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# The Pricing of Asian Options under Stochastic Interest Rates<sup>\*</sup>

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#### The Pricing of Asian Options under Stochastic Interest Rates

#### Abstract

The purpose of this paper is to analyse the effect of stochastic interest rates on the pricing of Asian options. It is shown that a stochastic, in contrast to a deterministic, development of the term structure of interest rates has a significant influence.

The price of the underlying asset, e.g. a stock or oil, and the prices of bonds are assumed to follow correlated two dimensional Ito processes. The averages considered in the Asian options are calculated on a discrete time grid, e.g. all closing prices on Wednesdays during the lifetime of the contract. The value of an Asian option will be obtained through the application of Monte Carlo simulation, and for this purpose the stochastic processes for the basic assets need not to be severely restricted. However to make comparison with published results originating from models with deterministic interest rates we will stay within the setting of a Gaussian framework.

### Keywords

Asian Options, Forward Risk Adjusted Measure, Monte Carlo Simulation.

#### 1 Introduction

The basic economic setting in which pricing of Asian options has been analysed is characterized by an underlying asset which adheres to a geometric Brownian motion and by a deterministic development of the bond market. No easily implementable closed form solution to the pricing problem has so far been developed in the literature. The suggested methods of pricing all builds on different schemes of approximations.

Kemna and Vorst (1990) show that the Asian option price, subject to the boundary condition characteristic for the option considered, is the solution to a second order partial differential equation in three variables, time, spot price of the underlying asset and the known information about the average value. Rather than solving the partial differential equation, Kemna and Vorst apply Kolmogorov's backward equation and obtain that the price of the Asian option can be written as the discounted expected value of the maturity payment of the option. To solve the pricing equation which involves knowledge of the distribution of a sum of correlated lognormal distributions Kemna and Vorst apply Monte Carlo simulation.

Carverhill and Clewlow (1990) solve the pricing equation applying Fast Fourier Transform techniques to obtain an approximation to the law of the average.

Levy (1992) argues that the sum of correlated lognormal random variables is well approximated by another lognormal distribution and applying Wilkinson's approximation a lognormal distribution with the first and second moment chosen in accordance to the correct distribution is applied as a surrogate. In Turnbull and Wakeman (1991) an Edgeworth expansion, involving the first four cumulants, is used to represent the approximating distribution by a lognormal distribution.

Vorst (1992) uses the fact that the geometric average is never greater than its corresponding aritmetic average, and due to the assumed geometric Brownian motion of the underlying asset the geometric average is also lognormal and the price of the geometric Asian option can be found in closed form. By means of this Vorst calculates a lower as well as an upper bound for the arithmetic Asian option, and then chooses in an ad hoc manner the price of the Asian option in a way which guarantees that the established bounds are fulfilled.

Geman and Yor (1993) succeed in obtaining a closed form solution for the Asian option but it is unfortunately of a very complicated form. To determine the price an inversion of a nontrivial Laplace transform has to be performed.

In this paper we will relax the assumption concerning the deterministic nature of the bond market but retain the geometric Brownian motion for the underlying asset. The stochastic interest rate environment will be assumed to be Gaussian which in accordance to e.g. Jamshidian (1991) and El Karoui, Lepage, Myneni and Viswanathana (1991) implies a lognormal distribution of the zero coupon bond prices. Pricing of standard options in this setting has been analysed in e.g. Amin and Jarrow (1992) and Amin and Bodurtha Jr. (1995). The pricing of Asian options and in particular the influence of the stochastic interest rate on the pricing will be analysed in this paper.

The schedule of the paper is as follows. In section 2, the notation and the definition of the contract is presented. Section 3 deals with the pricing of Asian options. A discussion of different numerical approaches is given in section 4. Section 5 contains the simulation result. Finally,

section 6 concludes.

## 2 Notation and definition of the contract

The following notation will be applied:

Xexercise price of the Asian option.  $t_n$ a date included in the average calculation,  $n = 1, 2, ..., N; t_o = 0.$  $t_N$ the maturity date of the option contract,  $t_N = T$ . S(t)the price of the underlying asset at time t. the price at date t of a zero coupon bond with maturity date t',  $t \leq t'$ . D(t,t')the arithmetic average of the spot prices at the date  $t_n; \quad n=1,\ldots,N$  .  $A(t_n) = \frac{1}{n} \sum_{i=1}^n S(t_i)$  $G(t_n) = \sqrt[n]{\prod_{i=1}^n S(t_i)}$  the geometric average of the spot prices at the date  $t_n$ ;  $n = 1, \ldots, N$ .  $V_A(T) = \max\left\{\frac{1}{N}\sum_{i=1}^N S(t_i) - X, 0\right\} = \max\left\{A(t_N) - X, 0\right\}$ the benefit from the arithmetic Asian option received at maturity date T.  $V_G(T) = \max\left\{ \sqrt[N]{\prod_{i=1}^{N} S(t_i) - X, 0} \right\} = \max\left\{ G(t_N) - X, 0 \right\}$ the benefit from the geometric Asian option received at maturity date T.

r(t) the instantaneous risk free rates of interest at time t.

Next the option prices at time  $t_0$ ,  $V_A(t_0)$  and  $V_G(t_0)$ , will be found in accordance to the absence of arbitrage possibilities in the financial market. We restrict ourself to the pricing of *European* type Asian call options where the averaging period still has to start. The value of an Asian option during the averaging period can be calculated the same way by adjusting the exercise price X, see e.g. Kemna and Vorst (1990) and Vorst (1992).

## 3 Pricing of the Asian option

Assume that the dynamics of the underlying asset S(t) is determined by a lognormal diffusion process with time dependent volatility. For the interest rate market we concentrate on a Gaussian term structure model<sup>1</sup>, which is well known from previous work by Jamshidian (1991) and El Karoui, Lepage, Myneni and Viswanathana (1991). Under the absence of arbitrage opportunities there exists a probability measure  $P^*$  such that the stochastic behaviour of both markets are related in the following way:

$$dS(t) = r(t)S(t)dt + \sigma_1(t)S(t)dW_1^*(t) + \sigma_2(t)S(t)dW_2^*(t),$$
  
$$dD(t,t') = r(t)D(t,t')dt + \sigma(t,t')D(t,t')dW_1^*(t),$$

 $<sup>^{1}</sup>$ As special cases we will discuss the Vasicek (1977) model and the continuous time limit of the Ho and Lee (1986) model

where  $W_1^*$  and  $W_2^*$  are independent standard Wiener processes. The volatility functions  $\sigma_1(t)$ ,  $\sigma_2(t)$ and  $\sigma(t, t')$  are assumed to be non-stochastic and satisfy the usual regularity conditions<sup>2</sup>, in particular  $\sigma(t, t) = 0$  and D(t, t) = 1 with probability 1. In other words we are working under the so called risk neutral martingale measure. Note that by  $\frac{\sigma_1(t)}{\sqrt{\sigma_1^2(t) + \sigma_2^2(t)}}$  the instantaneous correlation between both markets is determined. Due to the stochastic development of r(t), it will be convenient to work in the T-forward risk adjusted probability measure, denoted by  $P^T$ , where it is well known<sup>3</sup>, that the differential equations for  $\frac{D(t,t')}{D(t,T)}$  and  $\frac{S(t)}{D(t,T)}$  are respectively given by

$$d\left(\frac{D(t,t')}{D(t,T)}\right) = \frac{D(t,t')}{D(t,T)} \cdot (\sigma(t,t') - \sigma(t,T)) dW_1^T(t),$$
  
$$d\left(\frac{S(t)}{D(t,T)}\right) = \frac{S(t)}{D(t,T)} \cdot \left[(\sigma_1(t) - \sigma(t,T)) dW_1^T(t) + \sigma_2(t) dW_2^T(t)\right]$$

where  $W_1^T$  and  $W_2^T$  are independent standard Wiener processes under the  $P^T$  probability measure. The change to the forward risk adjusted measure  $P^T$  implies that the stochastic discounting is replaced by the time-t measurable discounting and in particular that

(1) 
$$\frac{S(t)}{D(t,T)} = E_t^T \left[ \frac{S(T)}{D(T,T)} \right] = E_t^T [S(T)]$$

in contrast to

$$S(t) = E_t \left[ \exp\left\{ -\int_t^T r(u) du \right\} S(T) \right]$$

under the risk neutral probability measure. The solutions of the above stochastic differential equations under the T-forward risk adjusted measure  $P^T$  are given by:

$$\begin{split} S(t) &= S(t_0) \cdot \frac{D(t,T)}{D(t_0,T)} \cdot \exp\left\{-\frac{1}{2} \int_{t_0}^t \left((\sigma_1(u) - \sigma(u,T))^2 + \sigma_2^2(u)\right) du \\ &+ \int_{t_0}^t (\sigma_1(u) - \sigma(u,T)) dW_1^T(u) + \int_{t_0}^t \sigma_2 dW_2^T(u)\right\}, \\ \frac{D(t,T)}{D(t_0,T)} &= \frac{D(t,t)}{D(t_0,t)} \cdot \exp\left\{\frac{1}{2} \int_{t_0}^t (\sigma(u,t) - \sigma(u,T))^2 du - \int_{t_0}^t (\sigma(u,t) - \sigma(u,T)) dW_1^T(u)\right\}. \end{split}$$

This allows us to express the the solution for the underlying asset as

$$S(t_{i}) = \frac{S(t_{0})}{D(t_{0}, t_{i})} \cdot \exp\left\{-\frac{1}{2}\int_{t_{0}}^{t_{i}} ((\sigma_{1}(u) - \sigma(u, T))^{2} + \sigma_{2}^{2}(u))du\right\}$$

$$(2) \qquad \cdot \exp\left\{\frac{1}{2}\int_{t_{0}}^{t_{i}} (\sigma(u, t_{i}) - \sigma(u, T))^{2}du\right\}$$

$$\cdot \exp\left\{\int_{t_{0}}^{t_{i}} (\sigma_{1}(u) - \sigma(u, t_{i}))dW_{1}^{T}(u) + \int_{t_{0}}^{t_{i}} \sigma_{2}(u)dW_{2}^{T}(u)\right\}$$

 $<sup>^{2}</sup>$ In a general setup we could allow for stochastic volatility functions, see e.g. Geman, El Karoui and Rochet (1995), but for the continuous time numerical procedure we will be forced to restrict ourselves to non-stochastic functions.

<sup>&</sup>lt;sup>3</sup>In a similar economic context see e.g. Nielsen and Sandmann (1995).

The value of an Asian option with the discrete average  $A(t_N)$  is determined by

(3) 
$$V_A(t_0) = D(t_0, T)E^T \left[ \max \left\{ A(t_N) - X, 0 \right\} \right].$$

Under the specified  $S_t$ -process and the Gaussian interest rate dynamics, we know that the arithmetic average is determined by a sum of correlated lognormal distributed variables. So far, there exists no closed form expression for the distribution of such a sum. Therefore, numerical techniques have to be applied to approximate the value  $V_A(t_0)$  of an Asian option. Observe that (2) turns itself into a much simpler equation if  $\sigma(u,t) = 0 \forall u \leq t \forall t$  corresponding to a non stochastic development of the term structure of interest rates. In this case easily implementable techniques are available in the literature. In the following section these methods will be extended to include the Gaussian term structure model, and we show that major differences appear. Then in section 5, applying the formal analysis of section 4, we show that the parameter which mainly influences the pricing of Asian options is the correlation between the underlying asset and the term structure.

## 4 Numerical approximation for Asian options

In a similar economic setting, Carverhill and Clewlow (1990) solve the pricing equation for an Asian option by applying the Fast Fourier Transformation technique in order to calculate the distribution of the arithmetic average. Their idea is to rewrite the equation of the underlying asset such that  $S(t_i) = S(t_{i-1}) \cdot a^T(t_{i-1}, t_i)$  which implies that the arithmetic average can be reformulated as

$$A(T) = S(t_0) \left[ 1 + a^T(t_0, t_1) \left[ 1 + a^T(t_1, t_2) \left[ 1 + \dots + a^T(t_{N-2}, t_{N-1}) \left[ 1 + a^T(t_{N-1}, T) \right] \dots \right] \right] \right],$$

where the random variables  $a^{T}(t_{i-1}, t_i)$  in their case are pairwise independent. It can easily be seen that for a Gaussian term structure model the coefficients  $a^{T}(t_{i-1}, t_i)$  are defined as:

$$a^{T}(t_{i-1}, t_{i}) := \frac{D(t_{0}, t_{i-1})}{D(t_{0}, t_{i})} \cdot \exp\left\{-\frac{1}{2}\int_{t_{i-1}}^{t_{i}} (\sigma_{1}(u) - \sigma(u, T))^{2} + \sigma_{2}^{2}(u)du\right\}$$

$$(4) \qquad \cdot \exp\left\{\frac{1}{2}\int_{t_{0}}^{t_{i}} (\sigma(u, t_{i}) - \sigma(u, T))^{2}du - \frac{1}{2}\int_{t_{0}}^{t_{i-1}} (\sigma(u, t_{i-1}) - \sigma(u, T))^{2}du\right\}$$

$$\cdot \exp\left\{\int_{t_{i-1}}^{t_{i}} \sigma_{2}(u)dW_{2}^{T}(u) + \int_{t_{i-1}}^{t_{i}} (\sigma_{1}(u) - \sigma(u, t_{i}))dW_{1}^{T}(u)\right\}$$

$$\cdot \exp\left\{-\int_{t_{0}}^{t_{i-1}} (\sigma(u, t_{i}) - \sigma(u, t_{i-1}))dW_{1}^{T}(u)\right\},$$

which implies that the stochastic variables  $a^{T}(t_{i-1}, t_{i})$  are not pairwise independent unless

$$\sigma(u, t_i) = \sigma(u, t_{i-1}) \quad \forall u \le t_{i-1} < t_i \implies \sigma(u, t) = 0 \quad \forall u \le t.$$

For this reason the Fast Fourier Transformation cannot be applied to calculate the distribution of the arithmetic average.

Turnbull and Wakeman (1991) suggest to approximate the unknown density  $\rho^{T}$  of the sum of lognormal distributed variables by the following Edgeworth expansion:

(5) 
$$\rho^{T}(x) \approx f(x) + \frac{c_2}{2!} \frac{\partial^2 f(x)}{\partial x^2} - \frac{c_3}{3!} \frac{\partial^3 f(x)}{\partial x^3} + \frac{c_4}{4!} \frac{\partial^4 f(x)}{\partial x^4}$$

where f(x) denotes the lognormal density function, i.e.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_f} \frac{1}{x} \exp\left\{-\frac{(\ln x - \mu_f)^2}{2\sigma_f^2}\right\}$$
  
and  
$$c_2 = \mathcal{K}(2, \rho^T) - \mathcal{K}(2, f),$$
  
$$c_3 = \mathcal{K}(3, \rho^T) - \mathcal{K}(3, f),$$
  
$$c_4 = \mathcal{K}(4, \rho^T) - \mathcal{K}(4, f) + 3c_3^2.$$

 $\mathcal{K}(i, f) = E_f[(X - E_f[X])^i]$  equals the i-th central moment with respect to the lognormal distribution given by f, resp.  $\mathcal{K}(i, \rho^T)$  with respect to the unknown distribution given by  $P^T$ . To calculate these moments, the first four non-central moments of the average A(T) must be computed. The parameters  $\mu_f$  and  $\sigma_f$  are chosen such that the first two non-central moments under both measures are identical. Given the moments and a vanishing error term, the value of the arithmetic Asian option at time  $t_0$  is approximated by:

$$D(t_0, T) \quad \cdot E^T \left[ \max \left\{ A(T) - X, 0 \right\} \right]$$
(6) 
$$\approx \quad D(t_0, T) \quad \cdot \left\{ e^{\mu_f + \sigma_{f/2}^2} N(d) - XN(d - \sigma_f) + \frac{c_2}{2!} f(X) - \frac{c_3}{3!} \frac{\partial f}{\partial x}(X) + \frac{c_4}{4!} \frac{\partial^2 f}{\partial x^2}(X) \right\}$$

with  $d = \frac{\mu_f - \ln(X) + \sigma_f^2}{\sigma_f}$  and N(.) denoting the standard normal distribution. Since the  $a^{T}(t_{i-1}, t_i)$  in (4) are stochastic dependent variables, it is not possible to calculate

Since the  $a^{T}(t_{i-1}, t_i)$  in (4) are stochastic dependent variables, it is not possible to calculate the moments of A(T) as in Turnbull and Wakeman (1991). A generalized but much slower algorithm is given in the Appendix.

Based on the strong relationship between the arithmetic and the geometric average, Vorst (1992) suggests an alternative approximation of the arbitrage price for an Asian option and furthermore derives upper and lower bounds for these prices. The Vorst (1992) approximation and the bounds on the price of the Asian option are given by

$$D(t_0, T) \left( e^{m_G + \frac{1}{2}\sigma_G^2} N(d_1) - XN(d_1 - \sigma_G) \right)$$

$$\leq D(t_0, T) E^T \left[ \max \left\{ A(T) - X, 0 \right\} \right]$$

$$\approx D(t_0, T) \left( e^{m_G + \frac{1}{2}\sigma_G^2} N(d_2) - X'N(d_2 - \sigma_G) \right)$$

$$\leq D(t_0, T) \left( e^{m_G + \frac{1}{2}\sigma_G^2} N(d_1) - XN(d_1 - \sigma_G) + E^T [A(T)] - E^T [G(T)] \right),$$

where

$$\begin{aligned} d_1 &= \frac{m_G - \ln(X) + \sigma_G^2}{\sigma_G}, \quad d_2 &= \frac{m_G - \ln(X') + \sigma_G^2}{\sigma_G}, \\ X' &= X - (E^T[A(T)] - E^T[G(T)]), \\ m_G &= E^T[\ln G(T)] \\ \sigma_G^2 &= V^T[\ln G(T)] \end{aligned} \right\} \Rightarrow E^T[G(T)] = \exp\left\{m_G + \frac{1}{2}\sigma_G^2\right\}. \end{aligned}$$

Thus the Vorst (1992) approximation only involves the computation of the first moment for the arithmetic average and the mean and variance of the logarithmic geometric average. We notice that the approximation is derived by transforming the probability measure of a lognormal distribution with support  $\mathbb{R}^+$  to a lognormal distribution with support  $[E^T[A(T)] - E^T[G(T)], \infty[$ . Since the support of the random variable A(T) is  $\mathbb{R}^+$  the distance  $E^T[A(T)] - E^T[G(T)] > 0$  is important for the approximation error. Furthermore the discounted difference is an upper bound for the approximation error.

## 4.1 Arithmetic and geometric averages under the $P^T$ measure

To derive the Vorst (1992) approximation for the arbitrage price of an Asian option the expectation under the T-forward risk adjusted measure of the arithmetic and geometric averages have to be calculated. Due to the stochastic behaviour of the interest rate, the computation of the values is different from the one Vorst (1992) proposed. Furthermore it will turn out that the behaviour of the expected values depends crucially on the term structure model. Although this is mainly the case if we consider an unrealistic long time to maturity of the Asian option, this is a critical point with respect to the assumption of lognormal bond prices respectively a Gaussian term structure model. On the other hand we have to assume lognormality of bond prices to derive the closed form expressions for the expected values of these averages in a straightforward manner. The following theorems do summarize the results for these averages<sup>4</sup>.

**Theorem 1** Let  $\underline{T}(N) := \{0 = t_0 < t_1 < ... < t_N = T\}$  be a fixed discretization of the time axis and suppose that the time T-forward price dynamics of the underlying asset is given by (2). The expected value of the arithmetic mean under the T-forward risk adjusted measure is given by:

$$E^{T}[A(T)] = \frac{S(t_{0})}{N} \sum_{i=1}^{N} \frac{1}{D(t_{0}, t_{i})} \cdot exp\left\{\int_{t_{0}}^{t_{i}} \left[\sigma_{1}(u) - \sigma(u, t_{i})\right] \left[\sigma(u, T) - \sigma(u, t_{i})\right] du\right\}$$

If moreover the grid size is given by  $\Delta t = t_{i+1} - t_i = \frac{T}{N}$  and the initial term structure is integrable and bounded away from zero we have

$$\lim_{\Delta t \to 0} E^{T}[A(T)] = \frac{S(t_{0})}{T} \int_{t_{0}}^{T} \frac{1}{D(t_{0}, u)} \cdot exp\left\{\int_{t_{0}}^{v} [\sigma_{1}(u) - \sigma(u, v)][\sigma(u, T) - \sigma(u, v)]du\right\} dv.$$

The consequence of a stochastic interest rate implied by Theorem 1 is interesting. Suppose that the interest rate is deterministic, then Theorem 1 implies that

$$E^{T}[A(T)] = \frac{S(t_{0})}{N} \sum_{i=1}^{N} \frac{1}{D(t_{0}, t_{i})} \to \frac{S(t_{0})}{T} \int_{0}^{T} \frac{1}{D(t_{0}, u)} du \quad \text{for} \quad \Delta t \to 0.$$

In the case of a flat initial term structure, i.e.  $D(t_0, t) = \exp\{-rt\}$ , which is usually assumed within the Black-Scholes framework, this implies that the forward value of the expected arithmetic mean is strictly increasing in T with:

$$\lim_{\Delta t \to 0} E^T[A(T)] = \frac{S(t_0)}{T} \int_0^T e^{ru} du = \frac{S(t_0)}{T} \frac{1}{r} [e^{rT} - 1] \to \infty \quad \text{for} \quad T \to \infty.$$

<sup>&</sup>lt;sup>4</sup>For convenience to the reader the proofs are given in the Appendix.

