An Extension of a Theorem by Mitjushin and Polterovich to Incomplete Markets

by

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Abstract

This paper generalises the Mitjushin-Polterovich Theorem to the case of economies with incomplete financial markets where utility functions are of the von Neumann-Morgenstern-type. We thus give a sufficient condition on the joint distribution of the asset payoffs and the endowments which guarantees strict monotonicity of individual demand functions if the utility functions display small relative risk aversion.

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1 Introduction

The issue of uniqueness of a general equilibrium has been one of the most widely studied in General Equilibrium Theory. Such uniqueness would ensure that the prices given by the competitive solution to the allocation problem provide a well-determined theory of value. On the same grounds, uniqueness of the equilibrium also is a very desirable property of the general equilibrium model with incomplete markets (GEI-model), which finds wide applications in the theory of finance. Unfortunately, it is well-known that the notorious structure theorems by Sonnenschein, Debreu and Mantel (1973, 1974, 1974, respectively) carry over to the situation where markets are incomplete[1]. Even in the simplest case of a GEI-economy where there is only one consumption good, the standard model used in the finance literature, uniqueness of the equilibrium will therefore in general fail to hold. Despite of the importance of this issue, Mas-Colell's assertion that "...a systematic study of uniqueness conditions for incomplete market models is missing" (1991a) remains true even for the case with only one good. This paper forms an initial attempt to close this gap.

For certain special cases, uniqueness of the equilibrium in the one-good *GEI*-model has been established. For example, it is a standard and not so difficult exercise to show uniqueness for quadratic preferences. This is the simplest version of the so-called CAPM-model due to Sharpe (1964) and Lintner (1965)[2]. Becker (1995) shows uniqueness with non-collinear endowments in the case where there are only two assets and utilities are of the Cobb-Douglas-type[3]. Hens (1995) has pointed out that uniqueness can be derived via the Mitjushin-Polterovich-Theorem if the individual endowments are collinear and lie in the span of the asset matrix, and if, in addition, the agents' utility functions are characterised by small risk aversion[4].

This paper gives a sufficient condition on the joint distribution of asset payoffs and individual endowments, which allows to generalise the Mitjushin-Polterovich-Theorem to the case where the endowments might not be spanned by the assets' payoff vectors. This condition is closely related to the lattice structure of the linear space spanned by the asset payoffs[5]. Furthermore, it is

[1] see Hens (1991), Bottazzi and Hens (1996) and Gottardi and Hens (1996).

[2] For general CAPM-economies, however, Nielsen (1988) has shown that there may well be multiple equilibria. Bottazzi, Hens and Löffler (1995) have proven that the CAPM assumptions do not impose any additional restrictions besides the Tobin separation property and are hence compatible with any number of equilibria.

[3] His argument can be easily extended to cover von Neumann-Morgenstern utility functions with relative risk aversion not bigger than one.

[4] For a similar result see Madrigal and Smith (1995).

[5] A first observation of the importance of this structure in GEI-economies was made by

established that collinearity of the endowments still suffices to translate this condition into a sufficient condition for uniqueness of the GEI-equilibrium. We also show that our condition covers special cases for uniqueness known so far and that it comprises a large class of GEI-economies.

The paper is organised as follows. The model is described in section 2. We briefly recall the relation between monotonicity and uniqueness as well as the Theorem by Mitjushin-Polterovich in section 3. Section 4 contains the main body of the paper, and presents the theorem yielding our sufficient condition. The relevance and connexions of this result are illustrated by a set of examples given in section 5. Section 6 then concludes the paper.

2 The Model

Our model is the standard GEI-model with one consumption good. Thus, there are two periods $t = 0, 1$ with uncertainty in period 1, which is modelled by S possible states $s = 1, \ldots, S$. Consumption takes place in the second period only[6]. There are I individuals $i = 1, \ldots, I$ having utility functions $U^i : \mathbb{R}^S_+ \longrightarrow \mathbb{R}$. Frequently, we will denote index sets and their cardinalities by the same letter, for example $I = \{1, \ldots, I\}$. The following assumption, which states that the individuals' utility functions are of the von Neumann-Morgenstern type, will be maintained throughout:

Assumption (U) - (VNM): For every $i \in I$ the individual utility function is given by $U^i(x) = \sum_{i=1}^{I} \pi^i_s u^i_s(x_s)$, with $\pi^i >> 0$ and $\sum_{s=1}^{S} \pi^i_s = 1$, where $u^i_s : \mathbb{R}_{++} \rightarrow \mathbb{R}$ is $\mathcal{C}^2$, strictly monotone and strictly concave for $s = 1, ..., S$. Furthermore, the closure of the set $\{x \in \mathbb{R}^S_+ | \ \mathrm{U^i(x)} \geq \mathrm{U^i(y)}\}$ is contained in $\mathbb{R}^S_{++}$ for every $y \in \mathbb{R}^S_{++}$ and for every $i \in I$.

