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Kernel Estimation in Regime-Varying Regression Models

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Abstract:
We analyze the asymptotic behaviour of kernel estimators provided the underlying regression functions (called regimes here) are time-dependent. We show the uniform convergence in probability of the kernel estimators towards a Baire function that can be interpreted as a weighted average of the different regimes. Moreover, we propose consistent estimators for the different regimes for the case that the terms of the regimes are known.

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Chapter 1

Introduction

In general, the assumption of a time-independent correlation between two or more economic variables is not too realistic. On the one hand, the correlations considered are only partial - some more or less important influencing variables are left out of consideration. Changes of these possibly hidden variables could have lasting impacts on partial systems of other observable variables. On the other hand, time itself might have an influence. An example are day-of-the-week effects which are observed in financial markets.

Although this insight is neither new nor original, a standard assumption met in most papers on kernel estimation of regression functions is the strict stationarity of the underlying stochastic processes. In this paper we therefore analyze the consequences of this sort of misspecification. For this purpose we stipulate an underlying stochastic process $\{y_t, \mathbf{x}_t\}_{t \in Z}$ in $\mathbf{R} \times \mathbf{R}^k$ with $E|y_t| < \infty \; \forall t \in \mathbf{Z}$ and assume that the Baire functions representing the conditional expectations $E\{y_t \mid \mathbf{x}_t\}$ are time-dependent. Various formulations of this sort of non-stationarity are conceivable. We deal with a very simple case assuming that there are r Baire functions representing r different regimes. We suppose that regime $j \in \{1, \ldots, r\}$ rules whenever t is in $\mathbf{Z}^j$, where the $\mathbf{Z}^j$ form a partition of $\mathbf{Z}$.

In the sequel we analyze the asymptotic behaviour of kernel estimators of the form

$$\hat{m}_T(\mathbf{x}) = \frac{\sum_{t=1}^{T} \psi(y_t) K\left(\frac{\mathbf{x}-\mathbf{x}_t}{\gamma_T}\right)}{\sum_{t=1}^{T} K\left(\frac{\mathbf{x}-\mathbf{x}_t}{\gamma_T}\right)}, \tag{1.1}$$

where $\psi : \mathbf{R} \rightarrow \mathbf{R}$ is a bounded and Lipschitz-continuous function that is introduced in order to avoid imposing implausible moment conditions on $\{y_t, \mathbf{x}_t\}_{t \in Z}$. The price we have to pay for using such a function consists in an asymptotic bias due to the identification of $E\{\psi(y_t) \mid \mathbf{x}_t = \mathbf{x}\}$ instead of $E\{y_t \mid \mathbf{x}_t = \mathbf{x}\}$. However, since on the one hand this bias should be small provided that ψ is chosen suitably, and on the other hand the estimator might become more robust, this trade-off seems to be favourable[1].

We demonstrate the uniform convergence in probability of estimator 1.1 towards a Baire function that can be interpreted as a weighted average of the different regimes. This result should be understood as a warning: Observing the convergence of estimator 1.1, we must not infer that the limiting function found is the true regression function that underlies a stationary process.

Moreover, we establish

$$\hat{m}_T^j(\mathbf{x}) = \frac{\sum_{t=1}^{T} I^j(t)\psi(y_t) K\left(\frac{\mathbf{x}-\mathbf{x}_t}{\gamma_T}\right)}{\sum_{t=1}^{T} I^j(t) K\left(\frac{\mathbf{x}-\mathbf{x}_t}{\gamma_T}\right)}, \tag{1.2}$$

where $I^j(t) := I(t \in \mathbf{Z}^j)$, as a consistent estimator for the function representing regime j for the case that the terms t of regime j are known.

The rest of this paper is organized as follows: In the subsequent chapter we formulate in detail the non-stationary regression model we shall deal with and give some definitions. In chapter 3 we present the consistency results stated above. Some auxiliary results used in the main body of this text are provided in the appendix.

[1] In a previous paper (cf. Cron(1995)) we used a truncation procedure based on the regressors in order to avoid an asymptotic bias. Unfortunately the argument used there fails if the estimation is based on fewer regressors than actually included in the true model.