Of course the  $t_0$  value of the expected arithmetic mean is strictly decreasing in T with

$$D(t_0, T)E^T[A(T)] = \frac{S(t_0)}{T}\frac{1}{r}[1 - e^{-rT}] \to 0 \text{ for } T \to \infty.$$

If the interest rate is stochastic, i.e.  $\sigma(t, t') > 0$  the situation is more complicated. Observe first that for a reasonable Gaussian term structure model the price volatility differential  $\sigma(u, T) - \sigma(u, v)$  should be either always positive<sup>5</sup> or negative  $\forall u \leq v \leq T$ . Due to the symmetry of the Brownian motion we therefore assume without loss of generality that  $\sigma(u, T) - \sigma(u, v) \geq 0 \quad \forall u \leq v \leq T$ . Therefore  $E^T[A(T)]$  is strictly increasing in  $\sigma_1(u)$ , and for non positive correlation, i.e.  $\sigma_1(u) \leq 0 \quad \forall u$  we have

$$E^{T}[A(T)] < \frac{S(t_{0})}{T} \int_{0}^{T} \frac{1}{D(t_{0}, u)} du$$

This means that the expected arithmetic mean for a non-stochastic interest rate is an upper bound for  $E^{T}[A(T)]$ . We therefore can expect lower option values due to stochastic interest rates in this situation.

If the correlation is positive, i.e.  $\sigma_1(u) > 0 \quad \forall u$  a sufficient condition for  $E^T[A(T)] > \frac{S(t_0)}{T} \int_0^T \frac{1}{D(t_0,u)} du$  is

$$\int_{t_0}^{v} \left[\sigma_1(u) - \sigma(u, v)\right] \left[\sigma(u, T) - \sigma(u, v)\right] du > 0 \quad \forall v < T.$$

In the case of a Vasicek (1977) model with constant mean reversion  $\alpha > 0$ , i.e  $\sigma(u, v) = \frac{\sigma}{\alpha} (1 - \exp\{-\alpha(v - u)\})$  and  $\sigma_1(u) = \sigma_1$  this is satisfied if  $\forall v < T$ 

$$\begin{array}{rcl} 0 & < & \displaystyle \frac{\sigma}{\alpha} \left( e^{-\alpha v} - e^{-\alpha T} \right) \left[ \displaystyle \frac{\sigma_1}{\alpha} \left( e^{\alpha v} - 1 \right) + \displaystyle \frac{\sigma}{\alpha^2} \left( 1 - \cosh(\alpha v) \right) \right] \\ \Longrightarrow & \displaystyle 2 \displaystyle \frac{\sigma_1}{\sigma} & > & \displaystyle \frac{2}{\alpha} \displaystyle \frac{\cosh(\alpha v) - 1}{e^{\alpha v} - 1} & \rightarrow & v \ \ \mbox{for} \quad \alpha \to 0. \end{array}$$

This indicates higher prices of Asian options due to stochastic interest rates for small time to maturities T. For  $\alpha \to 0$ , i.e. the Ho and Lee (1986) model this condition is satisfied for  $T < 2\frac{\sigma_1}{\sigma}$ . For time to maturities  $T > 2\frac{\sigma_1}{\sigma}$  simulations show that the expected arithmetic mean begins to decrease<sup>6</sup> for a long time period followed by an increase at around 80 years.

To calculate the expected value of the geometric mean we use that under the Gaussian term structure model the geometric mean is lognormally distributed. Therefore

$$E^{T}[G(T)] = \exp\left\{E^{T}[\ln G(T)] + \frac{1}{2} \cdot V^{T}[\ln G(T)]\right\}.$$

**Theorem 2** Suppose that the initial term structure  $D(t_0, \cdot) : [0, T] \to \mathbb{R}_{>0}$  is integrable and bounded away from zero. Let  $\underline{T}(N)$  be a fixed discretization of the time axis and suppose that S(t) is given by (2). The expected value and the variance of the logarithmic geometric mean

 $<sup>{}^{5}</sup>E.g.$  this is the case for the generalized Vasicek (1977) model and the continuous time limit of the Ho and Lee (1986) model.

<sup>&</sup>lt;sup>6</sup>This behaviour is illustrated in Figures 1 to 4.

under the  $P^{T}$  measure are given by

$$\begin{split} E^{T}[lnG(T)] &= lnS(t_{0}) - \frac{1}{N} \sum_{i=1}^{N} lnD(t_{0}, t_{i}) - \frac{1}{2N} \sum_{i=1}^{N} \int_{t_{0}}^{t_{i}} \left[\sigma_{1}^{2}(u) + \sigma_{2}^{2}(u)\right] du \\ &+ \frac{1}{2N} \sum_{i=1}^{N} \int_{t_{0}}^{t_{i}} \left(2\sigma(u, T) \left[\sigma_{1}(u) - \sigma(u, t_{i})\right] + \sigma^{2}(u, t_{i})\right) du \\ \lim_{\Delta t \to 0} E^{T}[lnG(T)] &= lnS(t_{0}) - \frac{1}{T} \int_{t_{0}}^{T} lnD(t_{0}, u) du - \frac{1}{2T} \int_{t_{0}}^{T} \int_{t_{0}}^{v} \left[\sigma_{1}^{2}(u) + \sigma_{2}^{2}(u)\right] du dv \\ &+ \frac{1}{2T} \int_{t_{0}}^{T} \int_{t_{0}}^{v} \left(2\sigma(u, T) \left[\sigma_{1}(u) - \sigma(u, v)\right] + \sigma^{2}(u, v)\right) du dv \end{split}$$

If the interest rate is deterministic, the volatility functions  $\sigma_{1/2}(.)$  are constant and the initial term structure is flat, i.e.  $D(t_0, t) = \exp\{-rt\}$  we get

$$\lim_{\Delta t \to 0} E^T [\ln G(T)] = \ln S(t_0) + \frac{1}{2}rT - \frac{1}{4} \left[\sigma_1^2 + \sigma_2^2\right] T.$$

Depending on the size of the volatility of the underlying asset,  $\sqrt{\sigma_1^2 + \sigma_2^2}$ , this either converges to plus or minus infinity as the time to maturity T approaches infinity.

Since the sign of  $E^{T}[\ln G(T)]$  for  $T \to \infty$  is determined by the last integral there is a strong tendency to reverse the above result in the case of a stochastic interest rate, i. e.  $\sigma(u, v) > 0$ . To illustrate this consider the Vasicek (1977) model with constant parameters. Then

$$\lim_{\Delta t \to 0} E^{T}[\ln G(T)] = \ln S(t_{0}) - \frac{1}{T} \int_{0}^{T} \ln D(t_{0}, u) du - \frac{1}{4} \left(\sigma_{1}^{2} + \sigma_{2}^{2}\right) T + \sigma_{1} \sigma \frac{\frac{1}{2} \alpha^{2} T^{2} + (\alpha T + 1) e^{-\alpha T} - 1}{\alpha^{3} T} + \sigma^{2} \frac{-\alpha^{2} T^{2} - 4\alpha T e^{-\alpha T} + \frac{3}{2} (1 - e^{-2\alpha T}) + aT}{4T \alpha^{4}}.$$

For a flat initial term structure this can be simplified to:

$$\lim_{\Delta t \to 0} E^{T}[\ln G(T)] = \ln S(t_0) + \frac{1}{2} \left( r - \frac{1}{2} \left( \sigma_1^2 + \sigma_2^2 \right) + \frac{\sigma_1 \sigma}{\alpha} - \frac{\sigma^2}{2\alpha^2} \right) T + \frac{\sigma^2}{4\alpha^3} + g(T)$$

where  $\lim_{T\to\infty} g(T) = 0$ . Therefore the Vasicek model approaches the same limit as in the determistic interest rate case for sufficiently large mean reversion coefficient  $\alpha$ . If instead the mean reversion coefficient  $\alpha$  is small, i.e. in the limit we get the Ho and Lee (1986) model, then

$$\lim_{\alpha \to 0} \lim_{\Delta t \to 0} E^{T}[\ln G(T)] = \ln S(t_{0}) - \frac{1}{T} \int_{0}^{T} \ln D(t_{0}, u) du - \frac{1}{4} \left(\sigma_{1}^{2} + \sigma_{2}^{2}\right) T + \frac{1}{3} \sigma_{1} \sigma T^{2} - \frac{1}{12} \sigma^{2} T^{3}$$
  
$$\to -\infty \quad \text{for} \quad T \to \infty,$$

as long as  $D(t_0, t) \ge \exp\{-kt^{\delta}\}$   $\forall t \text{ for some constants } k > 0 \text{ and } \delta < 3.$ 

**Theorem 3** Under the assumptions of Theorem 2 we have

$$\begin{split} V^{T}[lnG(T)] &= \frac{1}{N^{2}} \sum_{i=0}^{N-1} \left[ \int_{t_{i}}^{t_{i+1}} \left( (N-i) \cdot \sigma_{2}(u) \right)^{2} + \left( (N-i)\sigma_{1}(u) - \sum_{j=i+1}^{N} \sigma(u,t_{j}) \right)^{2} du \right] \\ \lim_{\Delta t \to 0} V^{T}[lnG(T)] &= \frac{1}{T^{2}} \int_{t_{0}}^{T} \left( T-u \right)^{2} \cdot \left( \sigma_{1}^{2}(u) + \sigma_{2}^{2}(u) \right) du \\ &- \frac{2}{T^{2}} \int_{t_{0}}^{T} \left[ \int_{u}^{T} (T-u) \cdot \sigma_{1}(u) \cdot \sigma(u,v) dv \right] du + \frac{1}{T^{2}} \int_{t_{0}}^{T} \left[ \int_{u}^{T} \sigma(u,v) dv \right]^{2} du \end{split}$$

Consider again the Vasicek (1977) model and assume that  $\alpha > 0$  and  $\sigma_{1/2}$  are constant. Solving in this situation the integral for the variance yields

$$\lim_{\Delta t \to 0} V^{T}[\ln G(T)] = \frac{1}{3} (\sigma_{1}^{2} + \sigma_{2}^{2})T - \sigma_{1}\sigma \cdot \left(\frac{2\alpha^{3}T^{3} - 3\alpha^{2}T^{2} - 6(\alpha T + 1)e^{-\alpha T} + 6}{3\alpha^{4}T^{2}}\right) + \sigma^{2} \cdot \left(\frac{2\alpha^{3}T^{3} - 12\alpha Te^{-\alpha T} - 3e^{-2\alpha T} + 6\alpha T(1 - \alpha T) + 3}{6\alpha^{5}T^{2}}\right) + \frac{1}{3} (\sigma_{1}^{2} + \sigma_{2}^{2})T - \frac{1}{4}\sigma_{1}\sigma T^{2} + \frac{1}{20}\sigma^{2}T^{3} \quad \text{for} \quad \alpha \to 0.$$

To clarify the impact of the stochastic interest rate consider as a border case a flat initial term structure and deterministic interest rate, which imply

$$\lim_{\Delta t \to 0} E^T[G(T)] = S(t_0) \cdot \exp\left\{\frac{1}{2}\left(r - \frac{1}{6}(\sigma_1^2 + \sigma_2^2)\right)T\right\}$$

and depending on the sign of  $r - \frac{1}{6}(\sigma_1^2 + \sigma_2^2)$  this either converges to zero or plus infinity for  $T \to +\infty$ . If instead the interest rate is stochastic, i.e.  $\sigma(t, t') > 0$  the convergence behaviour may be different. Consider once again the Vasicek model with constant  $\alpha > 0$  and  $\sigma_{1/2}$  and a flat initial term structure, then

$$\lim_{\Delta t \to 0} E^{T}[G(T)] = S(t_0) \cdot \exp\left\{\frac{1}{2}\left(r - \frac{1}{6}(\sigma_1^2 + \sigma_2^2) + \frac{\sigma_1\sigma}{3\alpha} - \frac{\sigma^2}{6\alpha^2}\right)T + \frac{\sigma_1\sigma}{2\alpha^2} - \frac{\sigma^2}{4\alpha^3} + g(T)\right\},$$

where  $\lim_{T\to\infty} g(T) = 0$ . If the mean reversion coefficient  $\alpha$  is large then the behaviour for  $T \to \infty$  of the Vasicek model and the determistic interest case is the same. If instead  $\alpha$  is small, then the expected geometric average under the T-forward risk adjusted measure converges to zero. As the extreme case consider the Ho and Lee model, i.e.

$$\lim_{\alpha \to 0} \lim_{\Delta t \to 0} E^T[G(T)] = S(t_0) \cdot \exp\left\{-\frac{1}{T} \int_{t_0}^T D(t_0, u) du - \frac{1}{12} (\sigma_1^2 + \sigma_2^2) T + \frac{5}{24} \sigma_1 \sigma T^2 - \frac{7}{120} \sigma^2 T^3\right\}$$

which converges to zero for  $T \to +\infty$  as long as  $D(t_0, t) \ge \exp\{-kt^{\delta}\} \ \forall t$  for k > 0 and  $\delta < 3$ .

Suppose that the total volatility of the underlying asset  $\sigma_S$  is fixed, i.e.  $\sigma_S = \sqrt{\sigma_1^2 + \sigma_2^2}$  is assumed to be constant. In this situation the expected value of the geometric average is a strictly increasing function in  $\sigma_1$ . In other words, fixing  $\sigma_S$  the expected geometric average under the *T*-forward risk adjusted measure increases in the instantaneous correlation.

To summarize our results at this point, Figures 1 to 4 do show some of the effects. In these figures we have chosen a flat initial term structure with  $D(t_0, t) = (1.06)^{-t}$ . Furthermore the volatility of the underlying asset is equal to 25 %, i.e.

$$\sigma_S^2 dt := V[dS(t)|S(t)] = \left(\sigma_1^2 + \sigma_2^2\right) dt = 0.25^2 dt.$$



Figure 1: Expected arithmetic averages for  $T \leq 3$  years, 120 realizations of the underlying asset per year with  $S(t_0) = 100$ ,  $\sigma_S = 25\%$  and  $\sigma = 10\%$ .



Figure 2: Expected geometric averages for  $T \leq 3$  years, 120 realizations of the underlying asset per year with  $S(t_0) = 100, \sigma_S = 25\%$  and  $\sigma = 10\%$ .

As model of the term structure we concentrate in Section 5 our price simulation of the Asian option on the continuous time limit of the Ho and Lee (1986) model. With respect to Figures 1 to 4 this model is the extreme case of the Vasicek (1977) model. We regard the Ho and Lee model as the most sensitive case. Therefore we set in Section 5 the price volatility of the zero coupon bonds equal to  $\sigma(u, v) = \sigma \cdot (v - u)$  with  $\sigma = 0.1$ . Furthermore note that by  $\sigma_1$  and  $\sigma_2$  the instantaneous correlation between the underlying asset and the term structure is defined by

(8) 
$$\rho := \frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} = \frac{\sigma_1}{\sigma_S}.$$



Figure 3: Expected arithmetic and geometric averages for  $T \leq 25$  years, 120 realizations of the underlying asset per year with  $S(t_0) = 100$ ,  $\sigma_S = 25\%$  and  $\sigma = 10\%$ .



Figure 4: Expected arithmetic and geometric averages for  $T \leq 4$  years, 120 realizations of the underlying asset per year with  $S(t_0) = 100$ ,  $\sigma_S = 25\%$  and  $\rho = 0$ .

It means that we can parametrise  $\sigma_1$  and  $\sigma_2$  in terms of the correlation such that the (total) volatility of the underlying asset is always equal to  $\sigma_S = 25\%$ :

(9)  

$$\begin{aligned}
\sigma_1 : [-1,+1] &\to [-\sigma_S, \sigma_S] \\
\rho &\mapsto \sigma_1(\rho) := \rho \sigma_S \\
\text{and} &\rho &\mapsto \sigma_2(\rho) := \sqrt{(1-\rho^2)} \sigma_S.
\end{aligned}$$

#### 5 Simulation results for Asian options

In this section we compare the different approximations proposed by Turnbull and Wakeman (1991) and Vorst (1992) for the pricing of Asian options with the results obtained by a Monte Carlo simulation. The starting point for the Monte Carlo simulation is the formulation of the asset price dynamics as in (4). In the case of the Ho and Lee (1986) model and constant volatility functions  $\sigma_{1/2}$  this can be reformulated to:

$$S(t_{i}) = S(t_{i-1}) \cdot \frac{D(t_{0}, t_{i-1})}{D(t_{0}, t_{i})} \cdot \exp\left\{-\frac{1}{3}\sigma^{2}\left[(t_{i-1} - T)^{2}t_{i-1} - (t_{i} - T)^{2}t_{i}\right]\right\}$$

$$(10) \quad \exp\left\{-\frac{1}{2}\left[\sigma_{1}^{2} + \sigma_{2}^{2}\right](t_{i} - t_{i-1}) - \frac{1}{2}\left[\sigma_{1}\sigma - \frac{1}{3}\sigma^{2}T\right]\left[(T - t_{i})^{2} - (T - t_{i-1})^{2}\right]\right\}$$

$$\cdot \exp\left\{\sigma_{1}\left(W_{1}^{T}(t_{i}) - W_{1}^{T}(t_{i-1})\right) + \sigma_{2}\left(W_{2}^{T}(t_{i}) - W_{2}^{T}(t_{i-1})\right)\right\}$$

$$\cdot \exp\left\{-\sigma\left[t_{i}W_{1}^{T}(t_{i}) - t_{i-1}W_{1}^{T}(t_{i-1}) - \int_{t_{i-1}}^{t_{i}} u \ dW_{1}^{T}(u)\right]\right\}$$

To simulate the last part of equation (10) we notice that

(11) 
$$\left[t_i W_1^T(t_i) - t_{i-1} W_1^T(t_{i-1}) - \int_{t_{i-1}}^{t_i} u \ dW_1^T(u)\right] = \int_{t_{i-1}}^{t_i} W_1^T(u) \ du$$

which is a normal distributed variable.

For the below simulations we have chosen  $\sigma = 10\%$ ,  $\sigma_S = 25\%$  and 120 time periods per year, i.e.  $\Delta t = 120^{-1}$ . Furthermore we have chosen four different maturity dates corresponding to T = 0.5, 1, and 3 years.

The approximation of the Asian option by Turnbull and Wakeman (1991) involves the computation of the non-central moments of the arithmetic mean up to order four. These moments can be calculated using the algorithms proposed in the Appendix. On the other hand we could estimate them by Monte Carlo simulation. Table 1 shows some results obtained by the algorithms and the Monte Carlo simulation. The Monte Carlo simulation leads to a reasonable approximation of the first and second moment and therefore also of the variance. Although the approximation of the higher moments is not as good, the skewness and the leptokursis of the unknown distribution are approximated quite satisfactorily. If not otherwise specified we use 100.000 paths and the antithetic technique<sup>7</sup> for the simulation, a flat initial term structure with  $D(t_0, t) = (1.06)^{-t}$ , and the initial value of the asset  $S(t_0) = 100$ .

In line with Theorems 1 to 3 Table 1 shows the increase of the expected arithmetic and geometric average as a function of the instantaneous correlation. Beyond this we see that the variance of the arithmetic and geometric average decreases as a function in  $\rho$ . Therefore we have two opposite effects which do influence the pricing of Asian options. The decrease of the variance of the geometric average is for the chosen parameter constellation a direct consequence of Theorems 2 and 3. However we should mention that there are parameter values for  $\sigma_S$ ,  $\sigma$  and T such that the variance is an increasing function in  $\rho$ . This is typically the situation if the

<sup>&</sup>lt;sup>7</sup>This implies in total 200.000 paths

time to maturity is extremely long. For the arithmetic average these findings are based on the implementation of the numerical procedure but so far no analytical results can be given.