Note that Assumption (U) implies that individual asset demand functions are continuously differentiable.[7] We also make the following interiority assumption on the endowments ω^i:

Assumption (E): $\omega^i >> 0$ for every $i \in I$.

Henrotte (1992) in a different context.

[6]This assumption is without loss of generality, since the first period can be interpreted as just another state (see Geanakoplos and Polemarchakis (1986)).

[7]see for example Magill and Quinzii (1996, Chapter 2, Lemma 11.5).

To enable the trading of endowments between uncertain states, there is a $S \times J$-matrix $A = (A_1, ..., A_J) \in \mathrm{IR}^{\mathrm{S}\times\mathrm{J}}$ of assets paying off an amount A_{sj} of the consumption good in state s. We assume without loss of generality that assets are not redundant, i.e. that A satisfies the rank condition rank $A = J$. When $J = S$, markets are said to be complete, while for $J < S$ they are incomplete. Assets can be traded without any short selling restrictions. Asset prices are denoted by a vector $q \in \mathrm{IR}^{\mathrm{J}}$. Thus, the i-th agent's decision problem is

$$(P^i) \qquad \max_{x\in \mathrm{IR}^{\mathrm{S}}_+} U^i(x)$$
$$\text{s.t.} \quad \exists \theta \in \mathrm{IR}^{\mathrm{J}} : \mathrm{q}^{\mathrm{T}}\theta \leq 0$$
$$\text{and } x = \omega^i + A\theta$$

The definition of an equilibrium in this context now is standard.

Definition 1: A tuple $((\overset{*}{x}{}^i, \overset{*}{\theta}{}^i)_{i=1}^I; \overset{*}{q}) \in \mathrm{IR}^{(\mathrm{S+J})\times \mathrm{I+J}}$ is called a GEI-equilibrium if

1. $(\overset{*}{x}{}^i, \overset{*}{\theta}{}^i)$ solves (P^i) given $\overset{*}{q}$ for every $i \in I$,
2. $\sum_{i=1}^I \theta^i = 0$.

Applying well-known results (see for example Magill and Quinzii (1996, Chapter 2, Theorem 10.5)) we can conclude that in our set-up GEI-equilibria always exist. Furthermore, we observe that any J-dimensional asset matrix $\tilde{A}$ which generates the same trading subspace as A, i.e. span$\tilde{A}$ = span$A$, will lead to the same set of GEI-equilibrium-allocations $\left\{(\overset{*}{x}{}^i)_{i=1}^I\right\}$ (see for example Magill and Quinzii (1996, Chapter 2, § 9)).

3 Complete Markets

3.1 Monotonicity and Uniqueness

When markets are complete, i.e. when span $A = \mathrm{IR}^{\mathrm{S}}$, then the maximisation problem (P^i) is just the standard problem known from GE-theory, namely

$$(\tilde{P}^i) \qquad \max_{x\in \mathrm{IR}^{\mathrm{S}}_+} U^i(x)$$
$$\text{s.t. } \pi x \leq \pi \omega^i,$$

with $\pi \in \mathbb{R}^S_{++}$ being the vector of state prices. If $\xi^i(\pi)$ denotes the solution to this problem then $z^i(\pi) := \xi^i(\pi) - \omega^i$ is the individual excess-demand function, while $z(\pi) := \sum_{i=1}^I z^i(\pi)$ denotes aggregate excess-demand. An important property of excess-demand is monotonicity with respect to a normalising vector.

Definition 2: The excess-demand function z is strictly monotone with respect to the normalising vector $e \geq 0, e \neq 0$, if $(z(\pi) - z(\tilde{\pi}))(\pi - \tilde{\pi}) < 0$ whenever $z(\pi) \neq z(\tilde{\pi})$ and $\pi \cdot e = \tilde{\pi} \cdot e = 1$.[8]

If aggregate excess-demand is strictly monotone with respect to some normalisation, equilibrium will be unique. Note that if endowments are chosen as the normalising vector, price changes which leave the value of the endowments unchanged are referred to as income-neutral price changes.

Clearly, the sum of excess-demands strictly monotone with respect to the same normalising vector again is strictly monotone. Hence, one can deduce strict monotonicity of aggregate excess-demand from strict monotonicity of the individual excess-demand functions if a common normalising vector can be chosen. A standard condition allowing for this procedure is given by the assumption that individual endowments are collinear, i.e. that there is some $\omega \in \mathbb{R}^S_{++}$ such that $\omega^i = \lambda^i \cdot \omega$ for $\lambda^i > 0, i = 1, ..., I$, and $\sum_{i=1}^I \lambda^i = 1$. Summarising this subsection, one arrives at the following theorem[9].

