Discussion Paper B–323

The Pricing of Asian Options under Stochastic Interest Rates*

J. Aase Nielsen † Klaus Sandmann ‡

May 1995. This version: March 13, 1996

*Financial support from the Danish Natural & Social Science Research Council by the first author and by the Deutsche Forschungsgemeinschaft Sonderforschungsbereich 303 at the University of Bonn by both authors, is gratefully acknowledged. Both authors are grateful for helpful comments of an anonymous referee. The plots were produced using GNUPLOT 3.5. Document typeset in LaTeX.

†Department of Operations Research, University of Aarhus, Bldg. 530, Ny Munkegade, DK-8000 Aarhus C, Denmark. E-mail: atsjan@mi.aau.dk

‡Department of Statistics, Rheinische Friedrich-Wilhelms-Universität Bonn, Adenauerallee 24–42, D-53113 Bonn, Germany. E-mail: sandmann@addi.or.uni-bonn.de

The Pricing of Asian Options under Stochastic Interest Rates

Abstract

The purpose of this paper is to analyse the effect of stochastic interest rates on the pricing of Asian options. It is shown that a stochastic, in contrast to a deterministic, development of the term structure of interest rates has a significant influence.

The price of the underlying asset, e.g. a stock or oil, and the prices of bonds are assumed to follow correlated two dimensional Ito processes. The averages considered in the Asian options are calculated on a discrete time grid, e.g. all closing prices on Wednesdays during the lifetime of the contract. The value of an Asian option will be obtained through the application of Monte Carlo simulation, and for this purpose the stochastic processes for the basic assets need not to be severely restricted. However to make comparison with published results originating from models with deterministic interest rates we will stay within the setting of a Gaussian framework.

Keywords

Asian Options, Forward Risk Adjusted Measure, Monte Carlo Simulation.

1 Introduction

The basic economic setting in which pricing of Asian options has been analysed is characterized by an underlying asset which adheres to a geometric Brownian motion and by a deterministic development of the bond market. No easily implementable closed form solution to the pricing problem has so far been developed in the literature. The suggested methods of pricing all builds on different schemes of approximations.

Kemna and Vorst (1990) show that the Asian option price, subject to the boundary condition characteristic for the option considered, is the solution to a second order partial differential equation in three variables, time, spot price of the underlying asset and the known information about the average value. Rather than solving the partial differential equation, Kemna and Vorst apply Kolmogorov's backward equation and obtain that the price of the Asian option can be written as the discounted expected value of the maturity payment of the option. To solve the pricing equation which involves knowledge of the distribution of a sum of correlated lognormal distributions Kemna and Vorst apply Monte Carlo simulation.

Carverhill and Clewlow (1990) solve the pricing equation applying Fast Fourier Transform techniques to obtain an approximation to the law of the average.

Levy (1992) argues that the sum of correlated lognormal random variables is well approximated by another lognormal distribution and applying Wilkinson's approximation a lognormal distribution with the first and second moment chosen in accordance to the correct distribution is applied as a surrogate. In Turnbull and Wakeman (1991) an Edgeworth expansion, involving the first four cumulants, is used to represent the approximating distribution by a lognormal distribution.

Vorst (1992) uses the fact that the geometric average is never greater than its corresponding aritmetic average, and due to the assumed geometric Brownian motion of the underlying asset the geometric average is also lognormal and the price of the geometric Asian option can be found in closed form. By means of this Vorst calculates a lower as well as an upper bound for the arithmetic Asian option, and then chooses in an ad hoc manner the price of the Asian option in a way which guarantees that the established bounds are fulfilled.

Geman and Yor (1993) succeed in obtaining a closed form solution for the Asian option but it is unfortunately of a very complicated form. To determine the price an inversion of a nontrivial Laplace transform has to be performed.

In this paper we will relax the assumption concerning the deterministic nature of the bond market but retain the geometric Brownian motion for the underlying asset. The stochastic interest rate enviroment will be assumed to be Gaussian which in accordance to e.g. Jamshidian (1991) and El Karoui, Lepage, Myneni and Viswanathana (1991) implies a lognormal distribution of the zero coupon bond prices. Pricing of standard options in this setting has been analysed in e.g. Amin and Jarrow (1992) and Amin and Bodurtha Jr. (1995). The pricing of Asian options and in particular the influence of the stochastic interest rate on the pricing will be analysed in this paper.

The schedule of the paper is as follows. In section 2, the notation and the definition of the contract is presented. Section 3 deals with the pricing of Asian options. A discussion of different numerical approaches is given in section 4. Section 5 contains the simulation result. Finally, section 6 concludes.

2 Notation and definition of the contract

The following notation will be applied:

X	exercise price of the Asian option.
t_n	a date included in the average calculation, $n = 1, 2, ..., N$; $t_o = 0$.
t_N	the maturity date of the option contract, $t_N = T$.
$S(t)$	the price of the underlying asset at time t.
$D(t, t')$	the price at date t of a zero coupon bond with maturity date t', $t \leq t'$.

$A(t_n) = \frac{1}{n} \sum_{i=1}^{n} S(t_i)$ the arithmetic average of the spot prices at the date t_n; $n = 1. \dots, N$.

$G(t_n) = \sqrt[n]{\prod_{i=1}^{n} S(t_i)}$ the geometric average of the spot prices at the date t_n; $n = 1. \dots, N$.

$V_A(T) = \max \left\{ \frac{1}{N} \sum_{i=1}^{N} S(t_i) - X, 0 \right\} = \max \left\{ A(t_N) - X, 0 \right\}$

the benefit from the arithmetic Asian option received at maturity date T .

$V_G(T) = \max \left\{ \sqrt[N]{\prod_{i=1}^{N} S(t_i)} - X, 0 \right\} = \max \left\{ G(t_N) - X, 0 \right\}$

the benefit from the geometric Asian option received at maturity date T .

$r(t)$ the instantaneous risk free rates of interest at time t.

Next the option prices at time t_0, $V_A(t_0)$ and $V_G(t_0)$, will be found in accordance to the absence of arbitrage possibilities in the financial market. We restrict ourself to the pricing of *European* type Asian call options where the averaging period still has to start. The value of an Asian option during the averaging period can be calculated the same way by adjusting the exercise price X, see e.g. Kemna and Vorst (1990) and Vorst (1992).

3 Pricing of the Asian option

Assume that the dynamics of the underlying asset $S(t)$ is determined by a lognormal diffusion process with time dependent volatility. For the interest rate market we concentrate on a Gaussian term structure model[1], which is well known from previous work by Jamshidian (1991) and El Karoui, Lepage, Myneni and Viswanathana (1991). Under the absence of arbitrage opportunities there exists a probability measure P^* such that the stochastic behaviour of both markets are related in the following way:

$$\begin{aligned} dS(t) &= r(t)S(t)dt + \sigma_1(t)S(t)dW_1^*(t) + \sigma_2(t)S(t)dW_2^*(t), \\ dD(t,t') &= r(t)D(t,t')dt + \sigma(t,t')D(t,t')dW_1^*(t), \end{aligned}$$

[1]As special cases we will discuss the Vasicek (1977) model and the continuous time limit of the Ho and Lee (1986) model

where W_1^* and W_2^* are independent standard Wiener processes. The volatility functions $\sigma_1(t), \sigma_2(t)$ and $\sigma(t,t')$ are assumed to be non-stochastic and satisfy the usual regularity conditions[2], in particular $\sigma(t,t) = 0$ and $D(t,t) = 1$ with probability 1. In other words we are working under the so called risk neutral martingale measure. Note that by $\frac{\sigma_1(t)}{\sqrt{\sigma_1^2(t)+\sigma_2^2(t)}}$ the instantaneous correlation between both markets is determined. Due to the stochastic development of $r(t)$, it will be convenient to work in the T-forward risk adjusted probability measure, denoted by P^T, where it is well known[3], that the differential equations for $\frac{D(t,t')}{D(t,T)}$ and $\frac{S(t)}{D(t,T)}$ are respectively given by

$$\begin{aligned} d\left(\frac{D(t,t')}{D(t,T)}\right) &= \frac{D(t,t')}{D(t,T)} \cdot (\sigma(t,t') - \sigma(t,T))dW_1^T(t), \\ d\left(\frac{S(t)}{D(t,T)}\right) &= \frac{S(t)}{D(t,T)} \cdot [(\sigma_1(t) - \sigma(t,T))dW_1^T(t) + \sigma_2(t)dW_2^T(t)] \quad , \end{aligned}$$

where W_1^T and W_2^T are independent standard Wiener processes under the P^T probability measure. The change to the forward risk adjusted measure P^T implies that the stochastic discounting is replaced by the time-t measurable discounting and in particular that

$$\frac{S(t)}{D(t,T)} = E_t^T\left[\frac{S(T)}{D(T,T)}\right] = E_t^T[S(T)] \tag{1}$$

in contrast to

$$S(t) = E_t\left[\exp\left\{-\int_t^T r(u)du\right\} S(T)\right]$$

under the risk neutral probability measure. The solutions of the above stochastic differential equations under the T-forward risk adjusted measure P^T are given by:

$$\begin{aligned} S(t) &= S(t_0) \cdot \frac{D(t,T)}{D(t_0,T)} \cdot \exp\left\{-\frac{1}{2}\int_{t_0}^{t} \left((\sigma_1(u) - \sigma(u,T))^2 + \sigma_2^2(u)\right) du \right. \\ &\quad \left. + \int_{t_0}^{t} (\sigma_1(u) - \sigma(u,T))dW_1^T(u) + \int_{t_0}^{t} \sigma_2 dW_2^T(u)\right\}, \\ \frac{D(t,T)}{D(t_0,T)} &= \frac{D(t,t)}{D(t_0,t)} \cdot \exp\left\{\frac{1}{2}\int_{t_0}^{t} (\sigma(u,t) - \sigma(u,T))^2 du - \int_{t_0}^{t} (\sigma(u,t) - \sigma(u,T))\, dW_1^T(u)\right\}. \end{aligned}$$

This allows us to express the the solution for the underlying asset as

$$\begin{aligned} S(t_i) &= \frac{S(t_0)}{D(t_0,t_i)} \cdot \exp\left\{-\frac{1}{2}\int_{t_0}^{t_i} ((\sigma_1(u) - \sigma(u,T))^2 + \sigma_2^2(u))du\right\} \\ &\quad \cdot \exp\left\{\frac{1}{2}\int_{t_0}^{t_i} (\sigma(u,t_i) - \sigma(u,T))^2 du\right\} \\ &\quad \cdot \exp\left\{\int_{t_0}^{t_i} (\sigma_1(u) - \sigma(u,t_i))dW_1^T(u) + \int_{t_0}^{t_i} \sigma_2(u)dW_2^T(u)\right\} \quad . \end{aligned} \tag{2}$$

[2]In a general setup we could allow for stochastic volatility functions, see e.g. Geman, El Karoui and Rochet (1995), but for the continuous time numerical procedure we will be forced to restrict ourselves to non-stochastic functions.

[3]In a similar economic context see e.g. Nielsen and Sandmann (1995).

The value of an Asian option with the discrete average $A(t_N)$ is determined by

$$V_A(t_0) = D(t_0,T)E^T\left[\max\left\{A(t_N) - X, 0\right\}\right]. \tag{3}$$

Under the specified S_t-process and the Gaussian interest rate dynamics, we know that the arithmetic average is determined by a sum of correlated lognormal distributed variables. So far, there exists no closed form expression for the distribution of such a sum. Therefore, numerical techniques have to be applied to approximate the value $V_A(t_0)$ of an Asian option. Observe that (2) turns itself into a much simpler equation if $\sigma(u,t) = 0 \forall u \leq t \forall t$ corresponding to a non stochastic development of the term structure of interest rates. In this case easily implementable techniques are available in the literature. In the following section these methods will be extended to include the Gaussian term structure model, and we show that major differences appear. Then in section 5, applying the formal analysis of section 4, we show that the parameter which mainly influences the pricing of Asian options is the correlation between the underlying asset and the term structure.

