The Dynamic (In)Stability of Backwards Induction[*]

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Abstract

The analysis of the replicator dynamic in generic perfect information games yields the following results. In the long run, players play a Nash equilibrium provided that initially all strategies are present. There is at most one “stable” component (formally, an interior asymptotically stable set), play in this component will follow the backwards induction path. Existence of such a component is guaranteed in games with at most three consecutive decision nodes. An example of a “longer” game is provided where some trajectories starting close to the backwards induction component lead away and never come back.

Keywords: perfect information, extensive-form game, Centipede Game, backwards induction, replicator dynamics, interior asymptotic stability.

JEL classification numbers: C72, C79.

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1 Introduction

The backwards induction procedure, whose origin can be traced to Zermelo (1913), is one of the fundamental techniques of game theory. However, its applicability is somewhat controversial, especially for games with many decision nodes but also for more elementary games such as the well-known Chain-Store Game of Selten (1965) that has one decision node for each of the two players. When this game is played once, the Nash equilibrium outcome, G^*, induced by the backwards indcution procedure, predicts the competitor enters the market and the monopolist cooperates. The controversy is over why the other Nash equilibrium outcome (i.e. the monopolist threatens to fight and the competitor does not enter the market) is not expected. In fact, G^* can be easily justified either by appealing to traditional arguments based on rationality or on dynamic grounds. We pursue the latter justification in this paper where we refer to this game as the Ultimatum Mini-Game, following the terminology of Gale et al. (1995).

One source of controversy can perhaps be better illustrated by reference to the Centipede Game (Rosenthal, 1981). Here the unique Nash equilibrium outcome specifies player one to move down at the first node and so all other decision nodes of the game are off the equilibrium path. Backwards induction relies on the assumption that players choose rationally off the equilibrium path. In particular, if a player is asked to move at a node that is not on the equilibrium path, rationality is assumed at all decisions following this node even though the only reason such a node is reached at all is because some decision must not have been rational. Reny (1993) formalizes this statement to conclude that, if there is a path with more than one move off the equilibrium path, then there is already a contradiction regarding the rationality of the players when rationality of the players is common knowledge. For the Centipede Game, this means there is a controversy unless its length is at most two.

Our paper focuses on whether the applicability of backwards induction to extensive-form two-player games with perfect information relies on the structure of the game (i.e. whether the number of moves off the equilibrium path matters) by analyzing dynamic stability properties of G^*. However, in contrast to the above literature, we choose to analyze this question using evolutionary arguments. More specifically, we analyze the continuous replicator dynamic of Taylor (1979) with respect to the game's normal from. This paradigm of biological and cultural evolution, derived originally without rationality considerations, has recently been shown to approximate various models of bounded rational learning. These models include many imitation scenarios (Björnerstedt and Weibull, 1993; Cabrales, 1993; Gale et al., 1995). In fact, under incomplete information, the replicator dynamic is the unique approximation when monotonicity is imposed on the learning process (Björnerstedt and Schlag, 1995). It must be emphasized at this point that learning in these models does not occur through observations but rather through the imitation of strategies. One would expect this strong assumption in the framework of extensive form games to facilitate learning the backwards induction solution (i.e., the subgame perfect equilibrium). However, we will see that, despite this assumption, the subgame perfect equilibrium can be difficult to learn. In fact, our results applied to the Centipede Game can be interpreted to say that the backwards induction outcome can be justified if the game has length three but not necessarily in all Centipede Games of greater length.

We now explain our model in more detail. We consider a finite extensive form game Γ without chance moves between two players who have complete knowledge of all choices taken on the path leading to their current decision point. That is, Γ is assumed to have a finite number of decision points, all of which are singleton information sets for one of the two players. We assume that payoffs are generic such that no player is ever indifferent between two actions. In such games we examine the trajectories of the continuous replicator dynamic of evolutionary game theory. The replicator dynamic is defined through the normal form of Γ. We are particularly interested in trajectories under this dynamic whose initial points have all pure strategies of both players present with positive frequency (i.e., the initial points are in the interior of the strategy simplex $\Delta^m \times \Delta^n$ defined in Section 2 below). Such interior starting points arise when mutations, that are very rare, are diverse in the sense that a positive proportion among the mutants plays each strategy. Long run properties of these trajectories are a major topic of dynamic evolutionary game theory. For instance, examples abound (e.g. Hofbauer and Sigmund, 1988) of general two-person normal form games where these trajectories need not converge. On the other hand, if the trajectory does converge, its limit point is a Nash equilibrium (Samuelson and Zhang, 1992).

Our first main result, Theorem 2 of Section 2, proves the convergence of any trajectory in the interior of Γ (given our genericity assumption). That is, provided that mutations are sufficiently rare and diverse, agents will learn to play a Nash equilibrium. For Γ with only one Nash equilibrium path, such as the Centipede Game of arbitrary (but finite) length, Theorem 2 implies every interior trajectory converges to a unique point in the Nash equilibrium component, G^*, of the normal form of Γ that contains the unique subgame perfect equilibrium given by backwards induction. In particular, the system evolves to one where the only strategy types remaining are those that follow this unique Nash equilibrium path when played against each other.

In general, Γ may have several Nash equilibrium paths, each corresponding to a Nash equilibrium component of the normal form. Another major concern of dynamic evolutiuonary game theory is the dynamic stability of these Nash equilibria. In this regard, our main results are that, generically, every pure strategy Nash equilibrium is stable (Theorem 3) and that G^* is the only interior asymptotically stable set if all paths in Γ have length at most three (Theorem 5). For details of these dynamic stability concepts, see Definitions 3 and 4 below.

As for the related literature on adaptive learning models and backwards induction, Nöldeke and Samuelson (1993) have analyzed a stochastic learning process in simple games where each individual moves at most once. They show for perfect information games that the subgame perfect equilibrium outcome corresponds to the unique long run outcome. As such their result coincides with our result on asymptotic stability in games of length 2. However, they do not consider longer games. Nor do Gale et al. (1995), who analyze the influences of drift in the Ultimatum Mini-Game. For certain properties of the drift they show that there are asymptotically stable outcomes close to the Nash equilibrium component that does not contain the subgame perfect equilibrium outcome. In a subsequent paper, Binmore and Samuelson (1995) show that this effect can also appear in the Centipede Game of length 3. Finally, Cressman (1993b) shows that all interior trajectories of the continuous replicator dynamic for symmetric games converge to the play of defect in a finitely repeated Prisoners' Dilemma played by symmetric players.

The paper is organized as follows. In Section 2 the limit outcomes of interior trajectories are analyzed. Section 3 concerns the stability of pure strategy Nash equilibria. In Section 4 we analyze interior attracting sets. Section 5 concerns interior asymptotic stability of the Nash equilibrium component in short games. In Section 6 we provide counterexamples and, finally, Section 7 provides a brief summary. Appendix A contains some of the more tedious proofs and Appendix B the relevant theory of centre manifolds for the linearization of the replicator dynamic.

2 Convergence to Nash Equilibrium

The replicator dynamic for a two-person extensive-form game, Γ, is based on the bimatrix payoff matrix of the game's normal form. Let $e_1, e_2 \ldots, e_m$ list the pure behavior strategies of player one (i.e. e_1 specifies a choice at each decision point of player one) and $f_i, \ldots, f_n$ list those of player two. Then (A, B) is the bimatrix normal form of Γ where A and B are $m \times n$ and $n \times m$

matrices respectively with entries A_{ij} and B_{ji} giving the payoff to player one and to player two respectively for the path of Γ that is followed when e_i plays f_j.

The (continuous) replicator dynamic, introduced by Taylor (1979),

$$\begin{aligned} \dot{p_i} &= p_i(e_i - p) \cdot Aq \text{ for } 1 \leq i \leq m, \\ \dot{q_j} &= q_j(f_j - q) \cdot Bp \text{ for } 1 \leq j \leq n, \end{aligned} \tag{1}$$

is a monotone selection dynamic (Samuelson and Zhang, 1992) on the strategy frequency vectors $p \in \Delta^m = \{(p_1, \ldots, p_m) \mid \sum p_i = 1, p_i \geq 0\}$ and $q \in \Delta^n$. Here the derivatives on the left-hand side are with respect to time; e_i is represented by the i^{th} unit coordinate vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ in Δ^m whose only nonzero entry is in the i^{th} component; f_j the corresponding unit vector in Δ^n; and the dot product on the right-hand side is the standard inner product in R^k given by $y \cdot z = \sum_{i=1}^{k} y_i z_i$ (hence $p \cdot Aq = \sum_i \sum_j p_i A_{ij} q_j$ and $q \cdot Bp = \sum_i \sum_j q_j B_{ji} p_i$).

The dynamic (1) describes an evolution on $\Delta^m \times \Delta^n$ that models two large, effectively infinite, populations whose individuals use pure strategies and compete in random pairwise interspecific contests with expected payoffs given by the entries of A and B. Then $p_i(t)$ is the frequency of individuals in population one who use pure strategy e_i at time t. The replicator dynamic reflects the intuitive property that the relative frequency of strategy types in population one (e.g. p_i relative to p_j) increases if and only if e_i has a higher expected fitness than e_j (i.e. $e_i \cdot Aq > e_j \cdot Aq$). Various models of individual learning have been found where the population change is approximated by the replicator dynamic (Björnerstedt and Weibull, 1993; Cabrales, 1993; Gale et al., 1995; Schlag, 1994). Evolutionary game theory is used to select a solution to the game by considering the long-run evolution of (1). This is illustrated by the following two examples based on the simplest non-trivial two-person extensive-form game with perfect information; namely, the game in Figure 1 where each player has one decision point together with two possible choices there.

