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The Pricing and Hedging of Options in Finitely Elastic Markets

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1 Introduction

Ever since derivative asset analysis started with the pathbreaking papers of Black and Scholes (1973) and Merton (1973), both academics and practioners were concerned about the strong assumptions this theory imposes on the markets for the underlying securities. Most of the work on derivative pricing assumes these markets to be complete, frictionless and perfectly elastic, which is of course a stylized picture of real security markets. Therefore recent research has studied the consequences of relaxing one or more of these hypothesis.

In this paper we study the pricing and hedging of derivatives assuming that markets are only finitely elastic. The framework of our analysis is a continuous-time version of the models proposed by Jarrow (1992) and Jarrow (1994). The class of economies considered in these papers is characterized by the interaction of a a "large trader" whose actions affect prices and many price taking "small traders". Jarrow (1992) finds conditions on the economy that exclude "market manipulation strategies," i.e. arbitrage opportunities for the large trader. Jarrow (1994) studies pricing and hedging of derivative securities by the large trader. He analyzes in detail a modified version of the of the binomial model introduced in (Cox, Ross, and Rubinstein 1979). He shows that the binomial option pricing model remains valid, but with a state dependent "volatility". Also the arguments used for the derivation of the model are different in his framework.

In this paper we extend his results in several ways. We show that even with continuous security trading it is possible to find hedging strategies for the large trader which have the potential to synthesize the payoff of certain derivative contracts including options. In the binomial model considered by Jarrow this question boils down to recursively solving a finite number of equations, but in our continuous time setting it becomes rather involved. Nonetheless working in continuous time allows us to give a rather succinct characterization of the solution to the option replication problem in finitely elastic markets in terms of a partial differential equation (PDE). The feedback effect of the large agent's trades on equilibrium prices causes this PDE to be non-linear. We provide conditions for existence and uniqueness of solutions and analyze the shape of the hedging strategies. It turns out that the qualitative properties of the hedge ratio are unchanged by the feedback effects. However, simulations demonstrate that there may be considerable quantitative differences. Our analysis also shows that in our setting the initial investment into the replicating strategy (the hedge cost per contract) depends on the total amount of contracts replicated by the large trader. Hence it is no longer obvious how options should be priced. To settle this issue we generalize the work of Jarrow (1994) and show that the synchronous market condition proposed in his paper — a condition relating the markets for the underlying and the derivative security — is sufficient to conclude that even in our framework the price of a derivative asset must be equal to the hedge cost. However, hedge cost and hence derivative prices do depend on the large trader's position in underlying and derivative asset.

We believe the extension of standard option pricing theory to finitely elastic markets to be interesting for a number of reasons. To begin with, recent work on feedback effects of dynamic hedging has shown that in finitely elastic markets perfect replication of option contracts is no longer feasible if investors restrict themselves to standard hedging strategies which do not account for the feedback effect of their implementation on market volatility; see for instance (Frey and Stremme 1995). This immediately raises the question, if there are more general general strategies that have the potential to replicate certain derivatives even in imperfectly elastic markets. Our study shows that the answer to this question is to the affirmative if certain restrictions on market liquidity and on the nonlinearities of the terminal payoff[1]not needed in the standard theory are satisfied. This is of interest in itself. Moreover, an analysis of these additional assumptions and a comparison of the hedging strategies with their classical counterparts sheds light on two important issues. It permits assessing the robustness of the traditional theory with respect to the elasticity assumption. It also relates certain market conditions to how well traditional option hedging performs under these conditions. Finally, as mentioned already in Jarrow (1994), one could use the fact that option prices depend on the large trader's position to explain certain deviations from the Black-Scholes formula on real options markets such as the smile pattern of implied volatility. We do not address these questions in this paper, but our analysis is a necessary prerequisite for tackling them.

Most of the literature dealing with option hedging and portfolio insurance in the context of imperfectly elastic markets focuses on the effects of dynamic trading strategies on the volatility of the underlying asset. Here we only mention the papers Grossman (1988), Brennan and Schwartz (1989), Gennotte and Leland (1990), Frey and Stremme (1995), Platen and Schweizer (1994) or Basak (1995). To my knowledge the previously mentioned work of Jarrow (1994) is the only study where a pricing theory for derivatives in such markets is developed.

The remainder of this paper is organized as follows: In the next section we introduce the framework for our analysis. The pricing of derivatives is discussed in Section 3. In Section 4 we characterize the solution of the replication problem by means of a nonlinear PDE. In Section 5 we carry out a detailed analysis of this PDE. The simulation results are presented in Section 6. Section 7 finally concludes.

2 The Model

Essentially our analysis uses the the framework proposed by Jarrow (1992) and Jarrow (1994), but in contrast to these papers we consider an economy with *continuous* security trading in some intervall $[0, T]$.

ASSETS: A risky asset, representing some stock, stock index or foreign exchange rate and a riskless bond which will be used as a numeraire trade in our economy. The price process of the stock, accounted in units of the numeraire, will be denoted by $X = (X_t)_{0 \leq t \leq T}$. For convenience we normalize the total supply of this asset to 1. We assume that there is also a market for a derivative securities on the stock[2] with maturity date T and payoff $c(X_T)$. The derivative asset is in zero net supply; its relative price process will be denoted by $C = (C_t)_{0 \leq t \leq T}$.