4 Numerical approximation for Asian options

In a similar economic setting, Carverhill and Clewlow (1990) solve the pricing equation for an Asian option by applying the Fast Fourier Transformation technique in order to calculate the distribution of the arithmetic average. Their idea is to rewrite the equation of the underlying asset such that $S(t_i) = S(t_{i-1}) \cdot a^T(t_{i-1}, t_i)$ which implies that the arithmetic average can be reformulated as

$$A(T) = S(t_0)\left[1 + a^T(t_0,t_1)\left[1 + a^T(t_1,t_2)\left[1 + \cdots + a^T(t_{N-2},t_{N-1})\left[1 + a^T(t_{N-1},T)\right]\ldots\right]\right]\right],$$

where the random variables $a^T(t_{i-1}, t_i)$ in their case are pairwise independent. It can easily be seen that for a Gaussian term structure model the coefficients $a^T(t_{i-1}, t_i)$ are defined as:

$$\begin{aligned}
a^T(t_{i-1},t_i) \quad := \quad & \frac{D(t_0,t_{i-1})}{D(t_0,t_i)} \cdot \exp\left\{-\frac{1}{2}\int_{t_{i-1}}^{t_i} (\sigma_1(u) - \sigma(u,T))^2 + \sigma_2^2(u)du\right\} \\
& \cdot \exp\left\{\frac{1}{2}\int_{t_0}^{t_i} (\sigma(u,t_i) - \sigma(u,T))^2 du - \frac{1}{2}\int_{t_0}^{t_{i-1}} (\sigma(u,t_{i-1}) - \sigma(u,T))^2 du\right\} \\
& \cdot \exp\left\{\int_{t_{i-1}}^{t_i} \sigma_2(u)dW_2^T(u) + \int_{t_{i-1}}^{t_i} (\sigma_1(u) - \sigma(u,t_i))\, dW_1^T(u)\right\} \\
& \cdot \exp\left\{-\int_{t_0}^{t_{i-1}} (\sigma(u,t_i) - \sigma(u,t_{i-1}))\, dW_1^T(u)\right\},
\end{aligned} \tag{4}$$

which implies that the stochastic variables $a^T(t_{i-1}, t_i)$ are *not* pairwise independent unless

$$\sigma(u,t_i) = \sigma(u,t_{i-1}) \quad \forall u \leq t_{i-1} < t_i \quad \Longrightarrow \quad \sigma(u,t) = 0 \quad \forall u \leq t.$$

For this reason the Fast Fourier Transformation cannot be applied to calculate the distribution of the arithmetic average.

Turnbull and Wakeman (1991) suggest to approximate the unknown density ρ^T of the sum of lognormal distributed variables by the following Edgeworth expansion:

$$\rho^T(x) \approx f(x) + \frac{c_2}{2!}\frac{\partial^2 f(x)}{\partial x^2} - \frac{c_3}{3!}\frac{\partial^3 f(x)}{\partial x^3} + \frac{c_4}{4!}\frac{\partial^4 f(x)}{\partial x^4} \tag{5}$$

where $f(x)$ denotes the lognormal density function, i.e.

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}\sigma_f}\frac{1}{x}\exp\left\{-\frac{(\ln x - \mu_f)^2}{2\sigma_f^2}\right\}, \\ \text{and} \qquad c_2 &= \mathcal{K}(2,\rho^T) - \mathcal{K}(2,f), \\ c_3 &= \mathcal{K}(3,\rho^T) - \mathcal{K}(3,f), \\ c_4 &= \mathcal{K}(4,\rho^T) - \mathcal{K}(4,f) + 3c_3^2. \end{aligned}$$

$\mathcal{K}(i,f) = E_f[(X - E_f[X])^i]$ equals the i-th central moment with respect to the lognormal distribution given by f, resp. $\mathcal{K}(i,\rho^T)$ with respect to the unknown distribution given by P^T. To calculate these moments, the first four non–central moments of the average $A(T)$ must be computed. The parameters μ_f and σ_f are chosen such that the first two non-central moments under both measures are identical. Given the moments and a vanishing error term, the value of the arithmetic Asian option at time t_0 is approximated by:

$$\begin{aligned} & D(t_0,T) \cdot E^T[\max\{A(T) - X, 0\}] \\ \approx\ & D(t_0,T) \cdot \left\{e^{\mu_f + \sigma_f^2/2} N(d) - X N(d - \sigma_f) + \frac{c_2}{2!} f(X) - \frac{c_3}{3!}\frac{\partial f}{\partial x}(X) + \frac{c_4}{4!}\frac{\partial^2 f}{\partial x^2}(X)\right\} \end{aligned} \tag{6}$$

with $d = \frac{\mu_f - \ln(X) + \sigma_f^2}{\sigma_f}$ and $N(.)$ denoting the standard normal distribution.

Since the $a^T(t_{i-1}, t_i)$ in (4) are stochastic dependent variables, it is not possible to calculate the moments of $A(T)$ as in Turnbull and Wakeman (1991). A generalized but much slower algorithm is given in the Appendix.

Based on the strong relationship between the arithmetic and the geometric average, Vorst (1992) suggests an alternative approximation of the arbitrage price for an Asian option and furthermore derives upper and lower bounds for these prices. The Vorst (1992) approximation and the bounds on the price of the Asian option are given by

$$\begin{aligned} & D(t_0,T)\left(e^{m_G + \frac{1}{2}\sigma_G^2} N(d_1) - X N(d_1 - \sigma_G)\right) \\ \leq\ & D(t_0,T) E^T[\max\{A(T) - X, 0\}] \\ \approx\ & D(t_0,T)\left(e^{m_G + \frac{1}{2}\sigma_G^2} N(d_2) - X' N(d_2 - \sigma_G)\right) \\ \leq\ & D(t_0,T)\left(e^{m_G + \frac{1}{2}\sigma_G^2} N(d_1) - X N(d_1 - \sigma_G) + E^T[A(T)] - E^T[G(T)]\right), \end{aligned} \tag{7}$$

where

$$d_1 = \frac{m_G - \ln(X) + \sigma_G^2}{\sigma_G}, \quad d_2 = \frac{m_G - \ln(X') + \sigma_G^2}{\sigma_G},$$

$$X' = X - (E^T[A(T)] - E^T[G(T)]),$$

$$\left.\begin{aligned} m_G &= E^T[\ln G(T)] \\ \sigma_G^2 &= V^T[\ln G(T)] \end{aligned}\right\} \Rightarrow E^T[G(T)] = \exp\left\{m_G + \tfrac{1}{2}\sigma_G^2\right\}.$$

Thus the Vorst (1992) approximation only involves the computation of the first moment for the arithmetic average and the mean and variance of the logarithmic geometric average. We notice that the approximation is derived by transforming the probability measure of a lognormal distribution with support $\mathbb{R}^+$ to a lognormal distribution with support $[E^T[A(T)] - E^T[G(T)], \infty[$. Since the support of the random variable $A(T)$ is $\mathbb{R}^+$ the distance $E^T[A(T)] - E^T[G(T)] > 0$ is important for the approximation error. Furthermore the discounted difference is an upper bound for the approximation error.

4.1 Arithmetic and geometric averages under the P^T measure

To derive the Vorst (1992) approximation for the arbitrage price of an Asian option the expectation under the T-forward risk adjusted measure of the arithmetic and geometric averages have to be calculated. Due to the stochastic behaviour of the interest rate, the computation of the values is different from the one Vorst (1992) proposed. Furthermore it will turn out that the behaviour of the expected values depends crucially on the term structure model. Although this is mainly the case if we consider an unrealistic long time to maturity of the Asian option, this is a critical point with respect to the assumption of lognormal bond prices respectively a Gaussian term structure model. On the other hand we have to assume lognormality of bond prices to derive the closed form expressions for the expected values of these averages in a straightforward manner. The following theorems do summarize the results for these averages[4].

Theorem 1 *Let* $\underline{\underline{T}}(N) := \{0 = t_0 < t_1 < ... < t_N = T\}$ *be a fixed discretization of the time axis and suppose that the time T-forward price dynamics of the underlying asset is given by (2). The expected value of the arithmetic mean under the T-forward risk adjusted measure is given by:*

$$E^T[A(T)] = \frac{S(t_0)}{N} \sum_{i=1}^{N} \frac{1}{D(t_0, t_i)} \cdot exp\left\{ \int_{t_0}^{t_i} [\sigma_1(u) - \sigma(u, t_i)] [\sigma(u, T) - \sigma(u, t_i)] \, du \right\}.$$

If moreover the grid size is given by $\Delta t = t_{i+1} - t_i = \frac{T}{N}$ *and the initial term structure is integrable and bounded away from zero we have*

$$\lim_{\Delta t \to 0} E^T[A(T)] = \frac{S(t_0)}{T} \int_{t_0}^{T} \frac{1}{D(t_0, u)} \cdot exp\left\{ \int_{t_0}^{v} [\sigma_1(u) - \sigma(u, v)] [\sigma(u, T) - \sigma(u, v)] \, du \right\} dv.$$

The consequence of a stochastic interest rate implied by Theorem 1 is interesting. Suppose that the interest rate is deterministic, then Theorem 1 implies that

$$E^T[A(T)] \quad = \quad \frac{S(t_0)}{N} \sum_{i=1}^{N} \frac{1}{D(t_0, t_i)} \quad \rightarrow \quad \frac{S(t_0)}{T} \int_{0}^{T} \frac{1}{D(t_0, u)} du \qquad \text{for} \qquad \Delta t \to 0.$$

In the case of a flat initial term structure, i.e. $D(t_0, t) = \exp\{-rt\}$, which is usually assumed within the Black-Scholes framework, this implies that the forward value of the expected arithmetic mean is strictly increasing in T with:

$$\lim_{\Delta t \to 0} E^T[A(T)] \quad = \quad \frac{S(t_0)}{T} \int_{0}^{T} e^{ru} du = \frac{S(t_0)}{T} \frac{1}{r} [e^{rT} - 1] \to \infty \quad \text{for} \quad T \to \infty.$$

[4] For convenience to the reader the proofs are given in the Appendix.

Of course the t_0 value of the expected arithmetic mean is strictly decreasing in T with

$$D(t_0,T)E^T[A(T)] = \frac{S(t_0)}{T}\frac{1}{r}[1-e^{-rT}] \to 0 \quad \text{for} \quad T\to\infty.$$

If the interest rate is stochastic, i.e. $\sigma(t,t')>0$ the situation is more complicated. Observe first that for a reasonable Gaussian term structure model the price volatility differential $\sigma(u,T)-\sigma(u,v)$ should be either always positive[5] or negative $\forall u\le v\le T$. Due to the symmetry of the Brownian motion we therefore assume without loss of generality that $\sigma(u,T)-\sigma(u,v)\ge 0 \quad \forall u\le v\le T$. Therefore $E^T[A(T)]$ is strictly increasing in $\sigma_1(u)$, and for non positive correlation, i.e. $\sigma_1(u)\le 0 \quad \forall u$ we have

$$E^T[A(T)] < \frac{S(t_0)}{T}\int\limits_0^T \frac{1}{D(t_0,u)}du \quad .$$

This means that the expected arithmetic mean for a non-stochastic interest rate is an upper bound for $E^T[A(T)]$. We therefore can expect lower option values due to stochastic interest rates in this situation.

If the correlation is positive, i.e. $\sigma_1(u)>0 \quad \forall u$ a sufficient condition for $E^T[A(T)] > \frac{S(t_0)}{T}\int\limits_0^T \frac{1}{D(t_0,u)}du$ is

$$\int_{t_0}^{v}\left[\sigma_1(u)-\sigma(u,v)\right]\left[\sigma(u,T)-\sigma(u,v)\right]du > 0 \quad \forall v<T.$$

In the case of a Vasicek (1977) model with constant mean reversion $\alpha>0$, i.e $\sigma(u,v) = \frac{\sigma}{\alpha}\left(1-\exp\{-\alpha(v-u)\}\right)$ and $\sigma_1(u)=\sigma_1$ this is satisfied if $\forall v<T$

$$\begin{aligned} 0 &< \frac{\sigma}{\alpha}\left(e^{-\alpha v}-e^{-\alpha T}\right)\left[\frac{\sigma_1}{\alpha}\left(e^{\alpha v}-1\right)+\frac{\sigma}{\alpha^2}\left(1-cosh(\alpha v)\right)\right] \\ \Longrightarrow \quad 2\frac{\sigma_1}{\sigma} &> \frac{2}{\alpha}\frac{cosh(\alpha v)-1}{e^{\alpha v}-1} \quad \to \quad v \quad \text{for} \quad \alpha\to 0. \end{aligned}$$

This indicates higher prices of Asian options due to stochastic interest rates for small time to maturities T. For $\alpha\to 0$, i.e. the Ho and Lee (1986) model this condition is satisfied for $T<2\frac{\sigma_1}{\sigma}$. For time to maturities $T>2\frac{\sigma_1}{\sigma}$ simulations show that the expected arithmetic mean begins to decrease[6] for a long time period followed by an increase at around 80 years.