Chapter 2

A regime-varying regression model

The following assumptions specify the model we deal with in the sequel:

(A. 1) $\{y_t, \mathbf{x}_t\}_{t \in Z}$ *is an integrable stochastic process in* $\mathbf{R} \times \mathbf{R}^k$ *that allows for a representation*

$$(y_t, \mathbf{x}_t) = \sum_{j=1}^{r} I^j(t) R^j(\epsilon_t, \epsilon_{t-1}, \ldots),$$

where the R^j *are Borel measurable mappings from the space of one sided infinite sequences of vectors in* $\mathbf{R}^p$ *into* $\mathbf{R} \times \mathbf{R}^k$ *and* $\{\epsilon_t\}_{t \in \mathbf{Z}}$ *is a strictly stationary,* φ*-mixing stochastic process in* $\mathbf{R}^p$.

(A. 2) *The distributions of the* $\mathbf{x}_t$*'s are absolutely continuous with respect to Lebesgue measure with densities*

$$f_t(x) = \sum_{j=1}^{r} I^j(t) f^j(x),$$

where the f^j *are continuous and bounded.*

(A. 3) *Define* $\mathcal{F}_{t,\tau} := \sigma(\epsilon_t, \ldots, \epsilon_{t-\tau+1})$ *for any* $\tau \in \mathbf{N}$. *We assume that*

$$E \left\| E\{(y_t, \mathbf{x}_t) \mid \mathcal{F}_{t,\tau}\} - (y_t, \mathbf{x}_t) \right\| = o(\mu^{-\tau})$$

for some $\mu > 1$.

Remark 1: We note that (A.1), which replaces the standard assumption of strict stationarity, and (A.2/3) are not independent. Strictly speaking, (A.2/3) can be completely determined from the shape of the R^j and the kind of distribution of ϵ_t. The description of this coherence, however, is part of the solution theory of the underlying models and beyond the scope of this paper.

Remark 2: Let $H : \mathbf{R} \times \mathbf{R}^k \to \mathbf{R}$ denote any Borel measurable function. We note that in view of (A.1) the subsequences

$$\{H(y_t, \mathbf{x}_t)\}_{t \in \mathbf{Z}^j} = \left\{H(R^j(\epsilon_t, \epsilon_{t-1}, \ldots))\right\}_{t \in \mathbf{Z}^j}$$

are strictly stationary (cf. Breiman(1968)).
Moreover,provided that $E\{|H(y_t, \mathbf{x}_t)|\} < \infty \; \forall t \in \mathbf{Z}$, the subsequences

$$\{E\{H(y_t, \mathbf{x}_t) \mid \mathcal{F}_{t,\tau}\}\}_{t \in \mathbf{Z}^j} = \left\{E\{H(R^j(\epsilon_t, \epsilon_{t-1}, \ldots)) \mid \mathcal{F}_{t,\tau}\}\right\}_{t \in \mathbf{Z}^j}$$

are strictly stationary for τ fixed.

Remark 3: Assumption (A.3) which is termed ν -stability (cf. Bierens(1990)) or near-epoch-dependence (NED) (cf. Davidson(1994)) in L_1-norm, is weaker than the usual mixing-assumptions found in the literature on kernel estimation of dependent processes.

We conclude this chapter giving some definitions:

$$\begin{aligned}
T^j &:= T^j(T) = \sum_{t=1}^{T} I^j(t), \\
t^j &:= \min_{\mathbf{Z}^j{}_+} t, \\
h^j(T) &:= \frac{T^j}{T}, \\
h^j &:= \lim_{T \to \infty} h^j(T).
\end{aligned}$$

The last definition contains the implicit assumption that $h^j(T)$ has a limit. Finally, we define

$$\begin{aligned}
m^j(\mathbf{x}) &:= E\{\psi(y_{t^j}) \mid \mathbf{x}_{t^j} = \mathbf{x}\}, \\
g^j(\mathbf{x}) &:= m^j(\mathbf{x}) f^j(\mathbf{x}).
\end{aligned}$$

