Uniqueness of the Fair Premium for Equity-Linked Life Insurance Contracts*

J. Aase Nielsen † Klaus Sandmann ‡

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†Department of Operations Research, University of Aarhus, Bldg. 530, Ny Munkegade, DK-8000 Aarhus C, Denmark. E-mail: atsjan@mi.aau.dk

‡Department of Statistics, Rheinische Friedrich-Wilhelms-Universität Bonn, Adenauerallee 24–42, D-53113 Bonn, Germany. E-mail: sandmann@addi.or.uni-bonn.de

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Abstract

An equity-linked life insurance contract combines an endowment life insurance and an investment strategy with a minimum guarantee. The benefit of this contract is determined by the guaranteed amount plus a bonus equal to a call on the portfolio. This bonus is similar to an Asian option.

We analyze the relationship between the periodic insurance premium and its proportional share invested into the portfolio. For a general model of the financial risks we show the existence and uniqueness of an insurance premium. Furthermore the premium is strictly increasing and convex as a function of the share invested.

Keywords

Asian Option, Forward Risk Adjusted Measure, Life Insurance, Monte Carlo Simulation, Stochastic Interest Rates.

1 Introduction

An equity-linked life insurance contract combines in a particular way an endowment[1] life insurance and a portfolio investment stategy. The life insurance component guarantees in the case of maturity of the contract or the death of the insured his or her heirs a fixed amount of money, the so called guaranteed amount. This guaranteed amount may be constant for the term of the contract, or may be an initially specified function of time. Especially in combination with an investment strategy the insured person may choose a decreasing guaranteed amount in time. In this case the insured, or his or her heirs would benefit from a relatively high guaranteed amount in the case of an unexpected death within the first period of the contract. The decrease of this insured guaranteed amount over time would then be compensated by the value of the investment strategy. A variety of different contracts are described in Ekern and Persson [1995]. In particular this idea underlies an equity-linked life insurance. In addition to the life insurance component a proportion of the typically periodic premium is invested into a mutual fund. In the case of death of the insured or at maturity of the contract the benefit is determined by the maximum between the guaranteed amount and the market value of the portfolio. The payoff of the contract equals the guaranteed amount plus a non negative bonus. This bonus corresponds to a call option on the portfolio value with an exercise price equal to the guaranteed amount. Via the return of the investment strategy the insured participate in the economic growth of the particular fund. Existing equity-linked life insurance contracts are based on interest rate, equity indexes or real estate portfolios. Thus the bonus of the insurance contracts is not as such related to the acquisitions of the insurance company. In addition the portfolio value, i.e. the financial risk is insured with a lower value equal to the guaranteed amount. For an equity-linked life insurance contract the exercise of this financial insurance component is conditioned on the death of the insured. The contracts take into consideration the technical life insurance risk as well as financial risks.

In this paper we will analyse a contract with periodic premium payments[2]. At each premium date the insurance company invests a certain fraction of the premium in a specific mutual fund. The guaranteed amount will be a function of time and the periodic premium. The bonus has similarities to an Asian option with the underlying asset consisting of what currently have been invested in the fund at the premium payment dates. The exercise price for this Asian–like option is the guaranteed amount. The development in the financial marked which consists of the specified fund and bonds will be described by Itô processes. The model for the contract is very flexible as we through the choice of the share of the premium invested in the security and the composition of the guaranteed amount can create many different profiles for the possible cash flow of insurance contracts. For given contractual conditions we will analyse methods to the determination of the fair premium, which creates equality between the cost of the contract and the market value of benefit.

In the literature single and periodic equity-linked contracts have been analysed in Brennan and Schwartz (1976, 1979); Delbaen [1990]; Aase and Persson [1994]; Persson (1993); Bacinello and Ortu (1993a, 1993b, 1994); and Nielsen and Sandmann [1995]. However it is only in Bacinello

[1] As an alternative pure endowment or term contracts could have been analysed. The methodology developed in this paper would be valid for these contracts as well.

[2] The analysis covers the situation with a single initial premium payment. This case is much easier to solve than the periodic premium case as the path dependence introduced through the periodic investment in the mutual fund is eliminated.

and Ortu [1994] and Nielsen and Sandmann [1995] that the theory is extended to cover stochastic interest rate dynamics in the periodic premium case. Taking into consideration that the usual term of these insurance contracts is equal to 12 to 30 years it is necessary to incorporate a stochastic behaviour of the term structure of interest rates. In Nielsen and Sandmann [1995] we analysed in detail, applying Monte Carlo simulation, the case where the guaranteed amount was a function of just the periodic premium, and from the simulations some conjectures were raised. These conjectures will in this paper be analysed on a theoretically basis and extended to allow for guaranteed amounts depending as well on the time. The results will be supported by simulation experiments and concerning the specific Asian option analysis comparisons will be made to statements in the literature on the pricing of such assets.

The content of the paper is as follows. In Section 2 the notation and the specification of the contract we analyse are presented. Section 3 is devoted to an analysis of the stochastic model underlying the equity and bond markets. In Section 4 the fair premium problem is stated and a number of theorems are presented. In the theorems we develop the interrelationships between the fair premium, the functional specification of the guaranteed amount and the share of the premium deemed to be invested in the fund at each premium payment date. Then in Section 5 we apply approximation techniques in pricing Asian options and discuss their validity for the insurance situation. Section 6 contains numerical results based on simulations and on the approximation by Vorst [1992]. In Section 7 we finally conclude.

2 Notation and definition of the contract

An equity-linked life insurance contract is an agreement between a buyer and a seller, where the buyer is committed to pay, typically at periodic intervals and until the maturity of the contract or the death of the buyer whichever comes first, a predetermined premium to the seller. At maturity or death of the buyer, the seller is committed to deliver a payment in accordance to the agreement settled when the contract was written. This payment, the benefit, is the maximum of 1) a function depending on the periodic premium and on the history of the spot price of the underlying mutual fund from the date of settlement to the expiration date of the contract and 2) a non-random guaranteed amount also depending on the periodic premium. To precisely define the insurance contract we apply the following notation:

The value of the *mutual fund* at time t is denoted by

$S(t)$ — Relative to the value $S(t)$ the payoff of the equity-linked life insurance will be calculated at time t.

The equity-linked life insurance contract includes the specification of

K — the *periodic premium* paid by the insured,

a — the *share* of the periodic premium invested in a mutual fund, $0 \leq a \leq 1$,

t_i — the set of *premium payment dates*, $i = 0, 1, 2, ..., n-1; t_0 = 0$,

t_n — the *maturity date*, $t_n = T$, and

$g(t, K)$ — the *guaranteed amount*. The guaranteed amount is assumed to be a non–stochastic function of the periodic premium K and the time t.

The reference portfolio

relative to which the final payoff is calculated is defined as the portfolio obtained by investing an amount $a \cdot K$ at each of the dates $t_i, i = 0, 1, 2, ..., n-1$, in the fund with price process $S(t)$. The market value of the reference portfolio at time t_i equals

$$aK \sum_{j=0}^{i} \frac{S(t_i)}{S(t_j)} .$$

The benefit

from the insurance contract payed at time t if the contract terminates at this date is therefore determined by:

$$g(t,K) + V(t,t,K) = g(t,K) + \left[a \cdot K \cdot \sum_{i=0}^{n^*(t)-1} \frac{S(t)}{S(t_i)} - g(t,K) \right]^+$$

with $n^*(t) := \min \{j| \quad t_j \geq t\}$ and $[X - g(t,K)]^+ := \max\{X - g(t,K), 0\}$.

The *fair periodic premium*

K^* is defined as the periodic premium K for which the value at date t_0 of the benefit equals the value at the same date of the premium payments, where the latter could also be denoted the cost of the insurance contract.

The *term structure of interest rates* is determined by

$r(t)$ the instantaneous risk free rate of interest at time t, and

$D(t,t')$ the price at date t of a zero coupon bond with maturity date $t', t \leq t'$.

The benefit is the proceeds from a financial contract and its price at time t_0 will be found in accordance to the absence of arbitrage opportunities in the financial market. The share a is one of the parameters which should be negotiated when the contract is settled. The size of aK determines together with the sequence of prices of the mutual fund the value of the reference portfolio through which the obligations of the insurance company are given. However it should be pointed out that the insurance company has no obligation to invest the amount aK in the mutual fund at each premium payment date. The company[3] will of course invest not only aK but K in accordance to a strategy which takes into account the overall interests of the company and as an extreme situation this could mean that it have no investment in the mutual fund at all.

3 The Stochastic Model

In order to analyze an equity-linked life insurance contract we have to model three sources of uncertainty which will affect the premium of such a contract. First we have to consider the death probability. Usually the death – and survival distribution is determined by the insurer using historical data. We assume that the death distribution is stochastically independent from the financial risks which we will discuss in more detail shortly[4]. Furthermore we assume that the insurer is risk neutral with respect to mortality. Since the death distribution depends on the age of the insured person we implicitly assume that the insurer can perfectly diversify the death uncertainty within each group. For standard endowment life insurance contracts this is a

[3]No administrative costs are included in the analysis.

[4]More precisely, we assume that the death- resp. survival distribution is stochastically independent from the distribution of the mutual fund and the zero coupon bonds under the forward risk adjusted measure which will be introduced in this section.

usual and acceptable assumption. Looking at equity-linked life insurance contracts, we have at least to remember that the total amount of contracts is much smaller than those of the standard case, and it is not obvious that the law of large numbers can be applied[5].

While the death distribution determines the payment date, the two further sources of risk, firstly the price risk of the underlying mutual fund and secondly the interest rate risk, determine the size of the benefit, if the contract terminates. Moreover we will argue that these financial risks are only responsible for the bonus part of the insurance contract, i.e. the ex ante uncertain payment which the insurer will pay in addition to the guaranteed amount. Clearly the stochastic development of the underlying mutual fund will not affect the guaranteed amount, but since we have to consider the discounted value of the total payments, i.e. a pathwise discounting of the payments, it is not obvious that the interest rate risk only influences the bonus part of the contract. In order to clarify this we now explicitly model the two financial markets.

Let $(\Omega, \mathbb{F}, P^*)$ be a filtered probability space. For the financial markets a continuous time and complete market framework is assumed. More specific the mutual fund and the interest rate market are given by Itô processes of the form

$$
\begin{aligned}
dS(t) &= r(t)\,S(t)\,dt + \sigma_1(t)S(t)\,dW_1^*(t) + \sigma_2(t)\,S(t)\,dW_2^*(t) && (1)\\
dD(t,t') &= r(t)\,D(t,t')\,dt + \sigma(t,t')\,D(t,t')\,dW_1^*(t) \quad \forall t' \quad \forall t \le t' && (2)
\end{aligned}
$$

where $\{W_1^*(t)\}_t$ and $\{W_2^*(t)\}_t$ are independent one-dimensional Brownian motions under the measure P^*. In order to exclude arbitrage possibilities we have to assume certain regularity conditions. First note, that the volatility functions $\sigma_1(t)$ and $\sigma_2(t)$ for the mutual fund and $\sigma(t,t') \quad \forall t' \quad \forall t \le t'$ are not necessarily deterministic. Repeating the arguments of El Karoui, Lepage, Myneni, Roseau and Viswanathan [1991] the interest rate market has to satisfy the following regularity assumption

Assumption 1 *For any maturity $t' \in [0,T]$ we assume that the volatility process $\{\sigma(t,t')\}_{0\le t\le t'}$ of the default free zero coupon bond $D(t,t')$ satisfies the following conditions:*

i) $\{\sigma(t,t')\}_{0\le t\le t'}$ *is a continuous and adapted process with* $\sigma(t',t') = 0$.

ii) The partial derivative with respect to maturity $\frac{\partial \sigma(t,t')}{\partial t'}$ *is a continuous and adapted process* $\forall t \le t'$.

iii) $E_{P^*}\left[\exp\left\{\frac{1}{2}\int_0^{t'} \sigma(t,t')^2 dt\right\}\right] < \infty$.

iv) $E_{P^*}\left[\left|\frac{\partial \sigma(t,t')}{\partial t'}\right|^2\right]$ *is bounded on* $\{(t,t')|0 \le t \le t',\ \in [0,T]\}$.

v) There exists a predictable process $\{A(t)\}_t$ *with* $E_{P^*}\left[A(t)^2\right] < \infty$ *and* $E_{P^*}\left[\int_0^T A(t)^2 dt\right] < \infty$ *such that*

$$\left|\frac{\partial \sigma}{\partial t'}(t,t'+\delta) - \frac{\partial \sigma}{\partial t'}(t,t')\right| \le A(t)\cdot\delta \quad \forall t \le t' \quad \forall \delta > 0$$

[5] Alternatively we could include so called loading factors, i.e. consider different survival and death distributions. This would not change our theoretical analysis, but the quantitative results on which we report in section 6 would indicate higher premium values.