To calculate the expected value of the geometric mean we use that under the Gaussian term structure model the geometric mean is lognormally distributed. Therefore

$$E^T[G(T)] = \exp\left\{E^T\left[\ln G(T)\right]+\frac{1}{2}\cdot V^T[\ln G(T)]\right\}.$$

Theorem 2 *Suppose that the initial term structure $D(t_0,\cdot):[0,T]\to \mathbb{R}_{>0}$ is integrable and bounded away from zero. Let $\underline{\underline{T}}(N)$ be a fixed discretization of the time axis and suppose that $S(t)$ is given by (2). The expected value and the variance of the logarithmic geometric mean*

[5] E.g. this is the case for the generalized Vasicek (1977) model and the continuous time limit of the Ho and Lee (1986) model.

[6] This behaviour is illustrated in Figures 1 to 4.

under the P^T measure are given by

$$E^T[lnG(T)] = lnS(t_0) - \frac{1}{N}\sum_{i=1}^{N} lnD(t_0,t_i) - \frac{1}{2N}\sum_{i=1}^{N}\int_{t_0}^{t_i}\left[\sigma_1^2(u)+\sigma_2^2(u)\right]du$$
$$+\frac{1}{2N}\sum_{i=1}^{N}\int_{t_0}^{t_i}\left(2\sigma(u,T)\left[\sigma_1(u)-\sigma(u,t_i)\right]+\sigma^2(u,t_i)\right)du$$
$$\lim_{\Delta t\to 0} E^T[lnG(T)] = lnS(t_0) - \frac{1}{T}\int_{t_0}^{T} lnD(t_0,u)du - \frac{1}{2T}\int_{t_0}^{T}\int_{t_0}^{v}\left[\sigma_1^2(u)+\sigma_2^2(u)\right]dudv$$
$$+\frac{1}{2T}\int_{t_0}^{T}\int_{t_0}^{v}\left(2\sigma(u,T)\left[\sigma_1(u)-\sigma(u,v)\right]+\sigma^2(u,v)\right)dudv$$

If the interest rate is deterministic, the volatility functions $\sigma_{1/2}(.)$ are constant and the initial term structure is flat, i.e. $D(t_0,t)=\exp\{-rt\}$ we get

$$\lim_{\Delta t\to 0} E^T[\ln G(T)] = \ln S(t_0) + \frac{1}{2}rT - \frac{1}{4}\left[\sigma_1^2+\sigma_2^2\right]T.$$

Depending on the size of the volatility of the underlying asset, $\sqrt{\sigma_1^2+\sigma_2^2}$, this either converges to plus or minus infinity as the time to maturity T approaches infinity.

Since the sign of $E^T[\ln G(T)]$ for $T\to\infty$ is determined by the last integral there is a strong tendency to reverse the above result in the case of a stochastic interest rate, i. e. $\sigma(u,v)>0$. To illustrate this consider the Vasicek (1977) model with constant parameters. Then

$$\lim_{\Delta t\to 0} E^T[\ln G(T)] = \ln S(t_0) - \frac{1}{T}\int_0^T \ln D(t_0,u)du - \frac{1}{4}\left(\sigma_1^2+\sigma_2^2\right)T + \sigma_1\sigma\frac{\frac{1}{2}\alpha^2T^2+(\alpha T+1)e^{-\alpha T}-1}{\alpha^3 T}$$
$$+\sigma^2\frac{-\alpha^2T^2-4\alpha Te^{-\alpha T}+\frac{3}{2}(1-e^{-2\alpha T})+aT}{4T\alpha^4}.$$

For a flat initial term structure this can be simplified to:

$$\lim_{\Delta t\to 0} E^T[\ln G(T)] = \ln S(t_0) + \frac{1}{2}\left(r-\frac{1}{2}\left(\sigma_1^2+\sigma_2^2\right)+\frac{\sigma_1\sigma}{\alpha}-\frac{\sigma^2}{2\alpha^2}\right)T + \frac{\sigma^2}{4\alpha^3} + g(T) \quad ,$$

where $\lim_{T\to\infty} g(T)=0$. Therefore the Vasicek model approaches the same limit as in the determistic interest rate case for sufficiently large mean reversion coefficient α. If instead the mean reversion coefficient α is small, i.e. in the limit we get the Ho and Lee (1986) model, then

$$\lim_{\alpha\to 0}\lim_{\Delta t\to 0} E^T[\ln G(T)] = \ln S(t_0) - \frac{1}{T}\int_0^T \ln D(t_0,u)du - \frac{1}{4}\left(\sigma_1^2+\sigma_2^2\right)T + \frac{1}{3}\sigma_1\sigma T^2 - \frac{1}{12}\sigma^2T^3$$
$$\to -\infty \quad \text{for} \quad T\to\infty,$$

as long as $D(t_0,t)\geq \exp\{-kt^\delta\}$ $\forall t$ for some constants $k>0$ and $\delta<3$.

Theorem 3 *Under the assumptions of Theorem 2 we have*

$$V^T[lnG(T)] = \frac{1}{N^2}\sum_{i=0}^{N-1}\left[\int_{t_i}^{t_{i+1}}((N-i)\cdot\sigma_2(u))^2+\left((N-i)\sigma_1(u)-\sum_{j=i+1}^{N}\sigma(u,t_j)\right)^2 du\right]$$

$$\lim_{\Delta t\to 0} V^T[lnG(T)] = \frac{1}{T^2}\int_{t_0}^{T}(T-u)^2\cdot\left(\sigma_1^2(u)+\sigma_2^2(u)\right)du$$
$$-\frac{2}{T^2}\int_{t_0}^{T}\left[\int_u^T(T-u)\cdot\sigma_1(u)\cdot\sigma(u,v)dv\right]du+\frac{1}{T^2}\int_{t_0}^{T}\left[\int_u^T\sigma(u,v)dv\right]^2du$$

Consider again the Vasicek (1977) model and assume that $\alpha > 0$ and $\sigma_{1/2}$ are constant. Solving in this situation the integral for the variance yields

$$\lim_{\Delta t\to 0} V^T[\ln G(T)] = \frac{1}{3}(\sigma_1^2+\sigma_2^2)T-\sigma_1\sigma\cdot\left(\frac{2\alpha^3T^3-3\alpha^2T^2-6(\alpha T+1)e^{-\alpha T}+6}{3\alpha^4T^2}\right)$$
$$+\sigma^2\cdot\left(\frac{2\alpha^3T^3-12\alpha Te^{-\alpha T}-3e^{-2\alpha T}+6\alpha T(1-\alpha T)+3}{6\alpha^5T^2}\right)$$
$$\to \frac{1}{3}(\sigma_1^2+\sigma_2^2)T-\frac{1}{4}\sigma_1\sigma T^2+\frac{1}{20}\sigma^2T^3 \qquad \text{for} \qquad \alpha\to 0.$$

To clarify the impact of the stochastic interest rate consider as a border case a flat initial term structure and deterministic interest rate, which imply

$$\lim_{\Delta t\to 0} E^T[G(T)] = S(t_0)\cdot\exp\left\{\frac{1}{2}\left(r-\frac{1}{6}(\sigma_1^2+\sigma_2^2)\right)T\right\}$$

and depending on the sign of $r-\frac{1}{6}(\sigma_1^2+\sigma_2^2)$ this either converges to zero or plus infinity for $T\to+\infty$. If instead the interest rate is stochastic, i.e. $\sigma(t,t')>0$ the convergence behaviour may be different. Consider once again the Vasicek model with constant $\alpha>0$ and $\sigma_{1/2}$ and a flat initial term structure, then

$$\lim_{\Delta t\to 0} E^T[G(T)] = S(t_0)\cdot\exp\left\{\frac{1}{2}\left(r-\frac{1}{6}(\sigma_1^2+\sigma_2^2)+\frac{\sigma_1\sigma}{3\alpha}-\frac{\sigma^2}{6\alpha^2}\right)T+\frac{\sigma_1\sigma}{2\alpha^2}-\frac{\sigma^2}{4\alpha^3}+g(T)\right\},$$

where $\lim_{T\to\infty} g(T)=0$. If the mean reversion coefficient α is large then the behaviour for $T\to\infty$ of the Vasicek model and the determistic interest case is the same. If instead α is small, then the expected geometric average under the T-forward risk adjusted measure converges to zero. As the extreme case consider the Ho and Lee model, i.e.

$$\lim_{\alpha\to 0}\lim_{\Delta t\to 0} E^T[G(T)] = S(t_0)\cdot\exp\left\{-\frac{1}{T}\int_{t_0}^{T}D(t_0,u)du-\frac{1}{12}(\sigma_1^2+\sigma_2^2)T+\frac{5}{24}\sigma_1\sigma T^2-\frac{7}{120}\sigma^2T^3\right\}$$

which converges to zero for $T\to+\infty$ as long as $D(t_0,t)\geq\exp\{-kt^\delta\}\ \forall t$ for $k>0$ and $\delta<3$.

Suppose that the total volatility of the underlying asset σ_S is fixed, i.e. $\sigma_S=\sqrt{\sigma_1^2+\sigma_2^2}$ is assumed to be constant. In this situation the expected value of the geometric average is a strictly increasing function in σ_1. In other words, fixing σ_S the expected geometric average under the T-forward risk adjusted measure increases in the instantaneous correlation.

To summarize our results at this point, Figures 1 to 4 do show some of the effects. In these figures we have chosen a flat initial term structure with $D(t_0,t)=(1.06)^{-t}$. Furthermore the volatility of the underlying asset is equal to 25 %, i.e.

$$\sigma_S^2dt := V[dS(t)|S(t)] = \left(\sigma_1^2+\sigma_2^2\right)dt = 0.25^2dt.$$

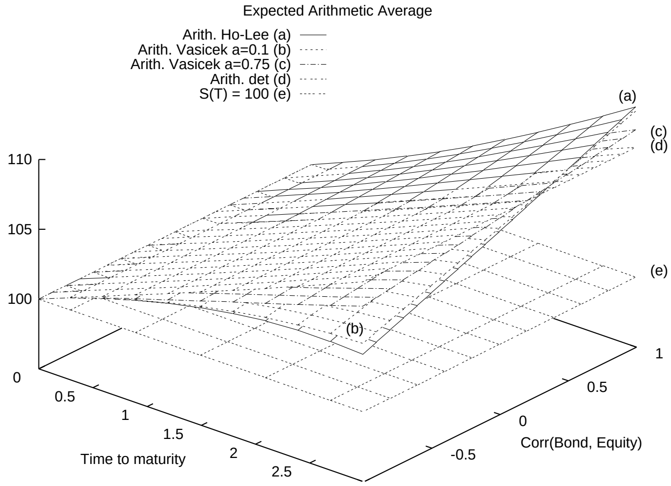

Figure 1: Expected arithmetic averages for $T \leq 3$ years, 120 realizations of the underlying asset per year with $S(t_0) = 100$, $\sigma_S = 25\%$ and $\sigma = 10\%$.

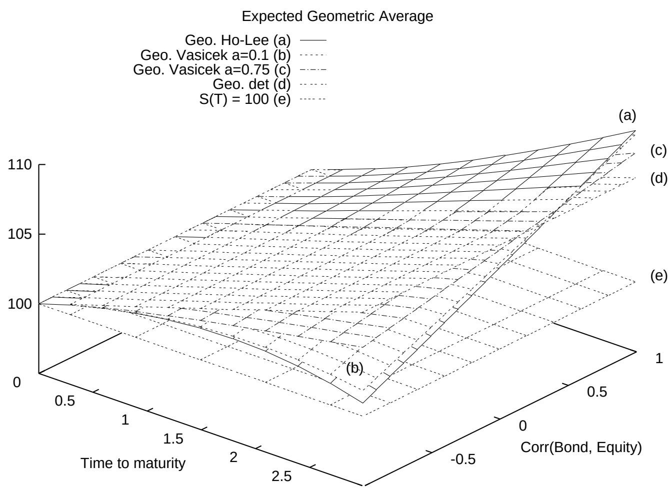

Figure 2: Expected geometric averages for $T \leq 3$ years, 120 realizations of the underlying asset per year with $S(t_0) = 100$, $\sigma_S = 25\%$ and $\sigma = 10\%$.