Chapter 3

Consistency results

We note that estimator 1.2 can be written as

$$\hat{m}_T^j(\mathbf{x}) = \frac{\hat{g}_T^j(\mathbf{x})}{\hat{f}_T^j(\mathbf{x})},$$

where

$$\begin{aligned} \hat{g}_T^j(\mathbf{x}) &:= \frac{1}{T}\gamma_T^{-k}\sum_{t=1}^{T} I^j(t)\psi(y_t)K\left(\tfrac{\mathbf{x}-\mathbf{x}_t}{\gamma_T}\right), \\ \hat{f}_T^j(\mathbf{x}) &:= \frac{1}{T}\gamma_T^{-k}\sum_{t=1}^{T} I^j(t)K\left(\tfrac{\mathbf{x}-\mathbf{x}_t}{\gamma_T}\right). \end{aligned}$$

For all $j \in \{1, \ldots, r\}$ we have the decomposition

$$\hat{g}_T^j(\mathbf{x}) - h^j g^j(\mathbf{x}) = G_T^{j,1}(\mathbf{x}) + G_T^{j,2}(\mathbf{x}) + G_T^{j,3}(\mathbf{x}),$$

where

$$\begin{aligned} G_T^{j,1}(\mathbf{x}) &:= E\{\hat{g}_T^j(\mathbf{x})\} - h^j g^j(\mathbf{x}), \\ G_T^{j,2}(\mathbf{x}) &:= E\{\hat{g}_T^j(\mathbf{x}) \mid \mathcal{F}_{t,\tau}\} - E\{\hat{g}_T^j(\mathbf{x})\}, \\ G_T^{j,3}(\mathbf{x}) &:= \hat{g}_T^j(\mathbf{x}) - E\{\hat{g}_T^j(\mathbf{x}) \mid \mathcal{F}_{t,\tau}\}, \end{aligned}$$

dropping τ for simplicity. The three terms are treated separately[1] in the subsequent Lemmata that hold for $j = 1, \ldots, r$. The validity of (A.1) to (A.3)

[1] The basic idea of the proof is due to BIERENS(1983).

is supposed without refer to explicitly. Moreover, we assume that $K(u)$ is a k-variate density with unbounded support and characteristic function

$$\beta(v) = \int_{\mathbf{R}^k} \exp(iv'u) K(u) du$$

satisfying

$$\int_{\mathbf{R}^k} \|v\| |\beta(v)| dv < \infty.$$

Lemma 1

Suppose that $C \subset \mathbf{R}^k$ is compact and $\lim_{T\to\infty} \gamma_T = 0$. Then we have

$$\lim_{T\to\infty} \sup_{\mathbf{x}\in C} \left| G_T^{j;1}(\mathbf{x}) \right| = 0.$$

Proof:

In view of remark 2 we can write

$$\begin{aligned}
& \left| E\{\hat{g}_T^j(\mathbf{x})\} - h^j g^j(\mathbf{x}) \right| \\
= & \left| E\left\{ \frac{1}{T} \gamma_T^{-k} \sum_{t=1}^{T} I^j(t) \psi(y_t) K\left(\tfrac{\mathbf{x}-\mathbf{x}_t}{\gamma_T}\right) \right\} - h^j g^j(\mathbf{x}) \right| \\
= & \left| h^j(T) \gamma_T^{-k} E\left\{ \psi(y_{t^j}) K\left(\tfrac{\mathbf{x}-\mathbf{x}_{t^j}}{\gamma_T}\right) \right\} - h^j g^j(\mathbf{x}) \right| \\
= & \left| h^j(T) \int_{\mathbf{R}^k} \gamma_T^{-k} g^j(\mathbf{z}) K\left(\tfrac{\mathbf{x}-\mathbf{z}}{\gamma_T}\right) d\mathbf{z} - h^j g^j(\mathbf{x}) \right| \\
\leq & \; h^j(T) \left| \int_{\mathbf{R}^k} \gamma_T^{-k} g^j(\mathbf{z}) K\left(\tfrac{\mathbf{x}-\mathbf{z}}{\gamma_T}\right) d\mathbf{z} - g^j(\mathbf{x}) \right| \\
+ & \left| g^j(\mathbf{x}) \right| \left| h^j(T) - h^j \right|.
\end{aligned}$$

Application of Proposition 1 of the appendix yields the result. ■

Remark 4: The suprema in the following Lemmata are random variables (cf. JENNRICH(1969),Lemma 1). We suppose from now on that $\tau = \tau(T)$.