With Assumption 1 the discounted zero coupon bond price processes are martingales under the measure P^*, i.e P^* is the risk neutral or equivalent martingale measure. The solution of the stochastic differential equation (2) is given by

$$D(t,t') = D(t_0,t') \cdot \exp\left\{ \int_0^t r(u)du - \frac{1}{2}\int_0^t \sigma(u,t')^2 du + \int_0^t \sigma(u,t') dW_1^*(u) \right\}. \qquad (3)$$

The spot rate process $\{r(t)\}_t$ is determined by the bond price processes, i.e.

$$r(t') = -\frac{\partial D(t_0,t')}{\partial t'} + \frac{1}{2}\int_0^{t'} \left(\frac{\partial}{\partial t'}\sigma(u,t')^2 \right) du + \int_0^{t'} \left(\frac{\partial}{\partial t'}\sigma(u,t') \right) dW_1^*(u). \qquad (4)$$

Assumption 2 *For the volatility processes of the mutual fund we assume:*

i) $\{\sigma_1(t)\}_t$ *and* $\{\sigma_2(t)\}_t$ *are one-dimensional, continuous and adapted processes.*

ii) $E_{P^*}\left[\exp\left\{\int_0^t \sigma_1^2(u) + \sigma_2^2(u)du\right\}\right] < \infty \quad \forall t \le T.$

iii) $E_{P^*}\left[\sigma_1^2(t) + \sigma_2^2(t)\right]$ *is bounded for* $t \in [0,T]$.

iv) There exists a predictable process $\{A(t)\}_t$ *with* $E_{P^*}\left[A(t)^2\right] < \infty$ *and* $E_{P^*}\left[\int_0^T A(t)^2 dt\right] < \infty$ *such that*

$$|\sigma_1(t+\delta) + \sigma_2(t+\delta) - \sigma_1(t) - \sigma_2(t)| \le A(t)\cdot\delta \quad \forall t \quad \forall \delta > 0.$$

The Assumptions 1 and 2 are sufficient for the combined mutual fund and interest rate market to be arbitrage free. Uniqueness of the measure P^* implies the completeness of the combined bond and mutual fund market. Furthermore the arbitrage price of a financial contract is determined as the expected discounted value under the risk neutral measure P^*. The independency between the Brownian motions $\{W_1^*(t)\}_t$ and $\{W_2^*(t)\}_t$ implies that the volatility process $\{\sigma_S(t)\}_t$ of the mutual fund is given by

$$\sigma_S^2(t) := \sigma_1^2(t) + \sigma_2^2(t), \qquad (5)$$

which is not necessarily deterministic. Furthermore $\{\sigma_1(t)\}_t$ and $\{\sigma_2(t)\}_t$ determine the correlation process between the mutual fund and the zero coupon bond $D(t,t')$ as

$$\rho(t) := \frac{\sigma_1(t)}{\sigma_S(t)} \qquad \forall t',\, \forall t \le t'. \qquad (6)$$

Vice versa, for a given specification of the volatility process of the underlying mutual fund and the correlation process $\{\rho(t)\}_t$ with $\rho(t) \in [-1,1]$ a.s. $\forall t \in [0,T]$ the processes $\{\sigma_1(t)\}_t$ and $\{\sigma_2(t)\}_t$ are determined by:

$$\sigma_1(t) \;:=\; \rho(t)\cdot\sigma_S(t) \quad \text{and} \quad \sigma_2(t) := \sqrt{1-\rho(t)^2}\cdot\sigma_S(t). \qquad (7)$$

Under the risk neutral measure P^* the pricing of a financial contract is done by calculating its expected discounted value. Since the discounting is stochastic, this implies a pathwise consideration, even if the value of the contract is path independent. As shown by Jamshidian (1989, 91) and Geman, El Karoui and Rochet [1995] this can be simplified by a change of measure technique, i.e. by using a zero coupon bond with maturity T as a numeraire for the stochastic processes $S(t)$ and $D(t,t')$, $t' \le T$. By Itô's Lemma we can rewrite the stochastic differential equations (1) and (2).

$$
\begin{aligned}
d\left(\frac{S(t)}{D(t,T)}\right) &= \frac{S(t)}{D(t,T)}\ \left([\,\sigma_1(t) - \sigma(t,T)\,]\, dW_1^T(t) \ + \ \sigma_2(t)\, d\, W_2^T(t)\right), && (8)\\
d\left(\frac{D(t,t')}{D(t,T)}\right) &= \frac{D(t,t')}{D(t,T)}\ [\,\sigma(t,t') - \sigma(t,T)\,]\, dW_1^T(t), && (9)
\end{aligned}
$$

where $(dW_1^T,\, dW_2^T) \ := \ ((dW_1^*(t) - \sigma(t,T))dt, dW_2^*(t))$ are standard Brownian motions under the so called T–forward risk adjusted measure P^T given by the Radon-Nikodym derivative

$$
\frac{dP^T}{dP^*} = \exp\left\{\int_{t_0}^{T} \sigma(t,T)dW_1^*(t) - \frac{1}{2}\int_{t_0}^{T} \sigma^2(t,T)dt\right\}. \qquad (10)
$$

This change of measure technique implies that the arbitrage price of a financial contract with payment at T is equal to the discounted expected value under the P^T measure. In particular the market value $V(t_0,T,K)$ at time t_0 of the bonus payable at time T equals

$$
V(t_0,T,K) := D(t_0,T)\cdot E_{P^T}\left[\left[a\cdot K\sum_{i=0}^{n-1}\frac{S(T)}{S(t_i)} - g(T,K)\right]^+\right]. \qquad (11)
$$

Furthermore the solution of the stochastic processes (8) and (9) implies that

$$
\begin{aligned}
\frac{S(T)}{S(t)} &= \frac{D(t_0,t)}{D(t_0,T)}\cdot\exp\left\{\int_{t_0}^{t}(\sigma(u,t)-\sigma(u,T))dW_1^T(u)\right\}\\
&\quad \exp\left\{\int_{t}^{T}(\sigma_1(t)-\sigma(u,T))dW_1^T(u) + \int_{t}^{T}\sigma_2(t)dW_2^T(u)\right\} \qquad (12)\\
&\quad \exp\left\{-\frac{1}{2}\int_{t_0}^{t}(\sigma(u,t)-\sigma(u,T))^2du - \frac{1}{2}\int_{t}^{T}\left((\sigma_1(t)-\sigma(u,T))^2+\sigma_2(t)^2\right)du\right\}.
\end{aligned}
$$

So far we have not restricted the volatility processes of the mutual fund and the term structure of interest rates to be deterministic. If we would do so, the mutual fund according to (1) would be a log-normal diffusion and the term structure model would belong to the Gaussian framework. As a consequence of this specific structure the ratio $\frac{S(T)}{S(t)}$ would be log-normally distributed under the T-forward risk adjusted measure. The theoretical results in Section 4 are not restricted to this specific case. But, it is this special distributional assumption which allows us in Section 5 to discuss different numerical techniques to approximate the solution of the fair premium problem.

4 The Fair Premium

Definition 3 *Consider an equity-linked life insurance contract determined by the time to maturity T, the age x of the insured person, his or her death distribution π_x, and the guaranteed amount $g:[0,T]\times I\!R^+ \to I\!R^+$. A periodic premium K^* at dates $t_i, i = 0,...,n-1$ is called a* fair premium *if the expected discounted value of the sum of the periodic premiums under the death* *distribution π_x is equal to the expected benefit of the contract under the death distribution, i.e if K^* is a solution to*

$$
\begin{aligned}
0 &= K^* \sum_{i=0}^{n-1} D(t_0,t_i)\left(1-\int_{t_0}^{t_i}\pi_x(t)dt\right) \\
&- \int_{t_0}^{T} D(t_0,t)g(t,K^*)\pi_x(t)dt - D(t_0,T)g(T,K^*)\left(1-\int_{t_0}^{T}\pi_x(t)dt\right) \\
&- \int_{t_0}^{T} V(t_0,t,K^*)\pi_x(t)dt - V(t_0,T,K^*)\left(1-\int_{t_0}^{T}\pi_x(t)dt\right),
\end{aligned} \tag{13}
$$

where

$$
\begin{aligned}
V(t_0,t,K^*) &= D(t_0,t)E_{P^t}\left[\left[aK^*\sum_{i=0}^{n^*(t)-1}\frac{S(t)}{S(t_i)} - g(t,K^*)\right]^+\right] \\
n^*(t) &= \min\{j|\quad t_j \geq t\}.
\end{aligned} \tag{14}
$$

The equations (13) and (14) explain the influence on the fair premium of the stochastic nature of the financial market and the death distribution. The pricing principle expressed in (14) shows that the price of a contingent claim is equal to the discounted expected value of its uncertain cash flow in the future, where the expectation has to be taken with respect to the forward risk adjusted measure. This pricing principle is valid when the share as well as the bond market is uncertain. Assume now that the bond market had a deterministic development. Then a similar pricing principle would be valid with the expectation taken under the risk neutral measure, P^*. It means that the influence of a stochastic versus a deterministic developing short term interest rate can be explained as the difference in pricing obtained through the application of the operators E_{P^t} and E_{P^*}. The difference between the two expected values of the Asian-like option gives us although in an implicit form the market price of interest rate risk. Denote the original/real probability measure by P. Assume again that the bond market is deterministic. Then the pricing principle similar to (14) but with the application of E_P instead of E_{P^*} would lead to prices where all components of market price of risk are neglected. The attention is next turned towards equation (13). It is through (13) that the interplay between the death distribution and the riskiness of the financial market enters the analysis. Had we used (13) and (14) with E_{P^t} relpaced by E_P then the fair premium would have been determined in accordance to the classical principle of equivalence.

The fair premium will in the following sequence of Propositions and Theorems be analysed to restrict the interrelationships between the share a, the periodic premium K and the function of the guaranteed amount $g(t,K)$. First note that the martingale property of the ratio $\frac{S(t)}{S(u)} \forall u \leq t$ under the t-forward risk adjusted measure P^t implies

$$
E_{P^t}\left[\sum_{i=0}^{n^*(t)-1}\frac{S(t)}{S(t_i)}\right] = \sum_{i=0}^{n^*(t)-1}\frac{D(t_0,t_i)}{D(t_0,t)}, \tag{15}
$$

Proposition 1 *Let $\pi_x(t)$ be the density function of the death distribution then*

$$
\begin{aligned}
\int_{t_0}^{T}\left(\sum_{i=0}^{n^*(t)-1} D(t_0,t_i)\right)\pi_x(t)dt + \left(\sum_{i=0}^{n-1} D(t_0,t_i)\right)\left(1-\int_{t_0}^{T}\pi_x(t)dt\right) \\
= \sum_{i=0}^{n-1}\left(D(t_0,t_i)(1-\int_{t_0}^{t_i}\pi_x(t)dt)\right).
\end{aligned}
$$

Proof:

$$\int_{t_0}^{T}\left(\sum_{i=0}^{n^*(t)-1} D(t_0,t_i)\right)\pi_x(t)dt = \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left(\sum_{j=0}^{i} D(t_0,t_i)\right)\pi_x(t)dt$$

$$= \sum_{i=0}^{n-1} D(t_0,t_i)\int_{t_i}^{T}\pi_x(t)dt = \sum_{i=0}^{n-1} D(t_0,t_i)\left[\left(1-\int_{t_0}^{t_i}\pi_x(t)dt\right)-\left(1-\int_{t_0}^{T}\pi_x(t)dt\right)\right]$$

□

Proposition 1 and equation (15) imply that the fair premium in the case of a guaranteed amount $g(t,K)=0 \quad \forall t \forall K$ is the solution of

$$0 = (1-a)K^*\sum_{i=0}^{n-1} D(t_0,t_i)\Big(1-\int_{t_0}^{t_i}\pi_x(t)dt\Big),$$

i.e. $\forall a \in [0,1[$ the value $K^*=0$ is the unique fair premium, and for $a=1$ any K is a fair premium in this situation. Let us now consider conditions on the functional form of a guaranteed amount which are sufficient for the existens of a unique solution K^* for the fair premium problem defined by equation (13) if the guaranteed amount is not identical to zero. In Nielsen and Sandmann [1995] we have derived the existence of a unique fair premium in the case of a time independent guaranteed amount $g(K)$ and a specific duration type Gaussian term structure model. In the general model for the financial markets we can now extend this to the multiplicative separable time dependent case, i.e. where

$$g(t,K) = F(t)\cdot h(K). \tag{16}$$

In the remaining part of this paper it is assumed that $g(t,K)$ can be separated in this form. To ease the economic interpretation of our expressions some simplified notation will be introduced. The fair premium problem (13) for a separable guaranteed amount consists basically of three terms which are differently related to the distributional assumptions discussed in the previous section.[6]

- The first term denotes the cost of the contract, i.e. the market value of the sum of the periodic premiums under the death distribution, which depends only on the death distribution and can be expressed by $K\cdot H$, where H is defined as

$$H := \sum_{i=0}^{n-1} D(t_0,t_i)\left(1-\int_{t_0}^{t_i}\pi_x(t)dt\right). \tag{17}$$

- The expected benefit of the contract can be separated in two terms. First the expected discounted guaranteed amount under the death distribution, which depends on the specific functional form of the guaranteed amount. Due to the separability assumption (16) we can reformulate this as the function $a\cdot K\cdot L\left(\frac{h(K)}{a\cdot K}\right)$ with

$$L(y) := y\left\{\int_{t_0}^{T} D(t_0,t)F(t)\pi_x(t)dt + D(t_0,T)F(T)\left(1-\int_{t_0}^{T}\pi_x(t)dt\right)\right\}. \tag{18}$$

[6] The notation will also be helpful in several of the proofs to follow.

The second part is the expected discounted bonus under the death distribution. This term is directly related to the reference portfolio and therefore depends on both types of the financial risks. Under the separability assumption (16) we can reformulate this benefit term as the function $a \cdot K \cdot U\left(\frac{h(K)}{a \cdot K}\right)$, with

$$\begin{aligned} U(y) &:= \int_{t_0}^{T} c(t_0,t,y)\pi_x(t)dt + c(t_0,T,y)\left(1 - \int_{t_0}^{T}\pi_x(t)dt\right), \qquad (19)\\ c(t_0,t,y) &:= D(t_0,t)E_{P^t}\left[\left[\sum_{i=0}^{n^*(t)-1}\frac{S(t)}{S(t_i)} - F(t)\cdot y\right]^+\right] \quad \forall t > t_0. \end{aligned}$$

The expected benefit of the contract under the death distribution is therefore determined by the sum of the functions defined by (18) and (19) and can be written as $a \cdot K \cdot R\left(\frac{h(K)}{a \cdot K}\right)$, with

$$R(y) := L(y) + U(y). \qquad (20)$$

With this notation and the separability of the guaranteed amount, the first formulation of fair premium problem (13) is equivalent to

$$f\left(\frac{h(K^*)}{a \cdot K^*}\right) = 0 \qquad (21)$$

if we assume that $a \neq 0$ and $K \neq 0$, where the function $f(\cdot)$ is defined by

$$f(y) := \frac{H}{a} - R(y) = \frac{H}{a} - L(y) - U(y). \qquad (22)$$

Observe that H is independent of K, a and g, and that the function $c(t_0,t,\cdot)$ fulfils the following conditions

1. $c(t_0,t,y)$ is continuous and strictly decreasing in y.
2. From equation 15 we know that $c(t_0,t,0) = \sum_{i=0}^{n^*(t)-1} D(t_0,t_i)$.
3. $\lim_{y\to+\infty} c(t_0,t,y) = 0$.
4. No arbitrage and the absolute continuity of the t-forward risk adjusted measure P^t with respect to the Lebesgue measure $\forall t$ imply that

   $$c(t_0,t,y_1) - c(t_0,t,y_2) < (y_2 - y_1)F(t)D(t_0,t) \text{ for } y_1 < y_2$$

   from where we obtain, that $U(y_1) - U(y_2) < L(y_2) - L(y_1)$ for $y_1 < y_2$.