We will consider the case where (T, L) is a Nash equilibrium strategy pair in the game in Figure 1 and assume that none of the payoffs to player one are equal (a generic assumption as discussed below). Thus $\alpha > \gamma$. Since a constant can be added to the payoffs of player one when playing one of the pure strategies of player two (or to the payoffs of player two) without affecting the dynamic (1) and the payoffs of both players can be multiplied (i.e., scaled) by the same positive constant without affecting the trajectories, for the long-run evolution we may assume $\alpha = \beta = 1$ and $\gamma = 0$ without

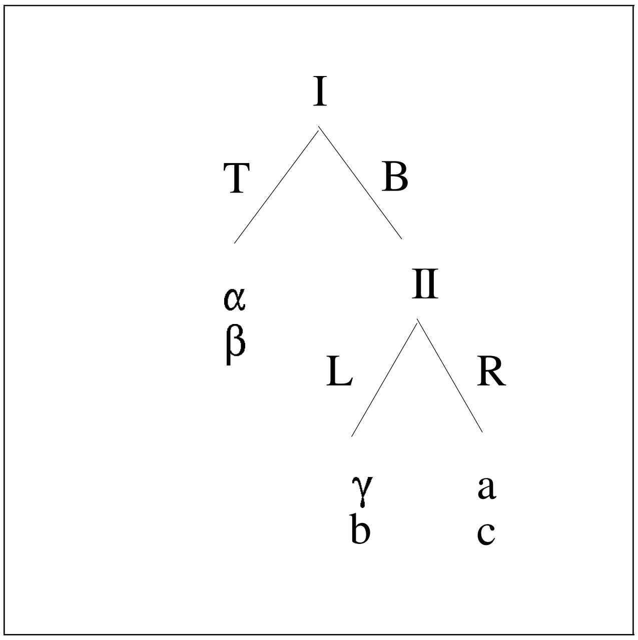

Figure 1: A simple extensive-form game.

loss of generality. If $a < 1$, then (T, q) is a Nash equilibrium for all $q \in \Delta^2$; the frequency of T is always increasing; and the frequency of L (respectively, R) is always increasing if and only if $b > c$ (respectively $b < c$). These monotonic flows are rather uninteresting from a dynamic perspective and so we will assume $a > 1$ from now on.

Example 1 *The Centipede Game of Length* 2.

Take $\alpha = \beta = 1$, $\gamma = 0$, $a > 1$ *and* $b > c$ *in Figure 1. The unique subgame perfect Nash equilibrium is* (T, L) *and it is contained in the unique Nash equilibrium component* $G^* = \left\{(T, (1-\lambda)L + \lambda R \mid 0 \leq \lambda \leq \frac{1}{a}\right\}$ *of the game's normal form*

	L	R
T	$1, 1$	$1, 1$
B	$0, b$	a, c

. (2)

These static results are consistent with the dynamic flow of (1) for this example given in Figure 2. Notice that the evolution of the frequency of T *is not monotonic. Nevertheless all interior trajectories converge to a unique point of* G^*. *Notice also that there is no interior trajectory that converges to* $\left(T, \frac{a-1}{a}L + \frac{1}{a}R\right)$ *or to* (T, L).

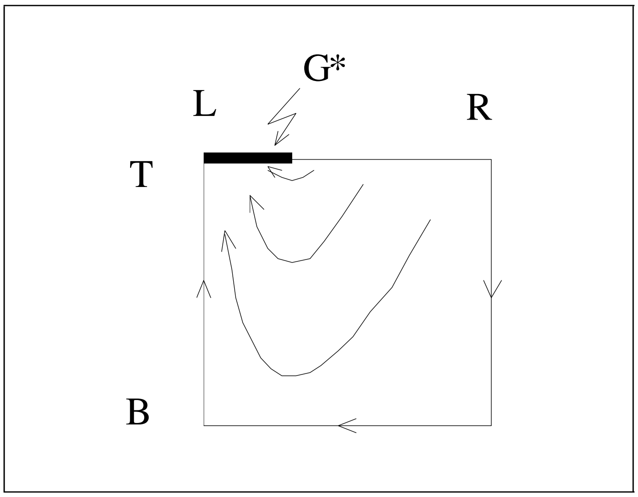

Figure 2: Trajectories for the Centipede Game of length 2.

Example 2 *The Chain-Store Game or Ultimatum Mini-Game*
Take $\alpha = \beta = 1$*,* $\gamma = 0$*,* $a > 1$ *and* $b < c$ *in the game in Figure 1. There are now two pure strategy Nash equilibria; namely, the unique subgame perfect Nash equilibrium,* (B, R)*, as well as* (T, L)*.* (B, R) *is a strict Nash equilibrium and so its Nash equilibrium component from (2) is a singleton set (i.e.,* $G^* = (B, R)$*). The Nash equilibrium component of* (T, L) *remains the set, now labelled* G*, given in Example 1. Again these results are reflected in the flow diagram of Figure 3. Every interior trajectory converges either to* G^* *or to a unique point in* G*.*

In this section, we examine the convergence of trajectories of (1) that start in the interior of $\Delta^m \times \Delta^n$. We are able to show (Theorem 2 below) that any such trajectory converges to a Nash equilibrium (p^*, q^*) of the normal form of any generic two-person extensive-form game with perfect information, Γ. The generic assumption is that no two paths of Γ have the same payoff for player one (or for player two). That is, there are no ties between different paths for either player. These Γ form an open dense set under any reasonable topology based on payoff entries.

The proof of Theorem 2 for Examples 1 and 2 is straightforward. In Example 2, R weakly dominates L for player two. A result of Akin (1980) (see Weibull (1995) for the bimatrix version) shows that either the frequency of strategy L vanishes in the limit or the frequencies of those pure strategies of players one that have higher payoff against R than against L vanish.

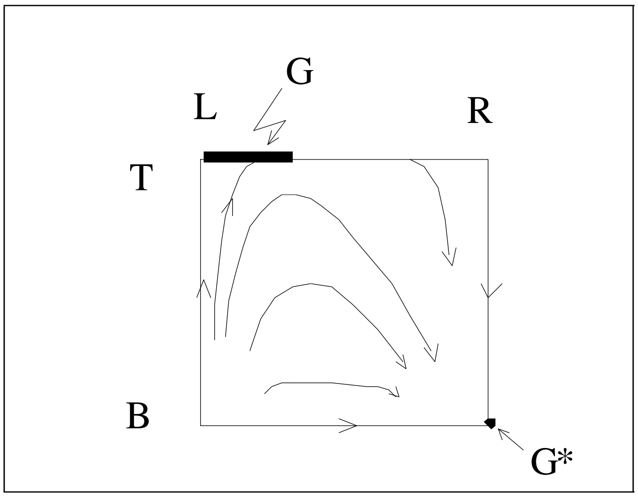

Figure 3: Trajectories for the Ultimatum Mini-Game.

Here, either $q_1 \to 0$ or $p_2 \to 0$ as $t \to \infty$. Since q_2 is always increasing for any interior trajectory, it is clear that "nothing wild can happen" and that each such trajectory either converges to (B, R) when $q_1 \to 0$ or to a point in $\{(p,q)\,|p_2 = 0\}$. In Example 1, the weak domination of R by L implies $q_2 \to 0$ or $p_2 \to 0$. In either case, the trajectory converges to a point where $p_2 = 0$. One might like to apply such elimination arguments repeatedly in a game with more decision points. However, a possible complication to the backwards induction approach cannot be ruled out a priori; namely, the set of ω limits (i.e., the limit points) of a trajectory may be larger than the set of ω limits for those trajectories that start at an ω limit of the trajectory. A typical example of such a possibility is illustrated by Weibull (Figure 5.5, 1995) on the boundary of the strategy space.

The method of proof of Theorem 2 for generic Γ is similar to that used by Cressman (1993b) for the symmetric finitely-repeated Prisoners' Dilemma Game where the analogue of the sets defined in Definition 1 were introduced. Note that it is sufficient to show that any $(p(0), q(0))$ in the interior of $\Delta^m \times \Delta^n$ converges to a single point (p^*, q^*) since such a point is automatically a Nash equilibrium under any monotone selection dynamic (Samuelson and Zhang, 1992). For the remainder of this section, fix an initial point $(p(0), q(0))$ in the interior of $\Delta^m \times \Delta^n$.