Figure 5: Densities of the arithmetic and geometric average with  $\sigma_S = 0.25$ , Ho-Lee term structure model with  $\sigma = 0.1$ ,  $\rho = -0.25$ , T = 3,  $D(t_0, t) = 1.06^{-t}$  and T = 3.

Both the Turnbull and Wakeman (1991) and the Vorst (1992) approximation of the Asian option can be interpreted as an approximation of the distribution resp. probability density function of the arithmetic mean of lognormal random variables. The approximation of Turnbull and Wakeman (1991) is given by (6) whereas the one used by Vorst (1992) is given by pricing formula (7). Since we price under the T-forward risk adjusted measure we can compare these approximations with the density function obtained by the Monte Carlo simulation. Note, that by multiplying with  $D(t_0, T)$  these functions do represent the implied state prices underlying the different numerical approximations. The influence of the correlation, which already can be seen in Table 1, seems to be quite important for the Turnbull and Wakeman (1991) approximation, as indicated by Figure 5. Furthermore the Vorst (1992) approximation seems to be better than the Turnbull and Wakeman (1991) approximation even if we do neglect the correlation term; but nevertheless there is an underestimation of lower and an overestimation of higher realizations relative to the Monte Carlo simulation.

Finally we can consider the pricing of Asian options. In addition to the antithetic technique we use the arbitrage price of a geometric average option as a control variate. Thus the Monte Carlo value for the Asian option is obtained by:

$$\hat{c}(T,X) = \frac{D(t_0,T)}{2M} \cdot \sum_{m=1}^{2M} \left[ \left[ \frac{1}{N \cdot T} \sum_{i=1}^{N \cdot T} S(t_i) - X \right]^+ - \left[ \sqrt[N \cdot T]{\prod_{i=1}^{N \cdot T} S(t_i)} - X \right]^+ \right] + g(T,X)$$

Corr(A,G)	79997		79997		79997		79997		79997		0.9993		0.9994		0.9994		0.9994		0.9994		0.9963		0.9966		0.9967		0.9969		0.9972	
$\sqrt{V^T[G]}$	10.7204	10.7133	10.6240	10.5977	10.5488	10.5199	10.4258	10.4415	10.3332	10.3226	15.6971	15.6633	15.3054	15.3463	15.1780	15.1310	14.9301	14.9122	14.6593	14.5772	32.1324	32.1776	30.6553	30.7818	29.7529	29.7982	28.6450	28.7667	26.9905	27.1176
$E^{T}\left[G^{2}\right]$	10354.0559	10353.7410	10356.3759	10355.2762	10357.5047	10356.2998	10356.6744	10357.3235	10359.3047	10358.8593	10703.1436	10701.0593	10705.1520	10707.5804	10714.6931	10711.9300	10717.3313	10716.2814	10727.5223	10722.8117	11961.6032	11967.0050	12019.4092	12033.9429	12073.7968	12078.7760	12110.2172	12123.7762	12178.3277	12191.5910
$E^{T}[G]$	101.1886	101.1878	101.2102	101.2075	101.2237	101.2207	101.2323	101.2339	101.2548	101.2537	102.2582	102.2532	102.3274	102.3331	102.3930	102.3864	102.4423	102.4398	102.5311	102.5198	104.5424	104.5543	105.2600	105.2921	105.7760	105.7868	106.2529	106.2838	107.0039	107.0338
$E^T \left[ A^4 \right]$	113436644.5	113425589.2	113380473.6	113327237.6	113312319.4	113261766.2	113169044.3	113196371.9	113115282.4	113098424.7	129243019.6	129135560.0	128539906.2	128678749.3	128535212.2	128376009.7	128132581.3	128074701.1	127872100.2	127625402.9	241959657.1	243068306.1	231532187.9	233337124.7	226733369.0	227225615.2	220558645.4	221395137.8	211977116.2	213142347.5
$E^{T} \left[ A^{3} \right]$	1080597.676	1080525.126	1080520.829	1080229.561	1080299.609	1080032.728	1079689.435	1079836.062	1079641.698	1079541.379	1168869.294	1168307.595	1166322.121	1167008.761	1166959.666	1166146.536	1165587.336	1165287.232	1165282.124	1164003.733	1652596.776	1656283.380	1628178.471	1634569.194	1618837.517	1620626.118	1602470.241	1607094.550	1582494.072	1587542.046
$\sqrt{V^T [A]}$	10.8028	10.7962	10.7050	10.6786	10.6283	10.5994	10.5039	10.5197	10.4090	10.3989	15.9543	15.9197	15.5474	15.5913	15.4180	15.3684	15.1611	15.1420	14.8807	14.7958	34.1924	34.2764	32.5037	32.6708	31.5002	31.5503	30.2406	30.3843	28.3842	28.5377
$E^{T}$ $\begin{bmatrix} A^{2} \end{bmatrix}$	10411.8468	10411.4805	10413.4785	10412.1707	10413.7378	10412.6312	10412.4734	10413.0920	10414.2529	10413.7838	10833.0688	10830.6568	10830.7307	10833.5306	10838.9151	10835.4523	10838.6593	10837.3788	10845.7130	10840.2773	12623.6395	12635.4959	12636.0680	12657.1062	12666.0588	12672.1006	12669.3503	12687.5629	12692.7980	12711.6293
$E^{T}[A]$	101.4650	101.4639	101.4834	101.4797	101.4927	101.4903	101.4995	101.5009	101.5180	101.5167	102.8520	102.8456	102.9029	102.9099	102.9621	102.9527	102.9990	102.9956	103.0741	103.0600	107.0258	107.0543	107.6084	107.6556	108.0453	108.0587	108.4198	108.4636	109.0281	109.0744
Method	simul.	exact																												
θ	-0.25		-0.10		0		0.10		0.25		-0.25		-0.10		0		0.10		0.25		-0.25		-0.1		0		0.1		0.25	
Maturity	0.5										1										33									

Table 1: Exact and simulated moments of the arithmetic average  $\mathrm{A}(\mathrm{T})$  and the geometric average  $\mathrm{G}(\mathrm{T})$ 

where the arbitrage price of the geometric average option is equal to

$$g(T,X) = D(t_0,T) \cdot \exp\left\{m_G(T) + \frac{1}{2}\sigma_G^2(T)\right\} N(d) - XN(d - \sigma_G(T))$$

(12) with 
$$m_G(T) = E^T [\ln G(T)]$$
  
 $\sigma_G^2(T) = V^T [\ln G(T)]$   
 $d = \frac{-\ln X + m_G(T) + \sigma_G^2(T)}{\sigma_G(T)}$ .

As before we choose M = 100.000, N = 120 and  $T \in \{0.5, 1, 3\}$ . Table 2 to 4 do summarize the results for some values of the exercise price X where the initial asset value  $S(t_0)$  is equal to 100.

The pricing of the Asian option is sensitive to the instantaneous correlation coefficient  $\rho$ . The arbitrage price of an Asian option obtained by the Vorst (1992) formula is decreasing in  $\rho$ . Define  $\rho$ (Vorst), as the implied correlation coefficient such that the Vorst (1992) solution equals the simulated value of the Monte Carlo simulation. As Tables 2 to 4 in the Appendix show, this implied correlation is not only substantially different for out-of-the-money options, but also for-in-the-money options from the one used by the Monte Carlo simulation. Furthermore we can conclude that for the out-of-the-money options the Turnbull and Wakeman (1991) approximation gives prices in excess of the other methods independently of the correlation coefficient. For a high correlation and out-of-the-money options the three methods give approximately equal prices. For other correlations the simulated prices of out-of-the-money options are between those obtained by the two approximation methods. Looking at deep-in-the-money options we furthermore observe that the Monte Carlo simulation leads to the highest prices. These conclusions are also obvious looking at the numerical results in Table 4. Taking e.g.  $\rho = -0.5$  and the exercise price equal to 115 the prices obtained by applying Turnbull-Wakeman, Vorst and the simulation are 9.31, 8.25 and 8.86 respectively. These differences are of a nonnegligible size. In general the Turnbull-Wakeman prices seem to be better supported by the simulations than the prices derived by the Vorst approximation. The same conclusion can be reached for a time to maturity of 2 years whereas the differences between the different methods are nonessential for smaller time to maturities.

The three last columns in Tables 2 to 4 represent the standard deviations of the simulated arithmetic Asian options and the geometric average options. Applying the control variate technique for the Asian options, the standard deviation  $\sigma_c(Asian)$  is on average equal to  $0.1 \cdot \sigma(Asian)$  where  $\sigma(Asian)$  refers to the standard deviation applying only the antithetic technique. These standard deviations are small meaning that we can have confidence in our pricing results.

To elaborate further on the comparison between the methods we turn our attention to Figures 6 to 8. Figures 6 and 7 illustrate the same situation but with exchanged x- and y-axis. Taking the lower bound derived by Vorst we consider the difference between the price approximations to this lower bound. For the exercise prices considered the Vorst approximation leads to prices which are lower than those obtained from the Monte Carlo simulation. The price surface for the Turnbull-Wakeman approximation crosses both of the other surfaces and is dominating in roughly half the area corresponding to the out-of-the-money options.

Finally Figure 8 shows the ratio of the simulated prices to the approximated prices measured



Figure 6: Difference between price approximation and the lower bound for an Asian option with 3 years to maturity, 120 realizations of the underlying asset per year,  $\sigma_S = 25\%$  and Ho-Lee term structure model with  $\sigma = 10\%$ .

in percentage. For in-the-money options the ratio between the simulated and the Turnbull-Wakeman prices is decreasing in  $\rho$ , whereas the opposite is shown in the case for out-of-the-money options. For out-of-the-money options the Vorst approximation is clearly dominated by the Turnbull-Wakeman approximation. Observe that major differences in the approximations appear for out-of-the-money option.

## 6 Conclusion

Taking expectation under the T-forward risk adjusted measure the behaviour of the expected arithmetic and geometric averages is strongly influenced by the stochastic model of interest rates. In particular for the Ho and Lee (1986) model we observe a discontinuity of the expected geometric mean. Under the regime of stochastic interest rates the expected geometric average converges, independent of the instantaneous correlation, towards zero, whereas in the deterministic case it approaches plus infinity as the time to maturity increases. In contrast to this the Vasicek (1977) model with a sufficiently large degree of mean reversion does not generate this unexpected behaviour. On the other hand the behaviour of the expected arithmetic mean depends on the instantaneous correlation. If the correlation is non positive the expected arithmetic mean under stochastic interest rates is bounded from below by the expected arithmetic mean under deterministic interest rates. In the case of positive instantaneous correlation between the term structure of interest rates and the underlying asset and for short time to maturities the expected arithmetic average is higher than compared to the situation under the deterministic interest rates. The mean reversion in the the Vasicek model once again has a positive effect on the behaviour of the expected arithmetic mean. In contrast to this, without mean reversion the expected arithmetic mean decreases for a large range of maturities.

Looking at the literature on Asian option pricing we considered the approximation methods



Figure 7: Difference between price approximation and the lower bound for an Asian option with 3 years to maturity, 120 realizations of the underlying asset per year,  $\sigma_S = 25\%$  and and Ho-Lee term structure model with  $\sigma = 10\%$ .



MC Price / Approximation of Asian Option Price: 3 years to maturity

Figure 8: Monte Carlo values as percentage of the respectively numerical approximation for an Asian option with 3 years to maturity, 120 realizations of the underlying asset per year,  $\sigma_S = 25\%$  and and Ho-Lee term structure model with  $\sigma = 10\%$ .

developed by Turnbull and Wakeman (1991) and Vorst (1992). We generalized these techniques to include the case of a Gaussian term structure model. These generalizations are only valid for a Gaussian model, since we have to preserve the lognormal structure of the underlying asset under the appropriate forward risk adjusted measure. From a pure theoretical point of view the Vorst approximation shows up a more reasonable behaviour than the Turnbull-Wakeman approximation. This is based on the strange behaviour of the correction term used by the Turnbull-Wakeman method.

To compare the pricing results, we implemented extensive Monte Carlo simulations. To

reduce the variance we used the antithetic and control variate technique where the geometric average option was used as the control variate. Our simulation gives for the pricing as well as for the approximation of the unknown probability density of the arithmetic mean quite reasonable fits. Comparing the probability densities implied by the Monte Carlo simulation to those implied by the two analytical approximations, we can conclude that the Turnbull-Wakeman method produces a completely unrealistic behaviour if we consider times to maturities extending 2 years. Furthermore this behaviour, which is due to the high order Edgeworth expansion goes from bad to worse for negative instantaneous correlation between the underlying asset and the bond market. In this respect the Vorst approximation behaves much nicer, but nevertheless indicates a serious underpricing.

As a general finding, with respect to the pricing of Asian options, we conclude that the instantaneous correlation of the underlying asset and the term structure of interest rates is the principal parameter of importance. The arbitrage price seems to be negatively related with the correlation coefficient. Considering the price of an Asian option as a function of the instantaneous correlation we conclude, that the increase in the expected value of the arithmetic average, as proven by Theorem 1, is completely compensated by the decrease of the variance of the arithmetic average. Our simulations indicate a clear underpricing by the Vorst method. The conclusion for the Turnbull-Wakeman approximation is less strict and depends on both the exercise price and the instantaneous correlation. For deep-out-of-the money options and independent of the instantaneous correlation the Turnbull-Wakeman approximation implies higher Asian option prices than those produced by our Monte Carlo simulation. Whereas for deep in-the-money options the opposite is true and the Vorst solution is even higher than the Turnbull-Wakeman approximation in this situation if we consider negative correlation. For out-of-the money options and high positive correlation all three methods are close to each other, whereas for a negative correlation and out-of-the-money options the results differ substantially.

The results for the pricing of Asian options under stochastic interest rates do depend on the time to maturity. Our calculations indicate that the influence of a stochastic interest rate is less pronounced for time to maturities smaller than 1 year.

#### Appendix

### Proof of Theorem 1

Due to the stochastic independence of  $W_1^T$  and  $W_2^T$  we know that:

$$E^{T}[S(t_{i})] = \frac{S(t_{0})}{D(t_{0}, t_{i})} \cdot \exp\left\{-\frac{1}{2}\int_{t_{0}}^{t_{i}} (\sigma_{1}(u) - \sigma(u, T))^{2} du\right\}$$
$$\cdot \exp\left\{\frac{1}{2}\int_{t_{0}}^{t_{i}} (\sigma(u, t_{i}) - \sigma(u, T))^{2} du + \frac{1}{2}\int_{t_{0}}^{t_{i}} (\sigma_{1}(u) - \sigma(u, t_{i}))^{2} du\right\}$$
$$= \frac{S(t_{0})}{D(t_{0}, t_{i})} \cdot \exp\left\{\int_{t_{0}}^{t_{i}} [\sigma_{1}(u) - \sigma(u, t_{i})] [\sigma(u, T) - \sigma(u, t_{i})] du\right\}$$

## **Proof of Theorem 2**

The definition of the geometric mean implies that

$$E^{T}[\ln G(T)] = \frac{1}{N} \sum_{i=1}^{N} E^{T}[\ln(S(t_{i}))]$$
  
where  $E^{T}[\ln(S(t_{i}))] = \ln(S(t_{0})) - \ln(D(t_{0}, t_{i}))$   
 $-\frac{1}{2} \int_{t_{0}}^{t_{i}} ((\sigma_{1}(u) - \sigma(u, T))^{2} + \sigma_{2}^{2}(u)) du + \frac{1}{2} \int_{t_{0}}^{t_{i}} (\sigma(u, t_{i}) - \sigma(u, T))^{2} du$   
 $= \ln(S(t_{0})) - \ln(D(t_{0}, t_{i})) - \frac{1}{2} \int_{t_{0}}^{t_{i}} (\sigma_{1}^{2}(u) + \sigma_{2}^{2}(u)) du$   
 $+\frac{1}{2} \int_{t_{0}}^{t_{i}} (2\sigma(u, T)[\sigma_{1}(u) - \sigma(u, t_{i})] + \sigma^{2}(u, t_{i})) du$ 

### **Proof of Theorem 3**

We have to calculate the variance of a sum of correlated stochastic integrals under the T-forward risk ajusted measure, i.e.

$$V^{T}\left[\frac{1}{N+1}\left(\sum_{i=0}^{N}\int_{t_{0}}^{t_{i}}(\sigma_{1}(u)-\sigma(u,t_{i}))dW_{1}^{T}(u)+\int_{t_{0}}^{t_{i}}\sigma_{2}(u)dW_{2}^{T}(u)\right)\right]$$

Since  $W_1^T$  and  $W_2^T$  are stochastically independent we can consider the variance of both sums of stochastic integrals separately. For the stochastic integrals with respect to  $W_2^T$  we immediately

obtain:

$$\begin{split} V^T \left[ \frac{1}{N} \sum_{i=1}^N \int_{t_0}^{t_i} \sigma_2(u) dW_2^T(u) \right] &= \frac{1}{N^2} V^T \left[ \sum_{i=0}^{N-1} (N-i) \int_{t_i}^{t_{i+1}} \sigma_2(u) dW_2^T(u) \right] \\ &= \frac{1}{N^2} \sum_{i=0}^{N-1} (N-i)^2 \int_{t_i}^{t_{i+1}} \sigma_2^2(u) du \end{split}$$

The same idea can be applied to the stochastic integrals with respect to  $W_1^T$ .

$$\begin{split} V^{T}\left[\frac{1}{N}\sum_{i=1}^{N}\int_{t_{0}}^{t_{i}}(\sigma_{1}(u)-\sigma(u,t_{i}))dW_{1}^{T}(u)\right] \\ &= \frac{1}{N^{2}}V^{T}\left[\sum_{i=0}^{N-1}\int_{t_{i}}^{t_{i+1}}(N-i)\sigma_{1}(u)dW_{1}^{T}(u)-\sum_{i=1}^{N}\int_{t_{0}}^{t_{i}}\sigma(u,t_{i})dW_{1}^{T}(u)\right] \\ &= \frac{1}{N^{2}}V^{T}\left[\sum_{i=0}^{N-1}\int_{t_{i}}^{t_{i+1}}(N-i)\sigma_{1}(u)dW_{1}^{T}(u)-\sum_{i=0}^{N-1}\int_{t_{i}}^{t_{i+1}}\sum_{j=i+1}^{N}\sigma(u,t_{j})dW_{1}^{T}(u)\right] \\ &= \frac{1}{N^{2}}\sum_{i=0}^{N-1}\int_{t_{i}}^{t_{i+1}}\left((N-i)\sigma_{1}(u)-\sum_{j=i+1}^{N}\sigma(u,t_{j})\right)^{2}du \end{split}$$

### Recursive algorithms for the non central moments

From the previous discussion we know that under the T-forward measure  $P^T$  the value of the underlying asset S(t) is determined by equation (2). Consider the stochastic part of this equation separately and for simplicity of the notation define:

$$\begin{split} Y_i &:= \int_{t_0}^{t_i} \left( \sigma_1(u) - \sigma(u, t_i) \right) dW_1^T(u) \quad , \\ Z_i &:= \int_{t_0}^{t_i} \sigma_2(u) \ dW_2^T(u) \quad , \\ \nu_{ij} &:= \exp\left\{ \int_{t_0}^{t_i} \left[ \sigma_1(u) - \sigma(u, t_i) \right] \left[ \sigma(u, t_j) - \sigma(u, t_i) \right] du \right\} \quad . \end{split}$$

Due to the underlying assumptions Y and Z are related in the following way:

- a)  $Y_i$  and  $Z_i$  are stochastic independent, and  $E^T[Y_i] = E^T[Z_i] = 0$ .
- b) For  $i \le j$ :  $E^T[Z_i, Z_j] = E^T[Z_i^2]$ .

c) For  $i \leq j$ :

$$\begin{split} & E^{T}[Y_{i}, Y_{j}] \\ = & E^{T}\left[\left(\int_{t_{0}}^{t_{i}} (\sigma_{1}(u) - \sigma(u, t_{i})dW_{1}^{T}(u)\right) \left(\int_{t_{0}}^{t_{j}} (\sigma_{1}(u) - \sigma(u, t_{i})) - (\sigma(u, t_{j}) - \sigma(u, t_{i}))dW_{1}^{T}(t_{j})\right)\right] \\ = & E^{T}\left[Y_{i}^{2}\right] - E^{T}\left[\left(\int_{t_{0}}^{t_{i}} (\sigma_{1}(u) - \sigma(u, t_{i})dW_{1}^{T}(u)\right) \left(\int_{t_{0}}^{t_{j}} (\sigma(u, t_{j}) - \sigma(u, t_{i}))dW_{1}^{T}(t_{j})\right)\right] \\ = & E^{T}\left[Y_{i}^{2}\right] - \int_{t_{0}}^{t_{i}} [\sigma_{1}(u) - \sigma(u, t_{i})][\sigma(u, t_{j}) - \sigma(u, t_{i})]du = -E^{T}\left[Y_{i}^{2}\right] - \ln\nu_{ij}\right]. \end{split}$$