As model of the term structure we concentrate in Section 5 our price simulation of the Asian option on the continuous time limit of the Ho and Lee (1986) model. With respect to Figures 1 to 4 this model is the extreme case of the Vasicek (1977) model. We regard the Ho and Lee model as the most sensitive case. Therefore we set in Section 5 the price volatility of the zero coupon bonds equal to $\sigma(u, v) = \sigma \cdot (v - u)$ with $\sigma = 0.1$. Furthermore note that by σ_1 and σ_2 the instantaneous correlation between the underlying asset and the term structure is defined by

$$\rho \quad := \quad \frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} = \frac{\sigma_1}{\sigma_S}. \tag{8}$$

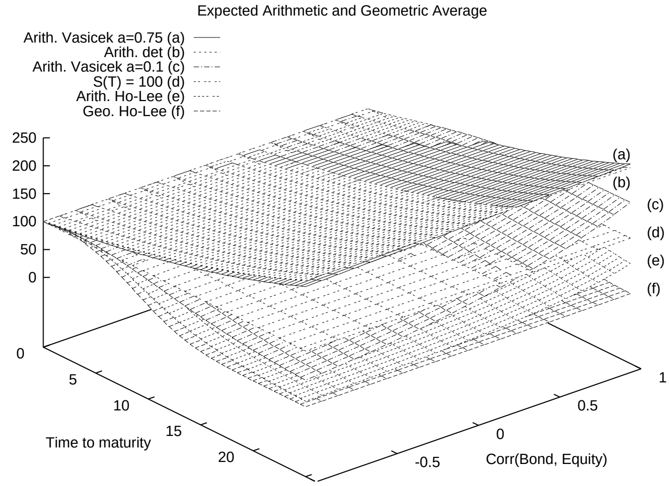

Figure 3: Expected arithmetic and geometric averages for $T \leq 25$ years, 120 realizations of the underlying asset per year with $S(t_0) = 100$, $\sigma_S = 25\%$ and $\sigma = 10\%$.

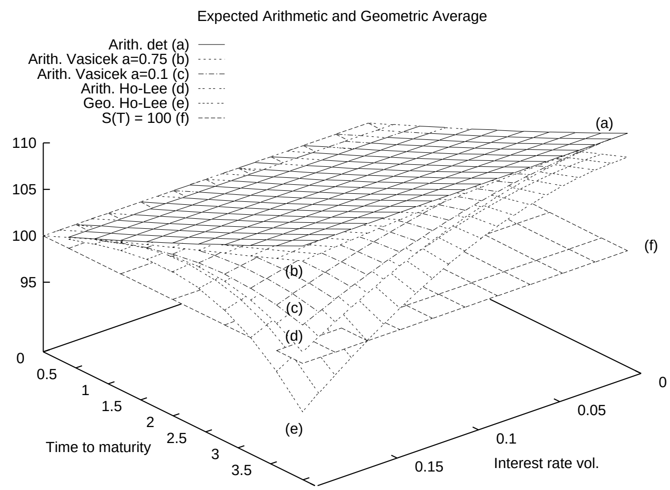

Figure 4: Expected arithmetic and geometric averages for $T \leq 4$ years, 120 realizations of the underlying asset per year with $S(t_0) = 100$, $\sigma_S = 25\%$ and $\rho = 0$.

It means that we can parametrise σ_1 and σ_2 in terms of the correlation such that the (total) volatility of the underlying asset is always equal to $\sigma_S = 25\%$:

$$
\begin{array}{rrcl}
 & \sigma_1 : [-1, +1] & \to & [-\sigma_S, \sigma_S] \\
(9) & \rho & \mapsto & \sigma_1(\rho) := \rho\sigma_S \\
\text{and} & \rho & \mapsto & \sigma_2(\rho) := \sqrt{(1-\rho^2)}\sigma_S.
\end{array}
$$

5 Simulation results for Asian options

In this section we compare the different approximations proposed by Turnbull and Wakeman (1991) and Vorst (1992) for the pricing of Asian options with the results obtained by a Monte Carlo simulation. The starting point for the Monte Carlo simulation is the formulation of the asset price dynamics as in (4). In the case of the Ho and Lee (1986) model and constant volatility functions $\sigma_{1/2}$ this can be reformulated to:

$$
\begin{aligned}
S(t_i) = \quad & S(t_{i-1}) \quad \cdot \frac{D(t_0,t_{i-1})}{D(t_0,t_i)} \cdot \exp\left\{-\frac{1}{3}\sigma^2\left[(t_{i-1}-T)^2 t_{i-1} - (t_i - T)^2 t_i\right]\right\} \\
(10) \qquad & \cdot\exp\left\{-\frac{1}{2}\left[\sigma_1^2+\sigma_2^2\right](t_i - t_{i-1}) - \frac{1}{2}\left[\sigma_1\sigma - \frac{1}{3}\sigma^2 T\right]\left[(T-t_i)^2-(T-t_{i-1})^2\right]\right\} \\
& \cdot\exp\left\{\sigma_1\left(W_1^T(t_i)-W_1^T(t_{i-1})\right)+\sigma_2\left(W_2^T(t_i)-W_2^T(t_{i-1})\right)\right\} \\
& \cdot\exp\left\{-\sigma\left[t_i W_1^T(t_i) - t_{i-1}W_1^T(t_{i-1}) - \int_{t_{i-1}}^{t_i} u\, dW_1^T(u)\right]\right\} \quad .
\end{aligned}
$$

To simulate the last part of equation (10) we notice that

$$
(11) \qquad \left[t_i W_1^T(t_i) - t_{i-1}W_1^T(t_{i-1}) - \int_{t_{i-1}}^{t_i} u\, dW_1^T(u)\right] = \int_{t_{i-1}}^{t_i} W_1^T(u)\, du
$$

which is a normal distributed variable.

For the below simulations we have chosen $\sigma = 10\%$, $\sigma_S = 25\%$ and 120 time periods per year, i.e. $\Delta t = 120^{-1}$. Furthermore we have chosen four different maturity dates corresponding to $T = 0.5, 1,$ and 3 years.

The approximation of the Asian option by Turnbull and Wakeman (1991) involves the computation of the non–central moments of the arithmetic mean up to order four. These moments can be calculated using the algorithms proposed in the Appendix. On the other hand we could estimate them by Monte Carlo simulation. Table 1 shows some results obtained by the algorithms and the Monte Carlo simulation. The Monte Carlo simulation leads to a reasonable approximation of the first and second moment and therefore also of the variance. Although the approximation of the higher moments is not as good, the skewness and the leptokursis of the unknown distribution are approximated quite satisfactorily. If not otherwise specified we use 100.000 paths and the antithetic technique[7] for the simulation, a flat initial term structure with $D(t_0,t) = (1.06)^{-t}$, and the initial value of the asset $S(t_0) = 100$.

In line with Theorems 1 to 3 Table 1 shows the increase of the expected arithmetic and geometric average as a function of the instantaneous correlation. Beyond this we see that the variance of the arithmetic and geometric average decreases as a function in ρ. Therefore we have two opposite effects which do influence the pricing of Asian options. The decrease of the variance of the geometric average is for the chosen parameter constellation a direct consequence of Theorems 2 and 3. However we should mention that there are parameter values for σ_S, σ and T such that the variance is an increasing function in ρ. This is typically the situation if the

[7]This implies in total 200.000 paths

time to maturity is extremely long. For the arithmetic average these findings are based on the implementation of the numerical procedure but so far no analytical results can be given.

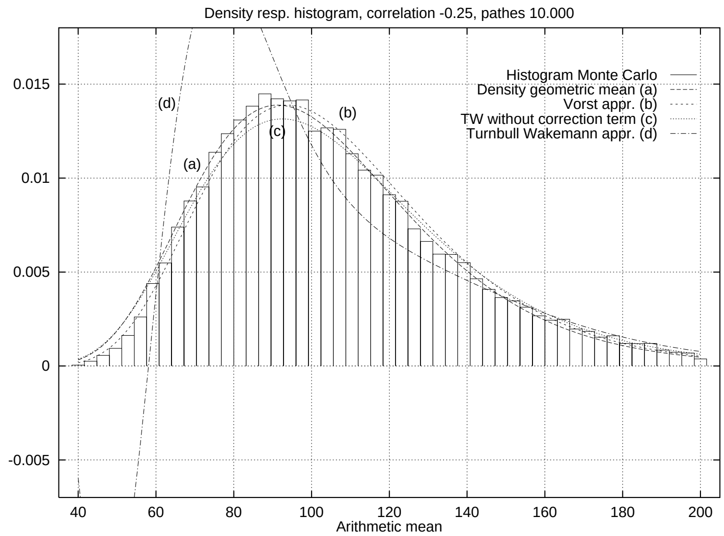

Figure 5: Densities of the arithmetic and geometric average with $\sigma_S = 0.25$, Ho-Lee term structure model with $\sigma = 0.1, \rho = -0.25, T = 3, D(t_0, t) = 1.06^{-t}$ and $T = 3$.

Both the Turnbull and Wakeman (1991) and the Vorst (1992) approximation of the Asian option can be interpreted as an approximation of the distribution resp. probability density function of the arithmetic mean of lognormal random variables. The approximation of Turnbull and Wakeman (1991) is given by (6) whereas the one used by Vorst (1992) is given by pricing formula (7). Since we price under the T–forward risk adjusted measure we can compare these approximations with the density function obtained by the Monte Carlo simulation. Note, that by multiplying with $D(t_0, T)$ these functions do represent the implied state prices underlying the different numerical approximations. The influence of the correlation, which already can be seen in Table 1, seems to be quite important for the Turnbull and Wakeman (1991) approximation, as indicated by Figure 5. Furthermore the Vorst (1992) approximation seems to be better than the Turnbull and Wakeman (1991) approximation even if we do neglect the correlation term; but nevertheless there is an underestimation of lower and an overestimation of higher realizations relative to the Monte Carlo simulation.

Finally we can consider the pricing of Asian options. In addition to the antithetic technique we use the arbitrage price of a geometric average option as a control variate. Thus the Monte Carlo value for the Asian option is obtained by:

$$\hat{c}(T, X) = \frac{D(t_0, T)}{2M} \cdot \sum_{m=1}^{2M} \left[\left[\frac{1}{N \cdot T} \sum_{i=1}^{N \cdot T} S(t_i) - X \right]^+ - \left[\sqrt[N \cdot T]{\prod_{i=1}^{N \cdot T} S(t_i)} - X \right]^+ \right] + g(T, X)$$