Definition 1 *Let E_1 and E_2 be the subsets of $\{e_1, e_2, \ldots, e_m\}$ given by $E_1 = \left\{e_i \,|\lim\sup_{t\to\infty} p_i(t) > 0\right\}$ and $E_2 = \{e_i \mid \int_0^\infty p_i(t) = \infty\}$. Similarly F_1 and F_2*

are subsets of $\{f_1, f_2, \ldots, f_n\}$ *with* $F_1 = \left\{ f_j \mid \limsup_{t\to\infty} q_j(t) > 0 \right\}$ *and* $F_2 = \{f_j \mid \int_0^\infty q_j(t)dt = \infty\}$*. Let* Γ' *be the extensive form game generated by* E_2 *and* F_2*. That is, the graph of* Γ' *is the subgaph of* Γ *whose paths are precisely those followed by some* e_i *playing against* f_j *where* $e_i \in E_2$ *and* $f_j \in F_2$*. The pure strategies of* Γ' *include all* e_i *and* f_j *with* $e_i \in E_2$ *and* $f_j \in F_2$ *when these strategies are restricted to the decision points of* Γ'*.*

Since we want to show that $\lim_{t\to\infty} p_i(t)$ and $\lim_{t\to\infty} q_j(t)$ exist for all i and j, it is somewhat surprising that we consider the sets E_2 and F_2. The compactness of $\Delta^m \times \Delta^n$ combined with the smoothness of the vector field for (1) imply $E_1 \subset E_2$ and $F_1 \subset F_2$. E_1 may not equal E_2 since it is possible $p_i(t)$ may converge to zero without the improper integral in E_2 being finite. Notice that the improper integrals in E_2 and F_2 are well-defined as an extended positive real number (i.e. either ∞ or a positive real number) since $p_i(t)$ is a continuous positive function for $t \geq 0$.

By our generic assumption, Γ' is a two-person extensive-form game with perfect information that has a unique backwards induction equilibrium outcome. We will see that e_i played against f_j follows this equilibrium path for all $e_i \in E_1$ and $f_j \in F_1$. Theorem 2 follows from this but first we need the following concept.

Definition 2 *Order the* N *decision points of* Γ' *as* $u_1, \ldots, u_N$ *where* $k < l$ *implies* u_l *does not follow* u_k *along a path in* Γ'*. In particular,* u_1 *is a final decision point of* Γ'*. For any pure strategy* e_i *of* Γ *and any* $1 \leq k \leq N$*, let* e_i^k *be the same strategy as* e_i *except that* e_i^k *chooses the dominant local strategy with respect to the backwards induction procedure applied to* Γ' *at any decision points of player one among* $u_1, \ldots, u_k$*. Similarly,* f_j^k *chooses the dominant strategy of* Γ' *at any player two decision points among* $u_1, \ldots, u_k$*.*

Theorem 1 *Suppose* $e_i \in E_2$ *and* $f_j \in F_2$*. Then, for any* $1 \leq k \leq N$*, the following are true:*

(i) $\lim_{t\to\infty} p_i^k(t)/p_i(t) = L^k$ *exists as a positive extended real number.*

(ii) $\lim_{t\to\infty} q_j^k(t)/q_j(t) = M^k$ *exists as a positive extended real number.*

(iii) If L^k *is finite, then* $e_i \cdot Af_j = e_i^k \cdot Af_j$ *for all* $f_j \in F_2$*.*

(iv) If M^k *is finite, then* $f_j \cdot Be_i = f_j^k \cdot Be_i$ *for all* $e_i \in E_2$*.*

(v) $e_i^k \in E_2$*.*

(vi) $f_j^k \in F_2$*.*

The proof of Theorem 1 is similar to that used in Cressman (1993b) for Prisoners' Dilemma Games and is given in Appendix A. To illustrate

Definition 2 and Theorem 1, consider the extreme case where all decision points of Γ' consist of single choices. In other words, Γ' has a single path. For this case $e_i^k = e_i$ and $f_j^k = f_j$ for all $1 \leq k \leq N$. Thus, Theorem 1 is obvious. Here it is also true that $E_1 = E_2$ and $F_1 = F_2$ since, assuming $e_1 \in E_1$ and $e_i \in E_2$,

$$\begin{aligned}
&\ln\frac{p_i(z)}{p_1(z)} - \ln\frac{p_i(0)}{p_1(0)} = \int_0^z \frac{d}{dt}\left(\ln\ p_i(t)/p_1(t)\right)dt \\
= &\int_0^z \left[(e_i \cdot Aq - p \cdot Aq) - (e_1 \cdot Aq - p \cdot Aq)\right]dt \\
= &\int_0^z (e_i - e_1) \cdot A \sum_{j=1}^{n} q_j(t) f_j dt \\
= &\int_0^z \sum_{f_j \in F_2} q_j(t)\left(e_i \cdot Af_j - e_1 \cdot Af_j\right)dt + \int_0^z \sum_{f_j \notin F_2} q_j(t)\left(e_i \cdot Af_j - e_1 \cdot Af_j\right)dt.
\end{aligned}$$

Since $e_i \cdot Af_j = e_1 \cdot Af_j$ for all $f_j \in F_2$ and $\int_0^\infty q_j(t)dt < \infty$ for $f_j \notin F_2$, $\lim_{z\to\infty} \ln\ p_i(z)/p_1(z)$ exists and hence $\lim_{t\to\infty} p_i(t)/p_1(t)$ exists as a positive real number, say L_i. Thus $i \in E_1$. Furthermore, $\lim_{t\to\infty}\left(\sum_{e_i \in E_1} p_i(t)\right) = 1$ and so $\lim_{t\to\infty} p_1(t) = 1/\sum_{e_i \in E_1} L_i$. That is, $p_1(t)$ converges as does $p_i(t)$ (to $L_i/\sum_{e_i \in E_1} L_i$ for $e_i \in E_1$ and, by definition, to 0 for $e_i \notin E_1$). Thus, in this extreme case, we have convergence of the trajectory to a Nash Equilibrium. Appendix A generalizes this method to Γ' that have some decision points with several choices.

Theorem 2 *Generically, every interior trajectory converges to a Nash equilibrium.*

Proof. (in Appendix A)

3 Stability

Now that we know each interior trajectory converges to a Nash equilibrium, this section and the following two examine which Nash equilibrium path is expected in the limit and, in particular, if trajectories that start near a Nash equilibrium path will stay near this path or at least converge to it. The set G^* in Example 1 and in Example 2 exhibit such stability properties as suggested in Figures 2 and 3; whereas G in Example 2 does not.

Definition 3 *Let G be a closed set of rest points for (1). G is called (interior) stable if, for every neighborhood O of G, there is another neighborhood U of G with $U \subset O$ and the trajectory of any initial point in U intersected with (the interior of) $\Delta^m \times \Delta^n$ remains inside O.*

In the context of continuous selection dynamics (e.g., the replicator dynamic), the notion of interior stability is equivalent to the concept of stability. This is due to the continuity of trajectories with respect to initial states. Interior stability was introduced in order to be consistent with Definition 4 below but will be dropped for the remainder of the paper.

Definition 3 can also be applied to single points or profiles (i.e., take G to be a singleton set). In the dynamical systems literature, a stable point is also called Lyapunov stable. For any monotone selection dynamic, if $G = \{(p^*, q^*)\}$, then the concept of stability is a refinement of the Nash equilibrium (Samuelson and Zhang (1992)). That is, (p^*, q^*) is a Nash equilibrium if it is stable.

The following theorem shows the connection between Nash equilibria and stable outcomes.

Theorem 3 *Generically, a pure strategy profile is a Nash equilibrium if and only if it is stable.*

Proof. The “if ” statement is due to Samuelson and Zhang (1992). For the other implication, let (e_1, f_1) be a pure strategy Nash equilibrium and let e_2 (f_2) be a pure best response for player one (two). By our generic assumption, e_2 can only differ from e_1 and f_2 from f_1 at nodes of Γ that are not reached on the equilibrium path. Hence $E_k(e_1, f_1) = E_k(e_2, f_1) = E_k(e_1, f_2) = E_k(e_2, f_2)$ for $k = 1, 2$ where $E_k(e_i, f_j)$ is the payoff to player k when e_i plays against f_j. Let E be the set of all points $(p, q) \in \Delta^n \times \Delta^m$ such that the support of p is contained in the best replies to f_1 and similarly for q (i.e., $\text{supp}(p) = \{i | p_i \neq 0\} \subseteq \{i | e_i \text{ is a best reply to} f_1\}$). Then E is a set of rest points for (1) by the above payoff equalities. In fact, E is the centre manifold for the dynamic at (e_1, f_1). By Appendix B, (e_1, f_1) is stable. ■

Example 3 *Take $\alpha = \beta = \gamma = b = 1$ and $a = c = 2$ in Figure 1. These parameter values do not satisfy our generic assumption. The subgame perfect equilibrium (B, R) is stable whereas the other Nash equilibrium (T, L) is not. This example shows that Theorem 3 is not true for all Γ.*

4 Interior Attracting Sets

Example 1 is a simple example of a game with a unique Nash equilibrium path. The only equilibrium outcome has player one choosing T (see Figure 2) . By Theorem 2, for a generic set of such Γ with a unique NE path, every initial point in the interior of $\Delta^m \times \Delta^n$ converges to a Nash equilibrium (p^*, q^*) of the game's normal form that follows this equilibrium path. That is, the set of Nash equilibria (i.e. the unique Nash equilibrium component) is globally interior attracting according to the following definition. [Definitions 3 and 4 follow Bhatia and Szegö (1970), applications can be found in Schlag (1993) and Binmore and Samuelson (1995).]

Definition 4 *Let G be a closed set of rest points in $\Delta^m \times \Delta^n$. Then*

(i) G is called (interior) attracting if G is contained in an open set O such that every initial point in O and also in (the interior of) $\Delta^m \times \Delta^n$ evolves under (1) to G (i.e. all ω limits are in G). G is globally (interior) attracting if all ω limits of trajectories starting in (the interior of) $\Delta^m \times \Delta^n$ are in G.

(ii) G is called (interior) asymptotically stable if it is (interior) attracting and stable. It is globally (interior) asymptotically stable if it is globally (interior) attracting and stable.