Finally define:

$$\sigma_i := \exp\left\{\frac{1}{2}V[Z_i + Y_i]\right\}$$

$$d_i := \frac{S_{t_0}}{D(t_0, t_i)} \cdot \exp\left\{\int_{t_0}^{t_i} [\sigma_1(u) - \sigma(u, t_i)] [\sigma(u, T) - \sigma(u, t_i)] du\right\}$$

$$\implies S(t_i) = d_i \cdot \sigma_i^{-1} \cdot \exp\left\{Y_i + Z_i\right\}$$

## Proposition

Under the assumptions on the process of the underlying asset  $S(t_i) = S_i$  we have  $\forall \ 0 \le i \le j \le k \le l < N$  and  $\forall \ \alpha, \gamma, \eta, \theta \in \mathbb{N}$ :

$$\begin{split} E^{T}[S_{i}^{\alpha}] &= d_{i}^{\alpha} \cdot \sigma_{i}^{\alpha(\alpha-1)} \\ E^{T}[S_{i}^{\alpha}S_{j}^{\gamma}] &= d_{i}^{\alpha} \cdot d_{j}^{\gamma} \cdot \sigma_{j}^{\gamma(\gamma-1)} \cdot \sigma_{i}^{\alpha(\alpha+2\gamma-1)} \cdot \nu_{ij}^{\alpha\gamma} \\ E^{T}[S_{i}^{\alpha}S_{j}^{\gamma}S_{k}^{\eta}] &= d_{i}^{\alpha} \cdot d_{j}^{\gamma} \cdot d_{k}^{\eta} \cdot \sigma_{i}^{\alpha(\alpha+2\gamma+2\eta-1)} \cdot \sigma_{j}^{\gamma(\gamma+2\eta-1)} \cdot \sigma_{k}^{\eta(\eta-1)} \cdot \nu_{ij}^{\alpha\gamma} \cdot \nu_{ik}^{\alpha\eta} \cdot \nu_{jk}^{\gamma\eta} \\ E^{T}[S_{i}^{\alpha}S_{j}^{\gamma}S_{k}^{\eta}S_{l}^{\theta}] &= d_{i}^{\alpha} \cdot d_{j}^{\gamma} \cdot d_{k}^{\eta} \cdot d_{l}^{\theta} \cdot \sigma_{i}^{\alpha(\alpha-1+2\gamma+2\eta+2\theta)} \cdot \sigma_{j}^{\gamma(\gamma-1+2\eta+2\theta)} \cdot \sigma_{k}^{\eta(\eta-1+2\eta)} \cdot \sigma_{l}^{\theta(\theta-1)} \\ & \cdot \nu_{ij}^{\alpha\gamma} \cdot \nu_{ik}^{\alpha\eta} \cdot \nu_{jk}^{\alpha\theta} \cdot \nu_{jk}^{\gamma\eta} \cdot \nu_{kl}^{\eta\theta} \cdot \lambda_{kl}^{\eta\theta} \cdot \lambda_{kl}^{\eta\theta} \end{split}$$

The algorithms will be derived by means of the following vector notations.

$$\begin{aligned} &d(i) &:= (d_i, \cdots, d_N) \in \mathbb{R}^{N+1-i}; \quad i = 1, ..., N \quad , \\ &\nu(i) &:= (\nu_{i,i+1}, \cdots, \nu_{i,N}) \in \mathbb{R}^{N-i}; \quad i = 1, ..., N-1 \quad . \end{aligned}$$

1. Moment

$$E^{T}\left[\frac{1}{N}\sum_{i=1}^{N}S_{i}\right] = \sum_{i=1}^{N}d_{i} = \langle d(1), 1 \rangle$$
.

#### 2. Moment

$$\begin{split} x(N) &:= E^{T}[S_{N}^{2}] = d_{N}^{2}\sigma_{N}^{2} \quad \text{and for } i = N - 1, ..., 1 \\ x(i) &:= E^{T}\left[\left(\sum_{j=i}^{N}S_{j}\right)^{2}\right] \\ &= E^{T}[S_{i}^{2}] + 2E^{T}\left[S_{i}\sum_{j=i+1}^{N}S_{j}\right] + E^{T}\left[\left(\sum_{j=i+1}^{N}S_{j}\right)^{2}\right] \\ &= d_{i}^{2}\sigma_{i}^{2} + 2 < d(i+1), \nu(i) > d_{i}\sigma_{i}^{2} + x(i+1) \\ &\implies E^{T}\left[\left(\frac{1}{N}\sum_{i=1}^{N}S_{i}\right)^{2}\right] = \frac{1}{N^{2}}x(1) \quad . \end{split}$$

### 3. Moment

$$\begin{split} x(N) &:= E^{T}[S_{N}^{3}] = d_{N}^{3} \sigma_{N}^{6} \quad \text{and for } i = N - 1, \cdots, 1 \\ x(i) &:= E^{T} \left[ \left( \sum_{j=i}^{N} S_{j} \right)^{3} \right] \\ &= E^{T}[S_{i}^{3}] + 3E^{T} \left[ S_{i} \left( \sum_{j=i+1}^{N} S_{j} \right)^{2} \right] + 3E^{T} \left[ S_{i}^{2} \left( \sum_{j=i+1}^{N} S_{j} \right) \right] + E^{T} \left[ \left( \sum_{j=i+1}^{N} S_{j} \right)^{3} \right] \\ &= d_{i}^{3} \sigma_{i}^{6} + 3a(i, i + 1) + 3 < d(i + 1, \nu(1)^{2}) > d_{i}^{2} \sigma_{i}^{6} + x(i + 1) \quad , \end{split}$$

where 
$$a(i, N) := d_i d_N^2 \sigma_i^4 \sigma_N^2 \nu_{iN}^2$$
 and for  $j = N - 1, \dots, i + 1$   
 $a(i, j) = a(i, j + 1) + d_i d_j^2 \sigma_i^4 \sigma_j^2 \nu_{ij}^2 + 2 \left( \sum_{l=j+1}^N d_l \nu_{il} \nu_{jl} \right) d_i d_j \sigma_i^4 \sigma_j \nu_{ij}$   
 $\implies E^T \left[ \left( \frac{1}{N} \sum_{i=1}^N S_i \right)^3 \right] = \frac{1}{N^3} x(1)$ .

## 4. Moment

$$\begin{split} x(N) &:= E^T \left[ S_N^4 \right] = d_N^4 \sigma_N^{12} \quad \text{and for } j = N - 1, \cdots, 1 \\ x(i) &:= E^T \left[ \left( \sum_{j=i}^N S_j \right)^4 \right] \\ &= E^T \left[ S_i^4 \right] + 4E^T \left[ S_i^3 \left( \sum_{j=i+1}^N S_j \right) \right] + 6E^T \left[ S_i^2 \left( \sum_{j=i+1}^N S_j \right)^2 \right] \\ &\quad + 4E^T \left[ S_i \left( \sum_{j=i+1}^N S_j \right)^3 \right] + E^T \left[ \left( \sum_{j=i+1}^N S_j \right)^4 \right] \\ &= d_i^4 \sigma_i^{12} + 4 < d(i+1), \nu(i)^3 > d_i^3 \sigma_i^{12} + 6c(i,i+1) + 4a(i,i+1) + x(i+1) \quad , \end{split}$$

where

$$\begin{split} a(i,N) &:= d_i d_N^3 \sigma_i^6 \sigma_N^6 \quad \text{and for } j = N - 1, \cdots, i + 1 \\ a(i,j) &:= d_i d_j \sigma_i^6 \sigma_j^6 + 3 \left( \sum_{l=j+1}^N d_l \nu_{il} \nu_{jl}^2 \right) d_i d_j^2 \sigma_i^6 \sigma_j^6 \nu_{ij}^6 + 3b(i,j,j+1) + a(i,j+1) \\ \text{with } b(i,j,N) &:= d_i d_j d_N^2 \sigma_i^6 \sigma_j^4 \sigma_N^2 \nu_{ij} \nu_{iN}^2 \nu_{jN}^2 \\ &\quad \text{and for } k = N - 1, \cdots, j + 1 \\ b(i,j,k) &:= d_i d_j d_k^2 \sigma_i^6 \sigma_j^4 \sigma_k^2 \nu_{ij} \nu_{ik}^2 \nu_{jk}^2 \\ &\quad + 2 \left( \sum_{l=k+1}^N d_l \nu_{il} \nu_{jl} \nu_{kl} \right) d_i d_j d_k \sigma_i^6 \sigma_j^4 \sigma_k^2 \nu_{ik} \nu_{ij} \nu_{jk} + b(i,j,k+1) \quad . \end{split}$$

 $\operatorname{and}$ 

$$\begin{split} c(i,N) &:= \quad d_i^2 d_N^2 \sigma_i^{10} \sigma_N^2 \nu_{iN}^4 \quad \text{and for } j = N - 1, \cdots, i + 1 \\ c(i,j) &:= \quad d_i^2 d_j^2 \sigma_i^{10} \sigma_j^2 \nu_{ij}^2 + 2 \left( \sum_{k=j+1}^N d_k \nu_{ik}^2 \nu_{jk} \right) d_i^2 d_j \sigma_i^{10} \sigma_j^2 \nu_{ij}^2 + c(i,j+1) \\ \implies \quad E^T \left[ \left( \frac{1}{N} \sum_{i=1}^N S_i \right)^4 \right] = \frac{1}{N^4} x(1) \quad . \end{split}$$

$(1) \sigma(geo)$	0.0298	0.0336	0.0336	0.0335	0.0315	0.024	0.0318	0.0319	0.0312	0.0261	0.0231	0.0261	0.0261	0.0258	0.0239	0.0226	0.0287	0.0288	0.0284	0.0248	0.0169	0.0212	0.0213	0.0209	0.0184	0.0293	0.0339	0.0339	0.0339	0.0313	0.0251	0.0309	0.0311	0.0307	
) $\sigma(asian$	0.0302	0.0341	0.0342	0.0341	0.0322	0.0243	0.0323	0.0325	0.0321	0.0271	0.0243	0.0272	0.0273	0.0271	0.0252	0.0232	0.0295	0.0298	0.0295	0.0261	0.0174	0.0217	0.0219	0.0215	0.019	0.0299	0.0348	0.0347	0.0347	0.0323	0.0263	0.0321	0.0323	0.032	
$\sigma_c(asian)$	0.0017	0.0014	0.0013	0.0013	0.0013	0.0011	0.001	0.0011	0.0012	0.0012	0.0013	0.0013	0.0013	0.0014	0.0014	0.0012	0.0012	0.0012	0.0013	0.0014	0.0014	0.0013	0.0013	0.0013	0.0013	0.0014	0.0013	0.0013	0.0013	0.0013	0.0017	0.0016	0.0016	0.0016	
exact geo	7.74791	4.77127	4.28304	3.83042	2.67894	7.71121	4.71601	4.22579	3.77184	2.62009	7.68896	4.68225	4.19082	3.73604	2.58418	7.67404	4.65949	4.16722	3.71189	2.55997	7.65904	4.63652	4.1434	3.68751	2.53554	7.6364	4.60165	4.10725	3.6505	2.4985	7.5983	4.54241	4.04578	3.58757	
MC geo	7.74766	4.7675	4.27917	3.82711	2.6778	7.69847	4.70145	4.21111	3.75753	2.60646	7.68769	4.68303	4.1916	3.73675	2.58348	7.67901	4.66503	4.17274	3.71752	2.56574	7.67187	4.64992	4.1567	3.70115	2.54928	7.62578	4.5891	4.09469	3.63787	2.48696	7.60108	4.54409	4.04726	3.58941	
$\rho(\text{Vorst})$	-0.49194	-0.58466	-0.59961	-0.61462	-0.65796	-0.24329	-0.33014	-0.34412	-0.3584	-0.39932	-0.09258	-0.1769	-0.19116	-0.20498	-0.24512	-0.00049	-0.08027	-0.09385	-0.10713	-0.1457	0.10674	0.02529	0.01191	-0.00088	-0.03931	0.25769	0.18048	0.16711	0.1543	0.11895	0.50159	0.4303	0.4187	0.40747	
Bu asian	8.02244	5.0458	4.55757	4.10495	2.95347	7.97941	4.98421	4.49399	4.04004	2.88829	7.95336	4.94665	4.45521	4.00044	2.84857	7.9359	4.92135	4.42908	3.97375	2.82183	7.91837	4.89584	4.40273	3.94684	2.79487	7.89193	4.85717	4.36277	3.90602	2.75402	7.84748	4.79159	4.29496	3.83675	
MC asian	7.94068	4.93514	4.44091	3.98237	2.81281	7.90103	4.87653	4.38022	3.92029	2.75019	7.87676	4.84063	4.34322	3.88243	2.71223	7.86186	4.81774	4.31942	3.85802	2.68749	7.84442	4.79252	4.29333	3.83126	2.66079	7.8197	4.755	4.25458	3.79172	2.62062	7.77942	4.69349	4.19054	3.72594	
Vorst	7.94196	4.91578	4.4174	3.95477	2.77472	7.90211	4.85782	4.35751	3.8936	2.71344	7.87797	4.82243	4.32094	3.85624	2.67605	7.86178	4.79859	4.29628	3.83104	2.65087	7.84551	4.77453	4.2714	3.80562	2.62546	7.82097	4.73803	4.23364	3.76704	2.58694	7.77968	4.67606	4.1695	3.70149	
$\mathrm{TW}$	7.93376	4.93275	4.43993	3.98267	2.81624	7.89456	4.87429	4.37932	3.92055	2.75339	7.87081	4.83861	4.34231	3.88262	2.71507	7.85487	4.81456	4.31736	3.85705	2.68925	7.83886	4.79031	4.29219	3.83125	2.66322	7.8147	4.75351	4.254	3.7921	2.62375	7.77404	4.69106	4.18915	3.72561	
log appr.	7.9689	4.94847	4.45046	3.98791	2.80654	7.9278	4.88923	4.38929	3.92547	2.744	7.90292	4.85308	4.35196	3.88735	2.70587	7.88624	4.82873	4.3268	3.86166	2.68019	7.86948	4.80417	4.30141	3.83573	2.65429	7.8442	4.76691	4.2629	3.7964	2.61502	7.80169	4.70369	4.19751	3.7296	
Bd asian	7.74791	4.77127	4.28304	3.83042	2.67894	7.71121	4.71601	4.22579	3.77184	2.62009	7.68896	4.68225	4.19082	3.73604	2.58418	7.67404	4.65949	4.16722	3.71189	2.55997	7.65904	4.63652	4.1434	3.68751	2.53554	7.6364	4.60165	4.10725	3.6505	2.4985	7.5983	4.54241	4.04578	3.58757	
Basis	95	100	101	102	105	95	100	101	102	105	95	100	101	102	105	95	100	101	102	105	95	100	101	102	105	95	100	101	102	105	95	100	101	102	
φ	-0.5					-0.25					-0.1					0					0.1					0.25					0.5				
$E^{T}[A(T)]$	101.4375					101.4639					101.4797					101.4903					101.5009					101.5167					101.5432				

Table 2: Approximation of Asian option prices, maturity 0.5 years

$E^T[A(T)]$	θ	$\operatorname{Basis}$	Bd asian	log appr.	$\mathrm{TW}$	Vorst	MC asian	Bu asian	$\rho(\text{Vorst})$	MC geo	exact geo	$\sigma_c({\rm asian})$	$\sigma({ m asian})$	$\sigma(\mathrm{geo})$
102.7386	-0.5	95	9.79466	10.26532	10.16694	10.17728	10.20054	10.37809	-0.56479	9.79425	9.79466	0.0037	0.0365	0.0346
_		100	7.03442	7.44313	7.38753	7.3434	7.40152	7.61785	-0.61652	7.03461	7.03442	0.0039	0.04	0.0381
		102	6.09747	6.47885	6.44374	6.3771	6.44898	6.6809	-0.63425	6.09769	6.09747	0.0037	0.0399	0.038
		103	5.66419	6.03157	6.00681	5.92941	6.00779	6.24762	-0.6427	5.66462	5.66419	0.0039	0.0395	0.0375
		110	3.24568	3.51401	3.55204	3.41897	3.5309	3.82911	-0.69714	3.24424	3.24568	0.0039	0.0355	0.0329
102.8456	-0.25	95	9.71383	10.16669	10.07827	10.08573	10.10862	10.27268	-0.31177	9.73001	9.71383	0.003	0.0446	0.0429
		100	6.91529	7.30728	7.25666	7.21468	7.27067	7.47414	-0.35739	6.93178	6.91529	0.0029	0.0455	0.0437
		102	5.96821	6.33319	6.30115	6.23854	6.3075	6.52706	-0.37299	5.98468	5.96821	0.0031	0.0457	0.0438
		103	5.53106	5.8822	5.8596	5.78712	5.86208	6.08991	-0.38027	5.54752	5.53106	0.0031	0.0456	0.0438
		110	3.10678	3.35996	3.39484	3.27182	3.37867	3.66563	-0.43052	3.1209	3.10678	0.0033	0.0417	0.0393
102.9099	-0.1	95	9.66397	10.10613	10.02358	10.02939	10.05	10.20806	-0.15454	9.676	9.66397	0.0044	0.0437	0.0411
		100	6.84104	7.223	7.17534	7.13464	7.18708	7.38513	-0.1978	6.85607	6.84104	0.0039	0.0459	0.0439
		102	5.88753	6.24268	6.21249	6.15226	6.21694	6.43162	-0.21204	5.90249	5.88753	0.0038	0.0464	0.0441
		103	5.44795	5.78934	5.76806	5.69848	5.76916	5.99204	-0.21926	5.46293	5.44795	0.0038	0.0459	0.0434
		110	3.02049	3.26457	3.29759	3.18052	3.28132	3.56458	-0.26582	3.0295	3.02049	0.0034	0.0424	0.0396
102.9527	0	95	9.63014	10.06518	9.98649	9.99122	10.01227	10.16438	-0.05498	9.64222	9.63014	0.0038	0.0722	0.0689
		100	6.79031	7.16558	7.11989	7.08003	7.13216	7.32454	-0.09543	6.80791	6.79031	0.0044	0.0773	0.0734
		102	5.83236	6.18095	6.15199	6.09333	6.15783	6.3666	-0.10957	5.85015	5.83236	0.0043	0.0774	0.0735
		103	5.39111	5.726	5.70559	5.63793	5.70827	5.92535	-0.11636	5.40844	5.39111	0.0045	0.0775	0.0734
		110	2.96165	3.19965	3.23146	3.11832	3.21762	3.49589	-0.1605	2.97169	2.96165	0.0055	0.0689	0.0638
102.9956	0.1	95	9.59584	10.02375	9.94889	9.95257	9.97322	10.12022	0.04673	9.6065	9.59584	0.003	0.0283	0.0266
		100	6.73854	7.10713	7.0634	7.02438	7.07508	7.26292	0.00898	6.75067	6.73854	0.0028	0.0316	0.03
_		102	5.77602	6.11805	6.09031	6.03321	6.0961	6.3004	-0.00466	5.78821	5.77602	0.0027	0.0314	0.0299
		103	5.33305	5.66145	5.64191	5.57615	5.64492	5.85743	-0.01143	5.34509	5.33305	0.0027	0.0316	0.03
		110	2.90173	3.13364	3.16423	3.05501	3.15165	3.4261	-0.05337	2.91167	2.90173	0.0024	0.0273	0.026
103.06	0.25	95	9.54349	9.9607	9.89148	9.89365	9.91133	10.05307	0.20532	9.53845	9.54349	0.0022	0.0529	0.0516
		100	6.65885	7.01743	6.97661	6.93885	6.98534	7.16844	0.16901	6.6505	6.65885	0.0027	0.0599	0.0582
		102	5.6892	6.02139	5.99549	5.94072	5.99853	6.19878	0.1568	5.6804	5.6892	0.0027	0.0606	0.0586
		103	5.24358	5.56221	5.54399	5.48107	5.54436	5.75316	0.1507	5.23521	5.24358	0.0028	0.0603	0.0582
		110	2.80969	3.03246	3.06125	2.95787	3.04759	3.31927	0.1116	2.80667	2.80969	0.0035	0.0518	0.0492
103.1674	0.5	95	9.45378	9.85317	9.79311	9.79294	9.81191	9.93868	0.45352	9.47801	9.45378	0.0019	0.0386	0.0374
_		100	6.52022	6.8621	6.82608	6.79043	6.83648	7.00511	0.4238	6.54821	6.52022	0.0024	0.0468	0.0451
		102	5.53786	5.85365	5.83077	5.77986	5.83638	6.02275	0.4137	5.56596	5.53786	0.0024	0.0476	0.0457
_		103	5.08753	5.3899	5.37384	5.31562	5.37702	5.57243	0.40881	5.1155	5.08753	0.0025	0.0475	0.0454
		110	2.65018	2.85765	2.88349	2.78974	2.87514	3.13507	0.37451	2.67273	2.65018	0.0033	0.0378	0.035