AGENTS: There are two different types of agents in this economy, a *large trader* or *speculator* and *small traders*. We do not give a detailed description of these types here. All that matters

[1] Even in finitely elastic markets *linear* payoffs can be replicated by using a static buy and hold strategy.

[2] In case there is more than one derivative security our analysis applies with only notational changes.

to us is that the trades of the speculator move prices whereas the small agents act as price takers. Some examples of economic models characterized by the interaction of a large trader and price takers are sketched below.

RELATIVE EQUILIBRIUM PRICES: Following Jarrow we do not give a fully specified model of the economy underlying our analysis. Instead we impose only existence and certain structural properties of a *reaction function* Ψ, which provides a reduced form equilibrium relationship between relative stock prices, the large trader's position in stock and derivative security and a fundamental state variable process $F = (F_t)_{0 \leq t \leq T}$ defined on some underlying filtered probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions. We assume that at time t the large trader has access to the information contained in $\mathcal{F}_t$. The price processes of the derivative will be specified later on; for the moment we only assume that it is a semimartingale.

Assumption (A.1) *1. The fundamental F is a geometric Brownian motion*[3] *with volatility η, i.e. it is a solution to the SDE $dF_t = \eta F_t dW_t$ for some constant $\eta > 0$ and a one-dimensional Brownian motion W on the filtered probability space $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{0 \leq t \leq T}$.*

2. The relative equilibrium stock price X_t at time t is given by $\Psi(t, F_t(\omega), \alpha_t, \gamma_t)$. Here α_t represents the stock position of the large trader, γ_t denotes his position in the derivative contract and the reaction function

$$\Psi : [0,T] \times \mathbb{R}_+ \times I_0 \times I_1 \to \mathbb{R}_+$$

is a smooth function with

- $0 < \Psi_f(t, f, \alpha, \gamma) := \frac{\partial}{\partial f} \Psi(t, f, \alpha, \gamma)$ *for all* $(t, f, \alpha, \gamma) \in [0,T] \times \mathbb{R}_+ \times I_0 \times I_1$.
- $0 < \Psi_\alpha(t, f, \alpha, \gamma) := \frac{\partial}{\partial \alpha} \Psi(t, f, \alpha, \gamma)$ *for all* $(t, f, \alpha, \gamma) \in [0,T] \times \mathbb{R}_+ \times I_0 \times I_1$.

Here the I_0 and I_1 represent some open intervals.

Note that the condition $\Psi_\alpha > 0$ reflects the market power of the large trader since it implies that his trades actually affect prices. Assumption (A.1) is consistent with different types of economies and equilibrium concepts. Two examples mentioned in (Jarrow 1992) are the model by Hart (1977) and the class of models considered for instance by Glosten and Milgrom (1985). In the Hart model one agent has market power because of his wealth, whereas in the latter class of models some agent moves prices because the others believe that he has superiour information. In these papers only stock and bond markets are considered. However, it will be shown below that a reaction function Ψ satisfying an additional structural hypothesis, the so-called *synchronous market condition*, is completely determined by the values $\{\Psi(t, f, \alpha, 0), (t, f, \alpha) \in [0,T] \times \mathbb{R}_+ \times I_0\}$. Hence the above models give rise to examples for reaction functions also in our setting. (A.1) is moreover satisfied by the temporary equilibrium models considered by Jarrow (1994) or Platen and Schweizer (1994) where

$$\Psi(t, f, \alpha, 0) = \exp(\lambda \alpha) \cdot f$$

[3] The choice of the dynamics for F is somewhat arbitrary; however assuming that the fundamental state variable process is a geometric Brownian motion will facilitate a comparison of our results to those of the standard Black-Scholes option pricing theory.

for some positive constant λ or in the model of Frey and Stremme (1995), where

$$\Psi(t, f, \alpha, 0) = \lambda \cdot f/(1-\alpha)\,.$$

TRADING STRATEGIES: A process $(\alpha, \beta, \gamma) = (\alpha_t, \beta_t, \gamma_t)_{0 \le t \le T}$ giving the speculator's holdings in stock, bond and derivative security will be termed an *admissible* trading strategy if it is an adapted RCLL process such that α and γ are semimartingales. This qualification not needed in the standard theory together with the smoothness of Ψ implies that the stock price process X is itself a semimartingale such that we may define gains from trade:

Definition 2.1 *Let (α, β, γ) be an admissible trading strategy. The* gains from trade *of this strategy up to time t are given by*

$$G_t := \int_0^t \alpha_s^- \, d\Psi(s, F_s, \alpha_s, \gamma_s) + \int_0^t \gamma_s^- \, dC_s$$

where α_s^- and γ_s^- denote the left continuous versions of α_s and γ_s, respectively. The value process *of this strategy is given by*

$$V_t := \alpha_t \cdot \Psi(t, F_t, \alpha_t, \gamma_t) + \beta_t + \gamma_t \cdot C_t\,. \tag{2.1}$$

The strategy is called selffinancing *if $V_t = V_0 + G_t$ for all $0 \le t \le T$.*

These definitions parallel the usual ones. In particular, every admissible trading strategy in stock and derivative asset can be turned into a selffinancing strategy by choosing an appropriate trading strategy in the bond. However, the feedback of the large trader's position into prices has an important consequence: linear combinations of selffinancing strategies need no longer be selffinancing.