Lemma 2
Define $\rho_T := \frac{1}{T}\sum_{m=0}^{T}\varphi^{\frac{1}{2}}(m)$*, where the* φ*'s are the mixing coefficients of the process* $\{\epsilon_t\}_{t\in\mathbf{Z}}$*. We have*

$$E\sup_{\mathbf{x}\in\mathbf{R}^k}\left|G_T^{j,2}(\mathbf{x})\right| = O\left(\gamma_T^{-k}\sqrt{\frac{\tau}{T}+\rho_T}\right).$$

Proof:
(Cf. Bierens(1983)). The inversion formula for characteristic functions and Proposition 2 of the appendix yield

$$E\sup_{\mathbf{x}\in\mathbf{R}^k}\left|G_T^{j,2}(\mathbf{x})\right| \le \left(\frac{1}{2\pi}\right)^k\int_{\mathbf{R}^k} w_T^j(v)|\beta(\gamma_T v)|dv,$$

where

$$w_T^j(v) := E\left|\frac{1}{T}\sum_{t=1}^{T} I^j(t)\left[E\{\psi(y_t)\exp(iv'\mathbf{x}_t)|\mathcal{F}_{t,\tau}\} - E\{\psi(y_t)\exp(iv'\mathbf{x}_t)\}\right]\right|.$$

We note that in view of remark 2 , for τ fixed, the subsequences $\{E\{\psi(y_t)\cos(v'\mathbf{x}_t)|\mathcal{F}_{t,\tau}\}\}_{t\in\mathbf{Z}^j}$ and $\{E\{\psi(y_t)\sin(v'\mathbf{x}_t)|\mathcal{F}_{t,\tau}\}\}_{t\in\mathbf{Z}^j}$ are strictly stationary and ,in addition, φ-mixing with

$$\varphi_\tau^*(m) = \begin{cases} 1 & \text{if } m < \tau \\ \varphi(m-\tau) & \text{otherwise}\end{cases}$$

Liapunov's inequality and Lemma 1 of Billingsley(1968), sec. 20, imply the rough estimation

$$\begin{aligned}\left(w_T^j(v)\right)^2 &\le var\left(\frac{1}{T}\sum_{t=1}^{T} I^j(t)E\{\psi(y_t)\cos(v'\mathbf{x}_t)|\mathcal{F}_{t,\tau}\}\right)\\ &+ var\left(\frac{1}{T}\sum_{t=1}^{T} I^j(t)E\{\psi(y_t)\sin(v'\mathbf{x}_t)|\mathcal{F}_{t,\tau}\}\right)\\ &\le 8\left(\frac{\tau}{T}+\rho_T\right)E\psi^2(y_{t^j}).\end{aligned}$$

The result follows easily. ∎

Lemma 3

$$E\sup_{\mathbf{x}\in\mathbf{R}^k}\left|G_T^{j,3}(\mathbf{x})\right| = o\left(\mu^{-\tau}\gamma_T^{-(k+1)}\right).$$

Proof:
As in the proof of Lemma 2 we can show that

$$E\sup_{\mathbf{x}\in\mathbf{R}^k}\left|G_T^{j,3}(\mathbf{x})\right| \le h^j(T)\left(\frac{1}{2\pi}\right)^k \int_{\mathbf{R}^k} W_T^j(v)|\beta(\gamma_T v)|dv,$$

where

$$W_T^j(v) := E\left|\psi(y_{t^j})\exp(iv'\mathbf{x}_{t^j}) - E\{\psi(y_{t^j})\exp(iv'\mathbf{x}_{t^j})|\mathcal{F}_{t^j,\tau}\}\right|.$$

