t_j)}\right)\right] \\ &= \frac{1}{1+v^*(i)} \sum_{j=0}^{v^*(i)} \left[\ln\left(\frac{D(t_0,t_j)}{D(t_0,\tau_{i+1})}\right) - \frac{1}{2}(\tau_{i+1}-t_j)^2\sigma^2 t_j \right. \qquad (17) \\ &\left. -\frac{1}{2}\int_{t_j}^{\tau_{i+1}} \left((\sigma_1 - (\tau_{i+1}-u)\sigma)^2 + \sigma_2^2\right) du \right] \\ \sigma_G^2(i) &= V^{\tau_{i+1}}\left[\ln\left(\sqrt[1+v^*(i)]{\prod_{j=0}^{v^*(i)} \frac{S(\tau_{i+1})}{S(t_j)}}\right)\right] \end{aligned}$$

which can be calculated with a similar recursive algorithm as the central moments (see Appendix). The fair premium K^* is then estimated from

$$\begin{aligned} 0 &= K^* \sum_{i=0}^{n-1} D(t_0,t_i)\left[1 - \sum_{j=0}^{i \cdot h-1} \pi(\tau_j)\right] \qquad (18) \\ &- g(K)\sum_{i=0}^{N-1} \pi(\tau_i) D(t_0,\tau_{i+1}) - g(K) D(t_0,t_n)\left(1 - \sum_{i=0}^{N-1}\pi(\tau_i)\right) \\ &- \sum_{i=0}^{N-1} \pi(\tau_i) D(t_0,\tau_{i+1}) \cdot \hat{C}(\tau_{i+1}, K^*, a) - \left(1 - \sum_{i=0}^{N-1}\pi(\tau_i)\right) D(t_0,\tau_N) \cdot \hat{C}(\tau_N, K^*, a) \end{aligned}$$

Due to the homogeneity of the bonus part, the right hand side is strictly monotonous increasing in K with a lower bound on K given by

$$\underline{K} = \frac{g(K)\sum\limits_{i=0}^{N-1} \pi(\tau_i) D(t_0,\tau_{i+1}) + g(K) D(t_0,t_n)(1 - \sum\limits_{i=0}^{N-1}\pi(\tau_i))}{\sum\limits_{i=0}^{n-1} D(t_0,t_i)\left[1 - \sum\limits_{j=0}^{i\cdot h-1}\pi(\tau_j)\right]} \qquad (19)$$

Finally, for the death distribution, we assume a mortality table adjusted with the Makeham formula

$$\begin{aligned} l_x &= b \cdot s^x \cdot g^{c^x} \qquad \text{with} \\ s &= 0.99949255 \\ g &= 0.99959845 \\ c &= 1.10291509 \\ b &= 1000401.71 \end{aligned} \tag{20}$$

which leads to

$$\begin{aligned} \pi_x(\tau_i) &= \frac{l_{x+\tau_i} - l_{x+\tau_i+\Delta\tau}}{l_x} \\ &\hat{=} \text{ the probability that a life-aged-}x\text{ will survive }\tau_i\text{ years and die within} \\ &\quad \text{the following }\Delta\tau\text{ years.} \end{aligned}$$

6 Simulation results

Within the Monte Carlo simulation we consider three different specifications for the initial term structure, i.e.

Scenario I	:	flat initial term structure	$D(t_0, \tau_i)$	$=$	$(1.06)^{-\tau_i}$
Scenario II	:	normal initial term structure	$D(t_0, \tau_i)$	$=$	$(0.06 + (1.02)^{\tau \frac{i}{15}})^{-\tau_i}$
Scenario III	:	invers initial term structure	$D(t_0, \tau_i)$	$=$	$(2.06 - (1.02)^{\tau \frac{i}{15}})^{-\tau_i}$

with $\tau_i < 15 \quad (years)$. All three scenarios imply non negative forward rates at time $t_0 = \tau_0 = 0$. For each scenario, we consider three possible maturities of the equity linked life insurance contract, i.e. $T = t_n \in \{10\ years, 12\ years, 15\ years\}$ where the payment of the premium ranges between yearly and monthly. The number of periods per year for each insurance contract is fixed to 12 which implies at the most 180 periods for the 15 year contract and $h = 1, 2, 6, 12$ for a yearly, $\frac{1}{2}$ yearly, quaterly resp. monthly payment frequency of the premium. The volatility parameters for all three scenarios are fixed by $\sigma = 8\%, \sigma_1 = 10\%$ and $\sigma_2 = 15\%$ which implies an instantaneous correlation with a zero coupon bond of $dSdD(t,T) = S \cdot D \cdot \sigma_1 \sigma (T-t)dt = S \cdot D \cdot 0.008(T-t)dt$ resp. with the spot rate process of $dSdr = S \cdot \sigma_1 \sigma dt = S \cdot 0.008dt$. Within each scenario, we run 10 independent Monte Carlo simulations each with $M = 1000$ paths[5] to calculate the fair premium and the standard deviations.
Table 2 shows, for the yearly payment frequency and the flat initial term structure, simulated initial moments and the calculated central moments applying the recursive algorithm given in the Appendix.

[5]This implies 2000 paths using antithetic technique for each simulation

Table 2: Simulated and exact central moments of $X(t_i) = \sum_{j=0}^{i-1} \frac{S(t_i)}{S(t_j)}$

t_i-years	method	$\mu = E[X(t_i)]$	$E[(X(t_i) - \mu)^2]$	$E[(X(t_i) - \mu)^3]$	$E[(X(t_i) - \mu)^4]$
2	simulated	2.183546634	0.1706166958	4.058493436E-2	1.026579508E-1
	exact	2.1836	0.1707642433	4.124530528E-2	1.054905504E-1
	sd	0.001518331538	0.00687513339	4.225170127E-3	1.075338954E-2
3	simulated	3.375831403	0.6538835535	4.163553347E-1	1.792371732E0
	exact	3.374616	0.6430792915	3.906301790E-1	1.674717935E0
	sd	0.004456675464	0.03319292174	6.282951028E-2	3.136366746E-1
4	simulated	4.641506035	2.082237824	3.114687107E0	2.238481570E1
	exact	4.63709296	2.034940946	2.955816318E0	2.040954491E1
	sd	0.009225995879	0.09360289184	3.322933273E-1	4.813023454E0
5	simulated	5.984845155	5.873241813	2.112293044E1	2.632730035E2
	exact	5.975318538	5.694434682	1.881511710E1	2.161130730E2
	sd	0.01789537053	0.3361507394	4.512128473E0	9.976128190E1
6	simulated	7.407485224	15.06282664	1.296532772E2	3.464251838E3
	exact	7.39383765	14.49150709	1.047013457E2	2.143898820E3
	sd	0.04120601035	1.431960579	6.320106493E1	3.729919880E3
7	simulated	8.929212895	36.3239221	6.773177647E2	3.567333166E4
	exact	8.897467909	34.41440139	5.347491129E2	2.153554003E4
	sd	0.0864395638	4.545117836	3.942398857E2	4.513553696E4
8	simulated	10.55897074	86.18790302	4.216454503E3	6.709244347E5
	exact	10.49131598	78.17208927	2.652587666E3	2.411826806E5
	sd	0.1611117614	17.48755385	4.752174780E3	1.480962452E6
9	simulated	12.28877093	203.876908	2.893606249E4	1.283438622E7
	exact	12.18079494	173.9390581	1.363833922E4	3.422616507E6
	sd	0.2629258015	71.89197513	4.803506106E4	3.294386670E7
10	simulated	14.20022701	558.7061554	3.283252224E5	4.783549105E8
	exact	13.97164264	388.3210822	7.848034592E4	7.242186418E7
	sd	0.48662079	459.9617218	8.135383092E5	1.418065301E9
11	simulated	16.26914928	1561.793191	2.538766323E6	7.822533985E9
	exact	15.8699412	892.1732186	5.526732855E5	2.722122064E9
	sd	0.8201196074	1871.018811	6.625288800E6	2.287502045E10
12	simulated	18.47356137	3891.534972	1.315146373E7	7.431135011E10
	exact	17.88213767	2168.52024	5.235719977E6	2.107606544E11
	sd	1.234842826	5696.180555	3.577773059E7	2.564215671E11
13	simulated	20.9962176	9533.295284	5.145572105E7	4.266375651E11
	exact	20.01506593	5744.509457	7.264082190E7	3.739511726E13
	sd	1.947522213	13293.61606	1.260760529E8	3.698755650E13
14	simulated	23.76235303	22763.88996	1.972781769E8	2.483869623E12
	exact	22.27596988	17096.68302	1.574873018E9	1.651355039E16
	sd	2.847674115	30694.12457	1.451536106E9	1.651106791E16
15	simulated	26.5110614	46683.09384	5.703046963E8	9.667355299E12
	exact	24.67252808	58815.91308	5.596388143E10	1.958121291E19
	sd	3.713532892	59546.28867	5.540675534E10	1.958120324E19