Theorem 2 *Suppose that the death distribution is determined by the density function $\pi_x(t)$ and that the stochastic processes $\{S(t)\}_{t\in[0,T]}$ and $\{D(t,t')\}_{t\in[0,t']}\forall t' \in [0,T]$ are satisfying the Assumptions 1 and 2.*

There exists a unique fair premium K^* for $a \in \,]0,1[$ if the guaranteed amount $g(t,K)$ can be decomposed such that

$$g(t,K) = F(t) \cdot h(K)$$

where

i) $F(t)$ *is a non-negative continuous and bounded function on* $[0, T]$

ii) $\tilde{h} = \frac{h(K)}{K}$ *is a continuous and bijective function from* $I\!R^+$ *to* $I\!R^+$.

Proof: Under the assumption made on $\tilde{h}(\cdot)$ it is sufficient to prove that there exists a unique y^* with $f(y^*) = 0$. The function $f(\cdot)$ fulfils

1. $\lim_{y \to +\infty} f(y) = -\infty$
2. for $y_1 < y_2$: $\quad f(y_2) - f(y_1) = -(L(y_2) - L(y_1)) + U(y_1) - U(y_2) < 0$
   where the last inequality is obtained through the application of property 4 for the function $c(t_0, t, \cdot)$. The inequality implies that $f(y)$ is a strictly decreasing in y.
3. By equation 15 we have that $f(0) = (\frac{1}{a} - 1) \cdot H > 0 \quad \forall a \in]0, 1[$.

Thus $f(y)$ is a continuous and strictly decreasing function in y and by the mean value theorem there exists a unique $y^* > 0$ with $f(y^*) = 0$. □

No arbitrage opportunities and a setting of equilibrium make it reasonable to introduce the condition on the g function that $g(t, K) = F(t) \cdot h(K) = F(t) \cdot \beta \cdot K$ where β is a positive constant and $F(t)$ is a continuous and bounded function on $[0, T]$. It means that the guaranteed amount is assumed to be homogeneous of degree one in the periodic premium. Typical examples of this form are

$$g(t, K) = \exp\{-\delta t\}\beta \cdot K \quad \text{or} \quad g(t, K) = \sum_{i=0}^{n^*(t)-1} \exp\{\delta(t - t_i)\} \beta \cdot K \tag{23}$$

where $\delta > 0$ is an internal discounting rate respectively growth rate guaranteed by the insurer and $\beta > 0$ is a constant. If now K^* is a fair premium and the function $g(t, K)$ is homogeneous of degree one in K then $2K^*$ is the fair premium for the situation where the guaranteed amount is multiplied by two. At a first glance Theorem 2 cannot be applied to this situation, but a reinterpretation of the Theorem immediately yields:

Corollary 3 *Under the assumptions of Theorem 2 and for a continuous and bounded function* $F(t) : [0, T] \to I\!R^+$ *there exist for each* $a \in]0, 1[$ *a unique non-trivial* $\beta^* > 0$, *the shape of the guaranteed amount, such that any non-negative premium* K *is a solution of the fair premium problem (21) with the guaranteed amount*

$$g(t, K) = F(t) \cdot \beta^* \cdot K.$$

Proof: A special case of Theorem 2 is the choice of the guaranteed amount $g(t, K) = F(t) \cdot \bar{g}$ for some fixed $\bar{g} > 0$. The fair premium is defined as the unique K^* which solve the equation $f(\frac{\bar{g}}{aK^*}) = 0$. Denoting $\beta^* = \frac{\bar{g}}{K^*} > 0$ we have the wanted result. □

The Corollary implies, that given the function $F(t)$, there exist for each death distribution (age) a unique function for the guaranteed amount which is homogeneous of degree one in the periodic premium. For these functions of the guaranteed amount the coefficient β^* has to be estimated. Given the value of β^* the fair premium of an equity-linked life insurance contract with a guaranteed amount $F(t) \cdot \bar{g}$ is just $K^* = \frac{\bar{g}}{\beta^*}$, where of course the value of β^* depends on all parameters of the life insurance contract and the distributional assumptions. We now can conclude the following qualitative result with respect to the shape β of the guaranteed amount:

Proposition 4 *Let the guaranteed amount of an equity-linked life insurance contract be equal to $g(t,K) = F(t)\cdot\beta\cdot K$ where $F : [0,T] \longrightarrow I\!R^+$ is a continuous and bounded function. Suppose that the solution of the fair premium problem (21), $\beta^*(a)$ as a function of the share $a \in]0,1[$, is differentiable, then the ratio $\frac{\beta^*(a)}{a}$ is a decreasing function in a.*

Proof: See the appendix.

The basic problem in estimating β^* from equation (21) is that we first have to approximate the options

$$E_{P^t}\left[\left[a\sum_{i=0}^{n^*(t)-1}\frac{S(t)}{S(t_i)} - F(t)\beta\right]^+\right]$$

for all $t \in [0,T]$ as a function of β and then secondly have to solve a zero point problem. As the above options can be interpreted as modified Asian options, we know that there exist no closed form solutions even if we would restrict the analysis to the situation under deterministic volatility functions. In Nielsen and Sandmann [1995] we have considered the pricing of these options by modifying two approximation methods for Asian options suggested by Turnbull and Wakeman [1991] and Vorst [1992]. Comparing these approximations with a Monte Carlo simulation we reached the conclusion that neither of the methods are appropriate to estimate the coefficient β respectively to estimate the fair premium K^*. On the other hand the Monte Carlo simulation technique implies that we not only have to simulate all option values but in addition have to solve the zero point problem with respect to all simulated paths of the sums $\sum_{i=0}^{n^*(t)-1}\frac{S(t)}{S(t_i)}$. This implies a serious computational problem. To avoid this problem the zero point problem (21) can be turned into the following dual problem:

Theorem 5 *Consider the function*

$$a \;:\; I\!R^+ \longrightarrow [0,1] \text{ with } a(b) := \frac{H}{R(b)}$$

then $\beta^* = a(b)\cdot b$ is the unique solution for the fair premium problem (21) with guaranteed amount $g(t,K) = F(t)\cdot\beta^*\cdot K$ and the share $a(b)$ invested in the fund at each premium payment date. Furthermore $a(b)$ is decreasing and $\frac{1}{a(b)}$ is convex in b.

Proof: Note firstly that for $a(b)$ as defined above, $\beta^* = a(b)\cdot b$ is a solution to equation (21). Secondly by equation 15 we know that $a(0) = 1$ and $\lim_{b\to+\infty} a(b) = 0$. Furthermore consider the denominator of the function $a(b)$. Differentiating with respect to b yields as in the proof of Proposition 5 that the denominator is strictly increasing. Differentiating a second time gives that the denominator is also a convex function. Thus $a(b) \in]0,1[\ \forall\ b \in \mathbb{R}^+$. By Corollary 3 we furthermore know that the solution $\beta^* = a(b)\cdot b$ is unique. □

Theorem 5 expresses that if $\beta^* = a\cdot b$ is a solution of the problem (21) for some given $b \in \mathbb{R}^+$, then a must be equal to the value defined by $a(b)$. On the reverse for $a = a(b)$ we know that the β^* that solves equation (21) must be equal to $a(b)\cdot b$. Thus we can interpret $a(b)$ as the fair share. The advantage of this formulation is that the function $a(b)$ is much easier to calculate than the solution of the zero point problem (21). Furthermore the limits of the function $a(b)$ are quite intuitive since

1) $a(0) = 1$ implies that $\beta^* = a(0)\cdot 0 = 0$. This corresponds to a contract situation with no guaranteed amount. As we already know, the fair premium in this case is a K^* which satisfies $0 = (1-a)\cdot K\cdot H$. For $a = 1$ this is true $\forall K$.

2) $\lim_{b\to+\infty} a(b) = 0$ implies, that we consider an equity-linked life insurance contract with no investment in the portfolio. In this situation we know that the β^* satisfying equation (21) is just equal to

$$\beta^* = \frac{H}{\int_{t_0}^{T} D(t_0,t)\, F(t)\, \pi_x(t)\, dt + F(T)\, D(t_0,T)\, (1 - \int_{t_0}^{T} \pi_x(t)\, dt)}$$

which is by l'Hopital's rule just equal to the limit $\lim_{b\to\infty} a(b)\cdot b$.

Finally we get the following theorem

Theorem 6 *Suppose that the guaranteed amount of an equity-linked life insurance contract is equal to $g(t,K) = F(t)\cdot\bar{g}$ for some fixed $\bar{g} > 0$ and $F : [0,T] \longrightarrow \mathbb{R}^+$ be a continuous and bounded function. Under the distributional assumptions of Theorem 2 the fair premium K^* as a function of the share a is strictly increasing and convex.*

Consequently for a fixed periodic premium K and a guaranteed amount $g(t,K) = F(t)\cdot\beta^*\cdot K$ the shape β^*, as a function of the share a, is decreasing and concave.

Proof: See the appendix

5 Approximation Techniques in a Standard Gaussian Framework

We now address the numerical question of calculating the fair premium. As we have pointed out in Section 4 the fair premium K^* with separated guaranteed amount is obtained by calculating the function of the fair share $a(b) = \frac{H}{R(b)}$.

The numerical procedure which we are going to apply essentially consists of two steps. First we calculate the values of $a(b_i)$ for a grid of values $b_i \in \{b_1 < b_2 < \cdots\}$ and second, we approximate by interpolation the function $a(b)$. More precisely the procedure is as follows: Suppose that for a fixed $\bar{a} \in]0,1[$ we have $b_i < b_{i+1}$ such that

$$a_{i+1} := a(b_{i+1}) < \bar{a} < a(b_i) =: a_i.$$

The function $a(b)$ we approximate on this interval by the following decreasing function

$$\hat{x}(\bar{b}) := \frac{a_i a_{i+1} (b_{i+1} - b_i)^2}{\left(\sqrt{a_{i+1}}\, [b_{i+1} - \bar{b}] + \sqrt{a_i}\, [\bar{b} - b_i]\right)^2} \quad \text{for } \bar{b} \in [b_i, b_{i+1}]. \tag{24}$$

The inverse function[7]

$$\hat{x}^{-1}(\bar{a}) = \frac{\sqrt{a_i a_{i+1}}\ [b_{i+1} - b_i] - \sqrt{\bar{a}}\ \left[\sqrt{a_{i+1}}\, b_{i+1} - \sqrt{a_i}\, b_i\right]}{\sqrt{\bar{a}}\ \left[\sqrt{a_i} - \sqrt{a_{i+1}}\right]} \quad \text{for } \bar{a} \in [a_{i+1}, a_i] \tag{25}$$

determines for a given share $\bar{a}$ the approximation of $\bar{b}$. The fair premium of an equity-linked life policy with a guaranteed amount $g(t,K) = F(t)\cdot\bar{g}$, $\bar{g} \geq 0$ and the share equal to the interpolated value $\bar{a}$ is then approximated by

$$\hat{K}^*(\bar{a}) = \frac{\bar{g}}{\bar{a}\cdot\bar{b}}. \tag{26}$$

[7] Note that the function $\frac{1}{\hat{x}(\cdot)}$ is convex in b.

Given this two step procedure it remains to calculate the function $a(b)$ at sufficiently many points b_i, which implies the calculation of the function $R(\cdot)$ as defined by (20) at these points. As $R(\cdot)$ has many similarities in common with an Asian option we will as an initial step apply the approximation techniques known for these options to calculate the function $R(\cdot)$. Up to now we have not used any specific distributional assumption, i.e. the present results are valid for any specification of the dynamics for the mutual fund and the term structure of interest rates as long as the regularity conditions (Assumption 1 and 2) are satisfied. Since all known methods to approximate the arbitrage price of an Asian option are restricted to the case of a deterministic and moreover constant volatility structure we will from now on restrict ourself to the most simple situation, i.e.

$$\begin{aligned} \sigma_1(t) &= \sigma_1 \forall t \\ \sigma_2(t) &= \sigma_2 \; > 0 \; \forall t \\ \sigma(t,t') &= \sigma \cdot (t'-t) \; \forall t \; \in [0,t'], \forall t \; \in [0,T], \sigma > 0. \end{aligned} \tag{27}$$

As a consequence the fund follows a standard geometric Brownian motion and the term structure model is specified as the continuous time limit of the Ho and Lee [1986] model.