Our definition of the value of the speculator's portfolio in (2.1) is in principle appropriate only for points in time $t < T$; at the terminal date T one should consider the *liquidation value* of the portfolio given by $\alpha_T^- \cdot \Psi(T, F_T, 0) + \beta_T^- + \gamma_T^- \cdot \tilde{C}_T$ where $\tilde{C}_T$ is the price of the derivative if the large trader has liquidated his position; see Jarrow (1994). However, in order to avoid arbitrage opportunities for the small traders, we exclude trading strategies for the speculator leading to predictable jumps of the asset price processes. Hence we may allow for the large trader unwinding his position only if this does not induce jumps of the asset price processes at T. In that case the terminal value V_T defined in (2.1) and the liquidation value of a selffinancing strategy are identical.

One possible way to justify the assumption of continuous asset prices at the terminal date is as follows: Suppose that — as in most equilibrium models — at T the stock price is exogenously given and equal to the fundamental value F_T. As t approaches T the uncertainty about F_T is gradually removed for the price takers such that they become more and more aggressive. This implies the convergence $\Psi(t, f, \alpha, \gamma) \to f$ as $t \to T$. If the trading strategy of the speculator is bounded and if this convergence is locally uniform, asset prices are therefore continuous at the terminal date. Alternatively we might confine the speculator to using trading strategies which are continuous. If the speculator is replicating the payoff of derivatives — which is our primary concern in this paper — this amounts to assuming that there is physical delivery of the underlying security at the maturity date of the contract.

3 Derivative Pricing under the Synchronous Market Condition

In this section we give a theory for derivative pricing suitable for our framework. Our arguments are an extension of the pricing theory developed by Jarrow (1994).

Definition 3.1 *Let (α, β, γ) be an admissible selffinancing strategy for the speculator.*

1. We say that there are no opportunities for market manipulation *at the trading strategy (α, β, γ), if there is no other admissible selffinancing strategy that requires the same initial investment and that yields a strictly higher terminal value.*

2. Suppose we are given an arbitrary derivative contract with maturity date T and payoff $\tilde{c}(X_T)$. An adapted RCLL trading strategy $\xi = (\xi_t)_{0 \leq t \leq T}$ in the stock is said to replicate *the derivative contract* at the speculator's position (α, β, γ), *if $\tilde{c}(X_T)$ admits a representation*

$$\tilde{c}(X_T) = \tilde{c}_0 + \int_0^t \xi_s^- \, d\Psi(s, F_s, \alpha_s, \gamma_s) \tag{3.2}$$

where $\tilde{c}_0$ is constant and $X_T = \Psi(T, F_T, \alpha_T, \gamma_T)$. At time t the value of the corresponding hedge portfolio or, equivalently, the hedge cost of ξ equals $V_t^\xi := \tilde{c}_0 + \int_0^t \xi_s^- \, d\Psi(s, F_s, \alpha_s, \gamma_s)$.

Remarks: If the speculator prefers more terminal wealth to less, absence of market manipulation opportunities at his "optimal" strategy is a prerequisite for any economic equilibrium, independent of the details of the market structure. Note that the hedging strategy used by the speculator influences his stock position and hence equilibrium prices. Therefore to compute replicating strategies for the large trader one has to solve a fixed point problem.[4] This will become obvious in (3.5) below.

In standard option pricing theory it is argued that the price of a derivative contract must be equal to the hedge cost per contract, because otherwise agents could make infinite profits. Now we will see in the simulations of section 6, that the feedback effect of the speculator's trading causes the hedge costs to be non-linear in the number of replicated contracts. Therefore in finitely elastic markets the pricing argument from the standard theory doesn't work any longer and additional assumptions on the market structure are needed to arrive at a fully specified pricing theory for the derivative asset. To fill this gap Jarrow (1994) proposed the *synchronous market condition* which relates the markets for the stock and for the derivative asset. Essentially this condition states that equilibrium prices are unchanged, no matter if the large trader replicates the payoff of the derivative by dynamic trading in stock and bond or if he takes his position directly in the derivative contract. To motivate why this condition should hold Jarrow gives examples of reaction functions which do not satisfy this condition and for which there exist trading strategies for the large trader allowing for market manipulation possibilities of arbitrary size.

Definition 3.2 *1. Suppose that we are given an admissible selffinancing trading strategy (α, β, γ) for the large trader and a hedging strategy ξ replicating the payoff of the traded derivative contract at (α, β, γ). Then the market for the derivative security and for the stock are said to be* in synchrony at (α, β, γ), *if a.s. for all $0 \leq t \leq T$*

$$\Psi(t, F_t, \alpha_t + \gamma_t \cdot \xi_t, 0) = \Psi(t, F_t, \alpha_t, \gamma_t) \tag{3.3}$$

[4] In section 4 we will actually take a slightly different route to proving existence of replicating strategies.