Table 3: Approximation of Asian option prices, maturity 1 year

0.1638 0.138 0.1635 0.137	0.1625 0.1377 0.1615 0.1366		0.1575 0.1327	0.1575 0.1327 0.0902 0.0812	0.1575 0.1327 0.1575 0.1327 0.0902 0.0812 0.0898 0.0823	$\begin{array}{c} 0.1575 \\ 0.1327 \\ 0.0902 \\ 0.0898 \\ 0.0823 \\ 0.0902 \\ 0.0823 \end{array}$	0.1375 0.1327 0.1575 0.1327 0.0902 0.0812 0.0902 0.0823 0.0902 0.0828 0.0907 0.0828	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$
$\begin{bmatrix} 0.027 \\ 0.0260 \end{bmatrix}$	5 0.0269		7 0.027	7 0.027 9 0.0141	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	7 0.027 9 0.0141 1 0.0136 5 0.0135	7 0.027 9 0.0141 1 0.0136 5 0.0135 3 0.0133	7 0.027 9 0.0141 5 0.0136 3 0.0135 2 0.0133	7 0.027 9 0.0141 5 0.0136 3 0.0135 2 0.0133 5 0.0153	7 0.027 9 0.0141 5 0.0136 3 0.0135 2 0.0133 5 0.0153 4 0.0105	7         0.027           9         0.0141           1         0.0136           5         0.0135           3         0.0135           5         0.0135           6         0.0135           7         0.0135           8         0.0115           6         0.0105           6         0.0105           6         0.0112	7         0.027           9         0.0141           1         0.0136           5         0.0135           3         0.0135           5         0.0135           6         0.0153           5         0.0153           6         0.0105           4         0.0109           5         0.0112           1         0.0112	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	7         0.027           9         0.0141           5         0.0135           3         0.0135           4         0.0105           4         0.0106           1         0.0112           6         0.0112           1         0.0112           6         0.0112           6         0.0112	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	7         0.027           9         0.0141           5         0.0136           5         0.0135           2         0.0135           5         0.0135           6         0.0153           7         0.0153           8         0.0109           6         0.0112           1         0.0013           6         0.0112           6         0.0113           8         0.0121           8         0.0121	7         0.027           9         0.0141           5         0.0136           5         0.0135           2         0.0133           2         0.0133           4         0.0153           5         0.0112           6         0.0112           1         0.0112           6         0.0112           6         0.0112           8         0.0121           8         0.0121	7         0.027           9         0.0141           5         0.0135           3         0.0135           4         0.0105           5         0.0105           6         0.0112           1         0.0112           6         0.0112           8         0.0121           8         0.0121           8         0.0121           8         0.0121	7         0.027           9         0.0141           5         0.0136           5         0.0135           6         0.0105           7         0.0105           6         0.0112           1         0.0109           6         0.0112           6         0.0112           8         0.0121           8         0.0121           8         0.0121           8         0.0121           10.0138         0.0121	7         0.027           9         0.0141           5         0.0136           5         0.0135           6         0.0135           7         0.0135           8         0.0112           1         0.0109           6         0.0112           1         0.0112           8         0.0111           8         0.0121           8         0.0121           8         0.0123           0.0123         0.0121           0.0121         0.0121	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$
10.5528	9.87265 9.29282	0.00 1 1 1 1	7.51207	7.51207 12.29049	7.51207 12.29049 10.33201	7.51207 12.29049 10.33201 9.62285	7.51207 12.29049 10.33201 9.62285 8.95423	7.51207 12.29049 10.33201 9.62285 8.95423 7.17772	7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 12.19185	7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 12.19185 10.17474	7.51207 12.29049 10.33201 9.62285 8.95423 7.1772 12.19185 10.17474 9.44615	7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 12.19185 10.17474 9.44615 9.44615 8.76041	7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 12.19185 10.17474 9.44615 8.76041 6.94581 6.94581	7.51207 12.29049 10.33201 9.62285 8.95423 7.1772 12.19185 10.17474 9.44615 8.76041 6.94581 12.11736	7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 12.19185 10.17474 9.44615 8.76041 6.94581 6.94581 12.11736 112.11736	7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 12.19185 10.17474 9.44615 8.76041 6.94581 12.11736 12.11736 12.11736 9.31518 9.31518	7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 8.5423 7.17772 12.19185 10.17474 9.44615 8.76041 6.94581 12.11736 10.05776 9.31518 8.61718 8.61718	7.51207 12.29049 10.33201 9.62285 8.95285 8.95423 7.17772 12.19185 10.17474 9.44615 8.76041 6.94581 12.11736 12.11736 12.11736 9.31518 8.61718 8.61718 8.61718 8.61718	7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 8.95423 12.19185 10.17474 9.44615 8.76041 6.94581 12.11736 12.11736 12.11736 9.44615 8.76041 6.94581 12.11736 12.117377 12.11737 12.11757 12.117	7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 8.95423 10.17474 9.44615 8.76041 6.94581 12.11736 10.05776 9.31518 8.61778 8.61718 6.77578 8.61718 9.31518 9.31518 8.61718 9.31518	7.51207           12.29049           12.29049           10.33201           9.62285           8.95423           7.17772           12.19185           12.19185           10.17474           9.44615           8.76041           6.94581           12.19185           10.17474           9.44615           8.76041           6.94581           12.11736           10.05776           9.31518           8.61718           8.61718           9.931518           9.931518           9.9298           9.17224           9.17224	7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           12.19185           12.19185           12.19185           12.19185           12.19185           12.19185           9.44615           8.76041           6.94581           12.11736           12.11736           9.31518           8.61718           8.61718           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           8.61718           8.61718           9.17224           9.17224           9.46118	7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           9.62285           8.95423           7.17772           12.19185           10.17474           9.44615           8.76041           6.94581           12.11736           12.11736           12.11736           12.11736           12.11736           9.44615           9.44615           8.61718           8.61718           9.17254           9.17224           9.17224           8.41118           6.59171           6.59171	7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           9.62285           8.95423           7.17772           12.19185           10.17474           9.44615           8.76041           6.94581           12.11736           12.11736           9.31518           8.61778           9.31518           9.31518           9.203512           9.9298           9.17224           9.17224           8.46118           6.59171           11.89561	7.51207           7.551207           12.29049           10.33201           9.62285           8.95423           7.17772           12.19185           12.19185           10.17474           9.44615           8.76041           6.94581           12.11736           12.11736           9.31518           8.61778           9.31518           9.31518           9.31518           9.31518           9.31518           8.61778           9.9298           9.17224           9.17224           8.46118           6.59171           11.89561           9.71432           9.71432	7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           12.19185           12.19185           12.19185           12.19185           12.19185           12.19185           12.19185           12.19185           12.19185           9.44615           8.76041           6.94581           12.11736           12.11736           9.31518           8.61173           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.17224           9.17224           8.46118           6.59171           9.71432           9.71432           9.71432           9.32206	7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           12.19185           12.19185           12.19185           12.19185           12.19185           12.19185           12.19185           12.19185           12.19185           12.19185           12.19185           9.44615           8.76041           6.94581           12.11736           9.31518           8.46018           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.3206           8.1996           8.1996	7.51207 7.15772 12.29049 10.33201 9.62285 8.95423 7.17772 12.19185 10.17474 9.44615 8.76041 6.94581 12.11736 10.05776 9.31518 8.61718 6.77578 12.015776 9.31518 8.61718 6.77578 9.17224 9.2928 9.17224 8.46118 8.46118 8.46118 8.46118 8.33206 8.33206 8.1996 8.1996 8.1996 8.1996 8.1996 8.1996	7.51207           7.551207           12.29049           10.33201           9.62285           8.95423           7.17772           12.19185           10.17474           9.44615           8.76041           6.94581           10.17474           9.44615           8.76041           6.94581           12.19185           10.17474           9.44615           8.76041           6.94581           12.11736           9.31518           8.61778           9.31518           8.61778           9.231518           9.9298           9.17224           9.9298           9.12204           8.46118           6.59171           11.89561           9.71432           8.1996           8.1996           6.28526           11.61239	7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           9.62285           8.95423           7.17772           12.19185           10.17474           9.44615           8.76041           6.94581           12.11736           9.31518           8.76041           6.94581           12.11736           9.31518           8.61778           9.31518           9.31518           8.61778           9.31518           9.31518           8.61778           9.31518           9.31518           8.61778           9.31518           9.31518           8.61778           9.17224           8.46118           6.59171           9.71432           8.1996           6.285266           6.285266           9.27733           9.27733	7.51207           7.551207           12.29049           10.33201           9.62285           8.95423           7.17772           12.19185           12.19185           12.19185           12.19185           12.19185           12.19185           9.44615           8.76041           6.94581           12.11736           9.31518           8.61778           9.31518           9.31518           9.31518           9.31518           8.61778           9.31518           9.31518           9.31518           8.61718           12.03572           9.9298           9.17224           8.46118           6.59171           11.89561           9.71432           8.1996           8.1996           6.23526           9.27733           9.27733           8.44566	7.51207           12.1207           12.29049           10.33201           9.62285           8.95423           7.17772           12.19185           12.19185           12.19185           12.19185           12.19185           12.19185           9.44615           8.76041           6.94581           12.11736           9.345138           8.76041           6.94581           12.11736           9.31518           8.76041           6.94581           12.11736           9.31518           8.76117           12.035728           9.31518           8.61718           11.89561           9.71432           9.71432           8.93206           8.1996           6.285226           11.61239           9.27733           8.445666           7.67097
10.51678	9.837U3 9.1954	T00T.0	7.48166	7.48166 12.27695	7.48166 12.27695 10.3184	7.48166 12.27695 10.3184 9.60892	7.48166 12.27695 10.3184 9.60892 8.94057	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           12.13767	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           12.13767           10.12017	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           12.13767           10.12017           9.39162	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           7.16247           12.13767           10.12017           9.39162           8.70638	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           12.13767           12.13767           10.12017           9.39162           8.70638           8.70638           6.89349	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           12.13767           12.13767           9.39162           8.70638           8.70638           6.89349           12.10155	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           12.13767           12.13767           10.12017           9.39162           8.70638           6.89349           12.10155           10.04153	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           7.16247           12.13767           10.12017           9.39162           8.70638           6.89349           12.10155           12.10155           12.10155           12.10155           10.04153           9.29857	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           12.13767           12.13767           10.12017           9.39162           8.70638           8.70638           9.39162           8.70638           10.12017           9.39162           9.39162           9.39162           9.39163           8.70638           10.12017           9.39162           8.70638           9.39162           8.70638           8.70638           9.29857           9.29857           9.29857           8.60012	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           9.39162           8.70638           6.89349           12.10155           10.04153           9.29857           9.29857           8.60012           8.60012           8.60012           8.60012           8.60012           8.60012           8.60012	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.10155           8.70638           6.89349           12.10155           10.04153           9.29857           9.29857           8.60012           8.60012           6.75727           11.98024	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           10.12017           9.39162           8.70638           8.70638           9.39162           8.70638           9.39162           8.70638           9.39162           8.70638           9.39162           8.70638           6.89349           10.12017           9.39162           8.70638           6.89349           12.10155           10.04153           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           10.3184           9.60892           8.94057           7.16247           12.13767           10.12017           9.39162           8.70638           8.70638           8.70638           9.39162           8.70638           9.39162           8.70638           9.39162           8.70638           8.70638           6.89349           12.10155           10.04153           9.29857           8.60012           6.75727           11.98024           9.81756           9.17556	7.48166           7.48166           10.3184           9.60892           8.94057           7.16247           10.313767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           9.39162           8.70638           6.89349           12.10155           10.04153           9.29857           8.60012           6.75727           9.29857           9.87488           9.87488           9.1756           9.1756           9.11756           9.11756	7.48166           7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           12.13767           12.13767           12.13767           12.13767           12.13767           12.110155           8.70638           6.89349           12.10155           10.04153           9.29857           8.60012           6.75727           11.98024           9.11756           9.11756           8.40667           6.54057           6.54057	7.48166           7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           7.16247           10.12017           9.39162           8.70638           8.70638           6.89349           10.12017           9.39162           8.70638           6.89349           10.12017           9.39162           8.70638           6.89349           12.10155           10.04153           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.11756           8.40667           6.54057           6.54057           11.33408	7.48166           7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           10.3184           9.60892           8.94057           7.16247           12.13767           10.12017           9.39162           8.70638           8.70638           6.89349           12.10155           10.04153           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           8.60012           6.75727           11.98024           9.11756           8.40667           6.54057           11.83408           9.65104           9.65104	7.48166           7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           10.3184           9.60892           8.94057           7.16247           12.13767           10.12017           9.39162           8.70638           8.7001           6.89349           12.10155           10.04153           9.29857           8.60012           6.89349           11.98024           9.29857           8.60012           6.75727           11.98024           9.11756           8.40667           6.54057           11.83408           9.65104           8.87001	7.48166           7.48166           10.3184           9.60892           8.94057           7.16247           10.12017           8.94057           7.16247           10.12017           9.60892           8.94057           7.16247           10.12017           9.39162           8.70638           6.89349           12.10155           9.39162           8.70638           6.89349           12.10155           9.39162           8.70638           9.39162           9.29857           8.60012           6.75727           11.98024           9.11756           9.11756           9.11756           9.11756           9.11756           9.11756           9.11756           9.11756           9.11756           9.11756           9.11756           9.11756           9.11756           9.11756           9.11756           9.1178           9.1178	7.48166           7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           10.313767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           12.13767           9.39162           8.70638           6.89349           9.39162           8.70638           6.89349           9.29857           9.29857           9.29857           9.29857           8.60012           6.75727           11.98024           9.11756           8.40667           6.54057           6.54057           8.87001           8.13958           9.23175           6.23175	7.48166           7.48166           10.3184           9.60892           8.94057           9.60892           8.94057           7.16247           10.12017           9.39162           8.70638           8.70638           9.39162           8.70638           8.70638           8.70638           9.39162           9.39162           9.39162           8.70638           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.11756           8.40667           6.54057           6.54057           8.87001           8.13358           9.23175           11.56031	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           9.60892           8.94057           7.16247           10.3184           9.60892           8.94057           12.13767           10.12017           9.39162           8.70638           8.70638           8.70638           9.39162           8.70638           9.39162           8.70012           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.1756           8.40667           6.54057           11.83408           9.65104           8.13958           6.23175           111.56031           9.22561	7.48166           7.48166           10.3184           9.60892           8.94057           7.16247           10.3184           9.60892           8.94057           7.16247           10.12017           9.39162           8.70638           8.70638           8.70638           9.39162           9.39162           8.70638           8.70638           9.39162           9.39162           8.70638           9.29857           9.29857           9.29857           9.29857           8.70012           9.1756           11.83408           9.1756           8.13958           6.23175           11.56031           9.22561           8.3939	7.48166           12.27695           10.3184           9.60892           8.94057           7.16247           10.3184           9.60892           8.94057           7.16247           10.12017           9.39162           8.70638           8.70638           8.70638           8.70638           9.39162           8.70638           8.70638           9.39162           9.39162           8.70638           9.39162           8.70638           9.29857           9.29857           9.29857           9.29857           9.29857           9.29857           9.1756           8.40667           6.54057           11.83408           9.65104           8.37001           8.13958           6.23175           11.56031           9.22561           8.3939           7.61877