Similar to the discussion following Definition 3, Definition 4 can also be applied to single points or profiles (i.e., take G to be a singleton set). Following Samuelson and Zhang (1992), the concept of interior attracting is a refinement of the Nash equilibrium. That is, if (p^*, q^*) is interior attracting then it must be a Nash equilibrium.

Asymptotic stability is the strongest of the dynamic stability concepts introduced and exhibits a robustness that the others do not. For instance, if G is interior asymptotically stable, then any trajectory that starts near G will stay near and eventually converge to G. For this reason asymptotic stability is often used as an equilibrium selection technique to eliminate Nash equilibria in normal form games.

Notice that it is true in general that the entire space $\Delta^m \times \Delta^n$ is (interior) attracting if it were not for our requirement that G consist entirely of rest points. In our case, due to Theorem 2, we obtain the following corollary.

Corollary 1 *Generically, the set of all Nash equilibria is globally interior attracting.*

In the following, we will be interested in refinements of the concept of interior attracting. From Corollary 1, a Nash equilibrium component G will be interior asymptotically stable if each point in G is stable. This observation will be exploited repeatedly below to select Nash equilibria on dynamic grounds. First, we summarize attracting sets in Examples 1 and 2.

Although $G^* = \Big\{(T, (1-\lambda)L + \lambda R) \mid 0 \leq \lambda \leq \frac{1}{a}\Big\}$ is not attracting, it is globally interior attracting in Example 1. In Example 2, G is not interior attracting since there are initial points arbitrarily close to $\Big(T, (1-\frac{1}{a})L + \frac{1}{a}R\Big)$ that evolve to (B, R). Figure 3 shows $\{(B, R)\}$ is interior attracting in this case and is the only connected interior attracting set. In this example we see a close connection between interior attracting sets and the subgame perfect equilibrium. In the next theorem part of this relationship is revealed.

Theorem 4 *Generically, any interior attracting set contains the Nash equilibrium* (e_B, f_B) *that results from the backwards induction procedure. Moreover, any interior attracting set* G *that is connected must contain the Nash equilibrium component,* G^**, of* (e_B, f_B) *and, if* G *is (interior) asymptotically stable, then* $G = G^*$*.*

Example 1 has many connected interior attracting sets (e.g., $\{(T, (1-\lambda)L + \lambda R) \mid 0 \leq \lambda \leq 1\}$), all containing G^*, the Nash equilibrium component containing the subgame perfect equilibrium. However, G^* is the only interior asymptotically stable set.

In order to prove the above theorem we will need the following lemma.

Lemma 1 *Let* G *be a Nash equilibrium component of* Γ *and let* E *be a connected set of rest points that contains* G*. Then, generically, all strategies in* E *induce the same outcome path.*

Proof. (in Appendix A)

We know come to the proof of Theorem 4.

Proof. Let G be an interior attracting set that does not contain the subgame perfect equilibrium (e_B, f_B). Following Theorem 2, G contains a Nash equilibrium, say (p, q). We will first show that the Nash equilibrium component G' containing (p, q) is contained in G. Since all interior Nash equilibria (which are rest points) close to G must be in G by the assumption of interior attraction, G contains the relative interior of G'. Moreover, the fact that G is closed implies that $G' \subseteq G$.

Following Lemma 1, each element of G' must generate the same equilibrium path. W.l.o.g. assume that (e_1, f_1) is the pure strategy profile generating this path. It follows that (e_1, f_1) is a Nash equilibrium and that $(e_1, f_1) \neq (e_B, f_B)$.

Let α be the *last* decision node on the path generated by (e_1, f_1) at which the player making his decision does not follow the subgame perfect equilibrium path. Assume that this player is player one, that the decision made at α is L, and that the action taken by the subgame perfect equilibrium at α is R. Let Γ' denote the subgame that is played after node α is reached and player one plays R. Let e_2 be the strategy of player one that is identical to e_1 except that at α he plays R and on any path in Γ' he follows e_B. Let f_2 be the strategy of player two except that f_2 follows f_B on any path in Γ'. Since (e_1, f_1) is a Nash equilibrium and the payoffs are not degenerate, $E_1(e_1, f_1) > E_1(e_2, f_1)$. By the construction of α, R, e_2 and f_2; $E_1(e_1, f_2) < E_1(e_2, f_2)$. Since nodes in Γ' are not reached when player one plays L at α, $E_k(e_1, f_1) = E_k(e_1, f_2)$, $k = 1, 2$. Finally, since both players play a Nash equilibrium given they enter G, we obtain that (e_2, f_2) is a Nash equilibrium of the entire game. In particular, from genericity, it follows that $E_2(e_2, f_1) < E_2(e_2, f_2)$.

Let $\lambda_0 = \frac{E_1(e_2,f_2) - E_1(e_1,f_1)}{E_1(e_2,f_2) - E_1(e_2,f_1)}$ and $BR_1(q)$ be the best responses of player one to q. Then $BR_1(f_1) \subseteq BR_1((1-\lambda) f_1 + \lambda f_2)$ if $\lambda \leq \lambda_0$ and hence $\{(e_1, (1-\lambda) f_1 + \lambda f_2), 0 \leq \lambda \leq \lambda_0\}$ is a set of Nash equilibria. Moreover, $e_2 \in BR_1((1-\lambda_0) f_1 + \lambda_0 f_2)$. Consider trajectories on the face $\{e_1, e_2\} \times \{f_1, f_2\}$. The structure of the game on this face is that of the Ultimatum Mini Game (see Example 2). In particular, there are trajectories starting close to $(e_1, (1-\lambda_0) f_1 + \lambda_0 f_2)$ that converge to (e_2, f_2). Hence there are interior trajectories starting close to $(e_1, (1-\lambda_0) f_1 + \lambda_0 f_2)$ that come arbitrarily close to (e_2, f_2). Since (e_2, f_2) is a pure-strategy Nash equilibrium and hence stable by Theorem 3, these trajectories converge to a point (p, q) near (e_2, f_2). Since (e_2, f_2) is a limit point of these (p, q) and since G is closed and limit attracting, $(e_2, f_2) \in G$. Repeat this backwards induction argument for (e_2, f_2) in place of (e_1, f_1) and we eventually obtain the result $(e_B, f_B) \in G$.

We will now show that the Nash equilibrium component containing (e_B, f_B), G^*, is the only possible interior asymptotically stable set. Let G be an interior asymptotically stable set. From the proof so far, $(e_B, f_B) \in G$. It follows from the arguments made at the beginning of this proof that $G^* \subseteq G$. We will now show that $G = G^*$. By definition of G^* there exists an open neighborhood U of G^* such that the only Nash equilibria contained in U are in G^*. Assume that $(p, q) \in (U \cap G) \backslash G^*$. Since (p, q) is not a Nash equilibrium, w.l.o.g. let $e_i \in \Delta^m$ such that $e_i \cdot Aq > p \cdot Aq$. Since $(p, q) \in G$ and is hence a rest point, it follows that $p_i = 0$. So there exists an open neighborhood V of (p, q) such that $V \cap G \subseteq \{p_i = 0\}$. However, the fact that trajectories starting close to (p, q) must stay close to $V \cap G$ contradicts the fact that $e_i \cdot Aq > p \cdot Aq$ and hence $(U \cap G) \backslash G^* = \emptyset$. ■

Given the result of Theorem 4, it may be conjectured that every subgame perfect equilibrium is contained in a connected interior attracting set. The following sections derive conditions on the game for this to be true. It should be noted at this point that the conjecture is not true in general (see Example 5).

5 Dynamic Stability in Simple Games

Remark 1 summarizes the dynamic stability properties of the Nash equilibrium sets for Examples 1 and 2.

Remark 1 *(i) For Example 1 (i.e. with $b > c$), the set G^* containing the subgame perfect equilibrium (T, L) is globally interior asymptotically stable. Each point of G^* is stable and these are the only stable points.*

(ii) For Example 2 (i.e. with $b < c$) , the subgame perfect equilibrium (B, R) is asymptotically stable. $\{(B, R)\}$ is the unique connected interior attracting set. Each Nash equilibrium point other than $\left(T, (1 - \frac{1}{a})L + \frac{1}{a}R\right)$ is stable.

Proof. The only aspect of Remark 1 that is more than a straightforward calculation is to show the stability of $\left(T, (1 - \frac{1}{a})L + \frac{1}{a}R\right)$ when $b > c$. Using the fact that all trajectories converge to G, it is enough to show that the frequency of B can not grow substantially relative to the rate at which L grows when starting sufficiently close to $\left(T, (1 - \frac{1}{a})L + \frac{1}{a}R\right)$. From equation (1),

$$\frac{\dot{p}_B}{\dot{q}_L} = \frac{a p_T \left(q_R - \frac{1}{a}\right)}{q_L q_R (b - c)} .$$

That is, $\frac{\dot{p}_B}{\dot{q}_L}$ is small when p_B is small and $q_R \approx \frac{1}{a}$. ■

We now consider extensive form games in which at most three consecutive decisions are made, referred to as games of length ≤ 3.

The generic example in which each path has exactly length 3 and players move alternately (see also Remark 2 below) is the game in Figure 4 where the dashed lines indicate the subgame perfect choice at each decision point. Let $[c_1; c_2, c_3, c_4, c_5]$, where c_i is either l or r, denote the pure strategy of player one that chooses c_1 at the first decision point (stage one) and c_2, c_3, c_4, c_5 at the four decision points of the third stage ordered from left to right. Similarly $[C_1, C_2]$, where C_i is L or R, gives the strategy of player two at the two decision points for the second stage. Some payoffs have been normalized (as was done in Examples 1 and 2) so that the unique subgame perfect equilibrium is $([l; l, l, l, l], [L, L])$. That is, we assume $a < 1$, $z < x$, $c < 0$, $e < v$, $y < 1$, $w < u$.