The synchronous market condition *for the stock holds, if the markets for the stock and for the traded derivative contract are in synchrony at all admissible selffinancing trading strategies.*

2. Analogously we say that the price process C of the derivative asset satisfies the synchronous market condition if at all admissible selffinancing trading strategies C is unchanged by the variation (3.3) of the large trader's position.

We now assume the synchronous market condition to hold and analyze some of its consequences. First, from (3.3) it is easily seen that the equilibrium price process for the stock is completely determined by the values $\{\Psi(t, f, \alpha, 0), (t, f, \alpha) \in [0, T] \times \mathbb{R}_+ \times I_0\}$: Suppose that the strategy ξ replicates the payoff of the derivative at a certain trading strategy (α, β, γ) of the large trader. Then we get by applying (3.3)

$$\Psi(t, F_t, \alpha_t, \gamma_t) = \Psi\left(t, F_t, \alpha_t + \gamma_t \cdot \xi_t, 0\right) . \tag{3.4}$$

Substituting (3.4) into (3.2) we see that here ξ must solve the following equation

$$c\left(X_T\right) = c_0 + \int_0^T \xi_t^- \, d\Psi(t, F_t, \alpha_t + \gamma_t \cdot \xi_t, 0) . \tag{3.5}$$

As ξ appears in both, the integrand and the integrator of the stochastic integral it is not a priori clear if solutions to this equation exist. We will consider this question in sections 4 and 5 below. The synchronous market condition also helps to determine prices of the traded derivative contracts:

Proposition 3.3 *Suppose that the synchronous market condition holds for the stock and for the derivative security. Consider an admissible selffinancing trading strategy (α, β, γ) for the large trader. Denote by V^ξ the value process of a hedging strategy ξ that replicates the derivative contract at (α, β, γ). Then the absence of opportunities for market manipulation at (α, β, γ) implies that the price process of the derivative asset must be equal to V^ξ.*

Proof: Since ξ replicates the derivative contract we have from (3.5) that V^ξ is a semimartingale. Moreover, the synchronous market condition for the stock implies the synchronous market condition for V^ξ. To prove that C_t must be equal to V_t^ξ for all t we consider the following variation $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ of the speculator's strategy: For $s < t$ his position is unchanged; for $s \in [t, T]$ his holdings are given by $\tilde{\alpha}_s := \alpha_s + \xi_s$ and by $\tilde{\gamma}_s := \gamma_s - 1$. The new bondholdings from time t onward are determined by the condition that the new strategy be selffinancing. The terminal value $\tilde{V}_T$ of this new strategy is then given by

$$\tilde{V}_T = V_t + \int_t^T (\alpha_s^- + \xi_s^-) d\Psi(s, F_s, \alpha_s + \xi_s, \gamma_s - 1) + \int_t^T \gamma_s^- dC_s - (C_T - C_t) , \tag{3.6}$$

where $(V_t)_{0 \le t \le T}$ denotes the value process of the original strategy (α, β, γ). Now by the synchronous market condition

$$\Psi(s, F_s, \alpha_s + \xi_s, \gamma_s - 1) = \Psi(s, F_s, \alpha_s, \gamma_s)$$

Hence (3.6) equals

$$V_T + \int_t^T (\xi_s)^- d\Psi(s, F_s, \alpha_s, \gamma_s) - (C_T - C_t) = V_T + V_T^\xi - V_t^\xi - (C_T - C_t) = V_T + C_t - V_t^\xi$$

The proposition follows, because if $C_t \neq V_t$ either this or the converse variation would yield a higher terminal value than the original strategy. □

REMARKS: Note that the synchronous market condition is needed in the proof to ensure that relative equilibrium prices remain unchanged along the variation of the large trader's position. From (3.5) it is apparent that the hedging strategy ξ and hence also the price of the derivative security depends on the large trader's trading strategy $(\alpha.\beta,\gamma)$. We will come back to this issue in the simulations of section 6.

To complete our development of a pricing theory for derivatives in finitely elastic markets it remains to prove existence of replicating strategies, i.e. existence of solutions ξ to (3.5). We need some simplifying assumptions on the speculator's trading strategy (α,β,γ):

Assumption (A.2) *The speculator's stockholdings are given by* $\alpha = (\bar\alpha(t,F_t)_t)_{0\le t\le T}$, *where* $\bar\alpha$ *is a smooth and nondecreasing function on* $[0,T]\times\mathbb{R}_+$. *His position in the derivative is given by* $\gamma = (\bar\gamma(t,F_t)_t)_{0\le t\le T}$ *where* $\bar\gamma$ *is a smooth and nonnegative function on* $[0,T]\times\mathbb{R}_+$.