no.sol.	no.sol.	.100.011	no.sol.	no.sol. -0.7771	no.sol. -0.7771 -0.66499	no.sol. -0.7771 -0.66499 -0.64609	no.sol. -0.7771 -0.66499 -0.64609 -0.63365	no.sol. -0.7771 -0.66499 -0.64609 -0.64609 -0.63365 -0.63365	no.sol. -0.7771 -0.66499 -0.64609 -0.64609 -0.63365 -0.63755 -0.61735 -0.49727	no.sol. -0.7771 -0.66499 -0.64609 -0.64609 -0.64609 -0.64609 -0.64735 -0.49727 -0.42697	no.sol. -0.7771 -0.6499 -0.64609 -0.64609 -0.63365 -0.63365 -0.61735 -0.49727 -0.42697 -0.41482	no.sol. -0.7771 -0.64699 -0.64609 -0.64609 -0.64609 -0.64509 -0.61735 -0.49727 -0.49727 -0.41482 -0.41482 -0.41482 -0.41482 -0.41682 -0.42697 -0.40767 -0.40767 -0.40767 -0.40767 -0.40767 -0.40767 -0.407777 -0.407777 -0.4077777 -0.40777777 -0.4077777777777777777777777777777777777	no.sol. -0.7771 -0.66499 -0.664609 -0.64609 -0.64735 -0.49727 -0.49727 -0.41482 -0.41482 -0.40719 -0.3981	no.sol. -0.7771 -0.66499 -0.64409 -0.64409 -0.64609 -0.64609 -0.64609 -0.49727 -0.49727 -0.41482 -0.41482 -0.41482 -0.416719 -0.3981 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        -0.41482           -0.41482           -0.41482           -0.41482           -0.41482           -0.41482           -0.28636           -0.28636           -0.27683           -0.27683           -0.27683           -0.27749           -0.17749           -0.13721           -0.264444           -0.17749           -0.13098           -0.13098           -0.12704           -0.12704           -0.12704           -0.12704           -0.12704           -0.12704           -0.12704           -0.12704           -0.12704           -0.12704           -0.12704           -0.12704           -0.12704           -0.12704           -0.12704           -0.12704      -0.12321<	no.sol.           -0.7771           -0.64609           -0.64609           -0.64609           -0.64609           -0.64609           -0.64609           -0.64609           -0.64609           -0.64609           -0.64609           -0.640719           -0.42697           -0.42697           -0.41482           -0.41482           -0.41482           -0.33319           -0.28636           -0.34319           -0.34319           -0.34319           -0.34319           -0.34319           -0.34319           -0.34319           -0.34319           -0.34319           -0.34319           -0.34319           -0.34319           -0.36636           -0.13721           -0.13721           -0.13721           -0.12321           -0.12704           -0.12321           -0.13308           -0.137277           -0.137277           -0.36919           -0.36919	no.sol.           -0.7771           -0.64609           -0.644609           -0.644609           -0.644609           -0.644609           -0.64735           -0.49727           -0.49727           -0.41482           -0.41482           -0.41482           -0.41482           -0.41482           -0.41482           -0.41482           -0.3981           -0.334319           -0.34319           -0.334319           -0.334319           -0.334319           -0.334319           -0.334319           -0.33636           -0.334319     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12.84069 13.12054	12.16054	11 5177	11.5177 $9.79995$	$\frac{11.5177}{9.79995}$ $14.38953$	11.5177 $9.79995$ $14.38953$ $12.43104$	11.5177 $9.79995$ $14.38953$ $12.43104$ $11.72188$	11.5177 $9.79995$ $14.38953$ $12.43104$ $11.72188$ $11.05326$	$\begin{array}{c} 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.05326\\ 9.27675\end{array}$	$\begin{array}{c} 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.05326\\ 9.27675\\ 14.1763\end{array}$	$\begin{array}{c} 11.5177\\ 9.79995\\ 14.38953\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.72188\\ 11.72188\\ 11.75326\\ 9.27675\\ 14.1763\\ 12.15919\end{array}$	$\begin{array}{c} 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.05326\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ \end{array}$	$\begin{array}{c} 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.05326\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ 11.43061\\ 10.74487\end{array}$	11.5177 9.79995 14.38953 12.43104 11.72188 11.05326 9.27675 14.1763 12.15919 11.43061 11.43061 11.43061 10.74487 8.93026	$\begin{array}{c} 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.72188\\ 11.75326\\ 9.27675\\ 12.15919\\ 12.15919\\ 12.15919\\ 11.43061\\ 11.43061\\ 11.43061\\ 10.74487\\ 8.93026\end{array}$	$\begin{array}{c} 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.05326\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.43026\\ 8.93026\\ 11.96529\\ 11.96529\end{array}$	$\begin{array}{c} 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.72188\\ 11.75326\\ 9.27675\\ 14.1763\\ 12.15919\\ 12.15919\\ 11.43061\\ 11.43061\\ 11.43061\\ 10.74487\\ 8.93026\\ 11.96529\\ 11.96529\end{array}$	$\begin{array}{c} 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.65326\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ 11.43061\\ 11.43061\\ 10.74487\\ 8.93026\\ 11.22272\\ 11.22272\\ 11.22272\\ 10.52471\\ 10.52472\\ 10.52471\\ 10.52471\\ 10.52471\\ 10.52471\\ 10.52471\\ 10.52472\\ 10.52472\\ 10.52472\\ 10.52472\\ 10.52472\\ 10.52472\\ 10.52472\\ 10.52472\\ 10.52472\\ 10.52272\\ 10.52472\\ 10.52272\\ 10.52472\\ 10.52272\\ 10.5$	$\begin{array}{c} 11.5177\\ 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.72188\\ 11.75326\\ 9.27675\\ 12.15919\\ 12.15919\\ 11.43061\\ 12.15919\\ 11.43061\\ 12.15919\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.2272\\ 11.2272\\ 11.2272\\ 10.52471\\ 8.68331\\ 8.68331\end{array}$	$\begin{array}{c} 11.5177\\ 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.72188\\ 11.75326\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.6529\\ 11.96529\\ 11.96529\\ 11.96529\\ 11.96523\\ 11.965331\\ 13.86532\end{array}$	$\begin{array}{c} 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.72188\\ 11.75269\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.2272\\ 10.52471\\ 8.68331\\ 11.55999\\ 11.75999\end{array}$	$\begin{array}{c} 11.5177\\ 9.79955\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.72188\\ 11.65326\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.2272\\ 10.74487\\ 8.93026\\ 11.96529\\ 11.2272\\ 10.52471\\ 8.68331\\ 11.75999\\ 11.75999\\ 11.00244\end{array}$	$\begin{array}{c} 11.5177\\ 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.75326\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ 11.43061\\ 11.43061\\ 10.74487\\ 8.933026\\ 14.02489\\ 11.96529\\ 11.96529\\ 11.96529\\ 11.2272\\ 10.52471\\ 8.68331\\ 11.2572\\ 10.52471\\ 8.68331\\ 11.75999\\ 11.00244\\ 10.29137\end{array}$	$\begin{array}{c} 11.5177\\ 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.75326\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ 10.74487\\ 8.93026\\ 14.02489\\ 11.96529\\ 11.96529\\ 11.96529\\ 11.96529\\ 11.2272\\ 10.52471\\ 8.68331\\ 11.2272\\ 10.52471\\ 8.68331\\ 11.75999\\ 11.00244\\ 10.29137\\ 8.4219\\ \end{array}$	$\begin{array}{c} 11.5177\\ 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.72188\\ 11.75326\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ 10.74487\\ 8.93026\\ 11.43061\\ 10.74487\\ 8.93026\\ 11.2272\\ 10.52471\\ 8.68331\\ 11.25999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759888\\ 11.758888\\ 11.758888\\ 11.75888\\ 11.75888\\ 11.75888\\ 11.75888\\ 11.758$	$\begin{array}{c} 11.5177\\ 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.72188\\ 11.75326\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.43061\\ 11.2272\\ 10.74487\\ 8.93026\\ 11.96529\\ 11.96529\\ 11.2272\\ 10.52471\\ 8.68331\\ 11.75999\\ 11.7599\\ 11.759\\ 11.7599\\ 11.7598\\ 11.7588\\$	$\begin{array}{c} 11.5177\\ 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.65326\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ 10.74487\\ 8.93026\\ 11.43061\\ 10.74487\\ 8.93026\\ 11.22272\\ 11.22272\\ 11.22272\\ 11.25999\\ 11.25999\\ 11.00244\\ 10.29137\\ 8.4219\\ 11.00244\\ 10.29137\\ 8.4219\\ 11.42769\\ 11.42769\\ 11.42769\\ 11.42769\\ 10.64544\end{array}$	$\begin{array}{c} 11.5177\\ 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.72188\\ 11.75326\\ 9.27675\\ 12.15919\\ 11.43061\\ 12.15919\\ 11.43061\\ 12.15919\\ 11.65299\\ 11.65299\\ 11.2272\\ 12.5999\\ 11.2272\\ 12.59999\\ 11.7599\\ 11.7599$	$\begin{array}{c} 11.5177\\ 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.72188\\ 11.75326\\ 9.27675\\ 12.15919\\ 11.43061\\ 11.43061\\ 10.74487\\ 8.93026\\ 11.43061\\ 10.74487\\ 8.93026\\ 11.42769\\ 11.22272\\ 10.52471\\ 8.68331\\ 11.25729\\ 11.25999\\ 11.00244\\ 10.59137\\ 8.68331\\ 11.759999\\ 11.00244\\ 10.29137\\ 8.68331\\ 11.759999\\ 11.00244\\ 10.29137\\ 8.64219\\ 11.7599864\\ 7.99864\end{array}$	$\begin{array}{c} 11.5177\\ 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.72188\\ 11.75326\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ 10.74487\\ 8.93026\\ 11.43061\\ 11.22272\\ 10.74487\\ 8.93026\\ 11.96529\\ 11.965299\\ 11.965299\\ 11.955999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.759999\\ 11.7599984\\ 11.7299864\\ 13.12891\\ 13.12891\end{array}$	$\begin{array}{c} 11.5177\\ 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.72188\\ 11.752169\\ 14.1763\\ 12.15919\\ 11.43061\\ 10.74487\\ 8.93026\\ 11.43061\\ 11.42769\\ 11.2272\\ 10.52471\\ 8.68331\\ 11.2272\\ 11.25999\\ 11.25999\\ 11.75999\\ 11.75999\\ 11.75999\\ 11.75999\\ 11.75999\\ 11.7799864\\ 12.799864\\ 13.12891\\ 10.79384\\ 10.79384\end{array}$	$\begin{array}{c} 11.5177\\ 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.72188\\ 11.65326\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ 10.74487\\ 8.93026\\ 14.02489\\ 11.42699\\ 11.22272\\ 10.52471\\ 8.68331\\ 11.22272\\ 11.22272\\ 11.22272\\ 11.22272\\ 11.42699\\ 11.00244\\ 11.00244\\ 11.00244\\ 11.029137\\ 8.4219\\ 11.65999\\ 11.65999\\ 11.6799864\\ 11.42769\\ 10.64544\\ 9.91297\\ 7.99864\\ 13.12891\\ 10.79384\\ 9.96217\\ 9.96217\end{array}$	$\begin{array}{c} 11.5177\\ 11.5177\\ 9.79995\\ 14.38953\\ 12.43104\\ 11.72188\\ 11.65326\\ 9.27675\\ 14.1763\\ 12.15919\\ 11.43061\\ 10.74487\\ 8.93026\\ 14.02489\\ 11.42699\\ 11.22272\\ 10.52471\\ 8.93026\\ 11.22272\\ 11.22272\\ 11.22272\\ 11.22272\\ 11.43061\\ 11.222891\\ 11.00244\\ 11.00244\\ 11.00244\\ 11.00244\\ 11.02938\\ 11.42769\\ 11.65999\\ 11.029137\\ 8.4219\\ 11.75999\\ 11.67938\\ 9.96217\\ 9.96217\\ 9.18749\\ 9.96217\\ 9.18749\end{array}$
12.01588	11.31298	10.64707	10.64707 8.85919	$\frac{10.64707}{8.85919}$ $13.71022$	$\begin{array}{c} 10.64707\\ 8.85919\\ 13.71022\\ 11.69086\end{array}$	$\begin{array}{c} 10.64707\\ 8.85919\\ 13.71022\\ 11.69086\\ 10.95688\end{array}$	10.64707 8.85919 13.71022 11.69086 10.95688 10.26334	10.64707 8.85919 13.71022 11.69086 10.95688 10.95688 10.26334 8.41101	10.64707 8.85919 13.71022 11.69086 10.95688 10.26334 8.41101 13.55043 13.55043	10.64707 8.85919 13.71022 11.69086 10.95688 10.95688 10.26334 8.41101 13.55043 11.46965	10.64707 8.85919 13.71022 11.69086 10.95688 10.95688 10.26334 8.41101 13.55043 11.46965 11.46965 10.71493	10.64707 8.85919 13.71022 11.69086 10.95688 10.95688 10.26334 8.41101 13.55043 11.46965 10.71493 10.71493	10.64707 8.85919 13.71022 11.69086 10.95688 10.95688 10.26334 8.41101 13.55043 11.46965 10.71493 10.00313 8.10828	10.64707 8.85919 13.71022 11.69086 10.95688 10.95688 10.95688 10.26334 8.41101 13.55043 11.46965 10.71493 11.46965 10.71493 10.00313 8.10828 8.10828	10.64707 8.85919 13.71022 11.69086 10.95688 10.95688 10.95688 10.26334 8.41101 13.55043 11.46965 10.71493 11.46965 10.71493 10.00313 8.10828 13.44432 13.44432 13.44432	10.64707 8.85919 13.71022 11.69086 10.95688 10.95688 10.95688 10.26334 8.41101 13.46965 10.71493 11.46965 10.71493 10.71493 10.71493 10.71493 11.31908 13.44432 11.31908	10.64707 8.85919 8.85919 13.71022 11.69086 10.95688 10.95688 10.26334 8.41101 13.55043 11.46965 10.71493 11.46965 10.71493 10.00313 8.10828 10.00313 8.10828 11.31908 11.31908 10.54988 9.82506 9.82506	$\begin{array}{c} 10.64707\\ 8.85919\\ 8.85919\\ 13.71022\\ 11.69086\\ 10.95688\\ 10.95688\\ 10.95688\\ 10.26334\\ 8.41101\\ 13.55043\\ 13.55043\\ 11.46965\\ 13.55043\\ 11.46965\\ 10.71493\\ 11.46965\\ 13.44432\\ 11.31908\\ 8.10828\\ 10.54988\\ 9.82506\\ 7.90112\end{array}$	10.64707 8.85919 13.71022 11.69086 10.95688 10.95688 10.95688 10.26334 8.41101 13.55043 11.46965 11.46965 11.46965 11.46965 11.46965 11.46965 11.46965 11.46988 8.10828 13.44432 13.44432 11.31908 11.31908 10.54988 9.82506 7.90112 13.3137	$\begin{array}{c} 10.64707\\ 8.85919\\ 8.85919\\ 13.71022\\ 11.69086\\ 10.95688\\ 10.95688\\ 10.95688\\ 10.26334\\ 8.41101\\ 13.55043\\ 11.46965\\ 10.71493\\ 11.46965\\ 10.71493\\ 11.46965\\ 10.71493\\ 11.46965\\ 10.79112\\ 11.31908\\ 9.82506\\ 7.90112\\ 11.1406\end{array}$	$\begin{array}{c} 10.64707\\ 8.85919\\ 8.85919\\ 13.71022\\ 11.69086\\ 10.95688\\ 10.95688\\ 10.26334\\ 8.41101\\ 13.55043\\ 11.46965\\ 10.71493\\ 11.46965\\ 10.71493\\ 11.46965\\ 10.71493\\ 11.46965\\ 10.71493\\ 11.31908\\ 10.71493\\ 11.31908\\ 10.55948\\ 9.82506\\ 7.90112\\ 11.1406\\ 11.1406\end{array}$	$\begin{array}{c} 10.64707\\ 8.85919\\ 8.85919\\ 13.71022\\ 11.69086\\ 10.95688\\ 10.95688\\ 10.95688\\ 10.26334\\ 8.41101\\ 13.55043\\ 13.55043\\ 11.46965\\ 10.71493\\ 13.55043\\ 13.44432\\ 11.4088\\ 9.82506\\ 7.90112\\ 11.31908\\ 9.82506\\ 7.90112\\ 11.1406\\ 11.3137\\ 11.1406\\ 10.35593\\ 9.61703\\ 9.61703\end{array}$	$\begin{array}{c} 10.64707\\ 8.85919\\ 8.85919\\ 13.71022\\ 11.69086\\ 10.95688\\ 10.95688\\ 10.95688\\ 10.26334\\ 8.41101\\ 13.55043\\ 11.46965\\ 10.71493\\ 11.46965\\ 10.71493\\ 11.46965\\ 10.71493\\ 11.46965\\ 11.31908\\ 13.44432\\ 11.31908\\ 10.54988\\ 9.82506\\ 7.90112\\ 11.3196\\ 10.35593\\ 9.61703\\ 7.66058\end{array}$	$\begin{array}{c} 10.64707\\ 8.85919\\ 8.85919\\ 13.71022\\ 11.69086\\ 10.95688\\ 10.95688\\ 10.95688\\ 10.26334\\ 8.41101\\ 13.55043\\ 11.46965\\ 10.71493\\ 11.46965\\ 10.71493\\ 11.46965\\ 11.31908\\ 10.71493\\ 13.11908\\ 10.54988\\ 9.82506\\ 7.90112\\ 11.31908\\ 10.35593\\ 9.61703\\ 13.11084\\ 13.11084\\ 13.11084\end{array}$	10.64707           8.85919           8.85919           13.71022           11.69086           10.95688           10.26334           10.26334           10.26334           10.26334           11.46965           11.46965           10.71493           11.46965           10.71493           11.46965           10.71493           10.71493           10.71493           10.67133           8.10828           11.31908           9.10333           9.82506           7.90112           11.1406           11.1406           11.1406           11.1406           11.133337           9.61703           9.61703           7.66058           13.11084           10.85736	$\begin{array}{c} 10.64707\\ 8.85919\\ 8.85919\\ 13.71022\\ 11.69086\\ 10.95688\\ 10.95688\\ 10.26334\\ 8.41101\\ 13.55043\\ 13.55043\\ 11.46965\\ 10.71493\\ 13.55043\\ 13.44432\\ 10.0313\\ 8.10828\\ 9.82506\\ 7.90112\\ 11.31908\\ 9.82506\\ 7.90112\\ 11.3196\\ 10.35593\\ 9.61703\\ 7.66058\\ 10.35593\\ 9.61703\\ 7.66058\\ 10.35736\\ 10.85736\\ 10.85736\\ 10.04565\\ 10.04565\end{array}$	$\begin{array}{c} 10.64707\\ 8.85919\\ 8.85919\\ 13.71022\\ 11.69086\\ 10.95688\\ 10.95688\\ 10.95688\\ 10.26334\\ 8.41101\\ 13.55043\\ 11.46965\\ 10.71493\\ 11.46965\\ 10.71493\\ 13.44432\\ 11.406\\ 7.90112\\ 11.31908\\ 9.82506\\ 7.90112\\ 11.3196\\ 10.35593\\ 9.61703\\ 7.66058\\ 10.35593\\ 9.61703\\ 7.66058\\ 10.35536\\ 9.28356\\ 9.28356\\ 9.28356\\ 9.28356\end{array}$	$\begin{array}{c} 10.64707\\ 8.85919\\ 8.85919\\ 13.71022\\ 11.69086\\ 10.95688\\ 10.95688\\ 10.95688\\ 10.56334\\ 8.41101\\ 13.55043\\ 11.46965\\ 10.71493\\ 11.46965\\ 10.71493\\ 11.46965\\ 13.44432\\ 13.11908\\ 13.44432\\ 13.44432\\ 13.13908\\ 13.44432\\ 13.5593\\ 9.82506\\ 7.90112\\ 11.1406\\ 11.31908\\ 13.11084\\ 10.35593\\ 9.61703\\ 7.66058\\ 13.11084\\ 10.355736\\ 10.04565\\ 9.28356\\ 10.04565\\ 9.28356\\ 7.27788\end{array}$	$\begin{array}{c} 10.64707\\ 8.85919\\ 8.85919\\ 13.71022\\ 11.69086\\ 10.95688\\ 10.95688\\ 10.95688\\ 10.26334\\ 8.41101\\ 13.55043\\ 11.46965\\ 10.71493\\ 11.46965\\ 10.71493\\ 13.44432\\ 11.4908\\ 10.00313\\ 8.10828\\ 9.82506\\ 7.90112\\ 11.31908\\ 10.54988\\ 9.82506\\ 10.35593\\ 9.61703\\ 11.1406\\ 10.35593\\ 9.61703\\ 13.11084\\ 10.355736\\ 10.04565\\ 9.28356\\ 9.28356\\ 7.27788\end{array}$	$\begin{array}{c} 10.64707\\ 8.85919\\ 8.85919\\ 8.85919\\ 13.71022\\ 11.69086\\ 10.95688\\ 10.95688\\ 10.26334\\ 8.41101\\ 13.55043\\ 11.46965\\ 10.71493\\ 13.55043\\ 11.46965\\ 10.71493\\ 11.46965\\ 10.71493\\ 8.10828\\ 9.82506\\ 7.90112\\ 11.31908\\ 10.54988\\ 9.82506\\ 7.90112\\ 11.1406\\ 10.35593\\ 9.61703\\ 7.66058\\ 10.35593\\ 9.61703\\ 7.66058\\ 10.35593\\ 9.5736\\ 10.94565\\ 9.28356\\ 7.27788\\ 10.31589\\ 10.31588\\ 10.31589\\ 10.31589\\ 10.31589\\ 10.31589\\ 10.31588\\ 10.31589\\ 10.31589\\ 10.31589\\ 10.31589\\ 10.31589\\ 10.31589\\ 10.31589\\ 10.31589\\ 10.31589\\ 10.31589\\ 10.31589\\ 10.31589\\ 10.31589\\ 10.31589\\ 10.31588\\ 10.3158\\ 10.31588\\ 10.31588\\ 10.31588\\ 10.31588\\ 10.31588\\ 10.31588\\ 10.31588\\ 10.31588\\ 10.31588\\ 10.31588\\ 10.3158\\$	$\begin{array}{c} 10.64707\\ 8.85919\\ 8.85919\\ 8.85919\\ 13.71022\\ 11.69086\\ 10.95688\\ 10.95688\\ 10.95688\\ 10.26334\\ 8.41101\\ 13.55043\\ 13.55043\\ 11.46965\\ 10.71493\\ 13.44432\\ 10.71493\\ 8.10828\\ 13.44432\\ 10.71493\\ 13.44432\\ 11.31908\\ 9.82506\\ 7.90112\\ 11.31908\\ 10.54988\\ 9.82506\\ 7.90112\\ 11.31084\\ 10.35593\\ 9.61703\\ 7.66058\\ 10.04565\\ 9.28356\\ 7.27788\\ 9.45176\\ 9.45176\\ 9.45176\end{array}$	$\begin{array}{c} 10.64707\\ 8.85919\\ 8.85919\\ 13.71022\\ 11.69086\\ 10.95688\\ 10.95688\\ 10.95688\\ 10.26334\\ 8.41101\\ 13.55043\\ 11.46965\\ 11.46965\\ 11.46965\\ 13.44432\\ 13.44432\\ 11.406\\ 7.90112\\ 13.44432\\ 10.71493\\ 8.10828\\ 9.82506\\ 7.90112\\ 11.31908\\ 10.54988\\ 9.82506\\ 7.90112\\ 11.31084\\ 10.35593\\ 9.61703\\ 7.66058\\ 10.04565\\ 9.28356\\ 10.04565\\ 9.28356\\ 7.27788\\ 10.31589\\ 9.45176\\ 8.64413\end{array}$