Maturity	ρ	Method	$E^T[A]$	$E^T[A^2]$	$\sqrt{V^T[A]}$	$E^T[A^3]$	$E^T[A^4]$	$E^T[G]$	$E^T[G^2]$	$\sqrt{V^T[G]}$	$Corr(A,G)$
0.5	-0.25	simul.	101.4650	10411.8468	10.8028	1080597.676	113436644.5	101.1886	10354.0559	10.7204	0.9997
		exact	101.4639	10411.4805	10.7962	1080525.126	113425589.2	101.1878	10353.7410	10.7133	
	-0.10	simul.	101.4834	10413.4785	10.7050	1080520.829	113380473.6	101.2102	10356.3759	10.6240	0.9997
		exact	101.4797	10412.1707	10.6786	1080229.561	113327237.6	101.2075	10355.2762	10.5977	
	0	simul.	101.4927	10413.7378	10.6283	1080299.609	113312319.4	101.2237	10357.5047	10.5488	0.9997
		exact	101.4903	10412.6312	10.5994	1080032.728	113261766.2	101.2207	10356.2998	10.5199	
	0.10	simul.	101.4995	10412.4734	10.5039	1079689.435	113169044.3	101.2323	10356.6744	10.4258	0.9997
		exact	101.5009	10413.0920	10.5197	1079836.062	113196371.9	101.2339	10357.3235	10.4415	
	0.25	simul.	101.5180	10414.2529	10.4090	1079641.698	113115282.4	101.2548	10359.3047	10.3332	0.9997
		exact	101.5167	10413.7838	10.3989	1079541.379	113098424.7	101.2537	10358.8593	10.3226	
1	-0.25	simul.	102.8520	10833.0688	15.9543	1168869.294	129243019.6	102.2582	10703.1436	15.6971	0.9993
		exact	102.8456	10830.6568	15.9197	1168307.595	129135560.0	102.2532	10701.0593	15.6633	
	-0.10	simul.	102.9029	10830.7307	15.5474	1166322.121	128539906.2	102.3274	10705.1520	15.3054	0.9994
		exact	102.9099	10833.5306	15.5913	1167008.761	128678749.3	102.3331	10707.5804	15.3463	
	0	simul.	102.9621	10838.9151	15.4180	1166959.666	128535212.2	102.3930	10714.6931	15.1780	0.9994
		exact	102.9527	10835.4523	15.3684	1166146.536	128376009.7	102.3864	10711.9300	15.1310	
	0.10	simul.	102.9990	10838.6593	15.1611	1165587.336	128132581.3	102.4423	10717.3313	14.9301	0.9994
		exact	102.9956	10837.3788	15.1420	1165287.232	128074701.1	102.4398	10716.2814	14.9122	
	0.25	simul.	103.0741	10845.7130	14.8807	1165282.124	127872100.2	102.5311	10727.5223	14.6593	0.9994
		exact	103.0600	10840.2773	14.7958	1164003.733	127625402.9	102.5198	10722.8117	14.5772	
3	-0.25	simul.	107.0258	12623.6395	34.1924	1652596.776	241959657.1	104.5424	11961.6032	32.1324	0.9963
		exact	107.0543	12635.4959	34.2764	1656283.380	243068306.1	104.5543	11967.0050	32.1776	
	-0.1	simul.	107.6084	12636.0680	32.5037	1628178.471	231532187.9	105.2600	12019.4092	30.6553	0.9966
		exact	107.6556	12657.1062	32.6708	1634569.194	233337124.7	105.2921	12033.9429	30.7818	
	0	simul.	108.0453	12666.0588	31.5002	1618837.517	226733369.0	105.7760	12073.7968	29.7529	0.9967
		exact	108.0587	12672.1006	31.5503	1620626.118	227225615.2	105.7868	12078.7760	29.7982	
	0.1	simul.	108.4198	12669.3503	30.2406	1602470.241	220558645.4	106.2529	12110.2172	28.6450	0.9969
		exact	108.4636	12687.5629	30.3843	1607094.550	221395137.8	106.2838	12123.7762	28.7667	
	0.25	simul.	109.0281	12692.7980	28.3842	1582494.072	211977116.2	107.0039	12178.3277	26.9905	0.9972
		exact	109.0744	12711.6293	28.5377	1587542.046	213142347.5	107.0338	12191.5910	27.1176	

Table 1: Exact and simulated moments of the arithmetic average A(T) and the geometric average G(T)

where the arbitrage price of the geometric average option is equal to

$$
\begin{aligned}
g(T,X) &= D(t_0,T)\cdot\exp\left\{m_G(T)+\frac{1}{2}\sigma_G^2(T)\right\}N(d)-XN(d-\sigma_G(T)) \\
\text{(12)}\quad \text{with}\quad m_G(T) &= E^T[\ln G(T)] \\
\sigma_G^2(T) &= V^T[\ln G(T)] \\
d &= \frac{-\ln X+m_G(T)+\sigma_G^2(T)}{\sigma_G(T)} \quad .
\end{aligned}
$$

As before we choose $M = 100.000, N = 120$ and $T \in \{0.5, 1, 3\}$. Table 2 to 4 do summarize the results for some values of the exercise price X where the initial asset value $S(t_0)$ is equal to 100.

The pricing of the Asian option is sensitive to the instantaneous correlation coefficient ρ. The arbitrage price of an Asian option obtained by the Vorst (1992) formula is decreasing in ρ. Define ρ(Vorst), as the implied correlation coefficient such that the Vorst (1992) solution equals the simulated value of the Monte Carlo simulation. As Tables 2 to 4 in the Appendix show, this implied correlation is not only substantially different for out-of-the-money options, but also for-in-the-money options from the one used by the Monte Carlo simulation. Furthermore we can conclude that for the out-of-the-money options the Turnbull and Wakeman (1991) approximation gives prices in excess of the other methods independently of the correlation coefficient. For a high correlation and out-of-the-money options the three methods give approximately equal prices. For other correlations the simulated prices of out-of-the-money options are between those obtained by the two approximation methods. Looking at deep-in-the-money options we furthermore observe that the Monte Carlo simulation leads to the highest prices. These conclusions are also obvious looking at the numerical results in Table 4. Taking e.g. $\rho = -0.5$ and the exercise price equal to 115 the prices obtained by applying Turnbull-Wakeman, Vorst and the simulation are $9.31, 8.25$ and 8.86 respectively. These differences are of a nonnegligible size. In general the Turnbull-Wakeman prices seem to be better supported by the simulations than the prices derived by the Vorst approximation. The same conclusion can be reached for a time to maturity of 2 years whereas the differences between the different methods are nonessential for smaller time to maturities.

The three last columns in Tables 2 to 4 represent the standard deviations of the simulated arithmetic Asian options and the geometric average options. Applying the control variate technique for the Asian options, the standard deviation σ_c(Asian) is on average equal to $0.1 \cdot \sigma$(Asian) where σ(Asian) refers to the standard deviation applying only the antithetic technique. These standard deviations are small meaning that we can have confidence in our pricing results.

To elaborate further on the comparison between the methods we turn our attention to Figures 6 to 8. Figures 6 and 7 illustrate the same situation but with exchanged x- and y-axis. Taking the lower bound derived by Vorst we consider the difference between the price approximations to this lower bound. For the exercise prices considered the Vorst approximation leads to prices which are lower than those obtained from the Monte Carlo simulation. The price surface for the Turnbull-Wakeman approximation crosses both of the other surfaces and is dominating in roughly half the area corresponding to the out-of-the-money options.

Finally Figure 8 shows the ratio of the simulated prices to the approximated prices measured

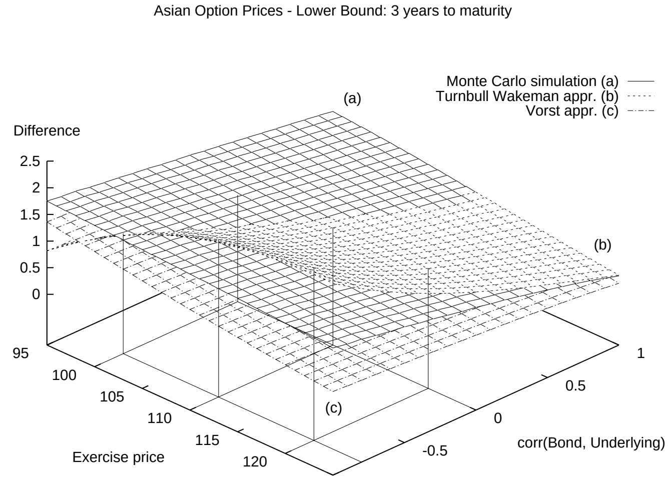

Figure 6: Difference between price approximation and the lower bound for an Asian option with 3 years to maturity, 120 realizations of the underlying asset per year, $\sigma_S = 25\%$ and Ho-Lee term structure model with $\sigma = 10\%$.

in percentage. For in-the-money options the ratio between the simulated and the Turnbull-Wakeman prices is decreasing in ρ, whereas the opposite is shown in the case for out-of-the-money options. For out-of-the-money options the Vorst approximation is clearly dominated by the Turnbull-Wakeman approximation. Observe that major differences in the approximations appear for out-of-the-money option.

6 Conclusion

Taking expectation under the T-forward risk adjusted measure the behaviour of the expected arithmetic and geometric averages is strongly influenced by the stochastic model of interest rates. In particular for the Ho and Lee (1986) model we observe a discontinuity of the expected geometric mean. Under the regime of stochastic interest rates the expected geometric average converges, independent of the instantaneous correlation, towards zero, whereas in the deterministic case it approaches plus infinity as the time to maturity increases. In contrast to this the Vasicek (1977) model with a sufficiently large degree of mean reversion does not generate this unexpected behaviour. On the other hand the behaviour of the expected arithmetic mean depends on the instantaneous correlation. If the correlation is non positive the expected arithmetic mean under stochastic interest rates is bounded from below by the expected arithmetic mean under deterministic interest rates. In the case of positive instantaneous correlation between the term structure of interest rates and the underlying asset and for short time to maturities the expected arithmetic average is higher than compared to the situation under the deterministic interest rates. The mean reversion in the the Vasicek model once again has a positive effect on the behaviour of the expected arithmetic mean. In contrast to this, without mean reversion the expected arithmetic mean decreases for a large range of maturities.

Looking at the literature on Asian option pricing we considered the approximation methods

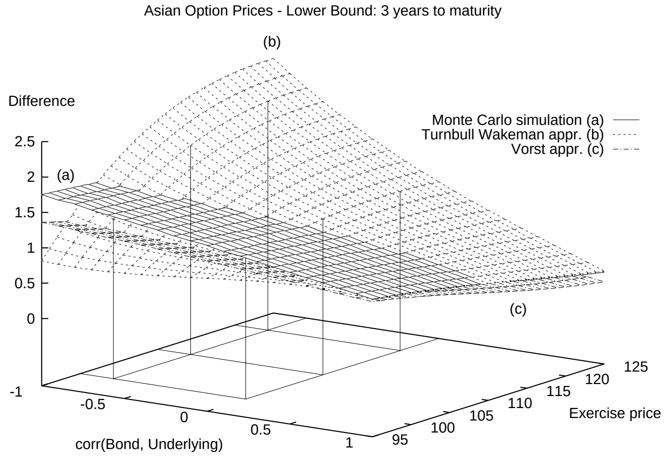

Figure 7: Difference between price approximation and the lower bound for an Asian option with 3 years to maturity, 120 realizations of the underlying asset per year, $\sigma_S = 25\%$ and and Ho-Lee term structure model with $\sigma = 10\%$.

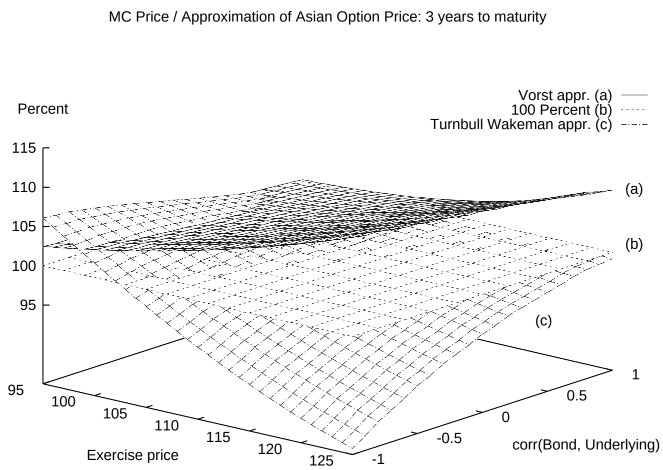

Figure 8: Monte Carlo values as percentage of the respectively numerical approximation for an Asian option with 3 years to maturity, 120 realizations of the underlying asset per year, $\sigma_S = 25\%$ and and Ho-Lee term structure model with $\sigma = 10\%$.

developed by Turnbull and Wakeman (1991) and Vorst (1992). We generalized these techniques to include the case of a Gaussian term structure model. These generalizations are only valid for a Gaussian model, since we have to preserve the lognormal structure of the underlying asset under the appropriate forward risk adjusted measure. From a pure theoretical point of view the Vorst approximation shows up a more reasonable behaviour than the Turnbull-Wakeman approximation. This is based on the strange behaviour of the correction term used by the Turnbull-Wakeman method.

To compare the pricing results, we implemented extensive Monte Carlo simulations. To reduce the variance we used the antithetic and control variate technique where the geometric average option was used as the control variate. Our simulation gives for the pricing as well as for the approximation of the unknown probability density of the arithmetic mean quite reasonable fits. Comparing the probability densities implied by the Monte Carlo simulation to those implied by the two analytical approximations, we can conclude that the Turnbull-Wakeman method produces a completely unrealistic behaviour if we consider times to maturities extending 2 years. Furthermore this behaviour, which is due to the high order Edgeworth expansion goes from bad to worse for negative instantaneous correlation between the underlying asset and the bond market. In this respect the Vorst approximation behaves much nicer, but nevertheless indicates a serious underpricing.