As the underlying fundamental state variable is a Markov process the assumption of the large trader's position depending only on the current value of F is quite palatable. Assumption (A.2) allows us to reduce (3.5) to a problem involving only stock and bond markets. We define a new function $\psi : [0,T]\times\mathbb{R}_+\times I \to \mathbb{R}_+$ by

$$\psi(t,f,\vartheta) := \Psi(t,f,\bar\alpha(t,f)+\bar\gamma(t,f)\cdot\vartheta,0)\,. \tag{3.7}$$

ψ can be interpreted as a reaction function depending only on the fundamental and on the speculator's stockholdings as given by ϑ. It is immediate from the definition of ψ in (3.7) that a trading strategy ξ solves (3.5) if and only if it is a solution to the following problem

$$c(\psi(T,F_T,\xi_T)) = c_0 + \int_0^T \xi_t^-\, d\psi(t,F_t,\xi_t)\,. \tag{3.8}$$

4 Perfect Option Replication in Finitely Elastic Markets

We now deal with (3.8) and give a characterization of solutions to this equation in terms of a nonlinear partial differential equation (PDE). In section 5 we then demonstrate that under some additional qualifications this PDE actually admits a solution. This is of interest for a number of reasons. To begin with, it shows that even a large agent whose trades move prices is able to synthesize the payoff of a derivative by dynamic trading. Moreover, this yields some insights on the robustness of the Black-Scholes theory with respect to the assumption of perfectly elastic markets. Finally, by settling this issue we provide the missing ingredient for the option pricing theory for finitely elastic markets developed in section 3.

In this section we work with the following assumptions on the reaction function ψ and the terminal payoff $c(X_T)$ of the derivative asset.

Assumption (A.3) *1. The relative equilibrium price* X_t *at time* t *is given by* $\psi(t,F_t(\omega),\alpha_t)$, *where* α_t *represents the stock position of the large trader at time* t *and where* $\psi : [0,T]\times\mathbb{R}_+\times I\to\mathbb{R}_+$ *is a smooth function with the following properties:*

- *For every compact set* $K \subset\subset I$ *there are constants* $0 < C_1 \le C_2 < \infty$ *such that* $C_1 \le \psi_f(t,f,\alpha) := \frac{\partial}{\partial f}\psi(t,f,\alpha) \le C_2 \quad \forall\,(t,f,\alpha)\in[0,T_f]\times\mathbb{R}_+\times K$

- $\psi_\alpha(t,f,\alpha) := \frac{\partial}{\partial\alpha}\psi(t,f,\alpha) > 0 \quad \forall\,(t,f,\alpha)\in[0,T_f]\times \mathbb{R}_+ \times I.$

Here I represents some suitably chosen open intervall with $(-1,1)\subseteq I$.

2. The function c belongs to the class $\mathcal{C}^3(\mathbb{R}_+)$. It is convex, the first derivative c' satisfies $|c'(x)| < 1 \quad \forall x > 0$, and the second derivative c'' has compact support in $\mathbb{R}_+$.

REMARKS: Assumptions (A.1) and (A.2) ensure that (A.3) is satisfied by the reaction function defined in (3.7). The assumption of c being convex is quite palatable, since the payoffs which are synthesized in practice are usually convex.[5] Assuming differentiability is more problematic, since this excludes the payoffs of ordinary options. However, idealized option contracts where the kinks have been smoothed are within the scope of our analysis. Moreover, we may interprete c as an idealized description of the aggregated payoff of a diversified option portfolio containing a multitude of contracts with many different strikes; see for instance (Frey and Stremme 1995). As we will see in Theorem 5.2, the restriction on $|c'|$ ensures that the large trader's demand will never exceed the total supply of the risky asset.

In the following we will represent the derivative's terminal payoff in the form $c(X_T) = \rho\cdot h(X_T)$ where ρ is such that $\sup\{|h'(x)|, x>0\} = 1$. According to our normalization X_T gives the price of the total supply of the stock. Therefore we can interprete ρ as the fraction of the total supply of X which is insured by the large trader's hedging strategy. Defining h_0 and $\tilde\xi$ by the relations $c_0 = \rho\cdot h_0$ and $\xi = \rho\cdot\tilde\xi$ we immediately get that (3.8) is equivalent to

$$h(X_T) = h_0 + \int_0^T \tilde\xi_s^-\, d\psi(s,F_s,\rho\cdot\tilde\xi_s)\,. \tag{4.9}$$

Guided by the form of the hedge ratio in the classical Black-Scholes model we seek a solution of the option replication problem (4.9) having the form $\tilde\xi_s(\omega) = \phi(s,F_s(\omega))$ for some function $\phi : [0,T]\times\mathbb{R}_+ \to \mathbb{R}$.[6] We shall always assume that ϕ belongs to the class $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}_+)$.[7] Clearly such a strategy is admissible. The resulting asset price process is then given by $(X^\phi(t,F_t))_{0\le t\le T}$, where X^ϕ is shorthand for the composite function $\psi(t,f,\phi(t,f))$. In particular X is continuous. Since the stock price is now a function of t and F_t, the gains from trade in (4.9) can be computed by Itô's lemma yielding

$$\begin{aligned} h(X^\phi(T,F_T)) &= h_0 + \int_0^T \phi(s,F_s)\cdot\frac{\partial}{\partial f}X^\phi(s,F_s)dF_s + \\ &+ \int_0^T \phi(s,F_s)\cdot\left(\frac{\partial}{\partial t}X^\phi(s,F_s) + \frac{1}{2}\eta^2F_s^2\frac{\partial^2}{\partial f^2}X^\phi(s,F_s)\right)ds\,. \end{aligned} \tag{4.10}$$