Define $y_{t^j,\tau} := E\{y_{t^j}|\mathcal{F}_{t^j,\tau}\}$ and $\mathbf{x}_{\mathbf{t^j},\tau} := E\{\mathbf{x}_{t^j}|\mathcal{F}_{t^j,\tau}\}$. Repeated application of Proposition 3 (see Appendix),the fact that $|e^{iu}-1|\le|u|$ and the boundedness and Lipschitz-continuity of ψ yield

$$\begin{aligned}
W_T^j(v) &\le 2E\left|\psi(y_{t^j})\exp(iv'\mathbf{x}_{t^j}) - E\{\psi(y_{t^j})|\mathcal{F}_{t^j,\tau}\}\exp(iv'\mathbf{x}_{t^j,\tau})\right| \\
&\le 2E\left|E\{\psi(y_{t^j})|\mathcal{F}_{t^j,\tau}\}\left[\exp(iv'\mathbf{x}_{t^j}) - \exp(iv'\mathbf{x}_{t^j,\tau})\right]\right| \\
&+ 2E\left|\exp(iv'\mathbf{x}_{t^j})\left[E\{\psi(y_{t^j})|\mathcal{F}_{t^j,\tau}\} - \psi(y_{t^j})\right]\right| \\
&\le C_1E\left|\exp(iv'\mathbf{x}_{t^j}) - \exp(iv'\mathbf{x}_{t^j,\tau})\right| \\
&+ C_2E\left|y_{t^j,\tau} - y_{t^j}\right| \\
&\le C\left[E\|\mathbf{x}_{t^j} - \mathbf{x}_{t^j,\tau}\|\|v\| + E|y_{t^j} - y_{t^j,\tau}|\right] \\
&= o\left(\mu^{-\tau}\right)(1+\|v\|).
\end{aligned}$$

The result follows easily.■

Remark 5: The preceding results remain valid if we replace $\psi(y_t)$ by 1.

The combination of the three Lemmata leads us to the main results announced (even though in reverse order):

Theorem 1
Suppose that $h^j > 0$ and $\rho_T^{\frac{1}{2}} = o\left(T^{-\epsilon k}\right)$ for some $\epsilon \in \left(0, \frac{1}{2k}\right)$. Then for $\gamma_T = T^{-\epsilon}$, any compact $C \subset \mathbf{R}^k$ and $\delta > 0$ we have

$$p\lim_{T\to\infty}\sup_{\mathbf{x}\in C, f^j(\mathbf{x})\ge\delta}|\hat{m}_T^j(\mathbf{x}) - m^j(\mathbf{x})| = 0.$$

Proof:
Choosing $\tau(T) = \left[T^{\frac{1}{2}-\epsilon k}\right]$,we can link Lemmata 2 and 3 to obtain

$$\lim_{T\to\infty} E \sup_{\mathbf{x}\in\mathbf{R}^k} \left|G_T^{j,2}(\mathbf{x}) + G_T^{j,3}(\mathbf{x})\right| = 0.$$

Combining this implication with Lemma 1 yields (see remark 5)

$$\begin{aligned}\lim_{T\to\infty} E \sup_{\mathbf{x}\in C} |\hat{g}_T^j(\mathbf{x}) - h^j g^j(\mathbf{x})| &= 0 \text{ and} \\ \lim_{T\to\infty} E \sup_{\mathbf{x}\in C} |\hat{f}_T^j(\mathbf{x}) - h^j f^j(\mathbf{x})| &= 0.\end{aligned}$$

Since we can write

$$|\hat{m}_T^j(\mathbf{x}) - m^j(\mathbf{x})| = \frac{1}{h^j f^j(\mathbf{x})}\left|\hat{m}_T^j(\mathbf{x})\left[h^j f^j(\mathbf{x}) - \hat{f}_T^j(\mathbf{x})\right] + \left[\hat{g}_T^j(\mathbf{x}) - h^j g^j(\mathbf{x})\right]\right|,$$

the result follows easily. ■

Theorem 2
Under the assumptions of Theorem 1, we have

$$p\lim_{T\to\infty} \sup_{\mathbf{x}\in C, \min_j f^j(\mathbf{x})\geq\delta} |\hat{m}_T(\mathbf{x}) - m(\mathbf{x})| = 0,$$

where

$$m(\mathbf{x}) := \sum_{j=1}^{r}\left(\frac{h^j f^j(\mathbf{x})}{\sum_{i=1}^{r} h^i f^i(\mathbf{x})}\right) m^j(\mathbf{x}).$$