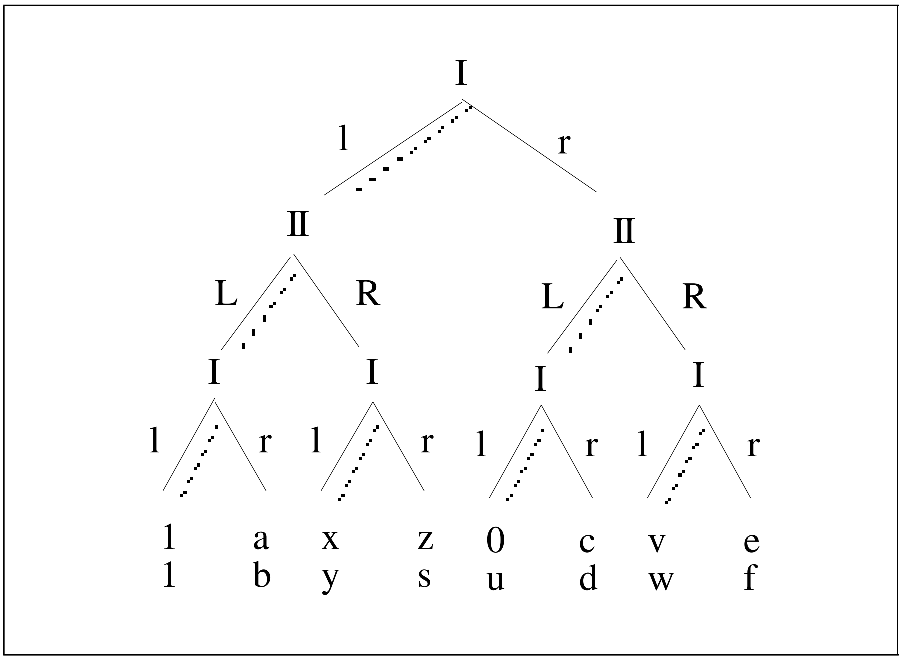

Figure 4: An extensive-form game of length 3.

The normal form of the game (actually its semi-reduced normal form (van Damme, 1991) that equates, for example, all duplicate strategies of the form $[l; l, l, c_4, c_5]$ and denotes them by $[l; l, l, *, *]$) is

	$[L,L]$	$[L,R]$	$[R,L]$	$[R,R]$
$[l; l, l, *, *]$	$1,1$	$1,1$	x,y	x,y
$[l; l, r, *, *]$	$1,1$	$1,1$	z,s	z,s
$[l; r, l, *, *]$	a,b	a,b	x,y	x,y
$[l; r, r, *, *]$	a,b	a,b	z,s	z,s
$[r; *, *, l, l]$	$0,u$	v,w	$0,u$	v,w
$[r; *, *, l, r]$	$0,u$	e,f	$0,u$	e,f
$[r; *, *, r, l]$	c,d	v,w	c,d	v,w
$[r; *, *, r, r]$	c,d	e,f	c,d	e,f

. (3)

This reduction does not change the dynamic stability analysis of the game (Cressman, 1993a). Specifically, if G^* is the Nash equilibrium component containing $[l; l, l, *, *]$, then G^* is interior asymptotically stable if and only if the Nash equilibrium component containing $[l; l, l, l, l]$ is interior asymptotically stable in the non-reduced normal form of Figure 4. It is straightforward to show

$$G^* = \left\{ (p,q) \in \Delta^8 \times \Delta^4 | p_2 \leq \frac{1-y}{s-y}, q_2 \leq \frac{1}{v}, p_1 + p_2 = q_1 + q_2 = 1 \right\}.$$

Theorem 5 *For generic games in Figure 4, G^* is the unique interior asymptotically stable set.*

Proof. Uniqueness follows from Theorem 4. The proof is complete if we show each point in G^* is stable. For this, we can eliminate all strategies that are not a best response for one of the players at some point in G^*. That is, the centre manifold at each point in G^* is contained in $\{(p,q) \in \Delta^8 \times \Delta^4 | p_1 + p_2 + p_5 = 1\}$. From (3), the corresponding normal form is

	f_1	f_2	f_3	f_4
e_1	$1,1$	$1,1$	x,y	x,y
e_2	$1,1$	$1,1$	z,s	z,s
e_3	$0,u$	v,w	$0,u$	v,w

(4)

where $z < x$, $y < 1$, $w < u$ and, generically, the sets $\{0, 1, x, z, v\}$ and $\{1, y, s, u, w\}$ both contain five distinct payoffs. The proof that each point of G^* is stable in $\Delta^3 \times \Delta^4$ for the normal form game of Figure 4 is quite technical and given in Appendix A. ■

Notice that the proof Theorem 5 shows that any element of G^* is stable, not only the entire set as required by Definition 3.

Remark 2 *Of course the extensive form game in Figure 4 is not the only one of length 3. However, it is easy to show that the general proof of Theorem 5 for any extensive form game of length at most three can be reduced to the above statement. Multiple decisions reduce to at most two decisions using center manifold theory. Similarly, one player moving twice in a row is equivalent to making one decision node where the player has multiple decisions. In this respect, it does not seem that the statement of Theorem 5 relies on the total length of the tree but on the number of decisions. This remark points to the fact that some slightly more general statement should hold. We conjecture that the interior asymptotic stability still holds as long as no player moves twice off the backwards induction equilibrium path. This will be a topic of future research.*

6 Counterexamples

Examples 4 and 5 in this section are both based on Centipede Games, the latter in combination with an Ultimatum Mini-Game. The Centipede Game

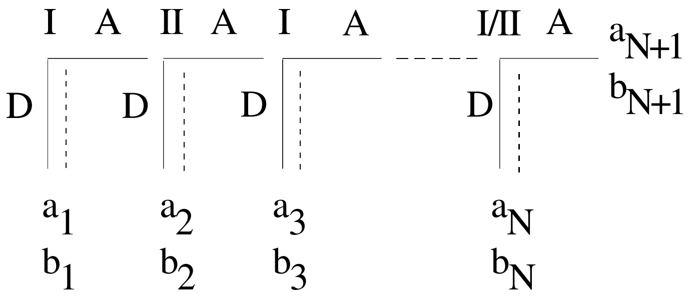

Figure 5: The Centipede Game of length N.

of length N is a game with N decision points that alternate between the two players and each player has two choices at their decision points as in Figure 5. Example 1 is the case $N = 2$. The payoffs of player one satisfy $a_{2i-1} > a_{2i}$ for $1 \leq i \leq N + 1$ and those of player two satisfy $b_{2j} > b_{2j+1}$ for $1 \leq j \leq N$.

The unique subgame perfect equilibrium is for both players to choose D at all their decision points. The only Nash equilibrium outcome has player one choosing D at the first decision point and its corresponding connected Nash equilibrium component, G^*, contains all Nash equilibria. Theorem 2 does not apply directly unless all a_i's are different and all b_j's are different. However, further analysis of the proof of Theorem 2 confirms G^* is globally interior attracting. The extra details are relatively easy and so omitted here. Furthermore, by Theorem 5 and Remark 1, G^* is globally interior asymptotically stable if $N = 2$ or $N = 3$.

Our first counterexample shows Theorem 5 need not be true for games where there is a path on which one player moves twice off the backwards induction equilibrium path.

Example 4 *A Centipede Game of length* 4. *Let* $N = 4$, $a_1 = 1$, $b_2 = 2$, $a_3 = 4$, $b_4 = 8$, $a_5 = b_5 = 6$ *and* $a_2 = a_4 = b_1 = b_3 = 0$. *The corresponding normal form is*

	$[D,*]$	$[A,D]$	$[A,A]$
$[D,*]$	$1,0$	$1,0$	$1,0$
$[A,D]$	$0,2$	$4,0$	$4,0$
$[A,A]$	$0,2$	$0,8$	$6,6$

.

In the following we will show that the unique Nash equilibrium component *(which of course contains the subgame perfect equilibrium) is interior attracting but not stable.*

Interior attracting follows directly from Corollary 1. Notice that $\{([D,*],(1-\lambda)[D,*]+\lambda[A,A])\}$ *is a Nash equilibrium if and only if* $\lambda \leq \frac{1}{6}$*. Moreover,* $AA \in BR_1\left(\frac{5}{6}[D,*]+\frac{1}{6}[A,A]\right)$*, in this case* $[A,A]$ *is called a toughest entrant (see Schlag (1993)). Restricting the game to the face* $\{[D,*],[A,A]\} \times \{[D,*],[A,A]\}$*, we have the Ultimatum Mini-Game of Example 2. It follows that trajectories starting close to* $\left([D,*],\frac{5}{6}[D,*]+\frac{1}{6}[A,A]\right)$ *in* $\{[D,*],[A,A]\} \times \{[D,*],[A,A]\}$ *lead to* $([A,A],[A,A])$*. Continuity of the trajectories with respect to their initial conditions implies that the Nash equilibrium component is not stable. Notice that the argument went through because* $a_5 > a_3$ *and* $b_5 > b_2$*. The result is robust to any Centipede Game of length* 4 *with these payoffs.*

What is the intuition behind this counterexample? In the Centipede Game of length 3*, starting in the subgame perfect equilibrium, mutations* A *in the strategy of player two can enter unpunished (i.e., without changing the best reply for player one and hence not changing the outcome). Once sufficiently many of player two play* A*, the best response of player one is to play* $[A,D]$*. However, this gives an incentive to player two to go back to playing* D*. Thus, the proportion of* A *among player two is held bounded. This is different in the Centipede game of length* 4*. Initially, the strategy* $[A,A]$ *of player two can enter unpunished. Once sufficiently many are in, the best response for player one is to play* $[A,A]$ *which gives more incentive for player two to play* $[A,A]$*. Thus, the entry of* $[A,A]$ *in population two cannot be prevented in the short run and the Nash equilibrium component fails to be stable.*

Given the above example one still might conjecture that the subgame perfect equilibrium is always contained in a connected interior attracting set. This is, for example, true in any generic game (e.g. the Centipede Game of length N) that has a unique Nash equilibrium outcome. The following example shows that the conjecture is not necessarily true in general when there are multiple Nash equilibrium outcomes in the game.