11.54178	10.80878	10 11470	10.11479 8.25434	$   \begin{array}{r}     10.11479 \\     8.25434 \\     13.3731 \\   \end{array} $	10.11479 8.25434 13.3731 11.27744	10.11479 8.25434 13.3731 11.27744 10.51579	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 13.24613\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 13.24613\\ 11.09276\\ 11.09276\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 13.24613\\ 11.09276\\ 11.09276\\ 10.31187\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 7.87758\\ 13.24613\\ 11.09276\\ 10.31187\\ 9.57535\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 13.24613\\ 13.24613\\ 13.24613\\ 11.09276\\ 10.31187\\ 9.57535\\ 7.61872\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 13.24613\\ 11.09276\\ 11.09276\\ 11.09276\\ 10.31187\\ 9.57535\\ 7.61872\\ 13.15227\\ 13.15227\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 7.87758\\ 13.24613\\ 11.09276\\ 11.09276\\ 11.09276\\ 11.09276\\ 11.09276\\ 13.25635\\ 7.61872\\ 13.15227\\ 13.15227\\ 10.95701\\ 10.95701\\ \end{array}$	$\begin{array}{c} 10.11479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 7.87758\\ 10.51575\\ 11.09276\\ 11.09276\\ 10.31187\\ 9.57535\\ 7.61872\\ 13.15227\\ 10.95701\\ 10.95701\\ 10.95701\\ 10.16223\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 10.51579\\ 13.24613\\ 13.24613\\ 11.09276\\ 10.31187\\ 9.57535\\ 7.61872\\ 9.57535\\ 7.61872\\ 10.35701\\ 10.95701\\ 10.16223\\ 9.41349\end{array}$	$\begin{array}{c} 10.11479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 13.24613\\ 11.09276\\ 11.09276\\ 11.09276\\ 10.31187\\ 9.57535\\ 7.61872\\ 13.15227\\ 10.95701\\ 10.16223\\ 9.41349\\ 7.43005\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 7.87758\\ 13.24613\\ 11.09276\\ 11.09276\\ 11.09276\\ 11.09276\\ 13.24613\\ 13.24613\\ 11.09276\\ 10.31187\\ 9.57535\\ 7.61872\\ 13.1527\\ 10.95701\\ 10.16223\\ 9.41349\\ 7.43005\\ 7.43005\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 11.09276\\ 11.09276\\ 13.24613\\ 11.09276\\ 13.25635\\ 7.61872\\ 9.57535\\ 7.61872\\ 9.57535\\ 7.61872\\ 9.57535\\ 7.61872\\ 9.57335\\ 7.43005\\ 10.16223\\ 9.41349\\ 7.43005\\ 13.05031\\ 10.80991\\ \end{array}$	$\begin{array}{c} 10.11479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 13.24613\\ 13.24613\\ 13.24613\\ 11.09276\\ 10.31187\\ 9.57535\\ 7.61872\\ 9.57535\\ 7.61872\\ 9.57535\\ 7.61872\\ 9.57535\\ 7.43005\\ 10.95701\\ 10.95701\\ 10.95701\\ 10.80991\\ 10.80991\\ 10.00022\end{array}$	$\begin{array}{c} 10.11479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 13.24613\\ 11.09276\\ 11.09276\\ 10.31187\\ 9.57535\\ 7.61872\\ 11.092701\\ 10.315227\\ 10.95701\\ 10.16223\\ 9.41349\\ 7.43005\\ 13.15227\\ 10.95701\\ 10.60991\\ 10.00022\\ 9.23842\\ 9.23842\\ 9.23842\end{array}$	$\begin{array}{c} 10.11479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 7.87758\\ 7.87758\\ 13.24613\\ 11.09276\\ 11.09276\\ 11.09276\\ 11.09276\\ 13.25635\\ 7.61872\\ 9.57535\\ 7.61872\\ 10.31187\\ 9.57535\\ 7.43005\\ 13.15227\\ 10.95701\\ 10.16223\\ 9.41349\\ 7.43005\\ 13.05031\\ 10.00022\\ 9.23842\\ 7.22679\end{array}$	$\begin{array}{c} 10.11479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 7.87758\\ 11.09276\\ 11.09276\\ 11.09276\\ 11.09276\\ 13.1877\\ 9.57535\\ 7.61872\\ 11.092761\\ 10.31187\\ 9.57535\\ 7.61872\\ 13.16223\\ 9.41349\\ 7.43005\\ 13.05031\\ 10.00022\\ 9.23842\\ 9.23842\\ 12.88069\\ 12.88069\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 7.87758\\ 11.09276\\ 11.09276\\ 11.09276\\ 11.09276\\ 11.09276\\ 11.09276\\ 12.43005\\ 7.43005\\ 10.16223\\ 9.41349\\ 7.43005\\ 10.16223\\ 9.41349\\ 7.43005\\ 10.00022\\ 9.23842\\ 7.22679\\ 12.88069\\ 12.88069\\ 12.88069\\ 12.88069\end{array}$	$\begin{array}{c} 10.11479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 13.24613\\ 11.09276\\ 11.09276\\ 11.09276\\ 11.092761\\ 11.092767\\ 10.31187\\ 9.57535\\ 7.61872\\ 13.15227\\ 10.95701\\ 10.16223\\ 9.41349\\ 7.43005\\ 13.15227\\ 10.95701\\ 10.00022\\ 9.23842\\ 7.22679\\ 112.88069\\ 10.56513\\ 9.73075\\ 9.73075\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 7.87758\\ 7.87758\\ 13.24613\\ 11.09276\\ 11.09276\\ 11.09276\\ 11.092761\\ 10.31187\\ 9.57535\\ 7.61872\\ 13.15227\\ 10.95701\\ 10.16223\\ 9.41349\\ 7.43005\\ 7.43005\\ 13.05031\\ 10.00022\\ 9.23842\\ 7.22679\\ 10.80991\\ 10.00022\\ 9.23842\\ 7.22679\\ 10.80991\\ 10.80991\\ 10.80991\\ 10.80991\\ 10.8069\\ 12.88069\\ 12.88069\\ 10.56513\\ 9.73075\\ 8.94746\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 7.87758\\ 7.87758\\ 10.515735\\ 11.09276\\ 11.09276\\ 11.092761\\ 11.092761\\ 12.16223\\ 9.57535\\ 7.61872\\ 13.15277\\ 10.95701\\ 10.16223\\ 9.41349\\ 7.43005\\ 13.05031\\ 10.00022\\ 9.23842\\ 7.22679\\ 12.88069\\ 10.56513\\ 9.73075\\ 8.94746\\ 6.89044\end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 7.87758\\ 7.87758\\ 11.09276\\ 11.09276\\ 11.092761\\ 11.092776\\ 13.1527\\ 7.61872\\ 9.57535\\ 7.61872\\ 13.25679\\ 13.16223\\ 9.41349\\ 7.43005\\ 13.05031\\ 10.00022\\ 9.23842\\ 7.43005\\ 13.05031\\ 10.00022\\ 9.23842\\ 7.22679\\ 12.88069\\ 12.88069\\ 12.88069\\ 12.88069\\ 12.88069\\ 12.54669\\ 8.94746\\ 6.89044\\ 12.54669\\ 12.5$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 7.87758\\ 11.09276\\ 11.09276\\ 13.24613\\ 11.09276\\ 11.092761\\ 11.09276\\ 12.513305\\ 7.43005\\ 13.15227\\ 10.95701\\ 10.6022\\ 9.57335\\ 7.43005\\ 10.56513\\ 9.41349\\ 12.88069\\ 12.88069\\ 12.88069\\ 12.88069\\ 12.88069\\ 12.54669\\ 6.89044\\ 12.54669\\ 12.54669\\ 12.54669\\ 12.54669\\ 10.07836\end{array}$	$\begin{array}{c} 10.111479\\ 8.255434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 7.87758\\ 7.87758\\ 13.24613\\ 11.09276\\ 11.09276\\ 11.09276\\ 11.09276\\ 13.15227\\ 9.57535\\ 7.61872\\ 13.15227\\ 10.95701\\ 10.95701\\ 10.95701\\ 10.923842\\ 7.43005\\ 13.152679\\ 10.923842\\ 7.43005\\ 10.97305\\ 8.94746\\ 6.89044\\ 12.54669\\ 10.07836\\ 9.19421\\ 9.19421\\ 9.19421\\ \end{array}$	$\begin{array}{c} 10.111479\\ 8.25434\\ 13.3731\\ 11.27744\\ 10.51579\\ 9.79628\\ 7.87758\\ 7.87758\\ 7.87758\\ 7.87758\\ 7.87758\\ 13.24613\\ 11.092761\\ 11.09276\\ 10.31187\\ 9.57535\\ 7.61872\\ 11.09276\\ 10.31672\\ 9.57535\\ 7.61872\\ 13.15227\\ 10.95701\\ 10.95701\\ 10.956513\\ 9.41349\\ 7.43005\\ 7.43005\\ 10.956513\\ 9.41349\\ 10.0022\\ 9.23842\\ 7.22679\\ 10.0022\\ 9.23842\\ 7.22679\\ 10.0022\\ 8.94746\\ 6.89044\\ 12.54669\\ 10.07836\\ 9.19421\\ 8.36812\\ 8.3$
12.00662	11.42102	10 86009	10.86009 9.31221	10.86009 9.31221 13.38613	10.86009 9.31221 13.38613 11.62523	10.86009 9.31221 13.38613 11.62523 10.98161	10.86009 9.31221 13.38613 11.62523 10.98161 10.36962	10.86009 9.31221 13.38613 11.62523 10.98161 10.36962 8.70664	10.86009 9.31221 13.38613 11.62523 11.62523 10.98161 10.36962 8.70664 13.25923	10.86009 9.31221 13.38613 11.62523 11.62523 10.98161 10.36962 8.70664 13.25923 11.38938	10.86009 9.31221 13.38613 11.62523 10.98161 10.36962 8.70664 13.25923 11.38938 11.38938	10.86009 9.31221 13.38613 11.62523 10.98161 10.36962 8.70664 13.25923 11.38938 10.7099 10.0663	10.86009 9.31221 13.38613 11.62523 10.36962 8.70664 13.25923 13.25923 11.38938 10.7099 10.0663 8.33219	10.86009 9.31221 13.38613 11.62523 10.98161 10.36962 8.70664 13.25923 11.38938 11.38938 10.7099 10.0663 8.33219	$\begin{array}{c} 10.86009\\ 9.31221\\ 13.38613\\ 11.62523\\ 11.62523\\ 10.98161\\ 10.36962\\ 8.70664\\ 13.25923\\ 11.38938\\ 11.38938\\ 10.7099\\ 10.7099\\ 10.7099\\ 13.16989\\ 13.16989\\ 13.16989\end{array}$	10.86009 9.31221 13.38613 11.62523 10.98161 10.36962 8.70664 11.38938 11.38938 11.38938 10.7099 10.7099 10.7099 11.22587 11.22587 11.22587	10.86009 9.31221 13.38613 11.62523 10.36962 8.70664 13.25923 13.25923 11.38938 10.7099 10.7099 10.7099 10.7099 10.7663 8.33219 11.22587 11.22587 11.22587 10.5219 9.8567 9.8567	$\begin{array}{c} 10.86009\\ 9.31221\\ 13.38613\\ 11.62523\\ 11.62523\\ 10.98161\\ 10.36962\\ 8.770664\\ 13.25923\\ 11.38938\\ 11.38938\\ 10.7099\\ 10.7099\\ 10.7099\\ 11.28938\\ 10.0663\\ 8.33219\\ 11.22587\\ 10.5219\\ 9.8567\\ 8.0742\\ 8.0772\end{array}$	$\begin{array}{c} 10.86009\\ 9.31221\\ 13.38613\\ 11.62523\\ 11.62523\\ 10.98161\\ 10.36962\\ 8.70664\\ 13.25923\\ 11.38938\\ 11.38938\\ 11.38938\\ 11.38938\\ 11.325923\\ 11.22587\\ 10.0663\\ 8.33219\\ 13.16989\\ 11.22587\\ 10.5219\\ 9.8567\\ 8.0742\\ 13.07511\\ 13.07511\end{array}$	10.86009 9.31221 13.38613 11.62523 10.98161 10.36962 8.70664 11.38938 11.38938 11.38938 10.7099 10.7099 11.22587 8.33219 10.0663 8.33219 11.22587 11.22587 8.07422 8.07711 11.05526	10.86009           9.31221           13.38613           11.62523           11.62523           10.98161           10.98161           10.36962           8.70664           11.325923           11.38938           11.38938           10.7099           10.7099           10.7099           11.22587           11.22587           10.5219           9.8567           9.8567           9.8567           10.5219           9.8567           10.5526           11.05526           10.35526	$\begin{array}{c} 10.86009\\ 9.31221\\ 13.38613\\ 11.62523\\ 11.62523\\ 10.98161\\ 10.36962\\ 8.770664\\ 13.25923\\ 11.38938\\ 10.7099\\ 10.7099\\ 10.7099\\ 11.38938\\ 8.33219\\ 11.22587\\ 10.5219\\ 9.8567\\ 8.0742\\ 11.25526\\ 10.3262\\ 9.63889\\ 9.63889\end{array}$	$\begin{array}{c} 10.86009\\ 9.31221\\ 13.38613\\ 11.62523\\ 11.62523\\ 10.98161\\ 10.36962\\ 8.70664\\ 13.25923\\ 11.38938\\ 13.25923\\ 11.38938\\ 10.0663\\ 8.33219\\ 10.7099\\ 13.16989\\ 11.22587\\ 10.7099\\ 13.16989\\ 13.16989\\ 11.22587\\ 10.5219\\ 9.8567\\ 8.0742\\ 11.05526\\ 10.3262\\ 9.63889\\ 7.80711\end{array}$	$\begin{array}{c} 10.86009\\ 9.31221\\ 13.38613\\ 11.62523\\ 10.98161\\ 10.68662\\ 8.70664\\ 13.25923\\ 11.38938\\ 11.38938\\ 10.7099\\ 10.7099\\ 10.7099\\ 10.7099\\ 11.22587\\ 10.7519\\ 9.8567\\ 8.0742\\ 11.22587\\ 11.22587\\ 10.5219\\ 9.8567\\ 8.07711\\ 11.05526\\ 10.3262\\ 9.63889\\ 9.63889\\ 7.807111\\ 12.91947\\ 12.91947\end{array}$	10.86009           9.31221           13.38613           11.62523           11.62523           10.98161           10.36962           8.70664           11.325923           11.38938           11.38938           11.38938           11.38938           11.325587           10.7099           11.22587           11.22587           11.22587           11.22587           11.22588           11.22588           11.22587           10.5219           9.8567           9.8567           8.0742           10.5219           9.8567           8.07511           11.05526           10.3262           9.63389           7.80711           12.91947           10.78167	$\begin{array}{c} 10.86009\\ 9.31221\\ 13.38613\\ 11.62523\\ 11.62523\\ 10.98161\\ 10.36962\\ 8.770664\\ 13.25923\\ 11.38938\\ 10.7099\\ 10.7099\\ 10.7099\\ 10.7099\\ 11.2587\\ 10.7699\\ 11.2587\\ 10.7699\\ 11.2587\\ 10.7611\\ 11.05526\\ 10.3262\\ 9.63889\\ 7.80711\\ 11.05526\\ 10.3262\\ 9.6389\\ 7.80711\\ 11.05526\\ 10.3262\\ 9.6389\\ 7.80711\\ 11.05526\\ 10.3262\\ 9.6389\\ 7.80711\\ 11.05526\\ 10.3262\\ 9.6389\\ 7.80711\\ 11.05526\\ 10.3262\\ 9.6389\\ 7.80711\\ 11.05526\\ 10.3262\\ 9.6389\\ 7.80711\\ 11.05526\\ 10.3262\\ 9.6389\\ 7.80711\\ 11.05526\\ 10.3262\\ 9.6389\\ 7.80711\\ 11.05526\\ 10.3262\\ 9.6389\\ 7.80711\\ 11.05526\\ 10.3262\\ 9.6389\\ 7.80711\\ 11.05526\\ 10.3262\\ 9.6389\\ 7.80711\\ 11.05526\\ 10.3262\\ 9.6389\\ 7.80711\\ 10.78167\\ 1$	$\begin{array}{c} 10.86009\\ 9.31221\\ 13.38613\\ 11.62523\\ 11.62523\\ 10.98161\\ 10.36962\\ 8.70664\\ 13.25923\\ 11.38938\\ 10.7099\\ 10.7099\\ 10.7611\\ 11.22587\\ 10.7619\\ 9.8567\\ 8.332199\\ 11.22587\\ 10.7519\\ 9.8567\\ 8.0742\\ 11.25526\\ 10.5219\\ 9.8567\\ 8.0772\\ 11.05526\\ 10.35262\\ 9.63889\\ 7.80711\\ 11.05526\\ 10.3562\\ 9.63889\\ 7.80711\\ 11.05526\\ 10.3562\\ 9.63889\\ 7.80711\\ 11.05526\\ 10.3586\\ 7.80711\\ 11.05526\\ 9.63889\\ 7.80711\\ 11.05526\\ 10.3562\\ 9.63889\\ 7.80711\\ 11.05526\\ 10.3562\\ 9.63889\\ 7.80711\\ 11.05526\\ 7.80711\\ 11.05526\\ 7.80711\\ 11.0568\\ 7.80711\\ 11.0568\\ 7.8072\\ 10.0012\\ 7.8072\\ 10.0012\\ 7.8072\\ 10.0012\\ 7.8072\\ 10.0012\\ 7.8072\\ 10.0012\\ 10.000\\ 10.0000$	$\begin{array}{c} 10.86009\\ 9.31221\\ 13.38613\\ 11.62523\\ 10.98161\\ 10.98161\\ 10.98161\\ 10.36962\\ 8.70664\\ 11.38938\\ 11.38938\\ 11.389389\\ 10.0663\\ 8.33219\\ 10.7699\\ 13.16989\\ 11.22587\\ 10.7699\\ 13.16989\\ 11.22587\\ 10.7619\\ 9.8567\\ 8.0742\\ 11.2587\\ 10.7516\\ 10.3262\\ 9.63889\\ 7.80711\\ 11.05526\\ 10.3262\\ 9.63889\\ 7.80711\\ 11.05526\\ 10.3262\\ 9.63889\\ 7.80711\\ 11.05526\\ 10.78167\\ 10.78167\\ 10.78167\\ 10.738409\\ 7.38409\end{array}$	$\begin{array}{c} 10.86009\\ 9.31221\\ 13.38613\\ 11.62523\\ 10.98161\\ 10.98161\\ 10.6862\\ 8.70664\\ 13.25923\\ 11.38938\\ 11.38938\\ 10.7699\\ 10.7699\\ 10.7699\\ 11.22587\\ 10.7699\\ 11.22587\\ 10.7699\\ 11.22587\\ 10.7619\\ 10.5219\\ 9.8567\\ 8.0742\\ 11.05526\\ 10.3262\\ 9.8567\\ 10.0135\\ 9.63889\\ 7.38409\\ 7.38409\\ 7.38409\end{array}$	$\begin{array}{c} 10.86009\\ 9.31221\\ 13.38613\\ 11.62523\\ 10.98161\\ 10.98161\\ 10.68662\\ 8.70664\\ 13.25923\\ 11.38938\\ 10.7099\\ 10.7099\\ 11.389389\\ 10.76919\\ 10.5219\\ 9.8567\\ 8.0742\\ 11.22587\\ 10.5219\\ 9.8567\\ 10.5219\\ 9.8567\\ 10.5219\\ 9.8567\\ 10.78167\\ 10.25821\\ 10.258$	$\begin{array}{c} 10.86009\\ 9.31221\\ 13.38613\\ 11.62523\\ 11.62523\\ 10.98161\\ 10.36962\\ 8.70664\\ 13.25923\\ 11.38938\\ 10.7099\\ 10.7099\\ 10.7619\\ 11.22587\\ 10.7619\\ 11.22587\\ 10.7619\\ 11.22587\\ 10.7619\\ 11.05526\\ 10.5219\\ 9.8567\\ 8.0742\\ 11.05526\\ 10.78167\\ 10.78167\\ 10.78167\\ 10.78167\\ 10.78167\\ 10.0135\\ 9.29179\\ 7.38409$	$\begin{array}{c} 10.86009\\ 9.31221\\ 13.38613\\ 11.62523\\ 11.62523\\ 10.98161\\ 10.36962\\ 8.70664\\ 13.25923\\ 11.38938\\ 11.38938\\ 10.7099\\ 10.7099\\ 10.7619\\ 10.75267\\ 10.7611\\ 11.05526\\ 10.5219\\ 9.8567\\ 8.0742\\ 11.055267\\ 10.76111\\ 11.055262\\ 9.63889\\ 7.80711\\ 11.055262\\ 9.63889\\ 7.80711\\ 11.055262\\ 9.29179\\ 7.38409\\ 7.384$
12.30691	11.57846	10~88647	10.88647 9.01797	$10.88647 \\9.01797 \\14.00437$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.93331\end{array}$	10.88647 9.01797 14.00437 11.93331 11.17685	$\begin{array}{c} 10.88647\\9.01797\\14.00437\\11.93331\\11.17685\\10.45998\end{array}$	10.88647 9.01797 14.00437 11.93331 11.17685 10.45998 8.53515	$\begin{array}{c} 10.88647\\9.01797\\14.00437\\11.93331\\11.17685\\10.45998\\8.53515\\8.53515\\13.81233\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.93331\\ 11.17685\\ 11.17685\\ 10.45998\\ 8.53515\\ 13.81233\\ 13.81233\\ 11.68442\\ \end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 13.81233\\ 13.81233\\ 11.68442\\ 11.68442\\ 10.9091\\ \end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.53515\\ 13.81233\\ 11.68442\\ 11.68442\\ 10.9091\\ 10.17562\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.53515\\ 13.81233\\ 13.81233\\ 11.68442\\ 13.81233\\ 11.68442\\ 10.9091\\ 10.17562\\ 8.21397\\ 8.21397\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.93331\\ 11.17685\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.53515\\ 13.81233\\ 13.81233\\ 13.81233\\ 11.68442\\ 13.81233\\ 11.68442\\ 10.17562\\ 8.21397\\ 8.21397\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 114.00437\\ 114.00437\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.53515\\ 13.81233\\ 13.81233\\ 13.81233\\ 11.68442\\ 11.68442\\ 10.17562\\ 8.21397\\ 13.67564\\ 11.50641\\ \end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.93331\\ 11.17685\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.53515\\ 11.68442\\ 11.68442\\ 11.68442\\ 10.17562\\ 8.21397\\ 13.67564\\ 11.50641\\ 11.50641\\ 11.50641\\ 10.71745\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.53515\\ 13.81233\\ 13.81233\\ 13.81233\\ 13.81233\\ 13.6764\\ 11.68442\\ 10.17562\\ 8.21397\\ 13.67564\\ 11.50641\\ 11.50641\\ 10.71745\\ 9.972\\ 9.972\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.53515\\ 13.81233\\ 13.81233\\ 13.81233\\ 13.87562\\ 8.21397\\ 10.17562\\ 8.21397\\ 10.17562\\ 8.21397\\ 11.50641\\ 11.50641\\ 11.50643\\ 11.50643\\ 7.9843\\ 7.9843\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.93331\\ 11.17685\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.53515\\ 13.81233\\ 13.81233\\ 13.81233\\ 13.67564\\ 11.56641\\ 10.17562\\ 8.21397\\ 13.67564\\ 11.50641\\ 10.71745\\ 9.972\\ 7.9843\\ 7.9843\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.93331\\ 11.17685\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.53515\\ 11.68442\\ 11.68442\\ 11.68442\\ 11.6991\\ 10.17562\\ 8.21397\\ 10.17562\\ 8.21397\\ 11.50641\\ 11.50641\\ 11.50641\\ 11.50641\\ 11.50641\\ 11.50641\\ 11.50641\\ 11.50641\\ 11.50641\\ 11.51745\\ 9.972\\ 7.9843\\ 13.53129\\ 11.31746\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.9331\\ 11.9331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.53515\\ 13.81233\\ 11.68442\\ 11.68442\\ 10.9091\\ 10.9091\\ 10.71762\\ 8.21397\\ 11.56641\\ 11.56641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5641\\ 11.5621\\ 31.53129\\ 11.31746\\ 11.31746\\ 11.31746\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.53515\\ 13.81233\\ 13.81233\\ 11.68442\\ 11.68442\\ 10.9091\\ 10.9091\\ 10.9091\\ 10.17562\\ 8.21397\\ 13.67564\\ 11.50641\\ 11.50641\\ 11.50641\\ 11.51745\\ 9.972\\ 7.9843\\ 13.53129\\ 11.31746\\ 10.51386\\ 9.75561\\ 9.75561\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 11.00437\\ 11.93331\\ 11.17685\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 12.81233\\ 13.81233\\ 13.67564\\ 10.17562\\ 8.21397\\ 13.675641\\ 10.17562\\ 8.21397\\ 13.67564\\ 11.50641\\ 11.50641\\ 10.771745\\ 9.972\\ 7.9843\\ 11.31746\\ 10.51386\\ 9.75561\\ 7.74046\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.17685\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.53515\\ 11.68442\\ 11.68442\\ 11.68442\\ 11.68442\\ 11.68442\\ 11.68442\\ 11.68442\\ 11.5664\\ 10.17562\\ 8.21397\\ 13.67564\\ 11.5764\\ 10.1745\\ 9.972\\ 7.9843\\ 13.53129\\ 11.31746\\ 10.51386\\ 9.75561\\ 7.74046\\ 13.2988\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.07855\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 11.68442\\ 11.68442\\ 11.68442\\ 11.68442\\ 11.6641\\ 10.17562\\ 8.21397\\ 13.67564\\ 11.50641\\ 11.50641\\ 11.50641\\ 11.51745\\ 9.972\\ 7.9843\\ 11.31746\\ 10.51386\\ 9.77561\\ 7.74046\\ 11.31746\\ 13.2988\\ 11.01063\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.93331\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.5351564\\ 11.68442\\ 10.9091\\ 10.9091\\ 10.17562\\ 8.21397\\ 13.67564\\ 11.50641\\ 11.50641\\ 11.50641\\ 11.51745\\ 9.972\\ 7.9843\\ 11.31746\\ 10.51386\\ 9.75561\\ 7.74046\\ 10.51386\\ 10.51386\\ 10.51386\\ 11.317467\\ 11.31746\\ 10.51386\\ 10.51386\\ 10.51386\\ 11.31746\\ 10.51386\\ 10.55561\\ 10.18274\\ 10.1063\\ 10.18274\\ 10.1063\\ 10.18274\\ 1$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.93331\\ 11.17685\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 13.67564\\ 11.6641\\ 10.17562\\ 8.21397\\ 13.67564\\ 11.56641\\ 10.771745\\ 9.972\\ 7.9843\\ 11.31746\\ 10.71745\\ 9.972\\ 7.9843\\ 11.31746\\ 10.51386\\ 9.75561\\ 7.74046\\ 11.31746\\ 10.51386\\ 9.75561\\ 7.74046\\ 11.31746\\ 10.51386\\ 9.75561\\ 7.74046\\ 11.31746\\ 10.51386\\ 9.75561\\ 7.74046\\ 11.31746\\ 9.75561\\ 7.74046\\ 11.31746\\ 9.75561\\ 7.74046\\ 11.31746\\ 9.75561\\ 7.74046\\ 11.31746\\ 9.7574\\ 9.40341\\ 9.40341\\ 9.40341\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 11.00437\\ 11.93331\\ 11.17685\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 13.81233\\ 13.67564\\ 10.17562\\ 8.21397\\ 13.675641\\ 11.50641\\ 10.17562\\ 8.21397\\ 13.53129\\ 11.51746\\ 10.71745\\ 9.972\\ 7.9843\\ 11.31746\\ 10.51386\\ 9.75561\\ 7.74046\\ 13.2988\\ 11.01063\\ 11.01063\\ 11.01063\\ 11.01063\\ 12.34399\\ 9.40341\\ 7.34399\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.17685\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.53515\\ 11.68442\\ 11.68442\\ 11.68442\\ 11.6141\\ 10.17562\\ 8.21397\\ 13.67564\\ 11.57643\\ 11.50641\\ 11.50641\\ 11.5145\\ 9.972\\ 7.9843\\ 13.53129\\ 13.53129\\ 13.53129\\ 13.53129\\ 11.31746\\ 10.17742\\ 13.53129\\ 11.31746\\ 10.18274\\ 9.40341\\ 7.34399\\ 12.86165\\ 12.86165\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.00437\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 11.68442\\ 11.68442\\ 11.68442\\ 10.17562\\ 8.21397\\ 10.51386\\ 9.75564\\ 11.50641\\ 11.50641\\ 11.50641\\ 11.51745\\ 9.972\\ 7.9843\\ 11.31746\\ 11.31746\\ 12.51386\\ 9.77561\\ 7.74046\\ 11.31746\\ 13.2988\\ 11.31746\\ 11.31766\\ 1$	$\begin{array}{c} 10.88647\\ 9.01797\\ 14.00437\\ 11.17685\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 8.53515\\ 13.67564\\ 11.68442\\ 10.17562\\ 8.21397\\ 13.67564\\ 11.56641\\ 11.56641\\ 11.57643\\ 11.5745\\ 9.972\\ 7.9843\\ 11.31746\\ 11.31746\\ 10.71745\\ 9.972\\ 7.9843\\ 11.31746\\ 10.71745\\ 9.972\\ 7.9843\\ 11.31746\\ 10.71745\\ 9.972\\ 7.9843\\ 11.31746\\ 10.71745\\ 9.972\\ 11.31746\\ 10.71745\\ 10.71745\\ 10.71745\\ 10.42201\\ 9.54503\\ 9.54503\end{array}$	$\begin{array}{c} 10.88647\\ 9.01797\\ 11.06457\\ 11.93331\\ 11.17685\\ 11.93331\\ 11.17685\\ 10.45998\\ 8.53515\\ 10.45998\\ 11.68442\\ 12.87564\\ 11.60641\\ 10.17562\\ 8.21397\\ 10.17562\\ 8.21397\\ 10.57349\\ 11.31746\\ 11.31746\\ 13.53129\\ 10.771745\\ 9.972\\ 7.9843\\ 11.31746\\ 11.31746\\ 11.31746\\ 12.86165\\ 11.31746\\ 12.86165\\ 10.42201\\ 9.54503\\ 8.72359\\ \end{array}$