As table 2 indicates, the Monte Carlo simulation implies reasonable estimations of the moments for the first 10 years. The standard error increases with the time to maturity and with the power of the moment. This is also true for the normal and invers initial term structure. Given the histogram of the Monte Carlo simulation for the distribution of the average, we can consider the difference between the probability distributions underlying the closed form analytic approximations suggested by Turnbull-Wakeman and Vorst. As shown by figure 2 to 5 the Turnbull-Wakeman approach already leads to an unreasonable approximation for the density for a maturity of 4 years. As already observed in section 5, this is due to the explosion of the c-coefficients. As a consequence, we get unreasonable values for the fair premium using the Turnbull-Wakeman approach.

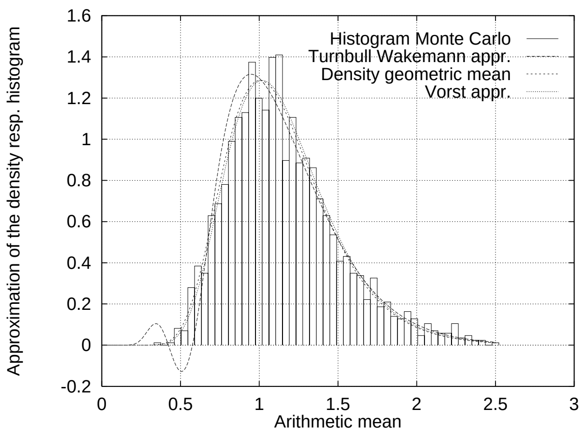

Figure 2: Simulated density of the arithmetic mean and approximations at $t = 4$ years with flat initial term structure and monthly frequency.

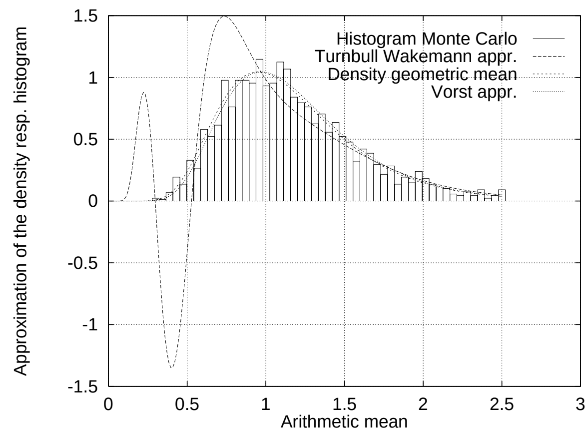

Figure 3: Simulated density of the arithmetic mean and approximations at $t = 5$ years with flat initial term structure and monthly frequency.

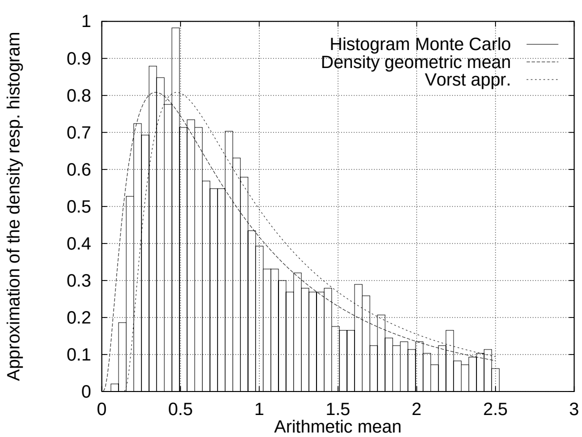

Figure 4: Simulated density of the arithmetic mean and approximations at $t = 10$ years with flat initial term structure and monthly frequency.

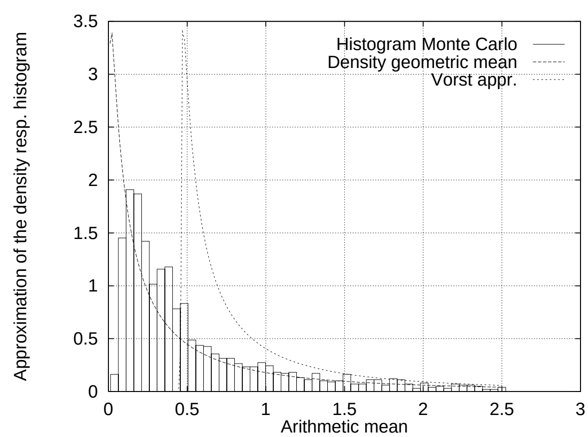

Figure 5: Simulated density of the arithmetic mean and approximations at $t = 15$ years with flat initial term structure and monthly frequency.

On the other hand, the performance of the Vorst approximation of the density coincide quit reasonable with the Monte Carlo simulation for maturities less than 10 years. If the maturity is higher than 10 years, the support of the density is substantial different from the estimation using our Monte Carlo simulation. Given that the Monte Carlo simulation overestimates the first moment of the true distribution it seems that the Vorst approximation induces a too large shift of the probability measure into high realisations if the maturity increases. For maturities above 10 years, the lower bound derived by Vorst is closer to the simulation result than the suggested approximation. Therefore, we expect that in terms of the fair premium, the result obtained by the Vorst approximation underestimates the fair premium. On the other hand, using the distribution of the geometric mean (lower bound derived by Vorst) we expect these to be close to the Monte Carlo simulation [6]. As table 3 shows, this is the case for all three initial term structures which we considered.