This representation gives rise to the following

Proposition 4.1 *Suppose we are given a strategy function $\phi\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}_+)$. Then ϕ satisfies equation (4.10) if there exists a function $H : [0,T]\times\mathbb{R}_+\to\mathbb{R}$ with the following properties:*

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[5] This is obvious for standard options and forwards, and it is even the defining characteristic of the so-called *portfolio insurance* strategies.

[6] If a solution ϕ of (4.9) exists, by inverting the reaction function we may of course represent it in the usual manner as a function of time and asset price.

[7] By this we mean that ϕ is once continuously differentiable in t and twice in f, both on the set $[0,T]\times\mathbb{R}_+$.

(i) H belongs to $\mathcal{C}([0,T]\times \mathbb{R}_+)\cap \mathcal{C}^{1,2}([0,T)\times \mathbb{R}_+)$.

(ii) H satisfies the terminal condition $H(T,f)=h(X^\phi(T,f)) \quad \forall f\in \mathbb{R}_+$

(iii) We have for the derivatives

$$\frac{\partial}{\partial f}H(t,f) = \phi(t,f)\cdot \frac{\partial}{\partial f}X^\phi(t,f) \tag{4.11}$$

$$\frac{\partial}{\partial t}H(t,f)+\frac{1}{2}\eta^2 f^2\frac{\partial^2}{\partial f^2}H(t,f) = \phi(t,f)\cdot\left(\frac{\partial}{\partial t}X^\phi(t,f)+\frac{1}{2}\eta^2 f^2\frac{\partial^2}{\partial f^2}X^\phi(t,f)\right) \tag{4.12}$$

At time t the value of the hedge portfolio corresponding to ϕ is then given by $H(t,F_t)$.

Proof: To proof this proposition simply apply Itô's Lemma to the function H and note that (4.11) and (4.12) imply (4.10). □

Under some technical conditions also the converse of Proposition 4.1 holds. This is of interest, since it will help us to provide a complete *characterization* of solutions to the hedge problem satisfying certain regularity conditions. Suppose we are given a solution $\phi(t,f)\in\mathcal{C}^{1,2}([0,T]\times \mathbb{R}_+)$ of the hedging problem. To construct a function H as in Proposition 4.1 we proceed as in the standard option pricing theory and compute the value process of the hedge portfolio as conditional expectation of the terminal payoff with respect to the equivalent martingale measure Q for X. Since the Markov property of the process F is preserved under the transition from P to Q, this conditional expectation is given by some function $H(t,F_t)$. If this function is sufficiently smooth it will fulfill the requirements of Proposition 4.1. We now give a formal proof. Define a function $\mu(t,f):[0,T]\times\mathbb{R}_+\to\mathbb{R}$ by

$$\mu(t,f):=\left(\frac{\partial}{\partial t}X^\phi(t,f)+\frac{1}{2}\eta^2 f^2\frac{\partial^2}{\partial f^2}X^\phi(t,f)\right)\cdot\left(\frac{\partial}{\partial f}X^\phi(t,f)\right)^{-1}$$

and a density process $Z=(Z_t)_{0\le t\le T}$ by

$$Z_t:=\exp\left(-\int_0^T \mu(s,F_s)\cdot(\eta F_s)^{-1}\,dW_s-\frac{1}{2}\int_0^T\left(\mu(s,F_s)\cdot(\eta F_s)^{-1}\right)^2\,ds\right).$$

Suppose that

$$E\left[\exp\left(\frac{1}{2}\int_0^T\left(\mu(s,F_s)\cdot(\eta F_s)^{-1}\right)^2\,ds\right)\right]<\infty \tag{4.13}$$

Then Z is a martingale according to the Novikov criterion. Hence we may define a new probability measure Q on $\mathcal{F}_T$ by setting $dQ/dP:=Z_T$. It follows from Girsanov's theorem that under Q the process

$$\tilde W_t:=W_t+\int_0^t\mu(s,F_s)\cdot(\eta F_s)^{-1}\,ds$$

is a Brownian motion. We note that F and X solve the following equations

$$dF_t = \eta F_t d\tilde W_t-\mu(t,F_t)dt\,, \tag{4.14}$$

$$dX_t = \eta\cdot F_t\cdot\frac{\partial}{\partial f}X^\phi(t,F_t)d\tilde W_t\,, \tag{4.15}$$

In particular the stock price process is a local martingale under Q. Now we may state the converse to Proposition 4.1:

Proposition 4.2 *Suppose that $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}_+)$ is a solution to the replication problem (4.10) and that the following conditions are satisfied.*