In Nielsen and Sandmann [1995] we have considered three different methods concerned with this problem: the Fast Fourier Transformation (FFT), the Turnbull and Wakeman [1991] and the Vorst [1992] approximation technique. We have shown that all of the three mentioned methods are for different reasons not suitable for the situation we are considering here. Firstly, due to the assumption of stochastic interest rates, the FFT technique cannot be applied. Secondly, the Turnbull and Wakeman [1991] approximation leads to a completely unusable approximation of the density of the arithmetic average

$$A(t) \;=\; \frac{1}{n^*(t)} \sum_{i=0}^{n^*(t)-1} \frac{S(t)}{S(t_i)} \tag{28}$$

given that the time to maturity of the insurance contract exceeds 5 years. Although we have generalized the Turnbull and Wakeman approximation to the case of stochastic interest rates [8] we must strongly reject this approximation in the case of an equity-linked life insurance policy (see also Figures 2 and 3). Thirdly, the Vorst [1992] approximation turns out to be a little bit more robust, but compared with the results from a Monte Carlo simulation in Section 6 also this approximation is not suitable for the determination of the fair premium. To clarify this we consider the Vorst approximation in more detail. Define in addition to the arithmetic average the geometric average by

$$G(t) \;=\; \sqrt[n^*(t)]{\prod_{i=0}^{n^*(t)-1} \frac{S(t)}{S(t_i)}} \;\leq A(t). \tag{29}$$

The relationship between these averages implies

$$\begin{aligned} D(t_0,t)E_{P^t}[[G(t)-Y]^+] &\leq D(t_0,t)E_{P^t}[[A(t)-Y]^+] \\ &\leq D(t_0,t)E_{P^t}[[G(t)-Y]^+] + D(t_0,t)\left(E_{P^t}[A(t)] - E_{P^t}[G(t)]\right). \end{aligned} \tag{30}$$

[8] For more details see Nielsen and Sandmann [1995]. The generalized algorithmus for the first four central moments are also given there.

Under the assumption of a log-normally distributed mutual fund and a Gaussian term structure model the geometric average $G(t)$ is log-normally distributed under the t-forward risk adjusted measure P^t. This implies that the t-forward value of the option on the geometric average is determined by

$$
\begin{aligned}
E_{P^t}[\max\{G(t)-Y,0\}] &= \exp\{m_G(t)+\frac{1}{2}\sigma_G^2(t)\}\ N(d) - Y N(d-\sigma_G(t)) \qquad (31)\\
\text{with} \qquad d &= \frac{m_G(t)-\ln Y+\frac{1}{2}\sigma_G^2(t)}{\sigma_G(t)}\,,\\
m_G(t) &= E_{P^t}[\ln G(t)] \quad \text{and} \quad \sigma_G^2(t) = V_{P^t}[\ln G(t)]\,,
\end{aligned}
$$

where $N(\cdot)$ denotes the standard normal distribution. In the case of a Gaussian term structure model Vorst [1992] provides us with upper and lower bounds for the Asians options which do imply lower and upper bounds for the fair premium. Furthermore Vorst suggest the following approximation for the t-forward value of an Asian Option:

$$
E_{P^t}[[A(t)-Y]^+] \approx \exp\{m_G(t)+\frac{1}{2}\sigma_G^2(t)\}\ N(z) - \tilde{Y}\cdot N(z-\sigma_G(t)) \qquad (32)
$$

$$
\text{with} \qquad \tilde{Y} = Y - (E_{P^t}[A(t)] - E_{P^t}[G(t)]) \quad \text{and} \quad z = \frac{m_G(t)-\ln\tilde{Y}+\frac{1}{2}\sigma_G^2(t)}{\sigma_G(t)}.
$$

The Vorst approximation implies that the value of an Asian option is approximated by the value of an option based on the geometric average, where the log-normal distribution of the geometric average has a support $[E_{P^t}[A(t)] - E_{P^t}[G(t)], +\infty[$. Remember that the unknown distribution of the arithmetic average has the support $\mathbb{R}^+$. Therefore the difference between the expected arithmetic and geometric average under the t-forward measure determines the approximation error. From equation 15 we already know that $E_{P^t}[A(t)] = \frac{1}{n^*(t)}\sum_{i=0}^{n^*(t)}\frac{D(t_0,t_i)}{D(t_0,t)}$ which for a flat initial term structure with $D(t_0,t_i) = \exp\{-rt_i\}, r > 0$ is strictly increasing. In this situation we can calculate the expected value of the geometric average by

Theorem 7 *If the stochastic processes $\{S(t)\}_{t\in[0,T]}$ and $\{D(t,t')\}_{t\in[0,t']}\forall t'\in[0,T]$ are solutions of the stochastic differential equations (8) resp. (9) with constant volatility coefficients then the expected value of the geometric average $G(t)$ under the t-forward risk adjusted measure is equal to*

$$
E_{P^t}[G(t)] = \exp\left\{m_G(t)+\frac{1}{2}\sigma_G^2(t)\right\}
$$

with

$$
\begin{aligned}
m_G(t) &= \frac{1}{n^*(t)}\sum_{i=0}^{n*(t)-1}\left[\ln\left(\frac{D(t_0,t_i)}{D(t_0,t)}\right)\right.\\
&\quad \left. -\frac{1}{2}\Big((\sigma_1^2+\sigma_2^2)(t-t_i) + (\sigma^2 t_i - \sigma_1\sigma)(t-t_i)^2 + \frac{1}{3}\sigma^2(t-t_i)^3\Big)\right],\\
\sigma_G^2(t) &= \frac{\sigma_1^2+\sigma_2^2}{n^*(t)^2}\sum_{i=0}^{n^*(t)-1}(i+1)^2(t_{i+1}-t_i) - \frac{\sigma_1\sigma}{n^*(t)^2}\sum_{i=0}^{n^*(t)-1}(i+1)^2(t_{i+1}-t_i)[2t-t_{i+1}-t_i]\\
&\quad +\frac{2\sigma_1\sigma}{n^*(t)^2}\sum_{i=0}^{n^*(t)-2}\left[\left(\sum_{j=i+1}^{n^*(t)-1}(j-i)(t_{j+1}-t_j)\right)(i+1)(t_{i+1}-t_i)\right]\\
&\quad +\frac{\sigma^2}{3n^*(t)^2}\sum_{i=0}^{n^*(t)-1}(i+1)^2\big((t-t_i)^3-(t-t_{i+1})^3\big)
\end{aligned}
$$

$$+\frac{\sigma^2}{n^*(t)^2}\sum_{i=0}^{n^*(t)-2}\left[\left(\sum_{j=i+1}^{n^*(t)-1}(j-i)(t_{j+1}-t_j)\right)^2(t_{i+1}-t_i)\right]$$
$$-\frac{\sigma^2}{n^*(t)^2}\sum_{i=0}^{n^*(t)-2}\left[\left(\sum_{j=i+1}^{n^*(t)-1}(j-i)(t_{j+1}-t_j)\right)(i+1)(t_{i+1}-t_i)[2t-t_{i+1}-t_i]\right].$$

For a constant discretization of the time axis, i.e. $\Delta t = t_{i+1}-t_i$, $t = t_{n^*(t)-1}+\Delta t$ *and a flat initial term structure this can be simplified to:*

$$\begin{aligned} E_{P^t}[G(t)] &= \exp\left\{\frac{1}{2}\,rt\,\frac{n(n+1)}{n^2}\right\}\cdot\exp\left\{-\frac{\sigma_1^2+\sigma_2^2}{12}\,t\,\frac{n^2-1}{n^2}\right\} \\ &\quad\cdot\exp\left\{\frac{\sigma_1\sigma}{24}t^2\frac{(n-1)(3n^2+7n+4)}{n^3}\right\} \\ &\quad\cdot\exp\left\{-\frac{\sigma^2}{720}\,t^3\,\frac{48n^4+75n^3-10n^2-75n-38}{n^4}\right\}. \end{aligned}$$

Proof: See the Appendix
Remarks:

- Consider a deterministic term structure, i.e. $\sigma = 0$. In this situation the expected value $E_{P^t}[G(t)]$ is strictly decreasing in the volatility $\sigma_S = \sqrt{\sigma_1^2+\sigma_2^2}$ of the underlying mutual fund. If furthermore we assume a flat initial term structure and an equidistant time grid we have
$$\lim_{\Delta t\to 0} E_{P^t}[G(t)] = \exp\{\frac{1}{12}(6r-\sigma_s^2)t\}$$
which is either decreasing (if $r < \frac{1}{6}\sigma_S^2$ which is usually not the case) or increasing (if $r > \frac{1}{6}\sigma_S^2$) in the maturity t of the insurance contract.

- If the interest rate is stochastic, i.e. $\sigma > 0$ the properties of the expected geometric average are different. Consider the situation of a flat initial term structure and an equidistant time grid Δt. For a fixed volatility $\sigma_S = \sqrt{\sigma_1^2+\sigma_2^2}$ of the underlying mutual fund the expected geometric average is strictly increasing in the correlation $\rho = \frac{\sigma_1}{\sigma_S}$. In addition for $t > 1$ the expected geometric average is a strictly decreasing function of the interest rate volatility $\sigma > 0$, i.e. a volatile interest rate market reduces the expected geometric average. Furthermore the expected geometric average converge towards zero in the maturity t of the insurance contract if $\sigma > 0$ and $n^*(t) > 1$ independent of the initial interest rate r.

- The expected arithmetic average is independent of the volatility functions $\sigma_1(\cdot), \sigma_2(\cdot)$ and $\sigma(\cdot)$ whereas the expected geometric average is strongly depending on these functions. For a flat initial term structure we know that
$$E_{P^t}[A(t)] = \frac{1}{n^*(t)}\sum_{i=0}^{n^*(t)-1}\exp\{r(t-t_i)\}$$
which for an equidistant time grid yields
$$\lim_{\Delta t\to 0} E_{P^t}[A(t)] = \frac{1}{t\cdot r}\left[e^{rt}-1\right] \longrightarrow +\infty \quad \forall t \to +\infty.$$

Thus the expected arithmetic average increases whereas the expected geometric average approach zero for time to maturity t going to infinity. In addition we can conclude from Theorem 7 that

$$\begin{aligned}\lim_{\Delta t \to 0} V_{P^t}[G(t)] &= \left(\exp\{-\frac{48}{120}\sigma^2 t^4\}\right)^2 \left(\exp\{-\frac{12}{360}\sigma^2 t^4\} - 1\right) \\ &\longrightarrow 0 \qquad \forall\ t \to \infty \text{ and } \sigma > 0.\end{aligned}$$

This implies that we expect, for an approximation method based on the geometric average, a systematic error for the pricing of the Asian option if the time to maturity is large. As a consequence we expect a systematic error in the pricing of the bonus of an equity-linked life insurance policy and the fair premium. Figure 1 summarizes some of the mentioned properties of the expected geometric and arithmetic average.

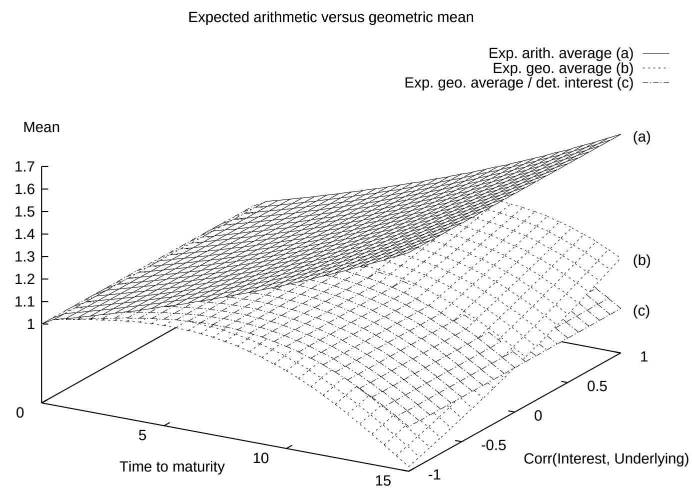

Figure 1: Expected arithmetic average A(t) and geometric average G(t) under the t - forward measure as a function of the maturity t and the correlation ρ for a flat initial term structure with $D(t_0, t) = \exp\{-0.0582 \cdot t\}$, $\sigma_s = 0.25, \sigma = 0.1$ and $\Delta t = t_{i+1} - t_i = \frac{1}{12}$.

6 Numerical Results

In this section we apply a Monte Carlo simulation to delineate our theoretical results in Sections 4 and 5. We will compare the results obtained by the Monte Carlo simulation with those derived by adopting the Vorst approximation. The life insurance policies, which we consider are characterized by maturities of 12, 15 and 18 years. As functions of the guaranteed amount we restrict ourself to two special types of a separable form (23), i.e.

$$\begin{aligned}\text{Model 1} \qquad g(t, K) &= \exp\{-\delta t\}\beta \cdot K \quad \text{for} \quad \delta = 0, 0.02 \text{ or } 0.035, \qquad (33)\\ \text{Model 2} \qquad g(t, K) &= \sum_{i=0}^{n^*(t)-1} \exp\{\delta(t - t_i)\}\beta \cdot K \quad \text{for } \delta = 0.025, 0.035 \text{ or } 0.045.\end{aligned}$$

In all the simulation we use a flat initial term structure with $D(t_0, t) = \exp\{-0.0582 \cdot t\}$ and assume a monthly payment frequency for the periodic premium. The death distribution is chosen to satisfy the Makeham formula (see e.g. F.Delbaen [1990]), i.e.