Example 5 *Consider the extensive form game in Figure 6.*

This game is a Centipede Game of length 3 *with an Ultimatum Mini-Game attached at the third decision node. In the following we will show that this game has no connected interior attracting set.*

The unique subgame perfect equilibrium in pure strategies is $([D,A],[D,A])$*. An additional Nash equilibrium in pure strategies is* $([A,D],[A,D])$*, which by Theorem 3 is Lyapunov stable. Testing for the stability of* $([D,A],[D,A])$ *against mutations of strategy* $[A,D]$ *in the population of player two, we find*

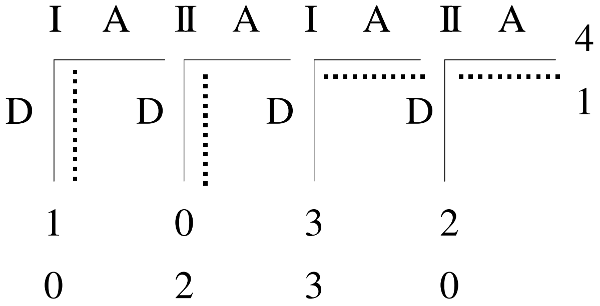

Figure 6: An extensive form game with the same structure as a Centipede Game.

$([D,A], (1-\lambda)[D,A] + \lambda [A,D])$ *to be a Nash equilibrium if and only if* $\lambda \leq \frac{1}{3}$. *In this context, the strategy of player one* $[A,D]$ *is called the toughest entrant. Considering the dynamics at* $\lambda = \frac{1}{3}$, *the centre manifold is the face* $M = \{[D,*], [A,D]\} \times \{[D,*], [A,D]\}$. *The corresponding normal form on this face is*

	$[D,*]$	$[A,D]$
$[D,*]$	$1,0$	$1,0$
$[A,D]$	$0,2$	$3,3$

.

This is the Ultimatum Mini-Game and hence there are trajectories starting close to $\left([D,A], \frac{2}{3}[D,A] + \frac{1}{3}[A,D]\right)$ *in* M *that converge to* $([A,D],[A,D])$. *Consequently, continuity of the paths of the trajectories with respect to initial starting conditions implies that there are trajectories starting in the interior of the entire space but sufficiently close to* $\left([D,A], \frac{2}{3}[D,A] + \frac{1}{3}[A,D]\right)$ *that lead arbitrarily close to* $([A,D],[A,D])$. *Since* $([A,D],[A,D])$ *is stable, trajectories going sufficiently close to* $([A,D],[A,D])$ *will converge to the Nash equilibrium component containing* $([A,D],[A,D])$. *Especially we have found trajectories that never return to the Nash equilibrium component containing the subgame perfect equilibrium. By Theorem 4, this is the only candidate for such a set and so we have the non existence of a connected interior attracting set or of an interior asymptotically stable set in this example.*

7 Discussion

For two-person extensive form games, Γ, with perfect information and no indifferences between paths, the unique subgame perfect Nash equilibrium given through the backwards induction procedure is often claimed to be the game's solution. Our paper examined the evolutionary basis for this claim. That is, can the subgame perfect Nash equilibrium (or, more specifically, the Nash equilibrium component, G^*, corresponding to this equilibrium outcome) be distinguished from other Nash equilibria on dynamic grounds? From this perspective, some of our results support this hypothesis while others do not. It is this ambiguity that is reflected in the title for our paper.

Let us say a Nash equilibrium "solves" Γ if there is a corresponding Nash equilibrium component that is either interior attracting and/or interior asymptotically stable. On the positive side, from Theorems 4 and 5, it appears that these dynamic stability properties can be used to distinguish G^*. To repeat their conclusions, Theorem 4 states that G^* is the only possible solution to Γ while Theorem 5 asserts that G^* is in fact the solution if the extensive form of the game is sufficiently simple. In particular, the Paradox of the Chain-Store Game is consistent with evolutionary arguments.

On the negative side, Examples 4 and 5 show G^* need not solve the game if Γ is sufficiently complex. Furthermore, Theorem 3 shows that one cannot distinguish the subgame perfect Nash equilibrium point by appealing to its (Lyapunov) stability since all other pure-strategy Nash equilibria satisfy the same property. On the other hand, Theorem 2 can be interpreted as saying that dynamic evolutionary game theory is consistent with static solutions of noncooperative game theory in the sense that every interior trajectory of (1) converges to some Nash equilibrium in the normal from of Γ. Most applications of dynamic evolutionary game theory to general noncooperative games are concerned with what happens near a Nash equilibrium (or set of Nash equilibria) with no expectation of this general convergence result.

As final comment, it is worth pointing out that the proofs of the two main results, Theorems 2 and 5, involve techniques that rely on original approaches developed independently by the respective coauthors who were both analyzing symmetric repeated Prisoners' Dilemma Games (Cressman, 1993b; Schlag, 1993). We feel this close connection to Prisoners' Dilemma Games in both theorems is more than a coincidence. It points to a fundamental relationship between the dynamics of extensive form games with perfect information and that of Prisoners' Dilemma Games.

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A The Proofs

Proof. (of Theorem 1) The proof is by induction on k. Let $e_i \in E_2$. For $k = 1$, a calculation similar to the one after the statement of Theorem 1 shows

$$\begin{aligned}\ln \frac{p_i^1(z)}{p_i(z)} - \ln \frac{p_i^1(0)}{p_i(0)} &= \int_0^z \sum_{e_j \in F_2} q_j(t)\,(e_i^1 \cdot Af_j - e_i \cdot Af_j)dt \\ &\quad + \int_0^z \sum_{e_j \notin F_2} q_j(t)(e_i^1 \cdot Af_j - e_i \cdot Af_j)dt. \qquad (5)\end{aligned}$$

Since u_1 is a final decision node, e_i^1 weakly dominates e_i, which means that $e_i^1 \cdot Af_j \geq e_i \cdot Af_j$ for any f_j. Hence, $\lim\limits_{z\to\infty} \ln p_i^1(z)/p_i(z)$ exists as a nonnegative extended real number and $\lim\limits_{t\to\infty} p_i^1(t)/p_i(t) = L^1$ where L^1 is a positive real number or ∞. If L^1 is finite, then $e_i^1 \cdot Af_j = e_i \cdot Af_j$ for all $f_j \in F_2$. In either case, $\int_0^\infty p_i^1(t)dt = \int_0^\infty \frac{p_i^1(t)}{p_i(t)} p_i(t)dt = \infty$ since $e_i \in E_2$ and this concludes the proof of parts (i), (iii) and (v) for $k = 1$. A similar proof comparing f_j^1 to f_j completes the other parts for $k = 1$.

Assume the six statements (i) - (vi) are true for some k with $1 \leq k < N$. Let $e_i \in E_2$. To show the statement for $k+1$, we first show $\lim\limits_{t\to\infty} p_i^{k+1}(t)/p_i^k(t)$ exists as a positive extended real number L_k^{k+1}. The analogue of (5) is now

$$\int_0^z \sum_{f_j \in F_2} q_j(t)(e_i^{k+1} \cdot Af_j - e_i^k \cdot Af_j)dt + \int_0^z \sum_{f_j \notin F_2} q_j(t)(e_i^{k+1} \cdot Af_j - e_i^k \cdot Af_j)dt. \quad (6)$$

If $e_i^{k+1} \cdot Af_j \geq e_i^k \cdot Af_j$ for all $f_j \in F_2$ then L_k^{k+1} exists and is finite if and only if $e_i^{k+1} \cdot Af_j = e_i^k \cdot Af_j$ for all $f_j \in F_2$.

Suppose $e_i^{k+1} \cdot Af_{j'} < e_i^k \cdot Af_{j'}$ for some $f_{j'} \in F_2$. Then the outcome path of e_i^k matched against $f_{j'}$ passes through u_{k+1} which must be a decision node of player one, hence $e_i^{k+1} \cdot Af_{j'}^k > e_i^k \cdot Af_{j'}^k$. Moreover, $f_{j'}$ does not follow the backwards induction path on Γ' on all nodes $u_1, .., u_k$. Let $u_l \in \{u_1, .., u_k\}$ be the node with the lowest index at which $f_{j'} \neq f_{j'}^l$. Then $f_{j'}^l \cdot Be_i^k > f_{j'} \cdot Be_i^k$ by the genericity assumption. Since $e_i \in E_2$ by assumption, $e_i^k \in E_2$ by the induction assumption of part (v). Moreover, part (iv) implies that $\lim\limits_{t \to \infty} q_{j'}^l / q_{j'} = \infty$ and part (vi) implies $f_{j'}^l \in F_2$. With $r \in \{1, .., n\}$ such that $q_r = q_{j'}^l$, for large t the term $q_r(t)(e_i^{k+1} \cdot Af_r - e_i^k \cdot Af_r)$ in (6) dominates $q_{j'}(t)(e_i^{k+1} \cdot Af_{j'} - e_i^k \cdot Af_{j'})$. Since the dominating term is positive, it follows that $L_k^{k+1} = \infty$.