(i) The SDE (4.14) is well-posed and the Novikov condition (4.13) holds.
(ii) Both, the asset price process X from (4.15) and the gains from trade $\int_0^t \phi(s,F_s)dX^\phi(s,F_s)$ are Q-martingales.
(iii) There is a solution $u \in \mathcal{C}([0,T] \times \mathbb{R}_+) \cap \mathcal{C}^{1,2}([0,T) \times \mathbb{R}_+)$ of the terminal value problem

$$\frac{\partial}{\partial t}u(t,f) + \frac{1}{2}\eta^2 f^2 \frac{\partial^2}{\partial f^2}u(t,f) - \mu(t,f)\frac{\partial}{\partial f}u(t,f) = 0\,, \quad u(T,f) = h(X^\phi(T,f)) \tag{4.16}$$

Then there exists a function $H \in \mathcal{C}([0,T] \times \mathbb{R}_+) \cap \mathcal{C}^{1,2}([0,T) \times \mathbb{R}_+)$ satisfying Proposition 4.1 (ii) and (iii).

Proof: Denote by $Q^{(t,f)}$ the law of the solution of the SDE (4.14) starting at time t with initial value equal to f. Since this SDE is well-posed we know that under $Q^{(t,f)}$ the coordinate process is a time-inhomogeneous Markov process, see for instance (Karatzas and Shreve 1988, Theorem 5.4.20). Defining $H(t,f)$ by

$$H(t,f) := E^{Q^{(t,f)}}[h(X^\phi(T,F_T))]$$

we therefore get $H(t,F_t) = E^Q[h(X^\phi(T,F_T))\,|\mathcal{F}_t]$. It follows from (4.14) and the Feynman-Kac representation theorem (Karatzas and Shreve 1988, Theorem 5.7.6) that H coincides with u and hence fulfills point (i) and (ii) of Proposition 4.1. Since ϕ solves the replication problem we moreover have

$$\begin{aligned} h(X^\phi(T,F_T)) &= \int_0^T \phi(s,F_s)dX^\phi(s,F_s) \\ &= E^Q[h(X^\phi(T,F_T)\,|\mathcal{F}_t] + \int_t^T \phi(s,F_s)dX^\phi(s,F_s) && (4.17) \\ &= H(t,F_t) + \int_t^T \phi(s,F_s)dX^\phi(s,F_s)\,, && (4.18) \end{aligned}$$

where (4.17) follows since the gains from trade are a martingale. On the other hand, since $h(X^\phi(T,f)) = H(T,f)$, Itô's Lemma yields

$$\begin{aligned} h(X^\phi(T,F_T)) &= H(t,F_t) + \int_t^T \frac{\partial}{\partial f}H(s,F_s)dF_s \\ &+ \int_t^T \left(\frac{\partial}{\partial t}H(s,F_s) + \frac{\eta^2}{2}F_s^2\frac{\partial^2}{\partial f^2}H(s,F_s)\right)ds \end{aligned} \tag{4.19}$$

By equating (4.18) and (4.19) we see that H satisfies also Proposition 4.1 (iii). □

We want to use Proposition 4.1 to construct a solution to the replication problem (4.10). If a function H with (4.11) and (4.12) exists we know that

$$\begin{aligned} \frac{\partial}{\partial t}H(t,f) &= \phi(t,f)\cdot\left(\frac{\partial}{\partial t}X^\phi(t,f) + \frac{1}{2}\eta^2 f^2 \frac{\partial^2}{\partial f^2}X^\phi(t,f)\right) \\ &- \frac{1}{2}\eta^2 f^2 \frac{\partial}{\partial f}\left(\phi(t,f)\cdot\frac{\partial}{\partial f}X^\phi(t,f)\right). \end{aligned} \tag{4.20}$$

Now if H exists it satisfies the identy $\frac{\partial}{\partial t}\frac{\partial}{\partial f}H(t,f) = \frac{\partial}{\partial f}\frac{\partial}{\partial t}H(t,f)$, yielding the following integrability condition for ϕ

$$\text{(4.21)} \quad \frac{\partial}{\partial t}\left(\phi(t,f)\cdot\frac{\partial}{\partial f}X^{\phi}(t,f)\right) = \frac{\partial}{\partial f}\Bigg[\phi(t,f)\cdot\left(\frac{\partial}{\partial t}X^{\phi}(t,f)+\frac{1}{2}\eta^2 f^2\frac{\partial^2}{\partial f^2}X^{\phi}(t,f)\right) - \frac{1}{2}\eta^2 f^2\frac{\partial}{\partial f}\left(\phi(t,f)\cdot\frac{\partial}{\partial f}X^{\phi}(t,f)\right)\Bigg]$$

Elementary but tedious computations given in Appendix A.1 now lead to the following

Lemma 4.3 *A function $\phi \in \mathcal{C}^{1,2}([0,T]\times \mathbb{R}_+)$ satisfies (4.21) if and only if it is a solution to the following PDE*

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\phi(t,f)+\frac{1}{2}\eta^2 f^2\left(1+2\rho\frac{\psi_\alpha}{\psi_f}\cdot\frac{\partial}{\partial f}\phi(t,f)\right)\frac{\partial^2}{\partial f^2}\phi(t,f) \\ &+ \frac{\eta^2}{\psi_f}\frac{\partial}{\partial f}\phi(t,f)\cdot\Bigg[f\cdot\psi_f-\psi_t+\frac{f^2}{2}\psi_{ff} \\ &+ \rho\frac{\partial}{\partial f}\phi(t,f)\left(f^2\psi_{\alpha f}+f\psi_\alpha\right)+\left(\rho\frac{\partial}{\partial f}\phi(t,f)\right)^2\frac{f^2}{2}\psi_{\alpha\alpha}\Bigg] \end{aligned} \qquad \text{(4.22)}$$

Here the arguments of t ψ and its derivatives are given by $(t,f,\rho\cdot\phi(t,f))$.