$$
\begin{aligned}
\pi_x(t_i) &= \frac{l_{x+t_i} - l_{x+t_i+\Delta t}}{l_x} &(34)\\
&\hat{=} \text{ probability that a life-aged-x will survive } t_i \text{ years and die}\\
&\quad\text{within the following } \Delta t \text{ years,}\\
\text{where} \quad l_x &= b \cdot s^x \cdot g^{e^x} ,\\
s &= 0.99949255 \, , g = 0.99959845 \, ,\\
c &= 1.10291509 \, , b = 1000401.71 \, ,
\end{aligned}
$$

and the age is fixed by $x = 30$ years. We assume that if the insured person dies at time $t \in [t_i, t_{i+1}[$ where we fixed $\Delta t = t_{i+1} - t_i = \frac{1}{12}$ then the insurer pays the guaranteed amount plus the bonus at time t_{i+1}. Therefore the estimation equation can be discretized with respect to the death distribution. As variance reduction we use an antithetic and control variate technique with the geometric average option as the control variate. The stochastic process of the underlying mutual fund and the zero coupon bonds are specified by the following simple volatility specifications

$$\sigma_1 = 0, \;\; \sigma_2 = 25\% \text{ and } \sigma(t, t') = 0.01(t' - t),$$

i.e. we concentrate on a situation where the correlation between the interest rate market and the mutual fund is equal to zero. The calculation of the fair premium based on a Monte Carlo simulation is determined by the two step procedure introduced in Section 5. Thus we simulate the function $a(\cdot)$ defined in Theorem 5 by

$$
\begin{aligned}
\hat{a}(b) &= \frac{H}{\hat{R}(b)} &(35)\\
\hat{R}(b) &= b\left(\sum_{i=0}^{n-1} D(t_0, t_{i+1}) F(t_i) \pi(t_i) + F(t_n) D(t_0, t_n)\left(1 - \sum_{i=0}^{n-1} \pi(t_i)\right)\right) &(36)\\
&\quad + \sum_{i=0}^{n-1} \hat{c}(t_0, t_i, b) \pi(t_i) + \hat{c}(t_0, t_n, b)\left(1 - \sum_{i=0}^{n-1} \pi(t_i)\right) ,\\
\hat{c}(t_0, t, b) &= \frac{D(t_0, t)}{2M} \sum_{m=1}^{2M} \left[\left[\sum_{j=0}^{n^*(t)-1} \frac{S_m(t)}{S_m(t_j)} - F(t) \cdot b\right]^+\right.\\
&\quad \left. - \, n^*(t) \cdot \left[\sqrt[n^*(t)]{\prod_{j=0}^{n^*(t)-1} \frac{S_m(t)}{S_m(t_j)}} - F(t) \cdot \frac{b}{n^*(t)}\right]^+\right]\\
&\quad + \, n^*(t) \cdot D(t_0, t) \cdot E_{P^t}\left[\left[G(t) - F(t) \cdot \frac{b}{n^*(t)}\right]^+\right]
\end{aligned}
$$

where M denotes the number of paths [9] and the value of the geometric average option is given by Equation (31). We set $\Delta t = t_{i+1} - t_i = \frac{1}{12}$ which implies that n is equal to $144, 180$ or 216 depending on the maturity of the insurance contract (12, 15 or 18 years). Note that we have to simulate between 144 and 216 option values for the mentioned maturities. To compare the Monte Carlo results we determine the boundary solutions and the approximation for Asian

[9] Using the antithetic technique this yields in total $2 \cdot M$ paths.

options suggested by Vorst [1992]. Instead of estimating the function $a(b)$ we solve the zero point problem (21) directly applying the closed form solutions for the related geometric average option as discussed in Section 5.

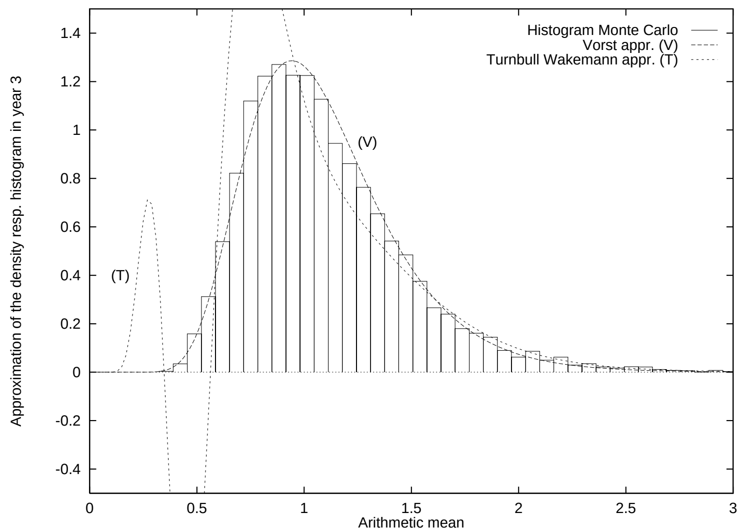

Figure 2: Approximation of the p.d.f. for the arithmetic average $A(t)$ after 3 years. Equity-linked life insurance contract with monthly payment frequency; flat initial term structure, age of the insured person = 30 years; $\sigma_1 = 0$; $\sigma_2 = 0.25$; $\sigma = 0.1$; Monte Carlo Simulation with $M = 6000$.

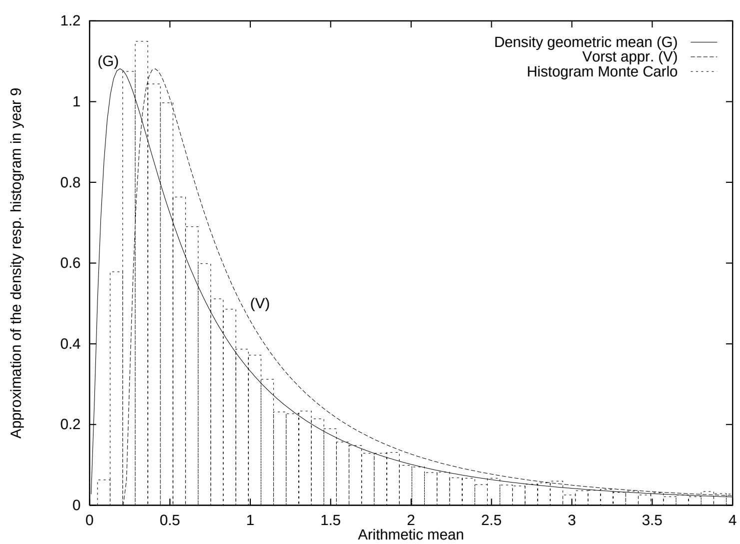

Figure 3: Approximation of the p.d.f. for the arithmetic average $A(t)$ after 9 years. Equity-linked life insurance contract with monthly payment frequency; flat initial term structure, age of the insured person = 30 years; $\sigma_1 = 0$; $\sigma_2 = 0.25$; $\sigma = 0.1$; Monte Carlo Simulation with $M = 6000$.

Figures 2 and 3 show the different approximations of the p.d.f. of the arithmetic average. As already mentioned in Section 5 the Turnbull and Wakeman [1991] method leads to a completely unreasonable approximation. The Vorst [1992] approximation seems to be more valid if we consider a time to maturity equal to 3 years. For the case of 9 years we recognize the shift of the density support measured by the difference between the expected arithmetic and geometric average. The p.d.f. for the geometric average represents the lower bound for the arithmetic average. The Vorst approximation seems to systematically overestimate the probability distribution. We therefore expect that the approximation of the fair premium based on the Vorst method will systematically underestimate the Monte Carlo result.

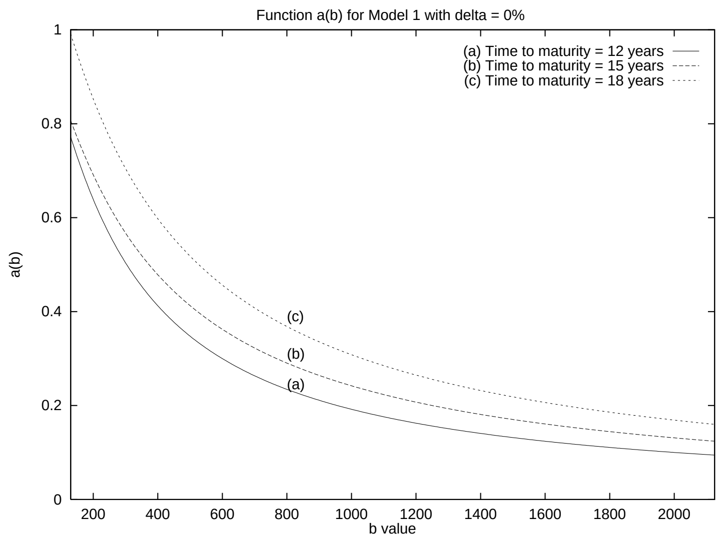

Figure 4: The function $a(b)$ in Model 1 with $\sigma = 0.0\%$ and different times to maturity for an equity-linked life insurance contract with monthly payment frequency; age 30 years, $\sigma_1 = 0,\ \sigma_2 = 0.25,\ \sigma = 0.1,\ M = 6000$, and a flat initial term structure.

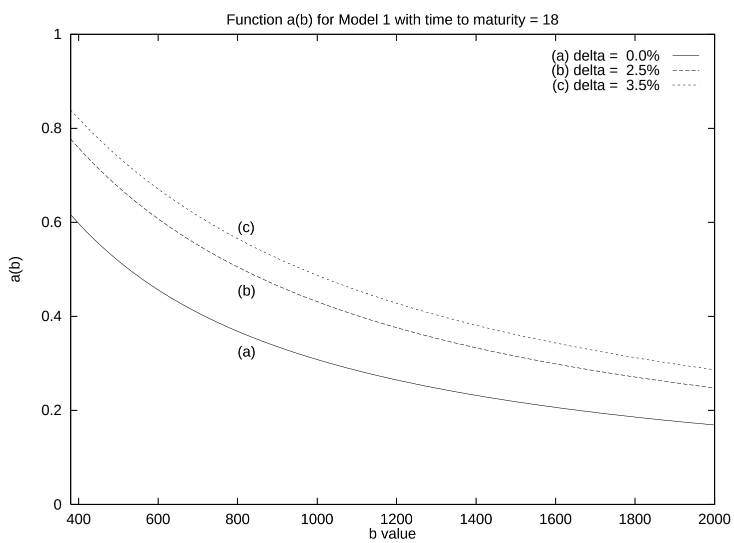

Figure 5: The function $a(b)$ in Model 1 with time to maturity 18 years and different value for the internal interest rate δ for an equity-linked life insurance contract with monthly payment frequency; age 30 years, $\sigma_1 = 0,\ \sigma_2 = 0.25,\ \sigma = 0.1,\ M = 6000$, and a flat initial term structure.

In Figures 4 and 5 we show the approximation of the function $a(b)$ defined in Theorem 5 for different parameter situations. For a fixed share $\bar{a} = a(b)$ the value of b is increasing in time to maturity. This implies a lower premium of the insurance contract if the time to maturity increases. Furthermore if the discount rate δ increases the fair premium for an otherwise equal life insurance contract decreases. A discount rate $\delta > 0$ in Model 1 implies that the guaranteed amount decreases in time, whereas $\delta = 0$ indicates a time independent guaranteed amount.

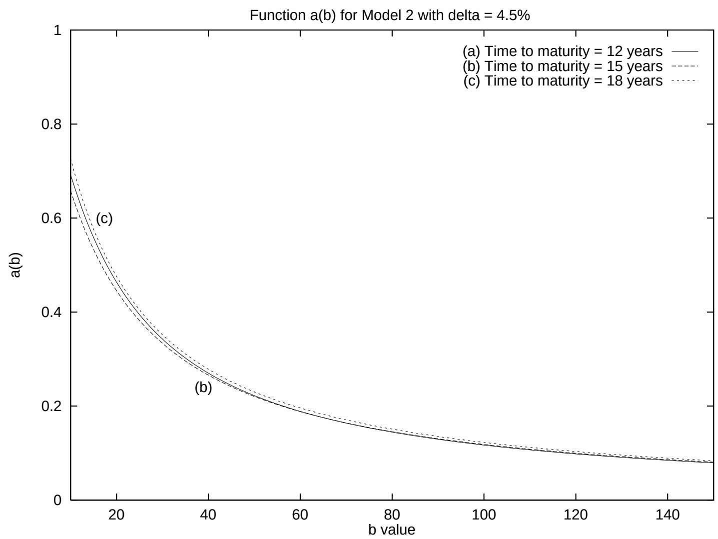

Figure 6: The function $a(b)$ in Model 2 with $\delta = 4.5\%$ and different times to maturity for an equity-linked life insurance contract with monthly payment frequency; age 30 years, $\sigma_1 = 0$, $\sigma_2 = 0.25$, $\sigma = 0.1$, $M = 6000$, and a flat initial term structure.

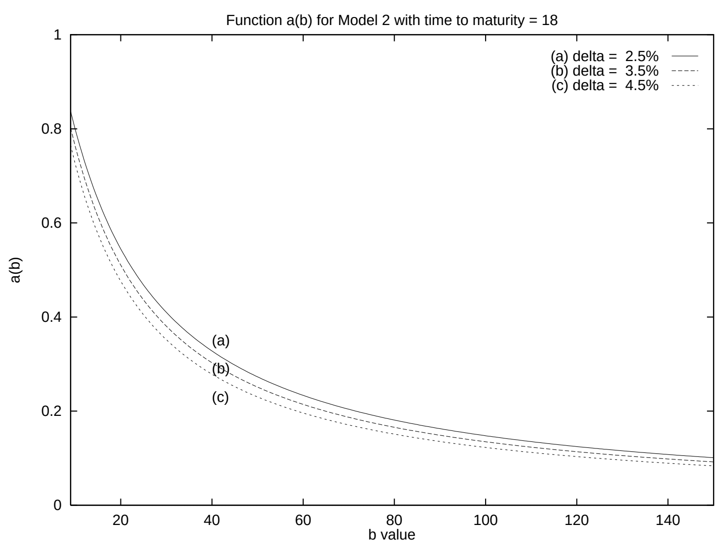

Figure 7: The function $a(b)$ in Model 2 with time to maturity 18 years and different values for the internal growth rate δ for an equity-linked life insurance contract with monthly payment frequency; age 30 years, $\sigma_1 = 0$, $\sigma_2 = 0.25$, $\sigma = 0.1$, $M = 6000$, and a flat initial term structure.