To summarize, we see that L_k^{k+1} exists as an extended real number and is finite if and only if $e_i^{k+1} \cdot Af_j = e_i^k \cdot Af_j$ for all $e_j \in F_2$. Since $L^{k+1} = L_k^{k+1} L^k$, parts (i), (iii) and (v) are true for $k+1$. The proof is complete by a similar method applied to parts (ii), (iv) and (vi). ■

Proof. (of Theorem 2)

Since $\lim\limits_{t \to \infty} \left(\sum\limits_{e_i \in E_1} p_i(t) \right) = 1$, $E_1 \neq \emptyset$. W.l.o.g., assume $e_1 \in E_1$. Then $e_1 \in E_2$ and so we can apply Theorem 1 to it. L^N is finite since $p_1(t)$ is infinitely often bounded away from 0. Theorem 1 (iii) implies $e_1 \cdot Af_j = e_1^N \cdot Af_j$ for all $j \in F_2$.

Repeating the above argument for $e_i \in E_1$, we obtain $e_i \cdot Af_j = e_i^N \cdot Af_j = e_1^N \cdot Af_j = e_1 \cdot Af_j$ for all $j \in F_2$. The same argument in the extreme case of Γ' having only one path given after the statement of Theorem 1 applies here to show $p_i(t)$ converges to $L_i / \sum\limits_{e_i \in E_1} L_i$ (where $L_i = \lim_{t \to \infty} p_i(t)/p_1(t)$ for $e_i \in E_1$ and to 0 if $e_i \notin E_1$). Here $L_i = \lim\limits_{t \to \infty} p_i(t)/p_1(t)$ is a finite positive number for all $e_i \in E_1$.

The proof of Theorem 2 is complete by the same method applied with E_1 and E_2 replaced by F_1 and F_2 respectively. ■

Proof. (of Lemma 1) Assume that payoffs are such that a player will never be indifferent between two actions at any node (i.e. the extensive form game is generic). We will first show that a Nash equilibrium puts probability one on a single outcome path. Let (p, q) be a Nash equilibrium and let α denote a node of the game such that

i) there are at least two decisions at α (say L and R) that are played with positive probability and

ii) the equilibrium path after passing node α is unique.

It follows that the player moving at α must be indifferent between L and R which contradicts the genericity assumption.

We will now show that a Nash equilibrium component is associated with a unique equilibrium path. Consider two Nash equilibria (p^1, q^1) and (p^2, q^2) in the same component generating two different payoffs to player k $(k \in \{1, 2\})$. Since they are in the same component, there exists a continuous function $\lambda : [0, 1] \to \Delta^m \times \Delta^n$ such that $\lambda(0) = (p^1, q^1)$, $\lambda(1) = (p^2, q^2)$ and $\lambda(x)$ is a Nash equilibrium for each $0 < x < 1$. The probability played on a particular outcome path varies continuously with λ, a contradiction of the fact that every Nash equilibrium places probability one on a single outcome path.

Moreover, if (p^2, q^2) in the above paragraph is a rest point in the same connected set of rest points as (p^1, q^1), then there exists $0 < \beta \leq 1$ such that one of the players achieves a payoff at $\lambda(\beta)$ that is not obtained on any outcome path. It follows that not both players are indifferent between all of the strategies in the support of $\lambda(\beta)$ and hence $\lambda(\beta)$ is not a rest point. ■

Proof. (of the remainder of Theorem 5) The proof is divided into cases that compare the payoffs s and v in (4) to the subgame perfect equilibrium payoff of 1 to both players. If $1 > v$ and $1 > s$, then $G^* = \{(p, q) \in \Delta^3 \times \Delta^4 | p_3 = q_3 = q_4 = 0\}$ is a strict equilibrium set (Balkenborg, 1994) and hence asymptotically stable.

If $1 < v$ and $1 > s$, the centre manifold at each point in G^* is contained in the face $\{(p, q) \in \Delta^3 \times \Delta^4 | q_3 = q_4 = 0\}$. On this face, e_1 and e_2 are duplicate strategies and the game reduces to the Centipede Game of Example 1. Each point of G^* is stable by Remark 1. Similarly, if $1 > v$ and $1 < s$, we need only consider the face $\{(p, q) \in \Delta^3 \times \Delta^4 | p_3 = 0\}$. The reduced normal form is again Example 1 with the row and column players interchanged and this implies stability.

For the remainder of the proof, assume $1 < v$ and $1 < s$. Then G^* is the rectangle $\{(p, q) \in \Delta^3 \times \Delta^4 | p_2 \leq \lambda, q_2 \leq \mu, p_1 + p_2 = 1 = q_1 + q_2\}$ where $\lambda = \frac{1-y}{s-y}$ and $\mu = \frac{1}{v}$. Note that λ and μ are both strictly between 0 and 1. If $(p, q) \in G^*$ and $p_2 < \lambda$, the centre manifold at (p, q) is contained in $\{(p, q) \in \Delta^3 \times \Delta^4 | q_3 = q_4 = 0\}$. The argument in the preceding paragraph applies and shows (p, q) is stable. Similarly, if $q_2 < \mu$, then (p, q) is stable.

All that is left to show is the stability of $(p, q) \in G^*$ where $p_2 = \lambda$ and $q_2 = \mu$. For the moment, we will ignore the existence of f_4 (i.e., assume $q_4 = 0$) and identify the point (p, q) in the interior of $\Delta^3 \times \Delta^3$ with the point $\left(\frac{p_2}{p_1+p_2}, p_3, \frac{q_2}{q_1+q_2}, q_3\right)$. We will show the stability of $(\lambda, 0, \mu, 0)$ in this new coordinate system. A convenient norm for these coordinates is the maximum norm, namely, $\|u - v\| = \max\{|u_i - v_i| \, |1 \leq i \leq 4\}$. We will show that, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\|p(t) - (\lambda, 0, \mu, 0)\| < \varepsilon$ for all $t \geq 0$ whenever $\|p(0) - (\lambda, 0, \mu, 0)\| < \delta$. Let $U_\beta(M) = \{u | \, \|u - v\| < \beta \text{ for some } \beta \in M\}$.

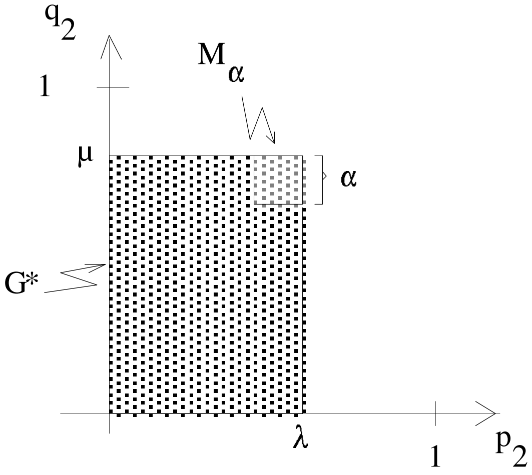

Figure 7: A Diagram of M_α.

e may assume $\varepsilon < \frac{1}{2} \min \{\lambda, \mu\}$. For each $\alpha \in (0, \frac{\varepsilon}{2}]$ and $\eta \in [\mu - \alpha, \mu]$, $(\lambda - \alpha, 0, \eta, 0)$ is stable and so there exists $\delta_0 > 0$ such that trajectories that start in $U_{\delta_0} (\lambda - \alpha, 0, \eta, 0)$, stay in $U_{\frac{\varepsilon}{2}} (\lambda - \alpha, 0, \eta, 0)$. By the compactness of $M'_\alpha = \{(\lambda - \alpha, 0, \eta, 0) \,|\eta \in [\mu - \alpha, \mu]\}$, there exists a $\delta_1 > 0$ such that trajectories starting in $U_{\delta_1} (M'_\alpha)$ stay in $U_{\frac{\varepsilon}{2}} (M'_\alpha)$. By symmetry, the same is true for $M''_\alpha = \{(\eta, 0, \mu - \alpha, 0) \,|\eta \in [\lambda - \alpha, \lambda]\}$. By construction, for each $0 < \alpha \leq \frac{\varepsilon}{2}$, $M_\alpha = M'_\alpha \cup M''_\alpha \subseteq U_\varepsilon (\lambda, 0, \mu, 0)$ is the subset of G^* in Figure 7.

e will show that trajectories that start sufficiently close to $(\lambda, 0, \mu, 0)$ either stay close to $(\lambda, 0, \mu, 0)$ or enter M_α for α sufficiently small.