Table 3.1: Fair premium with normal initial term structure for a life aged 30 and guaranteed amount 1000

maturity:		10		12		15	
a share	method	Vorst	simulated	Vorst	simulated	Vorst	simulated
0.40	down	68.6516		53.3759		37.6527	
	appr.	68.8651	70.6797	53.7024	56.5781	38.03	41.5548
	up	71.8648		57.7536		43.718	
0.45	down	69.7322		54.5246		38.6006	
	appr.	70.0649	72.3230	55.0134	58.5078	39.1424	43.4411
	up	73.5794		59.8496		46.0786	
0.50	down	71.0617		55.8769		39.6664	
	appr.	71.566	74.3160	56.5877	60.8115	40.4267	45.6367
	up	75.6654		62.3407		48.8596	
0.55	down	72.6825		57.4585		40.8633	
	appr.	73.4265	76.7379	58.4773	63.5548	41.9135	48.2176
	up	78.2027		65.3252		52.1766	
0,60	down	74.6473		59.3085		42.2084	
	appr.	75.7298	79.6930	60.7532	66.8638	43.6481	51.2870
	up	81.316		68.9362		56.1943	
0.65	down	77.0291		61.4792		43.7226	
	appr.	78.6019	83.3382	63.5261	70.8988	45.6916	55.0106
	up	85.1739		73.3839		61.1638	

[6] Again, the results for the normal and invers initial term structure are the same

Table 3.2: Fair premium with flat initial term structure for a life aged 30 and guaranteed amount 1000

maturity:		10		12		15	
a share	method	Vorst	simulated	Vorst	simulated	Vorst	simulated
0.40	down	75.3848		61.0836		46.5138	
	appr.	75.6078	77.7162	61.4431	65.0587	46.9655	53.3655
	up	78.7267		65.826		53.6159	
0.45	down	76.5879		62.4267		47.7266	
	appr.	76.9376	79.5277	62.9636	67.3776	48.374	56.2986
	up	80.5899		68.1959		56.49	
0.50	down	78.0677		64.0068		49.0917	
	appr.	78.5973	81.7472	64.7883	70.1401	50.0022	59.7956
	up	82.8595		71.0135		59.8774	
0.55	down	79.8737		65.8584		50.6293	
	appr.	80.6541	84.4338	66.9797	73.4436	51.8901	64.0342
	up	85.6234		74.3908		63.9189	
0.60	down	82.0651		68.03		52.3622	
	appr.	83.2047	87.7296	69.6206	77.4438	54.0968	69.3158
	up	89.0117		78.4816		68.8118	
0.65	down	84.7236		70.5823		54.3191	
	appr.	86.3781	91.8050	72.8407	82.3653	56.6984	76.1493
	up	93.2133		83.5144		74.866	

Table 3.3: Fair premium with invers initial term structure for a life aged 30 and guaranteed amount 1000

maturity:		10		12		15	
a share	method	Vorst	simulated	Vorst	simulated	Vorst	simulated
0.40	down	82.8088		69.9274		57.4674	
	appr.	83.0432	84.8639	70.3227	73.3564	58.0098	63.1250
	up	86.3096		75.1077		65.8741	
0.45	down	84.1441		71.4897		59.0054	
	appr.	84.5131	86.7543	72.0801	75.7489	59.7846	66.0432
	up	88.3375		77.7916		69.3858	
0.50	down	85.7889		73.3295		60.7406	
	appr.	86.3462	89.0468	74.1916	78.5892	61.8381	69.4521
	up	90.8096		80.9892		73.5256	
0.55	down	87.7942		75.4886		62.6987	
	appr.	88.6215	91.8318	76.7271	81.9763	64.2207	73.5038
	up	93.8237		84.8212		78.4654	
0.60	down	90.2321		78.0224		64.909	
	appr.	91.4378	95.2462	79.7849	86.0370	67.0077	78.3853
	up	97.5189		89.4613		84.4518	
0.65	down	93.1954		81.0086		67.4146	
	appr.	94.947	99.4537	83.513	90.9753	70.2985	84.4323
	up	102.1043		95.1791		91.8572	

In order to calculate the fair premium K^* we have to solve equation (**??**). This can be done by any iterative procedure, since the right side of (18) is strictly increasing. The results are given in tables 4 to 6 in the Appendix. As expected, the fair premium is monotonous in the share of the premium a invested into the index and in the age due to the death distribution. The standard deviation increases in the time to maturity and the frequency of the premium payment. Furthermore, the Monte Carlo simulation indicates a convex behavior of the fair premium with respect to the share $a \in [0.1]$.

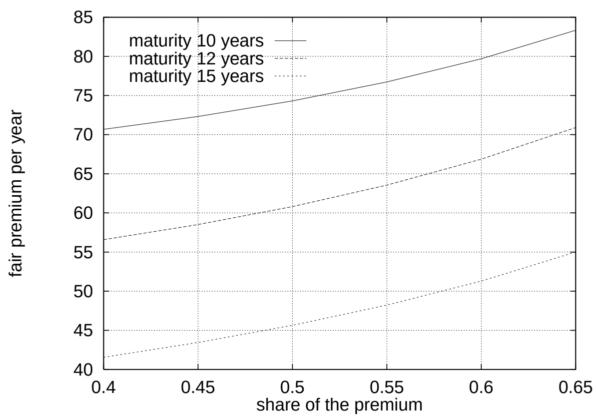

Figure 6: Simulated fair premium for a life aged 30 of the insurance contract with a guaranteed amount of 1000 as a function of the share a with maturities $T = 10, 12, 15$ years, normal initial term structure and annual premium payment.

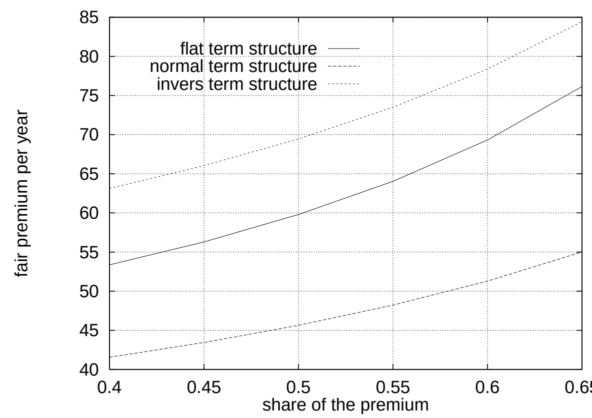

Figure 7: Simulated fair premium for a life aged 30 of the insurance contract with a guaranteed amount of 1000 as a function of the share a with maturity $T = 15$, years, normal, flat and invers initial term structure and annual premium payment.

From the point of view of the insurance company, this seems to be reasonable. The insurer has to guarantee the contract value $g(K)$. If the share a is relatively high, the insurer faces in addition to the mortality risk also the financial risk. To control this additional financial risk, he "maximizes" the part of the premium $(1-a)K$ not invested into the reference portfolio in the first periods. This results in a high premium which itself leads into an increase of the value of the reference portfolio. This implies that the Asian option will very soon be in the money and thus the insurer is compensated by a high expected bonus if the death event will not occur. The expected bonus at each time τ_{i+1} can be calculated by

$$E^{\tau_{i+1}}\left[\max\{aK^*\sum_{j=0}^{v^*(i)}\frac{S(\tau_{i+1})}{S(t_j)} - g(K^*), 0\}\right] \tag{21}$$

which is equal to the payment at time τ_{i+1} minus $g(K^*)$ in case of death between τ_i and τ_{i+1}. Table 7 shows the result for a 12 year contract with yearly payment frequency and normal initial term structure obtained by the Monte Carlo simulation.