The situation for Model 2 is different. The function of the guaranteed amount is increasing in time and in the internal growth rate δ. If the growth rate is fixed (say 4.5%) as in Figure 6 the fair premium depends on two opposite effects. As the time to maturity of the insurance contract increases the fair premium will decrease due to the expected delay in the payment of the benefit. On the other side, if the time to maturity increases the guaranteed amount will increase which on its own implies a higher premium. As Figure 6 indicates these two effects adds up in such a manner that the function $a(b)$ is nearly independent of the time to maturity. It

Table 1: Fair premiums in Model 1 with a fixed guaranteed amount of 10.000 at time 0

maturity	12			18		
delta	0.0%	2.5%	3.5%	0.0%	2.5%	3.5%
$G(T,K)$	10000.00	7408.1822	6570.4682	10000.00	6376.2815	5325.9180
$a = 0.3$	55.5872	41.3973	36.8048	32.2516	20.9222	17.6232
	(0.5219)	(0.3887)	(0.3455)	(0.7864)	(0.5101)	(0.4297)
$a = 0.35$	57.6269	42.9188	38.1585	33.4705	21.7139	18.2905
	(0.6312)	(0.4701)	(0.418)	(0.9521)	(0.6177)	(0.5203)
$a = 0.4$	59.9791	44.6741	39.7205	34.816	22.5875	19.0268
	(0.7508)	(0.5592)	(0.4972)	(1.1319)	(0.7344)	(0.6186)
$a = 0.45$	62.7108	46.7122	41.5343	36.3037	23.5546	19.8422
	(0.8831)	(0.6578)	(0.5849)	(1.3279)	(0.8615)	(0.7258)
$a = 0.5$	65.8899	49.0836	43.6442	37.9633	24.6325	20.7509
	(1.031)	(0.768)	(0.6829)	(1.5429)	(1.0011)	(0.8433)
$a = 0.55$	69.6356	51.8801	46.1333	39.8243	25.8423	21.7712
	(1.1985)	(0.8929)	(0.794)	(1.7804)	(1.1553)	(0.9733)
$a = 0.6$	74.1361	55.2377	49.1215	41.9256	27.2064	22.9207
	(1.3919)	(1.0371)	(0.9223)	(2.0447)	(1.3268)	(1.1178)
$a = 0.65$	79.5795	59.3016	52.7386	44.3083	28.7557	24.2274
	(1.6186)	(1.2062)	(1.0727)	(2.341)	(1.5193)	(1.28)
$a = 0.7$	86.3402	64.3469	57.2291	47.0475	30.5352	25.7278
	(1.8912)	(1.4094)	(1.2535)	(2.6769)	(1.7374)	(1.4639)
$a = 0.75$	94.9961	70.8121	62.9852	50.2187	32.5954	27.4648
	(2.2293)	(1.6617)	(1.4781)	(3.0615)	(1.9871)	(1.6743)
$a = 0.8$	106.6261	79.4971	70.7178	53.9374	35.0115	29.502
	(2.6689)	(1.9898)	(1.7701)	(3.5074)	(2.2767)	(1.9184)
$a = 0.85$	123.4692	92.0944	81.9388	58.366	37.8902	31.9301
	(3.2833)	(2.449)	(2.1789)	(4.0325)	(2.6179)	(2.2061)

The number in parentheses is the *standard deviation*

means that the fair premium for these contracts are very similar to each other. If the maturity of the contract is fixed as in Figure 7 the functions $a(b)$ are decreasing in the internal growth rate δ. This implies that the fair premium is increasing in the growth rate. This makes sense, if everything else remains equal these contracts differ in a monotone manner in the size of the guaranteed amount. In addition to these direct effects of the growth rate, there is an indirect effect, since an increase in the guaranteed amount implies a decrease of the bonus and therefore the value of the weighted options will decrease. This indirect effect will on average decrease the fair premium, and may be responsible, as Figure 6 suggests, for the small difference between the simulated functions $a(b)$ across different internal growth rates.

Table 1 and 2 show within Model 1 simulated values for the fair premium obtained by the Monte Carlo simulation. Since Model 1 implies a decreasing guaranteed amount if the internal growth rate δ exceeds zero we can either consider a fixed guaranteed amount at maturity of the contract (Table 1) or at the beginning of the contract (Table 2). In both cases the fair premium is an increasing function of the share a which is in accordance with Theorem 6. This

Table 2: Fair premiums in Model 1 with a fixed guaranteed amount of 10.000 at time T

maturity	12			18		
delta	0.0%	2.5%	3.5%	0.0%	2.5%	3.5%
$G(0,K)$	10000.00	13498.5881	15219.6156	10000.00	15683.1219	18776.1058
$a = 0.3$	55.5872	55.8805	56.0155	32.2516	32.8126	33.0895
	(0.5219)	(0.5246)	(0.5259)	(0.7864)	(0.8001)	(0.8068)
$a = 0.35$	57.6269	57.9343	58.0758	33.4705	34.0542	34.3425
	(0.6312)	(0.6346)	(0.6361)	(0.9521)	(0.9687)	(0.9769)
$a = 0.4$	59.9791	60.3037	60.4531	34.816	35.4243	35.7249
	(0.7508)	(0.7549)	(0.7568)	(1.1319)	(1.1517)	(1.1615)
$a = 0.45$	62.7108	63.0549	63.2136	36.3037	36.9409	37.2559
	(0.8831)	(0.888)	(0.8902)	(1.3279)	(1.3512)	(1.3627)
$a = 0.5$	65.8899	66.2559	66.4249	37.9633	38.6315	38.9621
	(1.031)	(1.0367)	(1.0394)	(1.5429)	(1.57)	(1.5835)
$a = 0.55$	69.6356	70.0308	70.2131	39.8243	40.5289	40.8778
	(1.1985)	(1.2053)	(1.2085)	(1.7804)	(1.8119)	(1.8275)
$a = 0.6$	74.1361	74.5631	74.761	41.9256	42.6681	43.0361
	(1.3919)	(1.4)	(1.4037)	(2.0447)	(2.0809)	(2.0989)
$a = 0.65$	79.5795	80.0488	80.2662	44.3083	45.098	45.4896
	(1.6186)	(1.6282)	(1.6326)	(2.341)	(2.3827)	(2.4034)
$a = 0.7$	86.3402	86.8593	87.1005	47.0475	47.8887	48.3067
	(1.8912)	(1.9025)	(1.9078)	(2.6769)	(2.7248)	(2.7486)
$a = 0.75$	94.9961	95.5864	95.8611	50.2187	51.1198	51.5682
	(2.2293)	(2.2431)	(2.2496)	(3.0615)	(3.1164)	(3.1437)
$a = 0.8$	106.6261	107.3098	107.6297	53.9374	54.9089	55.3933
	(2.6689)	(2.686)	(2.694)	(3.5074)	(3.5705)	(3.602)
$a = 0.85$	123.4692	124.3144	124.7077	58.366	59.4237	59.9524
	(3.2833)	(3.3058)	(3.3163)	(4.0325)	(4.1056)	(4.1421)

The number in parentheses is the *standard deviation*

is a general property within the diffusion framework considered. In addition the fair premium decreases remarkable with the discount rate δ if the guaranteed amount is fixed in the beginning. This is perfectly in line with the contract specifications in Model 1. If instead the guaranteed amount is fixed at the maturity of the contract an internal growth rate $\delta > 0$ implies that the guaranteed amount at time t_0 increases in δ. Thus in the case of the death of the insured person the insurer guarantees a higher amount as with $\delta = 0$. This results as Table 2 shows in an increasing premium. Nevertheless we have to note that this increase is of quite small magnitude. The reason is that a premature payment is not likely under the chosen death distribution and the fair premium is mainly determined by the payment at maturity, which in Table 2 is fixed for all contracts.

Tables 3 and 4 concentrate on the influence on the fair premium of the internal growth rate δ. As shown by Theorem 6 the fair premium is an increasing function of the share a. For a fixed guaranteed amount at time t_0 the monthly fair premium is an increasing function of the internal growth rate. Since we already know that mainly the terminal payment at maturity

Table 3: Fair premiums in Model 2 with a fixed guaranteed amount of 10.000 at time 0

maturity	12			18		
delta	2.5%	3.5%	4.5%	2.5%	3.5%	4.5%
$G(T,K)$	15170.0106	16175.6904	17271.952	24017.8275	26515.9529	29359.9221
$a = 0.3$	82.9338	88.4044	94.3666	74.7204	82.409	91.1558
	(0.7787)	(0.83)	(0.886)	(1.8219)	(2.0093)	(2.2226)
$a = 0.35$	85.9634	91.634	97.8138	77.5416	85.521	94.5984
	(0.9416)	(1.0038)	(1.0714)	(2.2059)	(2.4328)	(2.6911)
$a = 0.4$	89.457	95.3572	101.789	80.6594	88.9595	98.4032
	(1.1199)	(1.1937)	(1.2742)	(2.6224)	(2.8923)	(3.1993)
$a = 0.45$	93.5132	99.6819	106.4051	84.1041	92.7606	102.6081
	(1.3169)	(1.4038)	(1.4985)	(3.0763)	(3.3929)	(3.7531)
$a = 0.5$	98.243	104.7239	111.7879	87.9511	97.0049	107.3055
	(1.5372)	(1.6387)	(1.7492)	(3.5744)	(3.9424)	(4.361)
$a = 0.55$	103.802	110.65	118.1138	92.2614	101.7604	112.5683
	(1.7866)	(1.9045)	(2.0329)	(4.1246)	(4.5493)	(5.0324)
$a = 0.6$	110.4935	117.7834	125.7292	97.1423	107.1478	118.5311
	(2.0746)	(2.2115)	(2.3606)	(4.7376)	(5.2256)	(5.7807)
$a = 0.65$	118.5912	126.4187	134.9505	102.6721	113.2477	125.2813
	(2.4121)	(2.5713)	(2.7448)	(5.4246)	(5.9834)	(6.6191)
$a = 0.7$	128.662	137.1568	146.4167	109.038	120.2741	133.06
	(2.8182)	(3.0042)	(3.207)	(6.2041)	(6.8434)	(7.5709)
$a = 0.75$	141.5566	150.9113	161.1079	116.4222	128.4242	142.0857
	(3.3219)	(3.5414)	(3.7807)	(7.0974)	(7.8291)	(8.6619)
$a = 0.8$	158.9174	169.4251	180.8776	125.0941	137.9992	152.684
	(3.9777)	(4.2407)	(4.5273)	(8.1344)	(8.9736)	(9.9285)
$a = 0.85$	184.0775	196.2225	209.5478	135.4379	149.4225	165.3376
	(4.8949)	(5.2178)	(5.5723)	(9.3575)	(10.3237)	(11.4233)

The number in parentheses is the *standard deviation*

contributes to the fair premium this indicates in addition that the bonus at the maturity of the contract can on average not compensate for the higher guaranteed amount. In other words on average the embedded option ends out of the money for higher internal growth rates. The value of the embedded option is therefore small and does not effect the value of the premium in any significant manner.

If the guaranteed amount is fixed at the maturity of the contract a higher internal growth rate implies a lower guaranteed amount at the intermediate times. This should result in a lower premium, but as Table 4 shows this premium reduction is rather small. Again this is mainly based on the relative small contribution of the death distribution to the premium. In addition given the low guaranteed amount during the starting period of the contract we can expect that the value of the reference portfolio is likely to exceed the guaranteed amount, i.e. even in the case of a premature death of the insured person the payment will be quite similar for the different internal growth rates δ.

For a homogeneous guaranteed amount of the type $g(t, K) = F(t)\beta^* \cdot K$ Figures 8 and

Table 4: Fair premiums in Model 2 with a fixed guaranteed amount of 10.000 at time T

maturity	12			18		
delta	2.5%	3.5%	4.5%	2.5%	3.5%	4.5%
$G(0,K)$	659.1963	618.2116	578.9734	416.3574	377.1315	340.6004
$a = 0.3$	54.6696	54.6527	54.6357	31.1104	31.079	31.0477
	(0.5133)	(0.5131)	(0.513)	(0.7586)	(0.7578)	(0.757)
$a = 0.35$	56.6667	56.6492	56.6316	32.285	32.2527	32.2202
	(0.6207)	(0.6205)	(0.6203)	(0.9184)	(0.9175)	(0.9166)
$a = 0.4$	58.9697	58.9509	58.9331	33.5831	33.5494	33.5162
	(0.7382)	(0.738)	(0.7378)	(1.0919)	(1.0908)	(1.0897)
$a = 0.45$	61.6435	61.6245	61.6057	35.0174	34.983	34.9484
	(0.8681)	(0.8679)	(0.8676)	(1.2808)	(1.2796)	(1.2783)
$a = 0.5$	64.7613	64.7415	64.7222	36.6191	36.5836	36.5483
	(1.0133)	(1.013)	(1.0127)	(1.4882)	(1.4868)	(1.4854)
$a = 0.55$	68.4258	68.4051	68.3847	38.4137	38.3771	38.3408
	(1.1777)	(1.1774)	(1.177)	(1.7173)	(1.7157)	(1.714)
$a = 0.6$	72.8368	72.8151	72.7938	40.4459	40.4088	40.3717
	(1.3676)	(1.3672)	(1.3668)	(1.9725)	(1.9707)	(1.9689)
$a = 0.65$	78.1747	78.1535	78.1328	42.7483	42.7093	42.6708
	(1.59)	(1.5896)	(1.5892)	(2.2586)	(2.2565)	(2.2545)
$a = 0.7$	84.8134	84.792	84.7714	45.3988	45.3592	45.3203
	(1.8577)	(1.8572)	(1.8568)	(2.5831)	(2.5809)	(2.5787)
$a = 0.75$	93.3135	93.2951	93.2772	48.4732	48.4328	48.3944
	(2.1898)	(2.1893)	(2.1889)	(2.9551)	(2.9526)	(2.9503)
$a = 0.8$	104.7576	104.7406	104.7233	52.0838	52.0438	52.0042
	(2.6221)	(2.6216)	(2.6212)	(3.3868)	(3.3842)	(3.3817)
$a = 0.85$	121.343	121.3071	121.3226	56.3906	56.3519	56.314
	(3.2267)	(3.2257)	(3.2262)	(3.8961)	(3.8934)	(3.8908)

The number in parentheses is the *standard deviation*

9 represent the guaranteed amount at maturity of the contract as a function of the monthly premium for different values of the share a.