10.5528 0.07965	9.87265	XXX	9.22982 7.51207	9.22952 7.51207 12.29049	9.22962 7.51207 12.29049 10.33201	9.22852 7.51207 12.29049 10.33201 9.62285	$\begin{array}{c} 9.22952 \\ 7.51207 \\ 12.29049 \\ 10.33201 \\ 9.62285 \\ 8.95423 \end{array}$	9.22982 7.51207 12.29049 10.33201 9.62285 8.95423 7.17772	$\begin{array}{c} 9.22952 \\ 7.51207 \\ 12.29049 \\ 10.33201 \\ 9.62285 \\ 8.95423 \\ 7.1777 \\ 12.19185 \end{array}$	9.22982 7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 12.19185 10.17474	9.22952 7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 12.19185 10.17474 9.44615 9.44615	9.22952 7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 12.19185 10.17474 9.44615 9.44615 8.76041	9.22982 7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 12.19185 12.19185 12.19185 12.19185 10.17474 9.44615 9.44615 8.76041 6.94581	9.22982 7.51207 12.29049 10.33201 9.62285 8.95423 7.17722 12.19185 10.17474 9.44615 8.76041 6.94581 12.11736	9.22952 7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 12.19185 10.17474 12.19185 10.17474 9.44615 8.76041 6.94581 12.11736 12.11736	9.22952 7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 12.19185 10.17474 10.17474 10.17474 9.44615 8.76041 6.94581 6.94581 12.11736 10.05776 9.31518 9.31518	9.22952 7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 8.95423 10.17474 10.17474 9.44615 8.76041 6.94581 10.17464 8.76041 6.94581 11.11736 10.05776 9.31518 8.61718	9.22952 7.51207 12.29049 10.33201 9.62285 8.95423 7.1772 8.95423 7.1772 12.19185 10.17474 9.44615 8.76041 6.94581 10.17476 9.31518 8.76041 6.94581 12.11736 10.05776 9.31518 8.61718	9.22952 7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 10.17474 10.17474 10.17474 9.44615 8.76041 6.94581 12.11736 10.17776 9.31518 8.61718 8.61718 8.61778 9.31518	9.22952           7.51207           7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           12.19185           10.17474           9.62285           8.95423           7.17772           10.17474           9.64581           10.17474           9.44615           8.76041           6.94581           10.17576           9.31518           8.61718           8.61778           9.31518           9.31518           9.31518           9.31518           9.93512           9.9298	9.22952           7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           10.33201           9.44615           8.76041           6.94581           10.17474           9.44615           8.76041           6.94581           10.17576           9.31518           8.60776           9.31518           8.61778           9.31518           9.31518           9.31518           9.31518           9.31518           9.31518           9.31528           9.23151           9.23152           9.9298           9.17224	9.22952 7.51207 7.51207 12.29049 10.33201 9.62285 8.95423 7.17772 8.95423 10.17474 9.44615 8.76041 6.94581 10.17474 8.76041 6.94581 112.11736 10.05776 9.31518 8.61718 8.61718 8.61728 9.9298 9.9298 9.97224 8.46118	9.22982           7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           10.33201           9.62285           8.95423           7.1772           10.33201           9.62285           8.95423           7.1772           12.19185           12.19185           12.19185           10.17474           9.44615           8.76041           6.94581           10.17476           9.44615           8.76041           6.94581           10.17476           9.31518           9.31518           9.31518           9.33512           9.923512           9.923512           9.9298           9.17224           8.46118           6.59171	9.22952           9.22952           7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           10.33201           9.62285           8.95423           7.17772           10.33201           9.62285           8.95423           7.17772           10.17474           9.44615           8.76041           6.94581           10.17476           9.44615           8.76041           6.94581           10.17576           9.31518           8.61718           8.61718           6.77578           9.9298           9.17224           9.17224           8.46118           6.59171           11.89561	9.22952           7.51207           7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           9.42615           9.42615           10.17474           9.44615           8.76041           6.94581           10.17474           9.44615           8.76041           6.94581           10.17576           9.44615           8.76041           6.94581           10.17576           9.44615           8.76041           6.94581           12.11736           12.11736           9.31518           8.61718           6.77578           9.17224           9.17224           9.17224           8.46118           6.59171           11.89561           9.71432	9.22952           7.51207           7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           10.33201           9.62285           8.95423           7.17772           10.33201           9.62285           8.95423           7.17772           10.17474           9.44615           8.76041           6.94581           10.17476           9.44615           8.76041           6.94581           10.17576           9.44615           8.76041           6.94581           12.11736           9.31518           8.61718           9.23512           9.9298           9.17224           8.46118           6.59171           11.89561           9.71432           9.71432           9.71432	9.22952         7.51207         7.51207         10.33201         9.62285         8.95423         7.17772         9.62285         8.95423         7.17772         10.33201         9.62285         8.95423         7.17772         10.17474         9.44615         9.44615         8.76041         6.94581         10.17474         9.44615         8.76041         6.94581         10.17576         9.31518         8.61718         6.77578         9.17224         9.17224         9.17224         9.17224         8.46118         6.59171         11.89561         9.71432         8.93206         8.1996	9.22952           7.51207           7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           10.33201           9.62285           8.95423           7.17772           10.33201           9.62285           8.95423           7.17772           10.17474           9.44615           8.76041           6.94581           10.17476           9.44615           8.76041           6.94581           10.17476           9.31518           8.61718           6.94581           10.05776           9.31518           8.61718           8.61718           8.61718           9.17224           9.17224           9.17234           9.17234           9.1732           9.1732           9.1732           8.46118           6.59171           9.71432           8.1996           8.1996           8.1996	9.22952           7.51207           7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           9.62285           8.95423           7.17772           10.33201           9.62285           8.95423           7.1772           10.17474           9.44615           8.76041           6.94581           10.17474           9.44615           8.76041           6.94581           10.17474           9.44615           8.76041           6.94581           10.17474           9.31518           8.61718           8.61718           8.61718           9.33512           9.9298           9.17224           9.17224           8.46118           6.59171           11.89561           9.71432           8.1996           6.28526           9.11.61239	9.22952           7.51207           7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           9.44615           8.76041           9.44515           10.17474           9.44615           8.76041           6.94581           10.17474           9.44615           8.76041           6.94581           10.17474           9.44615           8.76041           6.94581           10.17474           9.44615           8.75041           6.94581           12.11736           9.231518           8.61718           6.77578           9.273312           9.2733           9.27733	9.22952           7.51207           7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           9.62285           8.95423           7.17772           10.33201           9.62285           8.95423           7.1772           12.19185           10.17474           9.44615           8.76041           6.94581           10.17476           9.44615           8.76041           6.94581           10.17576           9.44615           8.76041           6.94581           10.05776           9.33512           9.33512           9.23528           9.17224           8.46118           6.59171           11.89561           9.71432           8.1996           8.1996           8.1996           8.44566           8.44566	9.22952           9.22952           7.51207           12.29049           10.33201           9.62285           8.95423           7.17772           9.62285           8.95423           7.17772           10.33201           9.62285           8.95423           7.1772           12.19185           10.17474           9.44615           8.76041           6.94581           10.17476           9.44615           8.76041           6.94581           10.17576           9.44615           8.76041           6.94581           12.11736           9.31518           8.61718           6.77578           9.17224           9.17224           8.46118           6.59171           11.89561           9.71432           8.1996           8.1996           8.1996           8.44566           8.44566           8.44566
105	7.01 7.01	TUN	115	115 5 100	115           5         100           105	5 100 105 107	5 100 105 107 107 109	5 115 5 100 107 107 109 115	5 115 5 100 105 107 109 115 1 109	5 100 105 107 107 109 115 1 100 105	5 100 105 107 107 109 115 1 100 105 107	5 100 107 107 107 109 115 109 105 107 105	5 115 5 100 107 107 109 115 107 107 107 109	5 100 107 107 107 109 115 100 105 100 107 100 100	5 100 105 107 107 109 107 107 107 100 100 100 100	5 100 107 107 107 109 115 100 100 100 100 105 100	5 100 107 107 107 107 109 115 100 100 100 100 100 100 100 100	5         105           107         107           107         109           107         106           107         106           107         106           107         107           107         106           107         106           107         106           107         106           107         107           107         106           107         107           107         107           107         107	5         115           5         100           107         107           115         106           115         107           115         107           115         107           107         106           107         106           106         106           107         106           107         107           107         106           107         106           107         106	5 100 107 107 107 107 109 100 100 100 107 100 100 100 100 100 100	5         115           5         100           107         107           107         109           1115         100           101         100           102         103           103         103           1115         100           101         103           101         103           103         103           104         103           105         103           106         103           107         103           103         103	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	115           5         100           107         107           109         115           101         100           102         103           103         106           104         106           105         109           106         106           107         103           106         106           107         106           107         107           107         106           107         107           107         107           107         107           107         107           107         107           107         107           107         107	5         115           5         100           107         107           107         109           101         100           101         100           101         100           101         100           1010         100           1010         100           1010         100           100         100           100         100           100         100           100         100           100         100           100         100           100         100           100         100           100         100           100         100           100         100	5         115           5         100           107         107           107         109           1115         100           100         100           1115         100           107         100           1115         100           100         100           101         100           100         100           100         100           100         100           100         100           100         100           100         100           100         100           100         100           100         100           100         100           100         100           100         100	5         105           115         106           107         107           107         109           107         106           107         107           107         109           107         100           107         100           107         100           107         100           107         100           107         100           107         100           107         100           106         100           107         100           106         100           107         100           106         100           107         100	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	5         1115           107         107           107         107           115         107           107         109           115         107           107         107           107         107           107         107           107         107           107         107           107         107           107         107           107         107           107         107           107         107           107         107           107         107	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$
				)543 <u>-0.25</u>	0543 -0.25	0543 -0.25	0543 -0.25	0543 -0.25	0543 -0.25 6556 -0.1	0543 -0.25 6556 -0.1	0543 -0.25 6556 -0.1	0543 -0.25 6556 -0.1	0543 -0.25 6556 -0.1	0543 -0.25 6556 -0.1 0587 0	0543 -0.25 6556 -0.1 0587 0	0543 -0.25 6556 -0.1 0587 0	0543 -0.25 6556 -0.1 0587 0	0543         -0.25           5556         -0.1           0587         0	0543 -0.25 6556 -0.1 0587 0 4636 0.1	0543 -0.25 6556 -0.1 0587 0 1636 0.1	0543         -0.25           6556         -0.1           6587         0           7636         0.1	0543 -0.25 6556 -0.1 0587 0 4636 0.1	0543         -0.25           6556         -0.1           0587         0           4636         0.1	0543         -0.25           0556         -0.1           6556         -0.1           6587         0           744         0.25	0543         -0.25           6556         -0.1           6556         -0.1           6587         0           4636         0.1           4636         0.1           0744         0.25	0543         -0.25           6556         -0.1           6556         -0.1           6587         0           7436         0.1           0744         0.25	0543         -0.25           6556         -0.1           6556         -0.1           4636         0.1           744         0.25	0543         -0.25           6556         -0.1           6556         -0.1           4636         0.1           4636         0.1           0744         0.25	0543         -0.25           05566         -0.1           6556         -0.1           4636         0.1           4636         0.1           1016         0.25	0543         -0.25           6556         -0.1           6556         -0.1           4636         0.1           744         0.25           0744         0.25           1016         0.5	0543         -0.25           6556         -0.1           6556         -0.1           6556         -0.1           6587         0           744         0.25           0744         0.25           1016         0.5	0543         -0.25           6556         -0.1           6556         -0.1           6556         -0.1           74636         0           0744         0.25           .016         0.25

Table 4: Approximation of Asian option prices, maturity 3 years

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