Table 7: Development of the expected bonus of a 12 year insurance contract with guaranteed amount of 1000 for a life aged 30 , yearly premium payment and normal initial term structure

age 30		share a					
		0.40	0.45	0.50	0.55	0.60	0.65
year	premium	56.58	58.51	60.81	63.55	66.86	70.9
$t = 1$	bonus	0.0	0.0	0.0	0.0	0.0	0.0
	sd	0.0	0.0	0.0	0.0	0.0	0.0
$t = 2$	bonus	0.0	0.0	0.0	0.0	0.0	0.0
	sd	0.0	0.0	0.0	0.0	0.0	0.0
$t = 3$	bonus	0.0	0.0	0.0	0.0	0.0	0.0
	sd	0.0	0.0	0.0	0.0	0.0	0.0
$t = 4$	bonus	0.0	0.0	0.0	0.0	0.0	0.0
	sd	0.0	0.0	0.0	0.0	0.0	0.0
$t = 5$	bonus	0.0	0.0	0.0	0.0	0.0	0.01
	sd	0.0	0.0	0.00001	0.00009	0.00054	0.00304
$t = 6$	bonus	0.0	0.01	0.04	0.12	0.34	0.93
	sd	0.0002	0.00094	0.00365	0.01273	0.04061	0.1213
$t = 7$	bonus	0.17	0.44	1.03	2.24	4.57	8.93
	sd	0.00868	0.02453	0.06222	0.14726	0.3298	0.70953
$t = 8$	bonus	1.97	3.87	7.11	12.38	20.76	33.85
	sd	0.06534	0.14015	0.27904	0.53184	0.97855	1.75728
$t = 9$	bonus	9.03	15.1	24.0	36.74	54.77	80.2
	sd	0.2057	0.37453	0.64604	1.08387	1.77719	2.87232
$t = 10$	bonus	25.05	37.74	54.77	77.3	107.06	146.48
	sd	0.40586	0.66715	1.05258	1.63166	2.49153	3.77408
$t = 11$	bonus	51.53	72.52	99.1	132.58	174.88	228.76
	sd	0.61293	0.94257	1.40327	2.0665	3.01334	4.3772
$t = 12$	bonus	87.9	118.04	154.78	199.58	254.55	322.78
	sd	0.78635	1.15589	1.6546	2.35355	3.32683	4.69802

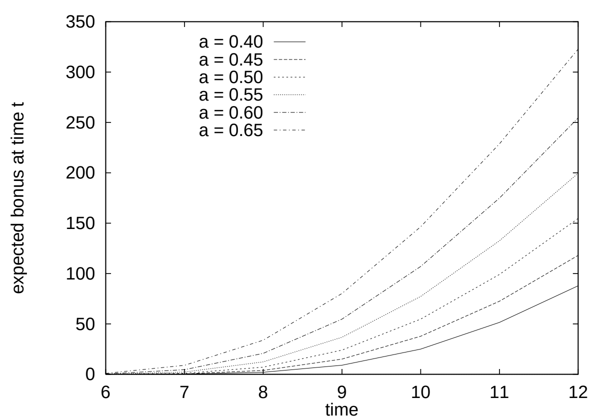

Figure 8: Expected bonus of a 12 year insurance contract for a life aged 30 with guaranteed amount of 1000, yearly premium payment, normal initial term structure, $\sigma = 0.08$, $\sigma_1 = 0.10$, and $\sigma_2 = 0.15$.

7 Conclusion

In an economy with stochastic development of the term structure of interest rates a model for the determination of the fair premium on an equity linked life insurance contract has been established. An essential part of the premium equation consists of a contingent claim with a character as an Asian option. However it was shown that the stochastic interest rate and the long time to maturity of the insurance contract prohibited the application of the "usual" solution methods: Edgeworth expansion or Fast Fourier transform. The approximation formula developed by Vorst (1992) exhibited a better performance than the two just mentioned for medium term contracts. To overcome the difficulties we applied and advocated for Monte Carlo simulations. The result obtained was compared to the Edgeworth and Vorst approximation and found to be preferable to these. Although the Monte Carlo simulations are more time consuming than the other methods we do not take it as a serious critical point against simulation as the fair premium only has to be calculated once when the contract is entered.

Appendix

Recursive algorithms for the first four non–central moments of a sum of n lognormal distributed variables. For simplicity we only give the recursive algorithm for the forward risk adjusted measure at time T. Define $\beta_i := \frac{S(T)}{S(t_i)}$ and for $i = 0, \dots, n-1$

$$\begin{aligned} \sigma_i &:= \exp\left\{\frac{1}{2}[(T-t_i)(\sigma_1^2+\sigma_2^2) + (t_i\sigma^2 - \sigma\sigma_1)(T-t_i)^2 + \frac{1}{3}\sigma^2(T-t_i)^3]\right\} \\ v_{i,j} &:= \exp\left\{\sigma(T-t_j)(t_j-t_i)[\frac{1}{2}\sigma(t_i+t_j)-\sigma_1]\right\} \quad \text{for } j = 0,\dots,i \\ d_i &:= \frac{D(t_0,t_i)}{D(t_0,T)} \end{aligned}$$

Proposition

$\forall\ 0 \le i \le j \le l \le n-1 \quad ; \quad \forall\ \alpha, \gamma, \eta \in \mathbb{N}$

(a) $E^T[\beta_i^\alpha] \ = \ d_i^\alpha \sigma_i^{\alpha(\alpha-1)}$

(b) $E^T[\beta_i^\alpha \beta_j^\gamma] \ = \ d_i^\alpha d_j^\gamma \sigma_i^{\alpha(\alpha-1)} \sigma_j^{\gamma(\gamma-1+2\alpha)} v_{i,j}^{\alpha\cdot\gamma}$

(c) $E^T[\beta_i^\alpha \beta_j^\gamma \beta_l^\eta] \ = \ d_i^\alpha d_j^\gamma d_l^\eta \sigma_i^{\alpha(\alpha-1)} \cdot \sigma_j^{\gamma(\gamma-1+2\alpha)} \cdot \sigma_l^{\eta(\eta-1+2\gamma+2\alpha)} v_{i,j}^{\alpha\cdot\gamma} v_{i,l}^{\alpha\cdot\eta} v_{j,l}^{\gamma\cdot\eta}$

Proof

Define $\beta_i \ = \ \frac{S(T)}{S(t_i)} \ = \ \mu_i \ \exp\{X_i + Y_i + Z_i\}$ with

$$\begin{aligned} \mu_i &:= d_i \exp\left\{-\frac{1}{2}(T-t_i)^2 t_i\sigma^2 - \frac{1}{2}\int_{t_i}^T \left((\sigma_1 - (T-u)\sigma)^2 + \sigma_2^2\right) du\right\} \ = \ d_i\sigma_i^{-1} \\ X_i &:= -(T-t_i)\sigma W_1^T(t_i) \\ Y_i &:= \int_{t_i}^T (\sigma_1 - (T-u)\sigma)\, dW_1^T(u) \\ Z_i &:= \int_{t_i}^T \sigma_2 dW_2^T(u) \end{aligned}$$