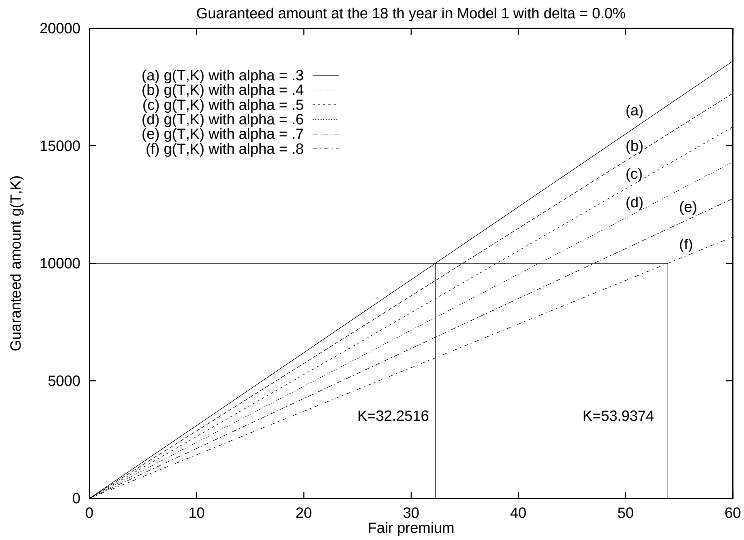

Figure 8: The guaranteed amount in Model 1 at the maturity of 18 years as a function of the periodic premium. Equity-linked life insurance contract with monthly payment; age 30 years, $\sigma_1 = 0$, $\sigma_2 = 0.25$, $\sigma = 0.1$, $M = 6000$, and a flat initial term structure.

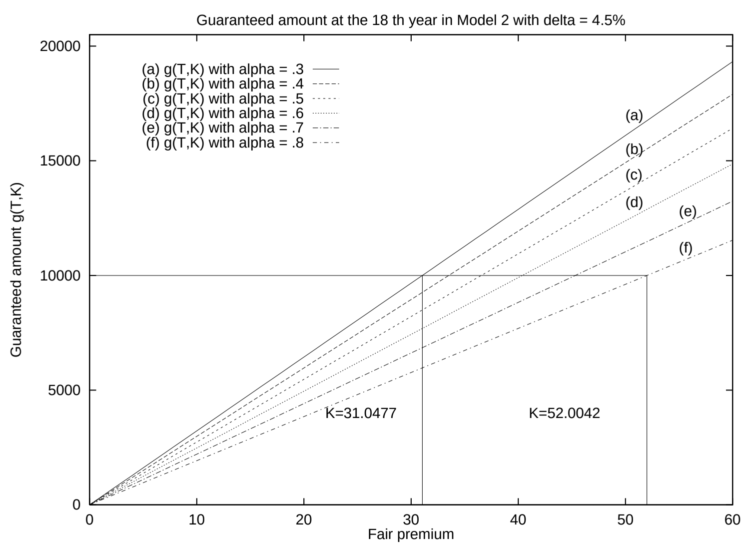

Figure 9: The guaranteed amount in Model 2 at the maturity of 18 years as a function of the periodic premium. Equity-linked life insurance contract with monthly payment; age 30 years, $\sigma_1 = 0$, $\sigma_2 = 0.25$, $\sigma = 0.1$, $M = 6000$, and a flat initial term structure.

For a fixed periodic premium K the guaranteed amount is shown to be a decreasing function of the share a which results from the decrease of β^* as a function of the share a. This property of β^* is by Theorem 6 satisfied for a general diffusion framework and not restricted to the constant volatility case we have simulated. As a consequence of this, the fair premium increases as a function of the share a for a fixed guaranteed amount. Since $\beta^*(\cdot)$ as a function of the share a is concave the same is true for the guaranteed amount.

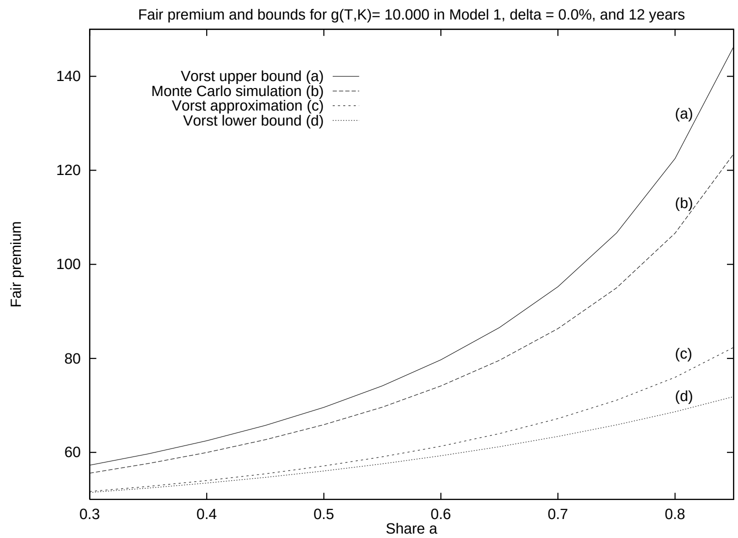

Figure 10: The fair premium and bounds implied by the Vorst approximation in Model 1 with a fixed guaranteed amount of 10.000 at the maturity of 12 years. Equity-linked life insurance with monthly payment frequency; age 30 years, $\sigma_1 = 0$, $\sigma_2 = 0.25$, $\sigma = 0.1$, $M = 6000$, and a flat initial term structure.

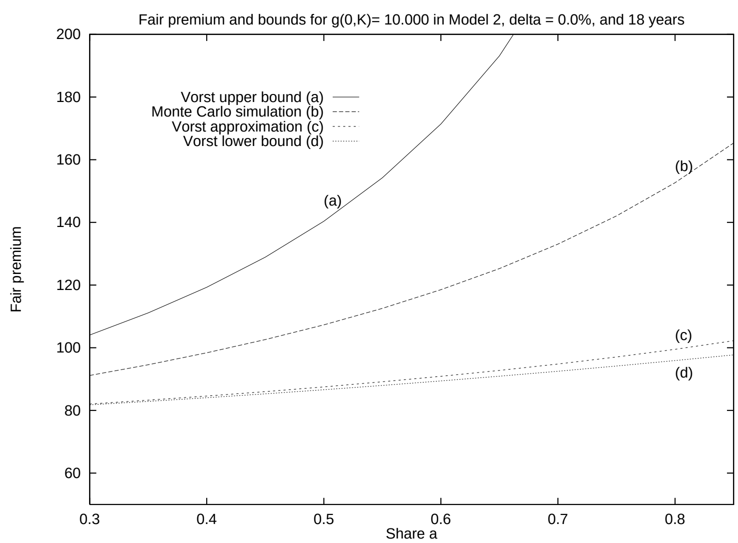

Figure 11: The fair premium and bounds implied by the Vorst approximation in Model 2 with a fixed initial guaranteed amount of 10.000 and a maturity of 18 years. Equity-linked life insurance with monthly payment frequency; age 30 years, $\sigma_1 = 0$, $\sigma_2 = 0.25$, $\sigma = 0.1$, $M = 6000$, and a flat initial term structure.

Finally Figures 10 and 11 show the fair premium as a function of the share a. In line with Theorem 6 the fair premium is a convex function with respect to the share a. Furthermore the upper bound for the premium, which corresponds to the lower bounds for the options given by Vorst [1992], is very weak. This indicates that the geometric average option is not suitable to approximate the bonus part. This result is perfectly in line with our discussion of the expected geometric average in Section 5. As expected from Figure 3 the Vorst approximation of the embedded options lead to a systematic overestimation of the embedded options and thus to a systematic underestimation of the fair premium. The size of this approximation error increases with the invested share a, since the value of the reference portfolio $a \cdot K \sum_{i=0}^{n^*(t)-1} \frac{S(t)}{S(t_i)}$ increases in a and therefore the option value will increase.

7 Conclusion

An equity-linked life insurance contract combines the technical risk of a life insurance with the financial risk of an investment strategy. More precisely the contract offers a life insurance policy in combination with an investment strategy together with a minimum guaranteed value. From this perspective the equity-linked life insurance contract represents an interesting contract which could have a promising market perspective. Since the investment strategy is based on a mutual fund, the insured person participates in the economic growth covered by the fund, whereas the downside risk is covered by the minimum guaranteed value. On the other hand this contract specification implies a complex payoff structure.

Due to the usually long time to maturity of life insurance policies the analysis of this contract has to take into consideration three different sources of uncertainty. Beside the technical risk specified by the death distribution of the insured two different financial risks are effecting the value of the insurance premium: First, the price risk of the underlying mutual fund and second, the interest rate risk. In the context of a general diffusion model for these financial risks the rational interrelationships between the periodic premium, the share of the premium invested in the mutual fund and the function specifying the guaranteed amount is analysed. As a main result it was found that the fair periodic premium is an increasing and convex function of the share invested in the mutual fund. This result is valid even if we allow for state dependent volatility processes.

The definition of the fair premium is based on the independence between the death distribution and the financial processes and, more important on the ability of the insurer to perfectly diversify the technical insurance risk. Given the at present relatively small markets for this contract this assumption seems to be questionable. One way to take care of this, would be to introduce loading factors on the death distribution.

The analysis of this study was concentrated on the properties and calculation of the fair periodic premium of an equity-linked life insurance contract. The mentioned results are obtained by the application of the arbitrage pricing theory. As an important tool we have used the change of measure technique and thus calculated the fair periodic premium under the forward risk adjusted measures. Despite the theoretical elegance this approach is first of all concentrated on the pricing of financial assets. It therefore does not directly address the hedging question which is of principal importance. Due to the complexity of the contract there is no straightforward answer to this problem. Like the fair premium the hedging problem, too, is related to the technical insurance risk and the financial risks. Even if we concentrate only on the financial risks the similarity between the insurance situation and the pricing of Asian options implies a number of problems. Firstly, the inappropriateness for the insurance situation of those methods traditionally used for the pricing of Asian options. The failure of these methods happens because of the non-deterministic bond market and because of the long time to maturity of the insurance contracts. Our simulation results have demonstrated this in the situation of a constant volatility structure. Secondly, due to the long time to maturity of the insurance contracts we should expect that the volatility processes are by no means deterministic. Even more problematic, the correlation process highly influences the contract. Thirdly, additional features such as the possibility of changing the fund or the right to cancel the contract may strongly influence the pricing and hedging of the contract.

We have the impression that the combination of the insurance theory and the finance theory is a fruitful and promising area for practical application and theoretical research. For the an equity-linked life insurance contract with periodic premium we have discussed some relevant aspects, but maybe unavoidable, more questions are raised than answered.

Appendix

Proof of Proposition 4

Proof: If $\beta^*(a)$ is a solution of $f\left(\frac{\beta^*(a)}{a}\right) = 0$ for $a \in]0,1[$, then $\beta^*(a)$ solves

$$\frac{H}{a} = R\left(\frac{\beta^*(a)}{a}\right).$$

This implies that $\frac{\partial}{\partial a}\left(R\left(\frac{\beta^*(a)}{a}\right)\right) = -\frac{H}{a^2} < 0$ and as $R(y) = L(y) + U(y)$ we have that

$$\begin{aligned}\frac{\partial\, R(y)}{\partial\, y} &= \frac{\partial L(y)}{\partial y} + \frac{\partial U(y)}{\partial y} \\ &= \frac{L(y)}{y} + \int_{t_0}^{T} \frac{\partial}{\partial y}\, c(t_0,t,y)\, \pi_x(t)\, dt + \frac{\partial}{\partial y}\, c(t_0,T,y)\left(1 - \int_{t_0}^{T} \pi_x(t)\, dt\right)\end{aligned}$$

with
$$\frac{\partial}{\partial y}\, c(t_0,t,y) = -D(t_0,t)\int_{yF(t)}^{+\infty} F(t)dQ_t(z) > -D(t_0,t)F(t) \quad \forall y > 0$$

where $Q_t(z)$ is the unknown distribution function of the sum $\sum_{i=0}^{n^*(t)-1}\frac{S(t)}{S(t_i)}$. Therefore

$$\frac{\partial\, U(y)}{\partial\, y} > -\int_{t_0}^{T} D(t_0,t)\, F(t)\, \pi_x(t)dt - D(t_0,T)\, F(T)\left(1 - \int_{t_0}^{T} \pi_x(t)\, dt\right) = -\frac{L(y)}{y}$$

which implies that $\frac{\partial\, R(y)}{\partial\, y} > 0$. Since

$$-\frac{H}{a^2} = \frac{\partial}{\partial a}\left(R\left(\frac{\beta^*(a)}{a}\right)\right) = \frac{\partial}{\partial a}\left(\frac{\beta^*(a)}{a}\right)\cdot \left.\frac{\partial R}{\partial y}\right|_{y=\frac{\beta^*(a)}{a}}$$

we obtain that $\frac{\beta^*(a)}{a}$ is decreasing in a. □

Proof of Theorem 6

Proof: Consider the guaranteed amount $g(t,K) = F(t)\cdot\beta^*\cdot K$ where β^* is a solution of (21). Then the fair premium for the above contract is equal to $K^* = \frac{\bar{g}}{\beta^*}$. Let $a_1 < a_2$ and $a_1, a_2 \in]0,1[$. Then there exists $b_1, b_2 \in \mathbb{R}$ such that the function $a(b)$ in Theorem 5 satisfies $a(b_1) = a_1$ and $a(b_2) = a_2$. Since $a(b)$ is a strictly decreasing function $a_1 < a_2$ implies that $b_1 < b_2$. Furthermore we can write

$$K^*(a_2) - K^*(a_1) = \frac{\bar{g}}{\beta^*(a_2)} - \frac{\bar{g}}{\beta^*(a_1)} = \bar{g}\,\frac{\beta^*(a_1) - \beta^*(a_2)}{\beta^*(a_2)\beta^*(a_1)}$$

where $$\beta^*(a_1) - \beta^*(a_2) \;=\; a(b_1)\cdot b_1 \;-\; a(b_2)\cdot b_2.$$