As a general finding, with respect to the pricing of Asian options, we conclude that the instantaneous correlation of the underlying asset and the term structure of interest rates is the principal parameter of importance. The arbitrage price seems to be negatively related with the correlation coefficient. Considering the price of an Asian option as a function of the instantaneous correlation we conclude, that the increase in the expected value of the arithmetic average, as proven by Theorem 1, is completely compensated by the decrease of the variance of the arithmetic average. Our simulations indicate a clear underpricing by the Vorst method. The conclusion for the Turnbull-Wakeman approximation is less strict and depends on both the exercise price and the instantaneous correlation. For deep-out-of-the money options and independent of the instantaneous correlation the Turnbull-Wakeman approximation implies higher Asian option prices than those produced by our Monte Carlo simulation. Whereas for deep in-the-money options the opposite is true and the Vorst solution is even higher than the Turnbull-Wakeman approximation in this situation if we consider negative correlation. For out-of-the money options and high positive correlation all three methods are close to each other, whereas for a negative correlation and out-of-the-money options the results differ substantially.

The results for the pricing of Asian options under stochastic interest rates do depend on the time to maturity. Our calculations indicate that the influence of a stochastic interest rate is less pronounced for time to maturities smaller than 1 year.

Appendix

Proof of Theorem 1

Due to the stochastic independence of W_1^T and W_2^T we know that:

$$\begin{aligned} E^T[S(t_i)] &= \frac{S(t_0)}{D(t_0,t_i)} \cdot \exp\left\{-\frac{1}{2}\int_{t_0}^{t_i} (\sigma_1(u)-\sigma(u,T))^2\,du\right\} \\ &\quad \cdot\exp\left\{\frac{1}{2}\int_{t_0}^{t_i} (\sigma(u,t_i)-\sigma(u,T))^2\,du + \frac{1}{2}\int_{t_0}^{t_i} (\sigma_1(u)-\sigma(u,t_i))^2\,du\right\} \\ &= \frac{S(t_0)}{D(t_0,t_i)} \cdot \exp\left\{\int_{t_0}^{t_i} [\sigma_1(u)-\sigma(u,t_i)]\,[\sigma(u,T)-\sigma(u,t_i)]\,du\right\} \end{aligned}$$

□

Proof of Theorem 2

The definition of the geometric mean implies that

$$\begin{aligned} E^T[\ln G(T)] &= \frac{1}{N}\sum_{i=1}^{N} E^T[\ln(S(t_i))] \\ \text{where}\quad E^T[\ln(S(t_i))] &= \ln(S(t_0)) - \ln(D(t_0,t_i)) \\ &\quad -\frac{1}{2}\int_{t_0}^{t_i} ((\sigma_1(u)-\sigma(u,T))^2+\sigma_2^2(u))du + \frac{1}{2}\int_{t_0}^{t_i} (\sigma(u,t_i)-\sigma(u,T))^2\,du \\ &= \ln(S(t_0)) - \ln(D(t_0,t_i)) - \frac{1}{2}\int_{t_0}^{t_i} \left(\sigma_1^2(u)+\sigma_2^2(u)\right)du \\ &\quad +\frac{1}{2}\int_{t_0}^{t_i} \left(2\sigma(u,T)[\sigma_1(u)-\sigma(u,t_i)]+\sigma^2(u,t_i)\right)du \end{aligned}$$

□

Proof of Theorem 3

We have to calculate the variance of a sum of correlated stochastic integrals under the T-forward risk ajusted measure, i.e.

$$V^T\left[\frac{1}{N+1}\left(\sum_{i=0}^{N}\int_{t_0}^{t_i}(\sigma_1(u)-\sigma(u,t_i))dW_1^T(u) + \int_{t_0}^{t_i}\sigma_2(u)dW_2^T(u)\right)\right] \quad .$$

Since W_1^T and W_2^T are stochastically independent we can consider the variance of both sums of stochastic integrals separately. For the stochastic integrals with respect to W_2^T we immediately obtain:

$$V^T\left[\frac{1}{N}\sum_{i=1}^{N}\int_{t_0}^{t_i}\sigma_2(u)dW_2^T(u)\right] = \frac{1}{N^2}V^T\left[\sum_{i=0}^{N-1}(N-i)\int_{t_i}^{t_{i+1}}\sigma_2(u)dW_2^T(u)\right]$$
$$= \frac{1}{N^2}\sum_{i=0}^{N-1}(N-i)^2\int_{t_i}^{t_{i+1}}\sigma_2^2(u)du$$

The same idea can be applied to the stochastic integrals with respect to W_1^T.

$$V^T\left[\frac{1}{N}\sum_{i=1}^{N}\int_{t_0}^{t_i}(\sigma_1(u)-\sigma(u,t_i))dW_1^T(u)\right]$$
$$= \frac{1}{N^2}V^T\left[\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}(N-i)\sigma_1(u)dW_1^T(u)-\sum_{i=1}^{N}\int_{t_0}^{t_i}\sigma(u,t_i)dW_1^T(u)\right]$$
$$= \frac{1}{N^2}V^T\left[\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}(N-i)\sigma_1(u)dW_1^T(u)-\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\sum_{j=i+1}^{N}\sigma(u,t_j)dW_1^T(u)\right]$$
$$= \frac{1}{N^2}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\left((N-i)\sigma_1(u)-\sum_{j=i+1}^{N}\sigma(u,t_j)\right)^2du$$

□

Recursive algorithms for the non central moments

From the previous discussion we know that under the T-forward measure P^T the value ot the underlying asset $S(t)$ is determined by equation (2). Consider the stochastic part of this equation seperately and for simplicity of the notation define:

$$Y_i := \int_{t_0}^{t_i}(\sigma_1(u)-\sigma(u,t_i))\,dW_1^T(u) \quad ,$$
$$Z_i := \int_{t_0}^{t_i}\sigma_2(u)\;dW_2^T(u) \quad ,$$
$$\nu_{ij} := \exp\left\{\int_{t_0}^{t_i}[\sigma_1(u)-\sigma(u,t_i)]\,[\sigma(u,t_j)-\sigma(u,t_i)]\,du\right\} \quad .$$

Due to the underlying assumptions Y and Z are related in the following way:

a) Y_i and Z_i are stochastic independent, and $E^T[Y_i]=E^T[Z_i]=0$.

b) For $i\le j$: $\quad E^T[Z_i,Z_j]=E^T[Z_i^2]$.

c) For $i \leq j$:

$$\begin{aligned}
& E^T[Y_i, Y_j] \\
= \quad & E^T \left[\left(\int_{t_0}^{t_i} (\sigma_1(u) - \sigma(u,t_i) dW_1^T(u) \right) \left(\int_{t_0}^{t_j} (\sigma_1(u) - \sigma(u,t_i)) - (\sigma(u,t_j) - \sigma(u,t_i)) dW_1^T(t_j) \right) \right] \\
= \quad & E^T \left[Y_i^2 \right] - E^T \left[\left(\int_{t_0}^{t_i} (\sigma_1(u) - \sigma(u,t_i) dW_1^T(u) \right) \left(\int_{t_0}^{t_j} (\sigma(u,t_j) - \sigma(u,t_i)) dW_1^T(t_j) \right) \right] \\
= \quad & E^T \left[Y_i^2 \right] - \int_{t_0}^{t_i} \left[\sigma_1(u) - \sigma(u,t_i) \right] \left[\sigma(u,t_j) - \sigma(u,t_i) \right] du = \quad E^T \left[Y_i^2 \right] - \ln \nu_{ij} \quad .
\end{aligned}$$

Finally define:

$$\begin{aligned}
\sigma_i &:= \exp \left\{ \frac{1}{2} V[Z_i + Y_i] \right\} \\
d_i &:= \frac{S_{t_0}}{D(t_0, t_i)} \cdot \exp \left\{ \int_{t_0}^{t_i} \left[\sigma_1(u) - \sigma(u,t_i) \right] \left[\sigma(u,T) - \sigma(u,t_i) \right] du \right\} \\
\Longrightarrow \qquad S(t_i) &= d_i \cdot \sigma_i^{-1} \cdot \exp \{ Y_i + Z_i \} \quad .
\end{aligned}$$

Proposition

Under the assumptions on the process of the underlying asset $S(t_i) = S_i$ we have $\forall\ 0 \leq i \leq j \leq k \leq l < N$ and $\forall\ \alpha, \gamma, \eta, \theta \in \mathbb{N}$:

$$\begin{aligned}
E^T[S_i^\alpha] &= d_i^\alpha \cdot \sigma_i^{\alpha(\alpha-1)} \\
E^T[S_i^\alpha S_j^\gamma] &= d_i^\alpha \cdot d_j^\gamma \cdot \sigma_j^{\gamma(\gamma-1)} \cdot \sigma_i^{\alpha(\alpha+2\gamma-1)} \cdot \nu_{ij}^{\alpha\gamma} \\
E^T[S_i^\alpha S_j^\gamma S_k^\eta] &= d_i^\alpha \cdot d_j^\gamma \cdot d_k^\eta \cdot \sigma_i^{\alpha(\alpha+2\gamma+2\eta-1)} \cdot \sigma_j^{\gamma(\gamma+2\eta-1)} \cdot \sigma_k^{\eta(\eta-1)} \cdot \nu_{ij}^{\alpha\gamma} \cdot \nu_{ik}^{\alpha\eta} \cdot \nu_{jk}^{\gamma\eta} \\
E^T[S_i^\alpha S_j^\gamma S_k^\eta S_l^\theta] &= d_i^\alpha \cdot d_j^\gamma \cdot d_k^\eta \cdot d_l^\theta \cdot \sigma_i^{\alpha(\alpha-1+2\gamma+2\eta+2\theta)} \cdot \sigma_j^{\gamma(\gamma-1+2\eta+2\theta)} \cdot \sigma_k^{\eta(\eta-1+2\eta)} \cdot \sigma_l^{\theta(\theta-1)} \\
&\quad \cdot \nu_{ij}^{\alpha\gamma} \cdot \nu_{ik}^{\alpha\eta} \cdot \nu_{il}^{\alpha\theta} \cdot \nu_{jk}^{\gamma\eta} \cdot \nu_{jl}^{\gamma\theta} \cdot \nu_{kl}^{\eta\theta} \quad .
\end{aligned}$$

The algorithms will be derived by means of the following vector notations.

$$\begin{aligned}
d(i) &:= (d_i, \cdots, d_N) \in \mathbb{R}^{N+1-i}; \quad i = 1, ..., N \quad , \\
\nu(i) &:= (\nu_{i,i+1}, \cdots, \nu_{i,N}) \in \mathbb{R}^{N-i}; \quad i = 1, ..., N-1 \quad .
\end{aligned}$$

1. Moment

$$E^T[\frac{1}{N} \sum_{i=1}^N S_i] = \sum_{i=1}^N d_i =< d(1), 1 > \quad .$$

2. Moment

$$\begin{aligned}
x(N) &:= E^T[S_N^2] = d_N^2\sigma_N^2 \quad \text{and for } i = N-1, ..., 1 \\
x(i) &:= E^T\left[\left(\sum_{j=i}^N S_j\right)^2\right] \\
&= E^T[S_i^2] + 2E^T\left[S_i \sum_{j=i+1}^N S_j\right] + E^T\left[\left(\sum_{j=i+1}^N S_j\right)^2\right] \\
&= d_i^2\sigma_i^2 + 2 < d(i+1), \nu(i) > d_i\sigma_i^2 + x(i+1) \\
&\Longrightarrow E^T\left[\left(\frac{1}{N}\sum_{i=1}^N S_i\right)^2\right] = \frac{1}{N^2}x(1) \quad .
\end{aligned}$$