These stochastic variables have expectation of zero and

(i) X_i and Z_j are in pairs stochastic independent $\forall\ t_i, t_j$

(ii) Y_i and Z_j are in pairs stochastic independent $\forall\ t_i, t_j$

(iii) X_i and Y_j are in pairs stochastic independent $\forall\ t_i \le t_j$

Furthermore we know that $\forall\ i \le j \le l$

$$\begin{aligned} E^T[X_iY_j] &= E[X_j^2] - (T-t_j)(t_j-t_i)[T-t_j-t_i]\sigma^2 \\ E^T[Y_iY_j] &= E[Y_j^2] \\ E^T[Z_iZ_j] &= E[Z_j^2] \\ E^T[X_jY_i] &= -\sigma(T-t_j)\int_{t_i}^{t_j} (\sigma_1 - (T-u)\sigma)du \\ &= \frac{1}{2}\sigma(T-t_j)(t_j-t_i)[\sigma(2T-t_i-t_j)-2\sigma_1] \end{aligned}$$

ad a)

$$\begin{aligned} E^T[\beta_i^\alpha] &= \mu_i^\alpha \cdot \exp\left\{\frac{1}{2}\alpha^2 V[X_i+Y_i+Z_i]\right\} \\ &= \mu_i^\alpha \cdot \sigma^{\alpha^2} \;=\; d_i^\alpha \sigma^{\alpha(\alpha-1)} \end{aligned}$$

ad b)

$$\begin{aligned} E^T[\beta_i^\alpha \beta_j^\gamma] &= \mu_i^\alpha \mu_j^\gamma E^T\left[\exp\left\{\alpha(X_i+Y_i+Z_i)+\gamma(X_j+Y_j+Z_j)\right\}\right] \\ &= \mu_i^\alpha \mu_j^\gamma \sigma_i^{\alpha^2} \sigma_j^{\gamma^2} \cdot \exp\left\{\alpha\gamma E^T[X_iX_j + X_jY_i + Y_iY_j + Z_iZ_j]\right\} \\ &= d_i^\alpha d_j^\gamma \sigma_i^{\alpha(\alpha-1)} \sigma_j^{\gamma(\gamma-1)} \sigma_j^{2\alpha\gamma} v_{ij}^{\alpha\gamma} \end{aligned}$$

ad c)

$$\begin{aligned} E^T[\beta_i^\alpha \beta_j^\gamma \beta_l^\eta] &= \mu_i^\alpha \mu_j^\gamma \mu_l^\eta \\ &\quad \cdot E^T\left[\exp\left\{\alpha(X_i+Y_i+Z_i)+\gamma(X_j+Y_j+Z_j)+\eta(X_l+Y_l+Z_l)\right\}\right] \\ &= \mu_i^\alpha \mu_j^\gamma \mu_l^\eta \cdot \sigma_i^{\alpha^2} \sigma_j^{\gamma^2} \sigma_l^{\eta^2} \\ &\quad \cdot \exp\Big\{\alpha\gamma E^T[X_iX_j + X_jY_i + Y_iY_j + Z_iZ_j] \\ &\quad + \alpha\eta E^T[X_iX_l + X_lY_i + Y_iY_l + Z_iZ_l] \\ &\quad + \gamma\eta E^T[X_jX_l + X_lY_j + Y_jY_l + Z_jZ_l]\Big\} \\ &= d_i^\alpha d_j^\gamma d_l^\eta \sigma_i^{\alpha(\alpha-1)} \sigma_j^{\gamma(\gamma-1)} \sigma_l^{\eta(\eta-1)} \cdot \sigma_j^{2\alpha\gamma} v_{i,j}^{\alpha\gamma} \cdot \sigma_l^{2\alpha\eta} v_{i,l}^{\alpha\eta} \cdot \sigma_l^{2\gamma\eta} v_{j,l}^{\gamma\eta} \end{aligned}$$

□

With the help of the following vector notation we can now give the recursive algorithms.

$$\begin{aligned} d(i) &:= (d_0,\ldots,d_i)^T \in \mathbb{R}^{i+1} && \forall\ i = 0,\ldots,n-1 \\ v(i) &:= (v_{0,i},\ldots,v_{i-1,i})^T \in \mathbb{R}^{i} && \forall\ i = 1,\ldots,n-1 \\ v^2(i) &:= (v_{0,i}^2,\ldots,v_{i-1,i}^2)^T \in \mathbb{R}^{i} && \text{resp. } v^3(i), v^4(i) \end{aligned}$$

1. Moment

$$E^T\left[\sum_{i=0}^{n-1}\beta_i\right] = \sum_{i=0}^{n-1} d_i = \langle d(n-1), 1\rangle$$

2. Moment

$$\begin{aligned} x(0) &:= d_0^2\sigma_0^2 \quad \text{and for } i = 1,\ldots,n-1 \\ x(i) &:= x(i-1) + d_i^2\sigma_i^2 + 2\langle d(i-1), v(i)\rangle d_i\sigma_i^2 \end{aligned}$$

$$\Rightarrow \quad E\left[\left(\sum_{i=0}^{n-1}\beta_1\right)^2\right] = x(n-1)$$

3. Moment

$$\begin{aligned} x(0) &:= d_0^3\sigma_0^6 \quad \text{and for } i=1,\ldots,n-1 \\ x(i) &:= x(i-1)+d_i^3\sigma_i^6+3\cdot\langle d(i-1),v(i)^2\rangle d_i^2\sigma_i^6+3\cdot a(i-1,i) \end{aligned}$$

where

$$\begin{aligned} a(0,i) &:= d_0^2 d_i\sigma_0^2\sigma_i^4 v_{0,i}^2 \quad \text{and for } j=1,\ldots,n-1 \\ a(j,i) &:= a(j-i,i)+d_j^2 d_i\sigma_j^2\sigma_i^4 v_{j,i}^2 \\ &\quad +2(\sum_{k=0}^{j-1} d_k v_{k,j} v_{k,i}) d_j\sigma_j^2 d_i\sigma_i^4 v_{j,i} \end{aligned}$$

$$\Rightarrow \qquad E^T\left[\left(\sum_{i=0}^{n-1}\beta_i\right)^3\right] = x(n-1)$$

4. Moment

$$\begin{aligned} x(0) &:= d_0^4\sigma_0^{12} \quad \text{and for } i=1,\ldots,n-1 \\ x(i) &:= x(i-1)+d_i^4\sigma_i^{12}+4\cdot a(i-1,i)+6\cdot c(i-1,i)+4\langle d(i-1),v(i)^3\rangle d_i^3\sigma_i^{12} \end{aligned}$$

where

I)

$$\begin{aligned} a(0,i) &:= d_0^3 d_i\sigma_0^6\sigma_i^6 v_{0,i}^3 \quad \text{and for } j=1,\ldots,i-1 \\ a(j,i) &:= a(j-1,i)+d_j^3 d_i\sigma_j^6\sigma_i^6 v_{j,i}^3+3\cdot b(j-1,j,i) \\ &\quad +3\cdot\left(\sum_{k=0}^{j-1} d_k v_{k,j}^2 v_{k,i}\right) d_j^2 d_k\sigma_j^6\sigma_k^6 v_{i,j}^2 \end{aligned}$$

where

$$\begin{aligned} b(0,j,i) &:= d_0^2 d_j d_i\sigma_0^2\sigma_j^4\sigma_i^6 v_{0,j}^2 v_{0,i}^2 v_{i,j} \quad \text{and for } j=1,\ldots,i-1 \\ b(k,j,i) &:= b(k-1,j,i)+d_k^2 d_j d_i\sigma_k^2\sigma_j^4\sigma_i^6 v_{k,j}^2 v_{k,i}^2 v_{j,i} \\ &\quad +2\left(\sum_{l=0}^{k-1} d_l v_{l,k} v_{l,j} v_{l,i}\right) d_k d_j d_i\sigma_k^2\sigma_j^4\sigma_i^6 v_{k,j} v_{k,i} v_{j,i} \end{aligned}$$

II)

$$\begin{aligned} c(0,i) &= d_0^2 d_i^2\sigma_0^2\sigma_i^{10} v_{0,i}^4 \quad \text{and for } j=1,\ldots,i-1 \\ c(j,i) &= c(j-1,i)+d_j^2 d_i^2\sigma_j^2\sigma_i^{10} v_{j,i}^4+2\left(\sum_{k=0}^{j-1} d_k v_{k,j} v_{k,i}^2\right) d_j d_i^2\sigma_j^2\sigma_i^{10} v_{j,i}^2 \end{aligned}$$

$$\Rightarrow \qquad E^T\left[\left(\sum_{i=0}^{n-1}\beta_i\right)^4\right] = x(n-1)$$