Theorem 1: Let markets be complete, i.e. let $\operatorname{rank} A = S$. Suppose $z^i(\pi)$ is strictly monotone with respect to income-neutral price changes for every $i = 1, ..., I$, and let individual endowments be collinear. Then there is a unique equilibrium.

3.2 Mitjushin-Polterovich-Theorem

An elegant device for showing strict monotonicity of individual excess demand with respect to individual endowments is suggested by the Mitjushin-Polterovich-Theorem (1978). The theorem states as a sufficient condition for strict monotonicity that a suitably chosen coefficient of risk aversion does not exceed a cer-

[8] Observe that it is not possible to dispense with the normalising factor. If $z(\pi) \neq 0$ there is always a $\tilde{\pi}$ such that $(z(\pi) - z(\tilde{\pi}))(\pi - \tilde{\pi}) > 0$.

[9] For a much more detailed discussion of these issues and for proofs of the various statements the reader is referred to Mas-Colell (1991b) and Mas-Colell et al. (1995, Chapter 17.F).

tain critical value[10]. Since we are mainly concerned with studying monotonicity of individual demand in this paper, we will henceforth drop the superscript i whenever possible.

Theorem 2 (Mitjushin-Polterovich): Suppose that the utility function $U(x) : \mathbb{R}^S_+ \to \mathbb{R}$ is such that the excess-demand function z derived from $(\tilde{P})$ is $\mathcal{C}^1$. If

$$\mathrm{MP}_U(x) := -\frac{x^T D^2 U(x) x}{x^T DU(x)} < 4 \, \forall x \in \mathbb{R}^S_+$$

then z is strictly monotone for income-neutral price changes.

Proof: see Mas-Colell (1991b)[11].

□

In the financial markets setting considered here the Mitjushin-Polterovich-coefficient MP has a nice interpretation in terms of risk aversion. Observe that assumption (U) imposes on individual utility the form of an expected utility standard in the finance literature, i.e. $U(x) = \sum_{s=1}^S \pi_s u_s(x_s)$. It is straightforward to show that

$$\forall x \in \mathbb{R}^S_+ : \mathrm{MP}_U(\mathrm{x}) < 4 \Leftrightarrow \forall s \in S \ \forall x_s \in \mathbb{R}_+ : \mathrm{MP}_{u_s}(\mathrm{x_s}) < 4,$$

where $\mathrm{MP}_{u_s}(x_s) = -\frac{u_s''(x_s)x_s}{u'(x_s)}$. But the last expression is just the agent's coefficient of relative risk aversion. Therefore, with complete markets, small risk aversion leads to strictly monotone individual demand functions.

4 Incomplete Markets

In this section we will derive a sufficient condition for the strict monotonicity of the individual excess demand function when utilities are of the von Neumann-Morgenstern type but when markets may be incomplete and endowments may be unspanned. By standard arguments and with collinear endowments this in turn translates into a sufficient condition for the uniqueness of the GEI-equilibrium.

[10] For a converse, see Kannai (1987). Also observe that in terms of the underlying preferences it suffices that some utility representation satisfies Mitjushin's and Polterovich's condition.

[11] Mas-Colell (1991b) also gives an example which shows that the number "4" occuring in the statement of the theorem "is very much the magic number as far as uniqueness is concerned" (Mas-Colell (1991b, p. 283)). It consists of a two-consumer-two-goods-economy having three equilibria, whilst the individual Mitjushin-Polterovich-coefficients can be chosen arbitrarily close to 4.

First, however, a result by Hens (1995) for the case of spanned endowments is briefly reviewed.

4.1 Spanned Endowments

With the endowments lying in the span of the payoff vectors of the asset matrix, Theorem 1 can be applied to derive strict monotonicity of an agent's asset demand. To see this, let $\bar{\theta}$ be a portfolio generating the agent's endowment, i.e. let $\omega = A\bar{\theta}$. Then the decision problem (P) can be rewritten as

$$
\begin{array}{lll}
(P') & & \max_{\theta \in \mathbb{R}^J} W(\theta) := U(A\theta) \\
& \text{s.t.} & q^T\theta \leq q^T\bar{\theta} \\
& & A\theta \geq 0
\end{array}
$$

Note the fact that the asset matrix may link payoffs across states implies that W does not inherit additive separability from U. For small risk aversion, however, Theorem 1 still yields strict monotonicity for income neutral price changes, because simple transformations show that $\text{MP}_W = \text{MP}_U$. Hence, in the important special case when endowments are spanned our question is settled[12].