3. Moment

$$\begin{aligned}
x(N) &:= E^T[S_N^3] = d_N^3\sigma_N^6 \quad \text{and for } i = N-1, \cdots, 1 \\
x(i) &:= E^T\left[\left(\sum_{j=i}^N S_j\right)^3\right] \\
&= E^T[S_i^3] + 3E^T\left[S_i\left(\sum_{j=i+1}^N S_j\right)^2\right] + 3E^T\left[S_i^2\left(\sum_{j=i+1}^N S_j\right)\right] + E^T\left[\left(\sum_{j=i+1}^N S_j\right)^3\right] \\
&= d_i^3\sigma_i^6 + 3a(i, i+1) + 3 < d(i+1, \nu(1)^2) > d_i^2\sigma_i^6 + x(i+1) \quad ,
\end{aligned}$$

$$\begin{aligned}
\text{where } a(i, N) &:= d_i d_N^2 \sigma_i^4 \sigma_N^2 \nu_{iN}^2 \quad \text{and for } j = N-1, \cdots, i+1 \\
a(i,j) &= a(i, j+1) + d_i d_j^2 \sigma_i^4 \sigma_j^2 \nu_{ij}^2 + 2\left(\sum_{l=j+1}^N d_l \nu_{il}\nu_{jl}\right) d_i d_j \sigma_i^4 \sigma_j \nu_{ij} \\
&\Longrightarrow E^T\left[\left(\frac{1}{N}\sum_{i=1}^N S_i\right)^3\right] = \frac{1}{N^3}x(1) \quad .
\end{aligned}$$

4. Moment

$$\begin{aligned}
x(N) &:= E^T\left[S_N^4\right] = d_N^4\sigma_N^{12} \quad \text{and for } j = N-1, \cdots, 1 \\
x(i) &:= E^T\left[\left(\sum_{j=i}^N S_j\right)^4\right] \\
&= E^T\left[S_i^4\right] + 4E^T\left[S_i^3\left(\sum_{j=i+1}^N S_j\right)\right] + 6E^T\left[S_i^2\left(\sum_{j=i+1}^N S_j\right)^2\right] \\
&\quad +4E^T\left[S_i\left(\sum_{j=i+1}^N S_j\right)^3\right] + E^T\left[\left(\sum_{j=i+1}^N S_j\right)^4\right] \\
&= d_i^4\sigma_i^{12} + 4 < d(i+1), \nu(i)^3 > d_i^3\sigma_i^{12} + 6c(i, i+1) + 4a(i, i+1) + x(i+1) \quad ,
\end{aligned}$$

where

$$\begin{aligned}
a(i,N) &:= d_i d_N^3 \sigma_i^6 \sigma_N^6 \quad \text{and for } j = N-1, \cdots, i+1 \\
a(i,j) &:= d_i d_j \sigma_i^6 \sigma_j^6 + 3 \left(\sum_{l=j+1}^{N} d_l \nu_{il} \nu_{jl}^2 \right) d_i d_j^2 \sigma_i^6 \sigma_j^6 \nu_{ij}^6 + 3b(i,j,j+1) + a(i,j+1) \\
\text{with } b(i,j,N) &:= d_i d_j d_N^2 \sigma_i^6 \sigma_j^4 \sigma_N^2 \nu_{ij} \nu_{iN}^2 \nu_{jN}^2 \\
& \quad\ \text{and for } k = N-1, \cdots, j+1 \\
b(i,j,k) &:= d_i d_j d_k^2 \sigma_i^6 \sigma_j^4 \sigma_k^2 \nu_{ij} \nu_{ik}^2 \nu_{jk}^2 \\
& \quad +2 \left(\sum_{l=k+1}^{N} d_l \nu_{il} \nu_{jl} \nu_{kl} \right) d_i d_j d_k \sigma_i^6 \sigma_j^4 \sigma_k^2 \nu_{ik} \nu_{ij} \nu_{jk} + b(i,j,k+1) \quad .
\end{aligned}$$

and

$$\begin{aligned}
c(i,N) &:= d_i^2 d_N^2 \sigma_i^{10} \sigma_N^2 \nu_{iN}^4 \quad \text{and for } j = N-1, \cdots, i+1 \\
c(i,j) &:= d_i^2 d_j^2 \sigma_i^{10} \sigma_j^2 \nu_{ij}^2 + 2 \left(\sum_{k=j+1}^{N} d_k \nu_{ik}^2 \nu_{jk} \right) d_i^2 d_j \sigma_i^{10} \sigma_j^2 \nu_{ij}^2 + c(i,j+1) \\
&\Longrightarrow \quad E^T \left[\left(\frac{1}{N} \sum_{i=1}^{N} S_i \right)^4 \right] = \frac{1}{N^4} x(1) \quad .
\end{aligned}$$

$E^T[A(T)]$	ρ	Basis	Bd asian	log appr.	TW	Vorst	MC asian	Bu asian	ρ(Vorst)	MC geo	exact geo	σ_c(asian)	σ(asian)	σ(geo)
101.4375	-0.5	95	7.74791	7.9689	7.93376	7.94196	7.94068	8.02244	-0.49194	7.74766	7.74791	0.0017	0.0302	0.0298
		100	4.77127	4.94847	4.93275	4.91578	4.93514	5.0458	-0.58466	4.7675	4.77127	0.0014	0.0341	0.0336
		101	4.28304	4.45046	4.43993	4.4174	4.44091	4.55757	-0.59961	4.27917	4.28304	0.0013	0.0342	0.0336
		102	3.83042	3.98791	3.98267	3.95477	3.98237	4.10495	-0.61462	3.82711	3.83042	0.0013	0.0341	0.0335
		105	2.67894	2.80654	2.81624	2.77472	2.81281	2.95347	-0.65796	2.6778	2.67894	0.0013	0.0322	0.0315
101.4639	-0.25	95	7.71121	7.9278	7.89456	7.90211	7.90103	7.97941	-0.24329	7.69847	7.71121	0.0011	0.0243	0.024
		100	4.71601	4.88923	4.87429	4.85782	4.87653	4.98421	-0.33014	4.70145	4.71601	0.001	0.0323	0.0318
		101	4.22579	4.38929	4.37932	4.35751	4.38022	4.49399	-0.34412	4.21111	4.22579	0.0011	0.0325	0.0319
		102	3.77184	3.92547	3.92055	3.8936	3.92029	4.04004	-0.3584	3.75753	3.77184	0.0012	0.0321	0.0312
		105	2.62009	2.744	2.75339	2.71344	2.75019	2.88829	-0.39932	2.60646	2.62009	0.0012	0.0271	0.0261
101.4797	-0.1	95	7.68896	7.90292	7.87081	7.87797	7.87676	7.95336	-0.09258	7.68769	7.68896	0.0013	0.0243	0.0231
		100	4.68225	4.85308	4.83861	4.82243	4.84063	4.94665	-0.1769	4.68303	4.68225	0.0013	0.0272	0.0261
		101	4.19082	4.35196	4.34231	4.32094	4.34322	4.45521	-0.19116	4.1916	4.19082	0.0013	0.0273	0.0261
		102	3.73604	3.88735	3.88262	3.85624	3.88243	4.00044	-0.20498	3.73675	3.73604	0.0014	0.0271	0.0258
		105	2.58418	2.70587	2.71507	2.67605	2.71223	2.84857	-0.24512	2.58348	2.58418	0.0014	0.0252	0.0239
101.4903	0	95	7.67404	7.88624	7.85487	7.86178	7.86186	7.9359	-0.00049	7.67901	7.67404	0.0012	0.0232	0.0226
		100	4.65949	4.82873	4.81456	4.79859	4.81774	4.92135	-0.08027	4.66503	4.65949	0.0012	0.0295	0.0287
		101	4.16722	4.3268	4.31736	4.29628	4.31942	4.42908	-0.09385	4.17274	4.16722	0.0012	0.0298	0.0288
		102	3.71189	3.86166	3.85705	3.83104	3.85802	3.97375	-0.10713	3.71752	3.71189	0.0013	0.0295	0.0284
		105	2.55997	2.68019	2.68925	2.65087	2.68749	2.82183	-0.1457	2.56574	2.55997	0.0014	0.0261	0.0248
101.5009	0.1	95	7.65904	7.86948	7.83886	7.84551	7.84442	7.91837	0.10674	7.67187	7.65904	0.0014	0.0174	0.0169
		100	4.63652	4.80417	4.79031	4.77453	4.79252	4.89584	0.02529	4.64992	4.63652	0.0013	0.0217	0.0212
		101	4.1434	4.30141	4.29219	4.2714	4.29333	4.40273	0.01191	4.1567	4.1434	0.0013	0.0219	0.0213
		102	3.68751	3.83573	3.83125	3.80562	3.83126	3.94684	-0.00088	3.70115	3.68751	0.0013	0.0215	0.0209
		105	2.53554	2.65429	2.66322	2.62546	2.66079	2.79487	-0.03931	2.54928	2.53554	0.0013	0.019	0.0184
101.5167	0.25	95	7.6364	7.8442	7.8147	7.82097	7.8197	7.89193	0.25769	7.62578	7.6364	0.0014	0.0299	0.0293
		100	4.60165	4.76691	4.75351	4.73803	4.755	4.85717	0.18048	4.5891	4.60165	0.0013	0.0348	0.0339
		101	4.10725	4.2629	4.254	4.23364	4.25458	4.36277	0.16711	4.09469	4.10725	0.0013	0.0347	0.0339
		102	3.6505	3.7964	3.7921	3.76704	3.79172	3.90602	0.1543	3.63787	3.6505	0.0013	0.0347	0.0339
		105	2.4985	2.61502	2.62375	2.58694	2.62062	2.75402	0.11895	2.48696	2.4985	0.0013	0.0323	0.0313
101.5432	0.5	95	7.5983	7.80169	7.77404	7.77968	7.77942	7.84748	0.50159	7.60108	7.5983	0.0017	0.0263	0.0251
		100	4.54241	4.70369	4.69106	4.67606	4.69349	4.79159	0.4303	4.54409	4.54241	0.0016	0.0321	0.0309
		101	4.04578	4.19751	4.18915	4.1695	4.19054	4.29496	0.4187	4.04726	4.04578	0.0016	0.0323	0.0311
		102	3.58757	3.7296	3.72561	3.70149	3.72594	3.83675	0.40747	3.58941	3.58757	0.0016	0.032	0.0307
		105	2.4356	2.54843	2.55682	2.52158	2.55499	2.68478	0.37292	2.43859	2.4356	0.0016	0.0285	0.0273