For the second moment of the sum of the logarithmic β_i a similar algorithm can be given. Set

$$\begin{aligned} \tilde\beta_i &:= \ln\beta_i-\ln\mu_i = X_i+Y_i+Z_i \\ \tilde\sigma_i^2 &:= 2\ln\sigma_i \\ \tilde v_{i,j} &:= \ln v_{i,j} \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad E^T[\tilde\beta_i\cdot\tilde\beta_j] &= E^T[\tilde\beta_j^2]+\tilde v_{i.j} \\ E^T[\tilde\beta_i^2\ ] &= \tilde\sigma_i^2 \end{aligned}$$

2. Moment

$$x(0) := \tilde{\sigma}_0^2 \quad \text{and for } i = 1, \ldots, n-1$$

$$x(i) := x(i-1) + \tilde{\sigma}_i^2 + 2\sum_{j=0}^{i-1}(\tilde{\sigma}_i^2 + \tilde{v}_{j,i})$$

$$\Rightarrow \qquad V^T\left[\sum_{i=0}^{n-1}\ln\beta_i\right] = V^T\left[\sum_{i=0}^{n-1}\tilde{\beta}_i\right] = x(n-1)$$

Alternative calculation for $V^T[lnG(t_n)]$ for an equidistant discretisation

$$\begin{aligned} V^T[\ln G(t_n)] &= \frac{1}{n^2}V^T\left[\sum_{i=0}^{n-1} -\sigma(T-t_1)W_1^T(t)i) + \int_{t_i}^{T}(\sigma_1) - (T-u)\sigma dW_t^T(u)\right] \\ &+ \frac{1}{n^2}V^T\left[\sum_{i=0}^{n-1}\int_{t_i}^{T}\sigma_2 dW_2^T(u)\right] \end{aligned}$$

I)

$$\begin{aligned} \frac{1}{n^2}V^T\left[\sum_{i=0}^{n-1}\int_{t_i}^{T}\sigma_2 dW_2\right] &= \frac{1}{n^2}V^T\left[\sum_{i=0}^{n-1}\sigma_2(W_2^T(T) - W_2^T(t_i))\right] \\ &= \frac{1}{n^2}V^T[\sum_{i=0}^{n-1}\sigma_2(i+1)[W_2^T(t_{i+1}) - W_2^T(t_i)]] \\ &= \frac{1}{n^2}\sigma_2^2\sum_{i=0}^{n-1}(i+1)^2(t_{i+1}-t_i) \\ &= \frac{1}{n^2}\sigma_2^2\sum_{i=0}^{n-1}(i+1)^2\Delta t \end{aligned}$$

II)

$$\begin{aligned} &\frac{1}{n^2}V^T\left[\sum_{i=0}^{n-1} -\sigma(T-t_i)W_1^T(t_i) + \int_{t_i}^{T}(\sigma_1 - (T-u)\sigma)dW_1^T(u)\right] \\ &= \frac{1}{n^2}V^T\left[\sum_{i=0}^{n-1} -\sigma\Delta t(n-i)W_1^T(t_i) + (i+1)\int_{t_i}^{t_{i+1}}(\sigma_1 - (T-u)\sigma)dW_1^T(u)\right] \\ &= \frac{1}{n^2}V^T\left[\sum_{i=0}^{n-1} -\sigma\Delta t \cdot a_{i+1}[W_1^T(t_{i+1} - W_1^T(t_i)] + (i+1)\int_{t_1}^{t_{i+1}}(\sigma_1 - (T-u)\sigma)dW_1^T(u)\right] \\ &= \frac{1}{n^2}\sum_{i=0}^{n-1}V^T\left[\int_{t_i}^{t_{i+1}}((\sigma_1 - (T-u)\sigma)(i+1) - \sigma\Delta t a_{i+1})dW_1^T(u)\right] \end{aligned}$$

where $a_n := 0$ and for $k = n-1, ..., 1 \qquad a_{n-k} := k + a_{n-k-1}$

With some standard reformulation this leads to

$$
\begin{aligned}
V^T[\ln G(t_n)] &= \frac{1}{n^2}\sum_{i=0}^{n-1}(i+1)^2\Delta t(\sigma_1^2+\sigma_2^2) \\
&- \frac{1}{n^2}\sum_{i=0}^{n-1}(i+1)^2\sigma_1\sigma[(T-t_i)^2-(T-t_{i+1})^2] \\
&+ \frac{1}{n^2}\sum_{i=0}^{n-1}(i+1)^2\frac{\sigma^2}{3}[(T-t_i)^3-(T-t_{i+1})^3] \\
&- \frac{2}{n^2}\sum_{i=0}^{n-1}a_{i+1}\cdot(i+1)\sigma_1\sigma\Delta t^2 \\
&+ \frac{1}{n^2}\sum_{i=0}^{n-1}a_{i+1}\cdot(i+1)\sigma^2\Delta t[(T-t_i)^2-(T-t_{i+1})^2] \\
&+ \frac{1}{n^2}\sum_{i=0}^{n-1}\alpha_{i+1}^2\Delta t^3\sigma^2
\end{aligned}
$$

which up to some possible simplifications of the first three terms is a linear problem.

Table 4: Monte Carlo simulation and standard deviation for the fair premium with normal initial term structure