4.2 A Generalised Mitjushin-Polterovich Theorem

Before stating our result we need to introduce some further notation.

For any subset $R \subseteq S$ with $|R| = J$ we let A_R be the $J \times J$-submatrix of A given by the payoffs of all the assets in the states contained in R, i.e.

$$A_R = (A_{sj})_{s \in R}^{j \in J}.$$

We denote the associated subvectors by $x_R = (x_s, s \in R)$ and ω_R. Correspondingly, we let $x_{\setminus R} := (x_s, s \notin R)$. If the submatrix A_R has full rank, i.e. if rank $A_R = J$, it can be inverted. We will denote this inverse matrix by A_R^{-1}. Note that by the rank condition there always exists at least one set $R \subseteq S$, $|R| = S$ such that A_R is invertible.

Let now A_R be such an invertible submatrix. Then the matrix A_R^{-1} can be used to transform the payoff-matrix A to the matrix $A(R) = (Id_R, A_{\setminus R}A_R^{-1})$ which generates the same space of possible net-trades. Id_R here denotes the $R \times R$-identity matrix. The following definition is crucial.

[12]For details see Hens (1995, chapter III.7.6).

Definition 3: Let $R \subset S, |R| = J$ be such that A_R is invertible. If $A(R) = (Id_R, A_{\setminus R}A_R^{-1}) \geq 0$ then R is called a fundamental set of states[13].

Thus, when there exists a fundamental set of states, the asset matrix can be transformed such that it consists of J assets being Arrow-Debreu-securities in J states and paying off non-negative amounts of the single commodity in the remaining $S - J$ states. Abramovich, Aliprantis and Polyrakis (1996) show that a matrix contains a fundamental set of states if and only if its column vectors span a lattice-subspace[14].

We can now state the additional assumption needed in order to extend the Mitjushin-Polterovich-Theorem to incomplete markets in the general case where endowments are not spanned by the payoff-vectors of the assets.

Assumption (A): There exists a fundamental set R of states, $R \subseteq S$, such that $\omega_s - A_s A_R^{-1}\omega_R \geq 0$ for every $s \in S\setminus R$.

Remark: Observe that Assumption (A) imposes joint restrictions on the asset matrix and the endowments. However, note that if (A) is satisfied for an arbitrary asset matrix $A \in \mathbb{R}^{S\times J}$ it is also satisfied for the transformed matrix $A(R) = (Id_R, A_{\setminus R}A_R^{-1})$. Hence, the market subspace spanned by the traded assets is the only thing which matters, and one could also assume without loss of generality that A is of the simple form $A = (Id, \tilde{A})$. As this does not lead to significant simplifications in the notation, however, we stick to presenting the more general case.

It turns out that the MP-property of a utility function holds for the incomplete markets case, if Assumption (A) is satisfied. This is captured by the following result. Note, that the assumption on the utility function is not ordinal; it suffices that it is satisfied by one utility representation of the preferences.

Proposition 1: Suppose (U) and (A) are satisfied, and let $R \subseteq S$ be the corresponding fundamental set of states. If $\mathrm{MP}_{u_s}(x) < 4$ for every $x \in \mathbb{R}_+$ and every $s \in S$, then the individual asset demand function f arising from the maximisation problem (P) is strictly monotone with respect to the normalising vector $A_R^{-1}\omega_R$ (i.e. for price changes q_1, q_2 such that $q_1^T A_R^{-1}\omega_R = q_2^T A_R^{-1}\omega_R$).

[13] The terminology is taken from Aliprantis, Brown and Werner (1996) who have independently arrived at applying this concept to the analysis of the standard GEI-model. They use fundamental states to suggest an optimal hedging portfolio for a non-redundant put option in an incomplete market.

[14] For a detailed mathematical discussion see Abramovich, Aliprantis and Polyrakis (1996).

Proof: see the appendix.

As mentioned before, it is well-known that strict monotonicity of the individual excess demand functions implies uniqueness of the equilibrium if the agents' endowments are collinear. A similar argument in the situation considered here leads to the following theorem which is the main result of our paper.

Theorem 3: Let $\omega := \sum_{i=1}^{I} \omega^i$ and assume (U) and (A). Let $R \subseteq S$, $|R| = S$, be the corresponding fundamental set of states. Suppose that

1. $\mathrm{MP}_{u_s^i}(x) < 4$ for every $x \in \mathbb{R}_+$, every $s \in S$ and every $i \in I$.
2. $\omega_R^i = \lambda^i \omega_R$ for some $\lambda^i > 0$, $i \in I$.

Then there is a unique GEI-equilibrium.

Proof: see the appendix.