From (4), e_1 (respectively, f_1) weakly dominates e_2 (respectively, f_2). Thus $\frac{p_2}{p_1+p_2}$ and $\frac{q_2}{q_1+q_2}$ are decreasing function in the interior of $\Delta^3 \times \Delta^3$. Specifically, from (1),

$$\frac{d}{dt}\left(\frac{p_2}{p_1+p_2}\right) = \frac{p_1\dot{p}_2 - \dot{p}_1 p_2}{(p_1+p_2)} = \frac{q_3 (z-x)(1-p_2-p_3)\, p_2}{(1-p_3)^2}$$

$$\frac{d}{dt}\left(\frac{q_2}{q_1+q_2}\right) = \frac{p_3 (w-u)(1-q_2-q_3)\, q_2}{(1-q_3)^2}.$$

The idea for the remainder of the proof is to show that p_3 and q_3 increase slowly (if at all) as $\frac{p_2}{1-p_3}$ and $\frac{q_2}{1-q_3}$ are decreasing, thereby forcing the trajectories either into some M_α or to stay close to $(\lambda, 0, \mu, 0)$. For this, consider the expression $h(\tau) = \frac{p_2}{1-p_3} + \frac{q_2}{1-q_3} + \tau(p_3 + q_3)$ for a fixed positive τ. A lengthy calculation yields the time derivative $\dot{h}(\tau) = p_3 I_1(\tau) + q_3 I_2(\tau) + p_3 q_3 \tau(c_2 p_2 + c_3 p_3 + d_2 q_2 + d_3 q_3)$ for some constants c_i and d_i where

$$\begin{aligned} I_1(\tau) &= \frac{(w-u)(1-q_2-q_3)q_2}{(1-q_3)^2} - \tau v(\mu - q_2)(1-p_3) \\ I_2(\tau) &= \frac{(z-x)(1-p_2-p_3)p_2}{(1-p_3)^2} + \tau(y-s)(\lambda - p_2)(1-q_3). \end{aligned}$$

For $\left(\frac{p_2}{1-p_3}, p_3\right)$ close to $(\lambda, 0)$ and $\left(\frac{q_2}{1-q_3}, q_3\right)$ close to $(\mu, 0)$, $I_1(\tau)$ and $I_2(\tau)$ are strictly negative.

In fact, it is clear that there exists an $\alpha > 0$ such that, for every $\tau > 0$, there exists $k > 0$ with $\dot{h}(\tau) \le 0$ whenever $\left(\frac{p_2}{1-p_3}, p_3\right)$ and $\left(\frac{q_2}{1-q_3}, q_3\right)$ satisfy

$$\frac{\lambda}{2} \le \frac{p_2}{1-p_3} < \lambda + k, \frac{\mu}{2} \le \frac{q_2}{1-q_3} < \mu + k, 0 \le p_3, q_3 \le \frac{3\alpha}{\tau}. \tag{7}$$

ithout loss of generality, we may assume $\alpha < \frac{\varepsilon}{2}$. Let δ_1 be the corresponding neighborhood of M_α in Figure 7.

Given $0 < \varepsilon < \frac{1}{2}\min\{\lambda, \mu\}$, we are now able to describe an appropriate δ-neighborhood of $(\lambda, 0, \mu, 0)$. Fix $\tau = \frac{3\alpha}{\delta_1}$, take $k > 0$ satisfying (7) and let $\delta_2 = \min\{k, \delta_1\}$. Then

$$C = \left\{ \begin{array}{c} (p, q) \in \Delta^3 \times \Delta^3 | h(\tau) < \lambda + \mu + \alpha, \\ -\alpha < \frac{p_2}{1-p_3} - \lambda < \delta_2, -\alpha < \frac{q_2}{1-q_3} - \mu < \delta_2 \end{array} \right\}$$

is an open neighborhood of $(\lambda, 0, \mu, 0)$ contained in $U_\varepsilon(\lambda, 0, \mu, 0)$ since $\frac{p_2}{1-p_3} > \lambda - \alpha$ and $\frac{q_2}{1-q_3} > \mu - \alpha$ imply $(p_3 + q_3)\tau < 3\alpha$ and $p_3 + q_3 < \delta_1$. Moreover, from (7), $\dot{h}(\tau) \le 0$ for all points in C. Since $\frac{p_2}{1-p_3}$ and $\frac{q_2}{1-q_3}$ are decreasing functions, trajectories that start in C can only leave C if, at some point on the trajectory, either $\frac{p_2}{1-p_3} = \lambda - \alpha$ or $\frac{q_2}{1-q_3} = \mu - \alpha$. However, if $\frac{p_2}{1-p_3} = \lambda - \alpha$ say, then $(p, q) \in M_\alpha$ since $\frac{q_2}{1-q_3} < \mu + \delta_2 \le \mu + \delta_1$. Thus, trajectories that start in C either remain in C or enter M_α. In either case, the trajectory $(p(t), q(t))$ satisfies $\left\| \left(\frac{p_2(t)}{1-p_3(t)}, p_3(t), \frac{q_2(t)}{1-q_3(t)}, q_3(t) \right) - (\lambda, 0, \mu, 0) \right\| < \varepsilon$ for all $t \ge 0$. The proof is completed by choosing a δ-neighborhood of $(\lambda, 0, \mu, 0)$ that is contained in C.

The above method can be adapted to the case where $q_4 \neq 0$. A straightforward (though lengthy) procedure shows that the suitably modified $h(\tau)$ has time derivative as before except that $I_1(\tau)$ and $I_2(\tau)$ become even more negative near $(p, q) \in G^*$ with $p_2 = \lambda$ and $q_2 = \mu$. This again forces the trajectory towards some M_α. ■

B Linearization and Centre Manifolds for the Replicator Dynamic

Suppose (p^*, q^*) is a rest point of the replicator dynamic (1). That is, $e_i \cdot Aq^* = p^* \cdot Aq^*$ for all i in the suport of p^* and $f_j \cdot Bp^* = q^* \cdot Bp^*$ for all j in the support of q^*. To linearize (1) about (p^*, q^*), let $p_i = p_i^* + x_i$ and $q_j = q_j^* + y_j$. The linear terms of $\dot{x}_i = \dot{p}_i$ and $\dot{y}_j = \dot{q}_j$ are

$$\begin{aligned} \dot{x}_i &= p_i^* \left[(e_i - p^*) \cdot Ay - x \cdot Aq^* \right] + x_i \left(e_i - p^* \right) \cdot Aq^* \\ \dot{y}_j &= q_j^* \left[(f_j - q^*) \cdot Bx - y \cdot Bp^* \right] + y_j \left(f_j - q^* \right) \cdot Bp^*. \end{aligned}$$

This can be written in the form $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = L \begin{bmatrix} x \\ y \end{bmatrix}$ where L is a square matrix of order $m + n$. Since $\Delta^m \times \Delta^n$ is invariant under (1), $X^m \times X^n$ is invariant under L where $X^k = \left\{ z \in \mathbb{R}^k \middle| \sum_{l=1}^{k} z_l = 0 \right\}$. Furthermore, $(p^*, 0)$ and $(0, q^*)$ are eigenvectors of L with corresponding eigenvalues $-p^* \cdot Aq^*$ and $-q^* \cdot Bp^*$ respectively.

For stability of (p^*, q^*), we are interested in the eigenvalues of L restricted to the $n + m - 2$ dimensional subspace $X^m \times X^n$ (i.e., the other eigenvalues of L). For instance, if (p^*, q^*) is a pure-strategy pair, say (e_1, f_1), these $n + m - 2$ eigenvalues are the diagonal entries $(e_i - e_1) \cdot Af_1$ and $(f_j - f_1) \cdot Be_1$ for $2 \leq i \leq m$ and $2 \leq j \leq n$. In particular, if (e_1, f_1) is a strict Nash equilibrium these diagonal entries are all negative and so (e_1, f_1) is asymptotically stable. This is a well-known result as is the fact that (e_1, f_1) is not Lyapunov stable if it is not a Nash equilibrium (from our perspective, the reason is that at least one eigenvalue is then positive).

Of course more interest here is in the case where (e_1, f_1) is a pure Nash equilbrium that is not strict. Then there are alternative pure best replies to e_1 or f_1, each corresponding to an eigenvalue of zero. Now linearization alone cannot determine the stability of (e_1, f_1). Stability is determined by the restriction of the dynamic (1) to a centre manifold at (e_1, f_1) (Cressman, 1992; iggins, 1990). In our situation, this centre manifold is an invariant manifold for (1) whose tangent space at (e_1, f_1) consists of points $(x, y) \in X^m \times X^n$ with $\text{supp}(x) \subseteq \{i | e_i \text{ is a best reply to } f_1\}$ and $\text{supp}(y) \subseteq \{j | f_j \text{ is a best reply to } e_1\}$. From the proof of Theorem 3, one such centre manifold is the set E defined there. Since each point in E is a rest point of (1), each point in E is Lyapunov stable with respect to trajectories that start in E. This implies that (e_1, f_1) is Lyapunov stable with respect to trajectories that start in $\Delta^m \times \Delta^n$ (Cressman, 1992).

For the proof of Lyapunov stability in Theorem 5, (p^*, q^*) in the backwards induction Nash equilibrium component, G^*, need not be a pure strategy. Nevertheless, the centre manifold at (p^*, q^*) is still contained in $B_1 \times B_2$ where B_1 is the face of Δ^m whose elements are best responses to q^* and B_2 are best responses to p^*. Again showing stability with respect to trajectories that start in $B_1 \times B_2$ is sufficient. This is the technique used in Appendix A to prove Theorem 5. Conversely, instability with respect to this set implies instability of (p^*, q^*) - a fact used in the analysis of Example 5.