maturity:		10				12				15			
a	age	1	2	4	12	1	2	4	12	1	2	4	12
0.40	25	70.4269	35.7158	17.9622	6.0149	56.2791	28.6543	14.3059	4.7819	41.1794	21.8929	10.489	3.541
		0.3662	0.1285	0.0906	0.0159	0.8222	0.5169	0.1542	0.0417	1.7221	2.2331	0.2415	0.1922
0.40	30	70.6797	35.8481	18.0301	6.0379	56.5781	28.8103	14.3854	4.8087	41.5548	22.0895	10.5877	3.5743
		0.3669	0.1287	0.0908	0.016	0.8248	0.5179	0.1546	0.0418	1.729	2.2376	0.2427	0.1927
0.40	35	71.0922	36.0643	18.141	6.0754	57.0651	29.0647	14.5153	4.8525	42.1665	22.4095	10.7486	3.6286
		0.3679	0.129	0.091	0.016	0.8283	0.5198	0.1552	0.042	1.7402	2.2444	0.2446	0.1934
0.45	25	72.0631	36.5033	18.351	6.1427	58.1984	29.6354	14.7697	4.9329	43.0516	23.1487	10.944	3.7024
		0.4475	0.1598	0.1076	0.0185	1.0147	0.6427	0.1871	0.0522	2.1613	3.0161	0.2998	0.245
0.45	30	72.323	36.6398	18.421	6.1663	58.5078	29.7968	14.8519	4.9607	43.4411	23.3521	11.0463	3.7369
		0.448	0.1601	0.1078	0.0185	1.0167	0.644	0.1877	0.0524	2.1691	3.0194	0.3012	0.2455
0.45	35	72.7478	36.8625	18.5353	6.205	59.012	30.0601	14.9861	5.0061	44.0757	23.6835	11.2131	3.7932
		0.4487	0.1609	0.108	0.0186	1.0188	0.646	0.1885	0.0526	2.1824	3.0242	0.3035	0.2462
0.50	25	74.0459	37.4629	18.824	6.2977	60.4907	30.8009	15.3156	5.1113	45.2323	24.6873	11.469	3.8908
		0.5425	0.1985	0.1286	0.0223	1.2407	0.7905	0.2285	0.0639	2.7185	4.1539	0.3716	0.313
0.50	30	74.316	37.6046	18.8966	6.3222	60.8115	30.9689	15.401	5.1401	45.6367	24.8978	11.5756	3.9267
		0.5437	0.1991	0.129	0.0224	1.2431	0.7918	0.2291	0.0641	2.7284	4.1537	0.3733	0.3135
0.50	35	74.7566	37.836	19.015	6.3624	61.3345	31.243	15.5405	5.1872	46.2954	25.2406	11.7493	3.9851
		0.5447	0.1999	0.1295	0.0225	1.247	0.7942	0.2302	0.0643	2.744	4.1526	0.376	0.3143
0.55	25	76.4572	38.6333	19.3955	6.4864	63.2204	32.194	15.9643	5.3228	47.795	26.6622	12.0842	4.1137
		0.656	0.2459	0.1579	0.0271	1.5365	0.9887	0.2815	0.0786	3.4429	5.9739	0.4622	0.4049
0.55	30	76.7379	38.7807	19.4711	6.512	63.5548	32.3695	16.0537	5.3529	48.2176	26.8789	12.1957	4.1511
		0.6563	0.2464	0.1583	0.0271	1.5399	0.9907	0.2825	0.0787	3.4546	5.963	0.4643	0.4054
0.55	35	77.1976	39.0214	19.5945	6.5539	64.1	32.6556	16.1996	5.4021	48.9048	27.2314	12.3773	4.212
		0.6579	0.2471	0.1591	0.0272	1.5453	0.994	0.2838	0.0791	3.472	5.9444	0.4677	0.4061
0.60	25	79.3983	40.0562	20.092	6.7179	66.5135	33.8682	16.7402	5.5762	50.845	29.4128	12.8093	4.3823
		0.8068	0.2941	0.1936	0.0326	1.9186	1.2554	0.3486	0.0967	4.3967	9.3168	0.5788	0.5332
0.60	30	79.693	40.2111	20.1713	6.7448	66.8638	34.052	16.8338	5.6079	51.287	29.628	12.9263	4.4214
		0.8079	0.2942	0.1941	0.0327	1.9219	1.2575	0.3496	0.0969	4.4089	9.2695	0.5815	0.5333
0.60	35	80.1738	40.4642	20.3009	6.7889	67.4361	34.352	16.9867	5.6596	52.0057	29.9787	13.1167	4.4852
		0.8103	0.2948	0.1949	0.033	1.928	1.2611	0.3512	0.0973	4.4278	9.1948	0.5856	0.5334
0.65	25	83.0275	41.8153	20.9495	7.0019	70.5296	35.9186	17.6811	5.8827	54.5451	34.2519	13.6786	4.7154
		0.9955	0.3654	0.2379	0.0413	2.4334	1.6173	0.4367	0.1212	5.7051	17.8544	0.736	0.7247
0.65	30	83.3382	41.9791	21.0338	7.0305	70.8988	36.1131	17.7803	5.9162	55.0106	34.4126	13.8022	4.7565
		0.9968	0.3663	0.2385	0.0413	2.438	1.6203	0.438	0.1215	5.7186	17.6057	0.7391	0.7242
0.65	35	83.8453	42.2467	21.1714	7.0772	71.5	36.43	17.942	5.9708	55.7662	34.6814	14.0033	4.8234
		0.9996	0.3677	0.2395	0.0414	2.4455	1.6247	0.4399	0.122	5.7388	17.2238	0.7441	0.7234

Table 5: Monte Carlo simulation and standard deviation for the fair premium with flatt initial term structure

maturity:		10				12				15			
a	age	1	2	4	12	1	2	4	12	1	2	4	12
0.40	25	77.4786	39.0672	19.6721	6.5723	64.783	32.4277	16.3585	5.4457	53.0312	25.6304	13.3549	4.3388
		0.4563	0.1624	0.0732	0.0273	1.1968	0.3429	0.1282	0.0558	2.9704	0.8279	0.4233	0.1314
0.40	30	77.7162	39.1928	19.7366	6.5941	65.0587	32.5719	16.4322	5.4706	53.3655	25.8037	13.4438	4.3687
		0.4568	0.1624	0.0734	0.0274	1.1977	0.3434	0.1285	0.0559	2.9726	0.8295	0.4237	0.1316
0.40	35	78.104	39.3979	19.8418	6.6298	65.5082	32.8074	16.5526	5.5113	53.9106	26.0867	13.5888	4.4176
		0.4569	0.1625	0.0737	0.0274	1.1989	0.3444	0.129	0.056	2.9761	0.8319	0.4242	0.1319
0.45	25	79.2831	39.9286	20.1011	6.7112	67.0941	33.4931	16.8949	5.618	55.9545	26.7673	14.0397	4.5343
		0.547	0.1924	0.0916	0.0327	1.4774	0.4219	0.1588	0.069	3.8289	1.0332	0.5352	0.1634
0.45	30	79.5277	40.0577	20.1674	6.7337	67.3776	33.6414	16.9709	5.6438	56.2986	26.9459	14.1314	4.5651
		0.5475	0.1925	0.0917	0.0327	1.4787	0.4225	0.1591	0.0691	3.831	1.0349	0.5356	0.1636
0.45	35	79.9274	40.2686	20.2757	6.7704	67.841	33.8838	17.0952	5.6858	56.8587	27.2374	14.281	4.6155
		0.5481	0.1928	0.092	0.0328	1.4805	0.4232	0.1595	0.0693	3.8322	1.0376	0.5363	0.164
0.50	25	81.4941	40.9758	20.6233	6.8806	69.8466	34.7487	17.5327	5.8226	59.4417	28.0867	14.8473	4.7614
		0.662	0.2335	0.1113	0.0389	1.8415	0.5139	0.1937	0.0847	4.977	1.2906	0.6793	0.2031
0.50	30	81.7472	41.1092	20.6919	6.9038	70.1401	34.9022	17.6115	5.8492	59.7956	28.2713	14.9422	4.7933
		0.663	0.2338	0.1114	0.039	1.8433	0.5145	0.1939	0.0849	4.976	1.2928	0.6795	0.2033
0.50	35	82.1604	41.3272	20.804	6.9418	70.6181	35.1528	17.7402	5.8927	60.3714	28.5725	15.0969	4.8454
		0.6638	0.2343	0.1116	0.039	1.8453	0.5155	0.1943	0.0851	4.9731	1.296	0.6798	0.2037
0.55	25	84.1711	42.2419	21.2549	7.0859	73.1402	36.2443	18.2905	6.0656	63.6718	29.628	15.8128	5.0278
		0.8056	0.2805	0.1318	0.0465	2.295	0.6294	0.2379	0.1037	6.5399	1.6176	0.8699	0.2528
0.55	30	84.4338	42.3807	21.3263	7.1101	73.4436	36.4039	18.3726	6.0934	64.0342	29.8195	15.9113	5.0609
		0.8058	0.2806	0.132	0.0466	2.2956	0.6298	0.2383	0.1039	6.5338	1.6194	0.8701	0.253
0.55	35	84.8626	42.6071	21.4429	7.1496	73.9387	36.6642	18.5066	6.1387	64.6246	30.1318	16.0716	5.1151
		0.8066	0.281	0.1322	0.0468	2.2971	0.6308	0.2389	0.1042	6.5245	1.6224	0.8699	0.2534
0.60	25	87.4552	43.789	22.0249	7.3368	77.1277	38.0447	19.2034	6.3567	68.9432	31.4665	16.9857	5.3455
		0.9842	0.3371	0.1597	0.0561	2.9026	0.7799	0.2974	0.1276	8.8294	2.0438	1.1263	0.3181
0.60	30	87.7296	43.9345	22.0997	7.3622	77.4438	38.2116	19.2893	6.3857	69.3158	31.6661	17.0878	5.3801
		0.9846	0.3376	0.1601	0.0562	2.9037	0.7811	0.298	0.1278	8.8147	2.0456	1.1258	0.3184
0.60	35	88.1786	44.1717	22.2219	7.4036	77.9591	38.4837	19.4294	6.4331	69.9211	31.9915	17.2541	5.4365
		0.9857	0.3383	0.1605	0.0563	2.9052	0.7829	0.2988	0.1281	8.7895	2.0484	1.1249	0.3187
0.65	25	91.5163	45.6982	22.975	7.6455	82.0351	40.2375	20.3162	6.7103	75.7743	33.6929	18.4439	5.7307
		1.2155	0.4189	0.1975	0.0685	3.7533	0.9785	0.3748	0.1599	12.4314	2.6182	1.4881	0.4056
0.65	30	91.805	45.8517	23.054	7.6724	82.3653	40.4126	20.4066	6.7409	76.1493	33.9013	18.5499	5.767
		1.2162	0.4195	0.1978	0.0686	3.753	0.9794	0.3755	0.1601	12.3929	2.6197	1.4865	0.4057
0.65	35	92.2754	46.1023	23.1831	7.7162	82.9032	40.6981	20.5539	6.7908	76.7592	34.2409	18.7223	5.826
		1.2178	0.4201	0.1983	0.0688	3.7528	0.9811	0.3766	0.1605	12.3304	2.6218	1.4837	0.4059