Proof:
Note that at least one h^j must be positive. Since we can write

$$\hat{m}_T(\mathbf{x}) = \frac{\sum_{j=1}^{r}\hat{g}_T^j(\mathbf{x})}{\sum_{j=1}^{r}\hat{f}_T^j(\mathbf{x})} \text{ and } m(\mathbf{x}) = \frac{\sum_{j=1}^{r} h^j g^j(\mathbf{x})}{\sum_{j=1}^{r} h^j f^j(\mathbf{x})},$$

we have

$$\begin{aligned}&|\hat{m}_T(\mathbf{x}) - m(\mathbf{x})| \\ =\ &\frac{1}{\sum_{j=1}^{r} h^j f^j(\mathbf{x})}\left|\hat{m}_T(\mathbf{x})\sum_{j=1}^{r}\left(h^j f^j(\mathbf{x}) - \hat{f}_T^j(\mathbf{x})\right) + \sum_{j=1}^{r}\left(\hat{g}_T^j(\mathbf{x}) - h^j g^j(\mathbf{x})\right)\right|,\end{aligned}$$

and the proof can be completed with the arguments used in the proof of Theorem 1. ■

Remark 6: Intuitively we would expect $\hat{m}_T(\mathbf{x})$ to converge pointwise to a weighted average of the $m^j(\mathbf{x})$ with weights h^j. Theorem 2 tells us however that the weights depend on $\mathbf{x}$ through the densities of the different regimes and are positively correlated with their own densities .

Appendix A

Proposition 1
Suppose that K is a k-variate density, $h : \mathbf{R}^k \to \mathbf{R}$ continuous and bounded, and $C \subset \mathbf{R}^k$ compact. Then we have

$$\lim_{\gamma \to 0} \sup_{\mathbf{x} \in C} \left| \int_{\mathbf{R}^k} \gamma^{-k} h(\mathbf{z}) K\left(\frac{\mathbf{x}-\mathbf{z}}{\gamma}\right) d\mathbf{z} - h(\mathbf{x}) \right| = 0.$$

Proof:
The proof is easy and therefore omitted.■

Proposition 2
Suppose that x and y are random vectors in $\mathbf{R}^{k_1}$,$\mathbf{R}^{k_2}$, respectively. Assume that $f : \mathbf{R}^{k_2} \times \mathbf{R}^k \to \mathbf{R}$ and $g : \mathbf{R}^k \to \mathbf{R}$ are Baire functions such that

$$\sup_{y \in \mathbf{R}^{k_2}} |f(y,t)| \leq g(t), \quad \int_{\mathbf{R}^k} g(t)dt < \infty.$$

Then we have

$$E\left\{ \int_{\mathbf{R}^k} f(y,t)dt \,|x \right\} = \int_{\mathbf{R}^k} E\{f(y,t)|x\}dt \qquad F_x - a.\ s.\ ,$$

where F_x denotes the distribution of x.

Proof:
Cf. Cron(1995).■

Proposition 3
Suppose that x is a random variable with $E|x|^p < \infty$ for some $p \geq 1$. Then we have for any $\mathcal{F}$-measurable y with $E|y|^p < \infty$

$$E|x - E\{x|\mathcal{F}\}|^p \leq 2^p E|x - y|^p.$$

Proof:
From the fact that $|x + y|^p \leq 2^{p-1}(|x|^p + |y|^p)$ and Jensen's inequality for conditional expectations we have

$$\begin{aligned} E|x - E\{x \mid \mathcal{F}\}|^p &\leq 2^{p-1}(E|x - y|^p + E|y - E\{x \mid \mathcal{F}\}|^p) \\ &= 2^{p-1}(E|x - y|^p + E|E\{y - x \mid \mathcal{F}\}|^p) \\ &\leq 2^p E|x - y|^p. \end{aligned}$$

■

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