Remark: Observe that endowments have to be collinear only for the subvector of the coordinates which corresponds to the fundamental set of states.

Interpretation: For both of the parts of Assumption (A) which have to be satisfied in addition to small risk aversion (condition 1.) there is an interpretation which sheds some light on their economic content.

- When there is a fundamental set R of states nonnegative consumption $x_R \geq 0$ in the fundamental states implies nonnegative payoffs $A_{\setminus R} A_R^{-1} x_R$ outside these states. Thus, the indirect utility function $U_R(x_R)$ defined by $U_R(x_R) := U\left((x_s)_{s \in R}; (\omega_s + A_s A_R^{-1}(x_R - \omega_R))_{s \notin R}\right)$ is monotone for all $x_R \geq 0$. Such monotonicity always is required in order to apply the Mitjushin-Polterovich Theorem.

- If $x_R \equiv 0$ then consumption would be given by $x = ((0)_{s \in R}; (\omega_s - A_s A_R^{-1} \omega_R)_{s \notin R})$. Hence, if the second part of Assumption (A) does not hold, then the indirect consumption set X_R given by $X_R := \{(x_s)_{s \in R}; (\omega_s - A_s A_R^{-1}(x_R - \omega_R))_{s \notin R}\}$ is not the J-dimensional positive orthant. This means in particular that for the corresponding individual maximisation problem[15] the classical survival assumption is violated: positive income in

[15] see the proof of Proposition 1.

the R fundamental states does not necessarily imply a non-empty individual budget set.[16]

5 Examples

Example 1 - Complete Markets: When markets are complete we have that $\operatorname{rank} A = J = S$, so that we can assume without loss of generality that A consists of the S Arrow-Debreu-securities, i.e. that A is the $S \times S$-identity matrix. Hence, the set R we are looking for can only be the entire set of states, i.e. $R = S = \{1, \ldots, S\}$. Trivially, R is a fundamental set.

Moreover, observe that in this case

$$\omega_s - A_s A_R^{-1} \omega_R = \omega_s - \omega_s = 0.$$

Hence, Assumption (A) is satisfied and Theorem 3 holds.

Example 2 - Spanned Endowments:

With spanned endowments there exists a portfolio $\bar{\theta} \in \mathbb{R}^J$ such that $\omega = A\bar{\theta}$. For every fundamental set of states $R \subseteq S$, $|R| = S$, one therefore finds that

$$\omega_s - A_s A_R^{-1} \omega_R = \omega_s - A_s A_R^{-1} A_R \bar{\theta} = 0,$$

for every $s \in S$. Hence, in this case, the additional condition on endowments is redundant. Note, however, that the result given in section 4.1 is not completely covered by Theorem 3, since it does not require the asset matrix to contain a fundamental set of states. Indeed, with spanned endowments, the indirect utility function expressed in terms of portfolio holdings is monotone whenever the asset payoffs are nonnegative.

Example 3

Becker (1995) has shown that if utility functions are of the Cobb-Douglas-type and if $J = 2$, i.e. if there are only two assets, then there is a unique equilibrium. His proof uses the implicit function theorem to establish strict monotonicity of aggregate asset demand. This proof can easily be extended to utility functions satisfying (U) and displaying a coefficient of relative risk aversion not bigger than one. In particular, his result does not depend on restictions imposed on endowments.

[16] For the rôle of the survival assumption in GEI-theory see Gottardi and Hens (1996).

It is interesting to compare this result with Theorem 3: If $J = 2$, then every asset matrix contains a fundamental set of states. This follows from the fact that in $\mathbb{R}^2$ the cone spanned by the asset payoffs is a lattice cone.[17]

Example 4 - Class of Asset Structures Yielding Monotonicity:

This example presents a large class of possible asset structures A which satisfy the sufficient condition (A) for arbitrary endowments.

Suppose that the asset matrix is such that in each state of the world exactly one asset pays off a positive amount of the consumption good, i.e. that

$$\forall s \in S \, \forall j \in J : \quad A_{sj} \neq 0 \Rightarrow \quad A_{sj} > 0 \ \wedge \ \forall k \in J \backslash \{j\} : \quad A_{sk} = 0. \tag{1}$$

Such asset structures might be called 'weakly separating'. Note that since assets may still payoff in several states, perfect separation between states might fail to hold. A simple example for such a structure is $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$. Also observe, that for asset structures of this kind any subset $R \subseteq S$, $|R| = J$ such that rank $A_R = J$ is a fundamental set of states. With a 'weakly separating' asset matrix A of this form, Assumption (A) can be seen to hold for arbitrary endowments.

Proposition 2: Suppose A satisfies (2). Then Assumption (A) is satisfied.

Proof: see the appendix.