We therefore have to prove that $a(b_1)\cdot b_1 \;>\; a(b_2)\cdot b_2 \quad \forall b_1, b_2$, i.e. we want to prove that the function $a(b)\cdot b$ is increasing in b. The condition we are looking for can then be written as

$$\begin{aligned} \frac{\partial}{\partial b}\Big(a(b)\cdot b\Big) &= \frac{\partial}{\partial b}\left(\frac{H\cdot b}{R(b)}\right) = H\cdot\frac{R(b) - b\cdot\frac{\partial R(b)}{\partial b}}{R^2(b)} > 0 \\ \Longleftrightarrow &\qquad R(b) - b\cdot\frac{\partial R(b)}{\partial b} > 0 \end{aligned}$$

which indeed is the case since

$$R(b) - b\cdot\frac{\partial R(b)}{\partial b} = U(b) - b\cdot\frac{\partial U(b)}{\partial b} > 0 \text{ and } \frac{\partial U(b)}{\partial b} < 0.$$

For the convexity consider for $h > 0$ and $a \in]0, 1-2h]$

$$\begin{aligned} & \frac{1}{h}\left(\frac{K^*(a+2h) - K^*(a+h)}{h} - \frac{K^*(a+h) - K^*(a)}{h}\right) \\ =\; & \frac{\bar{g}}{h^2}\left(\frac{1}{\beta^*(a+2h)} - \frac{2}{\beta^*(a+h)} + \frac{1}{\beta^*(a)}\right) \\ =\; & \frac{\bar{g}}{h^2}\left[\frac{\beta^*(a)[\beta^*(a+h) - \beta^*(a+2h)] - \beta^*(a+2h)[\beta^*(a) - \beta^*(a+h)]}{\beta^*(a)\beta^*(a+h)\beta^*(a+2h)}\right] \\ >\; & \frac{\bar{g}}{h^2}\beta^*(a)\left[\frac{[\beta^*(a+h) - \beta^*(a+2h)] - [\beta^*(a) - \beta^*(a+h)]}{\beta^*(a)\beta^*(a+h)\beta^*(a+2h)}\right] \\ =\; & \frac{-\bar{g}\beta^*(a)}{\beta^*(a)\beta^*(a+h)\beta^*(a+2h)}\left[\frac{\beta^*(a+2h) - \beta^*(a+h)}{h^2} - \frac{\beta^*(a+h) - \beta^*(a)}{h^2}\right] \end{aligned}$$

since $\beta^*(a)$ is decreasing in a and positive. Therefore it is sufficient to prove, that β^* is a concave function in a. As before this is equivalent to the concavity of the function $a(b)\cdot b$ for $b \in R_{\geq}0$.

$$\frac{\partial^2(a(b)\cdot b)}{\partial b^2} = H\cdot\frac{R(b)\left(-b\cdot\frac{\partial^2 R(b)}{\partial b^2}\right) - \left(R(b) - b\cdot\frac{\partial R(b)}{\partial b}\right)2\frac{\partial R(b)}{\partial b}}{R(b)^3} < 0$$

since $\frac{\partial R(b)}{\partial b} > 0$, $\frac{\partial^2 R(b)}{\partial b^2} > 0$ and $\left(R(b) - b\cdot\frac{\partial R(b)}{\partial b}\right) > 0$.

□

Proof of Theorem 7

Proof: For simplicity set $n := n^*(t)$. Due to the log-normality assumption the expected value of the logarithmic ratio $\frac{S(t)}{S(t_i)}$ is determined by:

$$\begin{aligned} E_{P^t}\left[\ln\left(\frac{S(t)}{S(t_i)}\right)\right] &= \ln\left(\frac{D(t_0,t_i)}{D(t_0,t)}\right) - \frac{1}{2}(t-t_i)^2\sigma^2 t_i - \frac{1}{2}\int_{t_i}^{t}\big((\sigma_1 - (t-u)\sigma)^2 + \sigma_2^2\big)du \\ &= \ln\left(\frac{D(t_0,t_i)}{D(t_0,t)}\right) - \frac{1}{2}(t-t_i)^2\sigma^2 t_i - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)(t-t_i) \\ &\quad + \frac{1}{2}(t-t_i)^2\sigma_1\sigma - \frac{1}{6}(t-t_i)^3\sigma^2\ . \end{aligned}$$

By summing up this yields the value of $m_G(t)$. The calculation of the variance involves several steps. First not that since the Brownian motions W_1^t and W_2^t are independent we can write:

$$\begin{aligned} V_{P^t}[lnG(t)] &= \frac{1}{n^2} V_{P^t}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t} \sigma_2 dW_2^t(u)\right] \\ &+\frac{1}{n^2} V_{P^t}\left[\sum_{i=0}^{n-1}\left((t-t_i)\sigma W_1^t(t_i)+\int_{t_i}^{t}\big((\sigma_1-(t-u)\sigma\big)dW_1^t(u)\right)\right] . \end{aligned}$$

The first variance term is the given by

$$V_{P^t}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t}\sigma_2 dW_2^t(u)\right] = V_{P^t}\left[\sum_{i=0}^{n-1}(i+1)\int_{t_i}^{t}\sigma_2 dW_2^t(u)\right] = \sum_{i=0}^{n-1}(i+1)^2\sigma_2^2(t_{i+1}-t_i) ,$$

whereas the calculation of the second term is more complicated. Note that for $W_1^t(t_0)=0$ we have

$$\begin{aligned} \sum_{i=0}^{n-1}(t-t_i)\sigma W_1^t(t_i) &= \sum_{i=0}^{n-1}(t-t_i)\sigma W_1^t(t_0) + \sum_{i=1}^{n-1}\left(\sum_{j=i}^{n-1}(t-t_j)\right)\sigma\left(W_1^t(t_i)-W_1^t(t_{i-1})\right) \\ &= \sum_{i=0}^{n-2}\left(\sum_{j=i+1}^{n-1}(t-t_j)\right)\sigma\left(W_1^t(t_{i+1})-W_1^t(t_i)\right) . \end{aligned}$$

Furthermore we know that

$$\sum_{j=i+1}^{n-1}(t-t_j) = \sum_{j=i+1}^{n-1}(j-i)(t_{j+1}-t_j)$$

which implies that

$$\sum_{i=0}^{n-1}(t-t_i)\sigma W_1^t(t_i) = \sum_{i=0}^{n-2}\left[\sum_{j=i+1}^{n-1}(j-i)(t_{j+1}-t_j)\right]\sigma\big(W_1^t(t_{i+1})-W_1^t(t_i)\big) .$$

For the second variance term we therefore can conclude

$$\begin{aligned} & V_{P^t}\left[\sum_{i=0}^{n-1}\left((t-t_i)\sigma W_1^t(t_i)+\int_{t_i}^{t}\big(\sigma_1-(t-u)\sigma\big)dW_1^t(u)\right)\right] \\ = \; & V_{P^t}\left[\sum_{i=0}^{n-2}\left(\sum_{j=i+1}^{n-1}(j-i)(t_{j+1}-t_j)\right)\sigma\Big(W_1^t(t_{i+1})-W_1^t(t_i)\Big)\right. \\ & \left. +\sum_{i=0}^{n-1}(i+1)\int_{t_i}^{t_{i+1}}\Big(\sigma_1-(t-u)\sigma\Big)dW_1^t(u)\right] \\ = \; & V_{P^t}\left[\sum_{i=0}^{n-2}\int_{t_i}^{t_{i+1}}\left(\sum_{j=i+1}^{n-1}(j-i)(t_{j+1}-t_j)\right)\sigma+(i+1)\Big(\sigma_1-(t-u)\sigma\Big)dW_1^t(u)\right. \\ & \left. +n\int_{t_{n-1}}^{t_n}\Big(\sigma_1-(t-u)\sigma\Big)dW_1^t(u)\right] \\ = \; & \sum_{i=0}^{n-2}\int_{t_i}^{t_{i+1}}\left[\left(\sum_{j=i+1}^{n-1}(j-i)(t_{j+1}-t_j)\sigma\right)+(i+1)\Big(\sigma_1-(t-u)\sigma\Big)\right]^2 du + n^2\int_{t_{n-1}}^{t}\Big(\sigma_1-(t-u)\sigma\Big)^2 du . \end{aligned}$$

This implies that

$$\begin{aligned}
& n^2 V_{P^t}\left[\ln G(t)\right] \\
= & \sum_{i=0}^{n-1}\left[(i+1)^2(\sigma_1^2+\sigma_2^2)(t_{i+1}-t_i)\right] \\
& +\sum_{i=0}^{n-1}\left[(i+1)^2\left[\sigma_1\sigma\left((t-t_{i+1})^2-(t-t_i)^2\right)-\frac{\sigma^2}{3}\left((t-t_{i+1})^3-(t-t_i)^3\right)\right]\right] \\
& +\sum_{i=0}^{n-2}\left[\left(\sum_{j=i+1}^{n-1}(j-i)(t_{j+1}-t_j)\right)^2\sigma^2(t_{i+1}-t_i)\right] \\
& +2\sum_{i=0}^{n-2}\left[\left(\sum_{j=i+1}^{n-1}(j-i)(t_{j+1}-t_j)\right)(i+1)\sigma\sigma_1(t_{i+1}-t_i)\right] \\
& +\sum_{i=0}^{n-2}\left[\left(\sum_{j=i+1}^{n-1}(j-i)(t_{j+1}-t_j)\right)(i+1)\sigma^2\left[(t-t_{i+1})^2-(t-t_i)^2\right]\right] .
\end{aligned}$$

We now assume that $t = t_{n^*(t)-1} + \Delta t = n^*(t)\cdot\Delta t$ where for simplicity we again set $n := n^*(t)$. With $E_{P^t}[G(t)] = \exp\left\{E_{P^t}[\ln G(t)] + \frac{1}{2}V_{P^t}[\ln G(t)]\right\}$ and $\sum_{i=0}^{n-1}(t-t_i) = \sum_{i=0}^{n-1}(i+1)(t_{i+1}-t_i)$ we can rewrite the equation for the expected geometric average as

$$\begin{aligned}
E_{P^t}[G(t)] \;=\; & \left(\prod_{i=0}^{n-1}\frac{D(t_0,t_i)}{D(t_0,t)}\right)^{\frac{1}{n}} \\
& \cdot\exp\left\{-\frac{\sigma_1^2+\sigma_2^2}{2n^2}\sum_{i=0}^{n-1}\big(n-(i+1)\big)(i+1)(t_{i+1}-t_i)\right\} \\
& \cdot\exp\left\{+\frac{\sigma_1\sigma}{2n^2}\sum_{i=0}^{n-1}\left[n(t-t_i)^2-(i+1)^2(t_{i+1}-t_i)[2t-t_{i+1}-t_i]\right]\right\} \\
& \cdot\exp\left\{+\frac{\sigma_1\sigma}{2n^2}\sum_{i=0}^{n-2}\left[\left(\sum_{j=i+1}^{n-1}(j-i)(t_{j+1}-t_j)\right)(i+1)(t_{i+1}-t_i)\right]\right\} \\
& \cdot\exp\left\{-\frac{\sigma^2}{2n^2}\sum_{i=0}^{n-1}\left[nt_i(t-t_i)^2+\frac{1}{3}n(t-t_i)^3+\frac{1}{3}(i+1)^2\big((t-t_{i+1})^3-(t-t_i)^3\big)\right]\right\} \\
& \cdot\exp\left\{+\frac{\sigma^2}{2n^2}\sum_{i=0}^{n-2}\left[\left(\sum_{j=i+1}^{n-1}(j-i)(t_{j+1}-t_j)\right)^2(t_{i+1}-t_i)\right]\right\} \\
& \cdot\exp\left\{-\frac{\sigma^2}{2n^2}\sum_{i=0}^{n-2}\left[\left(\sum_{j=i+1}^{n-1}(j-i)(t_{j+1}-t_j)\right)(i+1)(t_{i+1}-t_i)[2t-t_{i+1}-t_i]\right]\right\} .
\end{aligned}$$

Note first that

$$\left(\prod_{i=0}^{n-1}e^{r(t-t_i)}\right)^{\frac{1}{n}} = \exp\left\{\frac{rt}{2}\;\frac{n(n+1)}{n^2}\right\}.$$

The remaining sums are calculated as follows

$$\frac{1}{n^2}\sum_{i=0}^{n-1}\big[(n-(i+1))(i+1)\big]\Delta t = \frac{t}{6}\frac{n^2-1}{n^2} ,$$

$$\frac{1}{n^2}\sum_{i=0}^{n-1}\big[n(n-i)^2-(i+1)^2(2n-2i-1)\big]\Delta t^2 = \frac{1}{6}\frac{(n-1)(n+1)^2}{n}\Delta t^2 ,$$

$$\frac{1}{n^2}\sum_{i=0}^{n-2}\left[\left(\sum_{j=i+1}^{n-1}(j-i)\right)(i+1)\right]\Delta t^2 = \frac{\Delta t^2}{24}\frac{n-1}{n}\Big[n^2+3n+2\Big] ,$$

$$\frac{1}{n^2}\sum_{i=0}^{n-2}\left[\left(\sum_{j=i+1}^{n-1}(j-i)\right)^2\right]\Delta t^3 = \frac{\Delta t^3}{60}\frac{n+1}{n}\left[3n^3-3n^2-2n+2\right] ,$$

$$\frac{1}{n^2}\sum_{i=0}^{n-2}\left[\left(\sum_{j=i+1}^{n-1}(j-i)\right)(i+1)(2n-2i-1)\right]\Delta t^3 = \frac{\Delta t^3}{40}\frac{n+1}{n}\left[2n^3+3n^2-3n-2\right] ,$$

and finally

$$\frac{1}{n^2}\sum_{i=0}^{n-1}\left[nt_i(t-t_i)^2+\frac{1}{3}n(t-t_i)^3+\frac{1}{3}(i+1)^2\left((t-t_{i+1})^3-(t-t_i)^3\right)\right]$$
$$= \frac{\Delta t^3}{n\cdot 180}\left[24n^4+15n^3-20n^2-15n-4\right] .$$

□

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