Table 2: Approximation of Asian option prices, maturity 0.5 years

$E^T[A(T)]$	ρ	Basis	Bd asian	log appr.	TW	Vorst	MC asian	Bu asian	ρ(Vorst)	MC geo	exact geo	σ_c(asian)	σ(asian)	σ(geo)
102.7386	-0.5	95	9.79466	10.26532	10.16694	10.17728	10.20054	10.37809	-0.56479	9.79425	9.79466	0.0037	0.0365	0.0346
		100	7.03442	7.44313	7.38753	7.3434	7.40152	7.61785	-0.61652	7.03461	7.03442	0.0039	0.04	0.0381
		102	6.09747	6.47885	6.44374	6.3771	6.44898	6.6809	-0.63425	6.09769	6.09747	0.0037	0.0399	0.038
		103	5.66419	6.03157	6.00681	5.92941	6.00779	6.24762	-0.6427	5.66462	5.66419	0.0039	0.0395	0.0375
		110	3.24568	3.51401	3.55204	3.41897	3.5309	3.82911	-0.69714	3.24424	3.24568	0.0039	0.0355	0.0329
102.8456	-0.25	95	9.71383	10.16669	10.07827	10.08573	10.10862	10.27268	-0.31177	9.73001	9.71383	0.003	0.0446	0.0429
		100	6.91529	7.30728	7.25666	7.21468	7.27067	7.47414	-0.35739	6.93178	6.91529	0.0029	0.0455	0.0437
		102	5.96821	6.33319	6.30115	6.23854	6.3075	6.52706	-0.37299	5.98468	5.96821	0.0031	0.0457	0.0438
		103	5.53106	5.8822	5.8596	5.78712	5.86208	6.08991	-0.38027	5.54752	5.53106	0.0031	0.0456	0.0438
		110	3.10678	3.35996	3.39484	3.27182	3.37867	3.66563	-0.43052	3.1209	3.10678	0.0033	0.0417	0.0393
102.9099	-0.1	95	9.66397	10.10613	10.02358	10.02939	10.05	10.20806	-0.15454	9.676	9.66397	0.0044	0.0437	0.0411
		100	6.84104	7.223	7.17534	7.13464	7.18708	7.38513	-0.1978	6.85607	6.84104	0.0039	0.0459	0.0439
		102	5.88753	6.24268	6.21249	6.15226	6.21694	6.43162	-0.21204	5.90249	5.88753	0.0038	0.0464	0.0441
		103	5.44795	5.78934	5.76806	5.69848	5.76916	5.99204	-0.21926	5.46293	5.44795	0.0038	0.0459	0.0434
		110	3.02049	3.26457	3.29759	3.18052	3.28132	3.56458	-0.26582	3.0295	3.02049	0.0034	0.0424	0.0396
102.9527	0	95	9.63014	10.06518	9.98649	9.99122	10.01227	10.16438	-0.05498	9.64222	9.63014	0.0038	0.0722	0.0689
		100	6.79031	7.16558	7.11989	7.08003	7.13216	7.32454	-0.09543	6.80791	6.79031	0.0044	0.0773	0.0734
		102	5.83236	6.18095	6.15199	6.09333	6.15783	6.3666	-0.10957	5.85015	5.83236	0.0043	0.0774	0.0735
		103	5.39111	5.726	5.70559	5.63793	5.70827	5.92535	-0.11636	5.40844	5.39111	0.0045	0.0775	0.0734
		110	2.96165	3.19965	3.23146	3.11832	3.21762	3.49589	-0.1605	2.97169	2.96165	0.0055	0.0689	0.0638
102.9956	0.1	95	9.59584	10.02375	9.94889	9.95257	9.97322	10.12022	0.04673	9.6065	9.59584	0.003	0.0283	0.0266
		100	6.73854	7.10713	7.0634	7.02438	7.07508	7.26292	0.00898	6.75067	6.73854	0.0028	0.0316	0.03
		102	5.77602	6.11805	6.09031	6.03321	6.0961	6.3004	-0.00466	5.78821	5.77602	0.0027	0.0314	0.0299
		103	5.33305	5.66145	5.64191	5.57615	5.64492	5.85743	-0.01143	5.34509	5.33305	0.0027	0.0316	0.03
		110	2.90173	3.13364	3.16423	3.05501	3.15165	3.4261	-0.05337	2.91167	2.90173	0.0024	0.0273	0.026
103.06	0.25	95	9.54349	9.9607	9.89148	9.89365	9.91133	10.05307	0.20532	9.53845	9.54349	0.0022	0.0529	0.0516
		100	6.65885	7.01743	6.97661	6.93885	6.98534	7.16844	0.16901	6.6505	6.65885	0.0027	0.0599	0.0582
		102	5.6892	6.02139	5.99549	5.94072	5.99853	6.19878	0.1568	5.6804	5.6892	0.0027	0.0606	0.0586
		103	5.24358	5.56221	5.54399	5.48107	5.54436	5.75316	0.1507	5.23521	5.24358	0.0028	0.0603	0.0582
		110	2.80969	3.03246	3.06125	2.95787	3.04759	3.31927	0.1116	2.80667	2.80969	0.0035	0.0518	0.0492
103.1674	0.5	95	9.45378	9.85317	9.79311	9.79294	9.81191	9.93868	0.45352	9.47801	9.45378	0.0019	0.0386	0.0374
		100	6.52022	6.8621	6.82608	6.79043	6.83648	7.00511	0.4238	6.54821	6.52022	0.0024	0.0468	0.0451
		102	5.53786	5.85365	5.83077	5.77986	5.83638	6.02275	0.4137	5.56596	5.53786	0.0024	0.0476	0.0457
		103	5.08753	5.3899	5.37384	5.31562	5.37702	5.57243	0.40881	5.1155	5.08753	0.0025	0.0475	0.0454
		110	2.65018	2.85765	2.88349	2.78974	2.87514	3.13507	0.37451	2.67273	2.65018	0.0033	0.0378	0.035

Table 3: Approximation of Asian option prices, maturity 1 year

$E^T[A(T)]$	ρ	Basis	Bd asian	log appr.	TW	Vorst	MC asian	Bu asian	ρ(Vorst)	MC geo	exact geo	σ_c(asian)	σ(asian)	σ(geo)
106.0612	-0.5	100	12.42447	14.29415	13.59003	13.55219	13.94337	14.71236	no.sol.	12.3901	12.42447	0.0274	0.1657	0.1408
		105	10.5528	12.30691	12.00662	11.54178	12.01588	12.84069	no.sol.	10.51678	10.5528	0.027	0.1638	0.1388
		107	9.87265	11.57846	11.42102	10.80878	11.31298	12.16054	no.sol.	9.83703	9.87265	0.0269	0.1625	0.1377
		109	9.22982	10.88647	10.86009	10.11479	10.64707	11.5177	no.sol.	9.1954	9.22982	0.0272	0.1615	0.1366
		115	7.51207	9.01797	9.31221	8.25434	8.85919	9.79995	no.sol.	7.48166	7.51207	0.027	0.1575	0.1327
107.0543	-0.25	100	12.29049	14.00437	13.38613	13.3731	13.71022	14.38953	-0.7771	12.27695	12.29049	0.0141	0.0902	0.0812
		105	10.33201	11.93331	11.62523	11.27744	11.69086	12.43104	-0.66499	10.3184	10.33201	0.0136	0.0898	0.0823
		107	9.62285	11.17685	10.98161	10.51579	10.95688	11.72188	-0.64609	9.60892	9.62285	0.0135	0.0902	0.0827
		109	8.95423	10.45998	10.36962	9.79628	10.26334	11.05326	-0.63365	8.94057	8.95423	0.0133	0.0907	0.0828
		115	7.17772	8.53515	8.70664	7.87758	8.41101	9.27675	-0.61735	7.16247	7.17772	0.0153	0.0886	0.079
107.6556	-0.1	100	12.19185	13.81233	13.25923	13.24613	13.55043	14.1763	-0.49727	12.13767	12.19185	0.0105	0.0763	0.0711
		105	10.17474	11.68442	11.38938	11.09276	11.46965	12.15919	-0.42697	10.12017	10.17474	0.0109	0.0752	0.0696
		107	9.44615	10.9091	10.7099	10.31187	10.71493	11.43061	-0.41482	9.39162	9.44615	0.0112	0.0734	0.0676
		109	8.76041	10.17562	10.0663	9.57535	10.00313	10.74487	-0.40719	8.70638	8.76041	0.011	0.071	0.0648
		115	6.94581	8.21397	8.33219	7.61872	8.10828	8.93026	-0.3981	6.89349	6.94581	0.0097	0.0646	0.0595
108.0587	0	100	12.11736	13.67564	13.16989	13.15227	13.44432	14.02489	-0.34319	12.10155	12.11736	0.0121	0.1079	0.101
		105	10.05776	11.50641	11.22587	10.95701	11.31908	11.96529	-0.28636	10.04153	10.05776	0.0118	0.1086	0.1005
		107	9.31518	10.71745	10.5219	10.16223	10.54988	11.22272	-0.27683	9.29857	9.31518	0.0121	0.1079	0.0993
		109	8.61718	9.972	9.8567	9.41349	9.82506	10.52471	-0.27083	8.60012	8.61718	0.0121	0.1063	0.0973
		115	6.77578	7.9843	8.0742	7.43005	7.90112	8.68331	-0.26444	6.75727	6.77578	0.0127	0.0989	0.0879
108.4636	0.1	100	12.03512	13.53129	13.07511	13.05031	13.3137	13.86532	-0.17749	11.98024	12.03512	0.0138	0.0844	0.0737
		105	9.9298	11.31746	11.05526	10.80991	11.1406	11.75999	-0.13721	9.87488	9.9298	0.0156	0.0856	0.0737
		107	9.17224	10.51386	10.3262	10.00022	10.35593	11.00244	-0.13098	9.11756	9.17224	0.016	0.0853	0.0733
		109	8.46118	9.75561	9.63889	9.23842	9.61703	10.29137	-0.12704	8.40667	8.46118	0.0164	0.0847	0.0726
		115	6.59171	7.74046	7.80711	7.22679	7.66058	8.4219	-0.12321	6.54057	6.59171	0.0157	0.0835	0.0713
109.0744	0.25	100	11.89561	13.2988	12.91947	12.88069	13.11084	13.60898	0.04163	11.83408	11.89561	0.0161	0.1366	0.1234
		105	9.71432	11.01063	10.78167	10.56513	10.85736	11.42769	0.06863	9.65104	9.71432	0.0164	0.1385	0.1244
		107	8.93206	10.18274	10.0135	9.73075	10.04565	10.64544	0.07277	8.87001	8.93206	0.0163	0.1381	0.1236
		109	8.1996	9.40341	9.29179	8.94746	9.28356	9.91297	0.07497	8.13958	8.1996	0.0166	0.1373	0.1227
		115	6.28526	7.34399	7.38409	6.89044	7.27788	7.99864	0.07557	6.23175	6.28526	0.0181	0.1314	0.115
110.1016	0.5	100	11.61239	12.86165	12.61047	12.54669	12.73007	13.12891	0.36919	11.56031	11.61239	0.0118	0.0635	0.0562
		105	9.27733	10.42201	10.25821	10.07836	10.31589	10.79384	0.38507	9.22561	9.27733	0.0116	0.0683	0.0606
		107	8.44566	9.54503	9.41884	9.19421	9.45176	9.96217	0.38715	8.3939	8.44566	0.0121	0.0689	0.0604
		109	7.67097	8.72359	8.63483	8.36812	8.64413	9.18749	0.38808	7.61877	7.67097	0.0125	0.0684	0.059
		115	5.67239	6.57948	6.59334	6.22455	6.54226	7.1889	0.38772	5.62199	5.67239	0.0133	0.062	0.0509

Table 4: Approximation of Asian option prices, maturity 3 years

References

Amin, K. I. and Bodurtha Jr., J. N. (1995), Discrete–Time Valuation of American Options with Stochastic Interest Rates, *The Review of Financial Studies*.

Amin, K. I. and Jarrow, R. A. (1991), Pricing Foreign Currency Options under Stochastic Interest Rates, *Journal of International Money and Finance* **10**, 310–330.

Amin, K. I. and Jarrow, R. A. (1992), Pricing Options on Risky Assets in a Stochastic Interest Rate Economy, *Mathematical Finance* **2**(4), 217–237.

Carverhill, A. P. and Clewlow, L. J. (1990), Flexible Convolution, *Risk* pp. 25–29.

El Karoui, N., Lepage, C., Myneni, R. and Viswanathana, R. (1991), The Valuation and Hedging of Contingent Claims with Markovian Interest Rates, *Technical report*, University of Paris.

Geman, H. and Yor, M. (1993), Bessel Processes, Asian Options and Perpetuities, *Mathematical Finance* **Vol. 3**, 349–375.

Geman, H., El Karoui, N. and Rochet, J.-C. (1995), Changes of Numeraire, Changes of Probability Measure and Option Pricing, *Journal of Applied Probability* **32**, 443–458.

Heath, D., Jarrow, R. and Morton, A. (1992), Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation, *Econometrica* **60**(1), 77–105.

Ho, T. S. and Lee, S.-B. (1986), Term Structure Movements and Pricing Interest Rate Contingent Claims, *The Journal of Finance* **Vol. XLI**(No.5), 1011–1029.

Hull, J. and White, A. (1990a), Pricing Interest–Rate Derivative Securities, *The Review of Financial Studies* **3**(4), 573–592.

Hull, J. and White, A. (1990b), Valuing Derivative Securities Using the Explicit Finite Difference Method, *Journal of Financial and Quantitative Analysis* **25**(1), 87–100.

Jamshidian, F. (1989), An Exact Bond Option Formula, *The Journal of Finance* **44**(1), 205–209.

Jamshidian, F. (1991), Bond and Option Evaluation in the Gausian Interest Rate Model, *Research in Finance* **9**, 131–170.

Kemna, A. and Vorst, A. (1990), A Pricing Method for Options Based on Average Asset Values, *Journal of Banking and Finance* **Vol. 14**, 113–129.

Levy, E. (1992), The Valuation of Average Rate Currency Options, *Journal of International Money and Finance* **11**, 474–491.

Nielsen, J. and Sandmann, K. (1995), Equity–linked Life Insurance: A Model with Stochastic Interest Rates, *Insurance, Mathematics & Economics* **16**, 225–253.

Turnbull, S. M. and Wakeman, L. M. (1991), Quick Algorithm for Pricing European Average Options, *Journal of Financial and Quantitative Analysis* **Vol. 26**(No.3), 377–389.

Vasicek, O. (1977), An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* **5**, 177–188.

Vorst, A. (1992), Prices and Hedge Ratios of Average Exchange Rate Options, *International Review of Financial Analysis* **Vol. 1**(No.3), 179–193.