Thus, there is a large class[18] of asset matrices for which Proposition 1 yields strictly monotone demand functions for arbitrary endowments[19].

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[17] However, Theorem 3 which allows for coefficients larger than 1, needs some restrictions on individual endowments.

[18] Observe that it suffices that there exists one basis for span$A$ which – when written in matrix form – is weakly separating.

[19] It is interesting to note that adding assumption (2) to spanned endowments, (see section 4.1) yields the stronger conclusions on aggregate excess-demand contained in Hens and Löffler (1995). If A satisfies (2), the individual maximisation problem is additively separable in the portfolio holdings. Since under the hypothesis of spanning endowments can be expressed as a portfolio of the assets, the analysis carried out by Hens and Löffler applies.

Example 5

This example shows that a generalisation of Proposition 2 to the case of arbitrary trading spaces cannot be hoped for. Therefore, the condition which Assumption (A) imposes on endowments is necessary for general asset matrices.

Indeed, suppose $J = 2$ and $S = 3$ and let the asset matrix A be given by $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$. Suppose furthermore that endowments are such that $\omega_1 + \omega_2 > \omega_3$. We claim that Assumption (A) is not satisfied.

To see why this is the case, consider the three possible two-element-sets of indices $R_1 = \{1,2\}$, $R_2 = \{1,3\}$ and $R_3 = \{2,3\}$. Consider the $1 \times J$-vectors $\tilde{A}_i := A_{\setminus R_i} A_{R_i}^{-1}$ $(i = 1,2,3)$. Simple computations yield that

$$
\begin{aligned}
\tilde{A}_1 = &\quad (1\ 1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && = (1,1) && \geq 0 \\
\tilde{A}_2 = &\quad (0\ 1) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} && = (-1,1) && \not\geq 0 \\
\tilde{A}_3 = &\quad (1\ 0) \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} && = (1,-1) && \not\geq 0\ .
\end{aligned}
$$

Thus, R_1 is the unique fundamental set of indices. But, clearly, by hypothesis

$$
\omega_3 - (1,1)\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \omega_3 - (\omega_1 + \omega_2) < 0\ .
$$

□

Example 6

In this example we show that the condition imposed on excess endowments is necessary in order to guarantee that the indirect utility function U_R defined on consumption in a fundamental set of states has an MP-coefficient less than 4.

As in Example 4 suppose that $S = 3$, $J = 2$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$. Let utility in the states be given by $u_s(x_s) := \ln x_s$, $s = 1, ..., S$. We have seen above that the only fundamental set of states in this case is $\bar{R} = \{1,2\}$, and that $\omega_3 - A_3 A_{\bar{R}}^{-1} \omega_{\bar{R}} = \omega_3 - (\omega_1 + \omega_2)$. Now choose $\omega_1 + \omega_2 = 1$ and $\omega_3 = \frac{1}{6}$. Thus, excess endowment in state 3 is given by $\tilde{\omega}_3 := \omega_3 - A_3 \cdot A_{\bar{R}}^{-1} \cdot \omega_{\bar{R}} = -\frac{5}{6}$.

Evaluating the Mitjushin-Polterovich coefficient MP for indirect utility $U_{\bar{R}}(x_{\bar{R}})$ at $\bar{x}_{\bar{R}} = (\bar{x}_1, \bar{x}_2) = (\frac{1}{2}, \frac{1}{2})$ then yields:

$$\mathrm{MP} = \frac{\pi_1 + \pi_2 + \pi_3 \frac{(x_1+x_2)^2}{(\tilde{\omega}_3+x_1+x_2)^2}}{\pi_1 + \pi_2 + \pi_3 \frac{x_1+x_2}{\tilde{\omega}_3+x_1+x_2}} = \frac{\pi_1 + \pi_2 + 36\pi_3}{\pi_1 + \pi_2 + 6\pi_3} \; .$$

For, for example, $\pi_1 = \pi_2 = \frac{1}{6}$ and $\pi_3 = \frac{2}{3}$ this becomes

$$\mathrm{MP} = \frac{\frac{1}{3} + 24}{\frac{1}{3} + 4} = \frac{73}{13} > 4 \; .$$

Observe that since $\tilde{\omega}_3 + \bar{x}_1 + \bar{x}_2 > 0$, $\bar{x}_R$ is an admissible consumption bundle.

□

6 Conclusion

In this paper we have studied uniqueness of equilibria in GEI-economies where agents have utility functions of the von Neumann-Morgenstern type with small risk aversion. The important Mitjushin-Polterovich-Theorem was generalised to the situation with incomplete markets. We have derived a sufficient condition on the joint distribution of asset payoffs and endowments guarenteeing that even with incomplete markets and non-spanned endowments, individual asset demand functions are strictly monotone for income-neutral price changes. Together with collinear endowments this also becomes a sufficient condition for uniqueness of the GEI-equilibrium.