Table 6: Monte Carlo simulation and standard deviation for the fair premium with invers initial term structure

maturity:		10				12				15			
a	age	1	2	4	12	1	2	4	12	1	2	4	12
0.40	25	84.6415	42.7392	21.3791	7.1537	73.1092	37.0947	18.3242	6.1378	62.8455	32.9609	15.499	5.1609
		0.2388	0.1609	0.0655	0.0338	0.5723	0.5701	0.1167	0.0705	2.9689	2.9736	0.3347	0.1304
0.40	30	84.8639	42.8567	21.4395	7.1742	73.3564	37.2254	18.3912	6.1605	63.125	33.106	15.5747	5.1866
		0.2391	0.1612	0.0655	0.0338	0.5731	0.5702	0.1168	0.0706	2.962	2.968	0.3346	0.1305
0.40	35	85.2258	43.0488	21.5382	7.2078	73.7608	37.4388	18.5007	6.1977	63.5811	33.343	15.6983	5.2286
		0.2392	0.1614	0.0656	0.0339	0.5733	0.5705	0.1168	0.0707	2.9507	2.959	0.3345	0.1307
0.45	25	86.5271	43.6575	21.8115	7.2978	75.4958	38.3209	18.8664	6.3206	65.7597	34.807	16.1167	5.3628
		0.2852	0.1971	0.0799	0.041	0.7076	0.7077	0.1416	0.0863	3.7708	3.9039	0.4128	0.1619
0.45	30	86.7543	43.7782	21.8735	7.3189	75.7489	38.455	18.9352	6.344	66.0432	34.9536	16.1943	5.3892
		0.2853	0.1972	0.0798	0.0411	0.7074	0.7076	0.1417	0.0863	3.7602	3.8944	0.4126	0.162
0.45	35	87.126	43.9753	21.9748	7.3533	76.1632	38.6742	19.0476	6.3821	66.5062	35.1929	16.321	5.4323
		0.2864	0.1976	0.0799	0.0411	0.7078	0.7075	0.1419	0.0864	3.7427	3.8787	0.4123	0.1622
0.50	25	88.8132	44.7776	22.3399	7.4729	78.3287	39.7822	19.5084	6.5363	69.1633	37.052	16.8303	5.5956
		0.3431	0.2388	0.096	0.0495	0.867	0.8704	0.1736	0.1053	4.7989	5.1867	0.5109	0.2008
0.50	30	89.0468	44.9024	22.404	7.4947	78.5892	39.9203	19.5795	6.5604	69.4521	37.1987	16.91	5.6227
		0.3433	0.2391	0.0962	0.0495	0.867	0.8702	0.1739	0.1054	4.7838	5.1701	0.5108	0.2008
0.50	35	89.4293	45.106	22.5086	7.5302	79.0144	40.1457	19.6955	6.5997	69.9216	37.4378	17.0401	5.6669
		0.344	0.2396	0.0964	0.0497	0.8671	0.8702	0.1743	0.1056	4.7578	5.1429	0.5106	0.201
0.55	25	91.589	46.1397	22.9828	7.6868	81.708	41.5203	20.27	6.792	73.2112	39.8697	17.662	5.8659
		0.4172	0.2949	0.1171	0.0603	1.0678	1.0838	0.2151	0.129	6.1746	7.0838	0.6346	0.2491
0.55	30	91.8318	46.2687	23.0493	7.7095	81.9763	41.6628	20.3434	6.817	73.5038	40.0139	17.744	5.8938
		0.4173	0.2954	0.1172	0.0604	1.0677	1.0835	0.2154	0.1291	6.1509	7.0532	0.6344	0.2492
0.55	35	92.2263	46.4793	23.1578	7.7464	82.4136	41.8954	20.4633	6.8578	73.9798	40.2495	17.8778	5.9394
		0.4176	0.2959	0.1175	0.0605	1.0682	1.0832	0.2157	0.1293	6.1124	7.0037	0.634	0.2492
0.60	25	94.9933	47.7955	23.7661	7.9484	85.7602	43.6244	21.1768	7.0986	78.0905	43.5864	18.64	6.1828
		0.5025	0.3625	0.1411	0.0727	1.3214	1.3651	0.2662	0.1598	8.0834	10.1623	0.7946	0.3108
0.60	30	95.2462	47.9303	23.8355	7.972	86.037	43.7723	21.2529	7.1246	78.3853	43.7212	18.7245	6.2116
		0.5021	0.3628	0.1413	0.0728	1.3217	1.3647	0.2663	0.16	8.0472	10.1005	0.7941	0.3108
0.60	35	95.6574	48.1501	23.9487	8.0106	86.4887	44.0138	21.3773	7.167	78.866	43.9419	18.8624	6.2587
		0.5018	0.3632	0.1415	0.0729	1.3207	1.3639	0.2668	0.1601	7.9872	10.0018	0.7933	0.3108
0.65	25	99.1908	49.8481	24.7307	8.2689	90.6885	46.1982	22.2728	7.4693	84.138	49.0592	19.8047	6.5611
		0.6018	0.4478	0.1714	0.089	1.6575	1.7441	0.3296	0.2002	10.9106	16.207	1.0084	0.3936
0.65	30	99.4537	49.9895	24.8036	8.2937	90.9753	46.3522	22.3524	7.4964	84.4323	49.1606	19.8921	6.5911
		0.6008	0.448	0.1715	0.0891	1.6566	1.7432	0.3298	0.2003	10.851	16.0523	1.0074	0.3936
0.65	35	99.8839	50.2208	24.9227	8.3343	91.4421	46.6031	22.4824	7.5408	84.9117	49.3281	20.0347	6.6401
		0.6009	0.4487	0.1717	0.0893	1.6548	1.7417	0.33	0.2006	10.7539	15.8051	1.0058	0.3935

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