Examples show that the conditions given hold in fairly general circumstances. The question for a sufficient condition for arbitrary endowments or asset structures, however, remains unsolved. We leave its investigation for further research.

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Appendix

Proof of Proposition 1: Fix the fundamental set $R \subseteq S$. Without loss of generality we can assume that $R = \{1, \ldots, J\}$. Let $p \in \mathrm{IR}^J$ be defined by $p := (A_R^{-1})^T q$ and set

$$U_R(x_R) := U\left((x_s)_{s \in R}; (\omega_s - A_s A_R^{-1}(x_R - \omega_R))_{s \notin R}\right).$$

Then it is straightforward to verify that the maximisation problem (P'') given by

$$(P'') \qquad \max_{x_R \in \mathrm{IR}_+^J} U_R(x_R) \quad \text{s.t.} \quad p^T x_R = p^T \omega_R$$

is equivalent to the problem (P) introduced in section 2. Let $f(q)$ be the asset demand arising from (P), and let $h(p)$ denote the demand for state consumption which follows from (P''). Simple transformations now establish that $f(q)$ is (strictly) monotone with respect to a normalising vector e if and only if $h(p)$ is (strictly) monotone with respect to the normalising vector $A_R e$. Therefore, it suffices to show strict monotonicity of $h(p)$. This will be done by applying the Mitjushin-Polterovich-Theorem, Theorem 2, to the indirect utility function U_R.

Thus let $\tilde{A} := A_{\setminus R} A_R^{-1}$, and compute the gradient and the Hesse-matrix of U_R as

$DU_R(x_R) = (\pi_j u_j'(x_j) + \sum_{s=J+1}^S \pi_s \tilde{A}_{sj} u_s'(x_s))_{j=1}^J$,

$D^2 U_R(x_R)_{jj} = (\pi_j u_j''(x_j) + \sum_{s=J+1}^S \pi_s \tilde{A}_{sj}^2 u_s''(x_s))$ for $j = 1, ..., J$, and

$D^2 U_R(x_R)_{jk} = (\sum_{s=J+1}^S \pi_s \tilde{A}_{sj} \tilde{A}_{sk} u_s''(x_s)))$ for $j, k \in J, k \neq j$.

Therefore,

$$\begin{aligned} x_R^T D^2 U_R(x_R) x_R &= \sum_{j=1}^J \left[\pi_j u_j''(x_j) x_j^2 + \sum_{s=J+1}^S \pi_s u_s''(x_s) \tilde{A}_{sj} [\tilde{A}_{sj} x_j^2 + \sum_{k \neq j} \tilde{A}_{sk} x_j x_k]\right] \\ &= \sum_{j=1}^J \pi_j u_j''(x_j) x_j^2 + \sum_{s=J+1}^S \pi_s u_s''(x_s) (\tilde{A}_s x_R)^2 , \end{aligned}$$

and

$$x_R^T DU_R(x_R) = \sum_{j=1}^J \pi_j u_j'(x_j) x_j + \sum_{s=J+1}^S \pi_s u_s'(x_s) \tilde{A}_s x_R .$$

The condition $\mathrm{MP}_{U_R}(x_R) < 4$ therefore is equivalent to

$$\sum_{j=1}^J \pi_j x_j \left[-u_j''(x_j) x_j - 4u_j'(x_j)\right] + \sum_{s=J+1}^S \pi_s \tilde{A}_s x_R \left[-u_s''(x_R) \tilde{A}_s x_R - 4u_s'(x_s)\right] < 0 .$$

Define $\tilde{\omega}_s := \omega_s - \tilde{A}_s \cdot \omega_R$ for every $s \in S \backslash R$. Using the assumption on small risk aversion and the transformation $\tilde{\omega}_{\backslash R} + \tilde{A} x_R = x_{\backslash R}$, it can be derived that this follows from

$$\sum_{s=J+1}^{S} \pi_s \tilde{A}_s x_R \; u_s''(x_R) \tilde{\omega}_s < 0.$$

The last inequality now holds since $\tilde{A} \geq 0$ and $\tilde{\omega}_s \geq 0$ by Assumption (A). Theorem 2 (Mitjushin-Polterovich-Theorem) then finishes the proof of Proposition 1.

□

Proof of Theorem 3: Normalise asset prices q such that $q^T A_R^{-1} \omega_R = 1$. This is certainly possible due to the homogeneity of the budget constraint in (P''). By the collinearity assumption on individual endowments, we then only consider income neutral price changes for each agent. By Proposition 1, individual asset demand f^i is strictly monotone for such price changes. Therefore, aggregate asset demand $f := \sum_{i=1}^{I} f^i$ is strictly monotone with respect to all pairs of prices in the domain of prices. But this means that the equilibrium known to exist must be unique.

□

Proof of Proposition 2: The assumption made on A allows to partition the set of states according to which of the assets pays off in a particular state. Thus, letting $S_j := \{s \in S : A_{sj} \neq 0\}$, i.e. S_j being the set of states in which asset j pays off, one finds that $S = \cup_{j=1}^{J} S_j$ and that the sets S_j are pairwise disjoint. Note that any J-element set R containing exactly one state from each of the subsets S_j is a fundamental set of states.

For any $s, t \in S_j$ we now have that

$$x_s = (\omega_s - \frac{A_{sj}}{A_{tj}} \omega_t) + \frac{A_{sj}}{A_{tj}} x_t.$$

It now suffices to find a state $\bar{s}_j \in S_j$ for every $j \in J$ such that

$$\forall s \in S_j : \tilde{\omega}_s := \omega_s - \frac{A_{sj}}{A_{\bar{s}j}} \omega_{\bar{s}_j} \geq 0$$

To see the existence of such a state $\bar{s}_j$, consider the directed graph G with the states $s \in S_j$ as its vertices and insert an edge from vertex s to vertex t if $\frac{\omega_s}{\omega_t} > \frac{A_{sj}}{A_{tj}}$. If $\frac{\omega_s}{\omega_t} = \frac{A_{sj}}{A_{tj}}$ insert an edge between s and t with an arbitrary direction chosen. Obviously, then, in each vertex s one finds that $d^+(s) + d^-(s) :=$

in $-$ degree $+$ out $-$ degree $= |S_j| - 1$. (The in-degree of a vertex s is the number of edges directed towards s, while the out-degree is the number of edges directed away from s.) If now $d^-(\bar{s}) = 0$ for some vertex $\bar{s} \in S_j$ we could choose $\bar{s}_j = \bar{s}$, and we would be done.

We now claim that there is such a vertex $\bar{s} \in S_j$ with $d^-(\bar{s}) = 0$ (by possibly redirecting edges connecting vertices s, t where $\frac{\omega_s}{\omega_t} = \frac{A_{sj}}{A_{tj}}$). We prove this claim by induction on $L = |S_j|$.

The case $L = 2$ is trivial. For the general case suppose S_j, $|S_j| = L$, is a set of states with associated endowments ω_s, $s \in S_j$, such that in the associated directed graph G_L every vertex has out-degree at least 1, i.e. $d^-(s) \geq 1$ for every $s \in S_j$. Pick any vertex $s_n \in S_j$ and consider the reduced graph G_{L-1} on $L-1$ vertices obtained by eliminating s_n and the edges connected to s_n. By hypothesis there is a vertex $\bar{s}$ such that $d^-(\bar{s}) = 0$ in the graph G_{L-1} (possibly redirecting edges). Thus, one concludes that one can redirect edges in G_{L-1} such that $d^-(\bar{s}) = 1$ in the original graph G_L. From $d^-(s_n) \geq 1$ one infers the existence of a vertex $\hat{s} \in S_j \backslash \{s_n, \bar{s}\}$ such that there is an edge pointing from s_n to $\hat{s}$. The fact that $d^-(\bar{s}) = 0$ in G_{L-1}, i.e. $d^+(\bar{s}) = L - 2$ in G_{L-1}, then implies that there is a directed circle $s_n, \hat{s}, \bar{s}$ in G_L. This, however, implies that

$$\frac{\omega_{s_n}}{\omega_{\hat{s}}} \geq \frac{A_{s_n j}}{A_{\hat{s}j}} \geq 0, \ \frac{\omega_{\hat{s}}}{\omega_{\bar{s}}} \geq \frac{A_{\hat{s}j}}{A_{\bar{s}j}} \geq 0, \ \text{ and } \ \frac{\omega_{\bar{s}}}{\omega_{s_n}} \geq \frac{A_{\bar{s}j}}{A_{s_n j}} \geq 0,$$

from which one concludes that

$$\frac{\omega_{s_n}}{\omega_{\bar{s}}} = \frac{A_{s_n j}}{A_{\bar{s}j}}.$$

Hence one can redirect the edge pointing from $\bar{s}$ to s_n which yields $d^-(\bar{s}) = 0$ in G_L.

Picking $s_j = \bar{s}$ and repeating this argument for each set $S_j, j \in J$, yields the fundamental set $R = \{\bar{s}_j, j \in J\}$. By construction, R satisfies Assumption (A) for arbitrary endowments.

□

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