Remark: The PDE (4.22) is *quasilinear* in the terminology of Friedman (1964) or Ladyzenskaja, Solonnikov, and Ural'ceva (1968), that is the coefficients depend not only on time and space variables but also on the solution and its first derivative. If there are no feedback effects, that is if $\psi_\alpha \equiv 0$, and if moreover $\psi(t,f,0)=f \quad \forall\, t,f$ the PDE (4.22) boils down to the usual linear PDE which is satisfied by the hedge ratio in the Black-Scholes model.

In the next theorem we show how to construct solutions of the replication problem (4.10) as solutions to a terminal value problem involving the PDE (4.22) and give a characterization of all solutions that possess certain smoothness properties.

Theorem 4.4 *Suppose that Assumption (A.3) holds. Then a strategy function ϕ belonging to $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}_+)$ solves the hedging problem with feedback (4.10) if it solves the PDE (4.22) and satisfies the terminal condition*

$$\phi(T,f)=h'(X^{\phi}(T,f)) \quad \forall\, f>0 \qquad \text{(4.23)}$$

Conversely, if $\phi \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R}_+)$ solves the hedging problem and if moreover the assumptions from Proposition 4.2 are satisfied for the corresponding asset price process, ϕ is a solution to the terminal value problem (4.22), (4.23).

Proof: To prove the first statement we want to construct a function H satisfying the requirements of Proposition 4.1. If ϕ solves the PDE (4.22), equations (4.11) and (4.20) define a vector field that satisfies the integrability conditions. Since the domain $(0,T)\times\mathbb{R}_+\subset\mathbb{R}^2$ is convex there exists a function H — uniquely defined up to a constant — with (4.11), (4.20) and hence (4.12). Since the derivatives of ϕ are continuous functions on the set $[0,T]\times\mathbb{R}_+$ the derivative $\frac{\partial}{\partial t}H(t,f)$ given by (4.20) is bounded on every strip of the form $[T-\delta,T]\times K$ for $K\subset\subset\mathbb{R}_+$. Hence we may extend H to a continuous function on $[0,T]\times\mathbb{R}_+$. Moreover, H can be defined in such a way that $H(T,f_0)=h(X^{\phi}(T,f_0))$ for some $f_0\in\mathbb{R}_+$.

It remains to prove that $\frac{\partial}{\partial f}H(T,f) = h'(X^\phi(T,f))$, since then also requirement (iii) of Proposition 4.1 is fulfilled. Since the product $\phi(t,f)\cdot\frac{\partial}{\partial f}X^\phi(t,f)$ converges locally uniformly to $\phi(T,f)\cdot\frac{\partial}{\partial f}X^\phi(T,f)$ as $t\to T$ we get

$$\phi(T,f)\cdot\frac{\partial}{\partial f}X^\phi(T,f) = \lim_{t\to T}\phi(t,f)\cdot\frac{\partial}{\partial f}X^\phi(t,f) = \lim_{t\to T}\frac{\partial}{\partial f}H(t,f) = \frac{\partial}{\partial f}H(T,f)$$

On the other hand we have $\frac{\partial}{\partial f}h(X^\phi(T,f)) = h'(X^\phi(T,f))\cdot\frac{\partial}{\partial f}X^\phi(T,f)$ such that the terminal condition (4.23) yields the desired equality of the derivatives.

To prove the converse statement we first note that Proposition 4.2 implies the existence of a smooth function H with (4.11) and (4.20) such that ϕ satisfies the integrability conditions and solves hence the PDE (4.22). The terminal condition must hold since again $\frac{\partial}{\partial f}H(T,f)$ and $\frac{\partial}{\partial f}h(X^\phi(T,f))$ must be equal. □

Note that the terminal value $\phi(T,f)$, which reflects the special form of the replicated payoff, is given by the solution φ^* to the equation

$$h'(\psi(T,f,\rho\cdot\varphi)) = \varphi \tag{4.24}$$

Lemma 4.5 *Suppose that Assumption (A.3) holds. If then for some* $\delta > 0$

$$\rho\cdot\sup\left\{h''(\psi(T,f,\rho\cdot\varphi))\cdot\psi_\alpha(T,f,\rho\cdot\varphi),\ f\in\mathbb{R}_+,\varphi\in I\right\} < 1-\delta \tag{4.25}$$

there exists for every $f\in\mathbb{R}_+$ *a unique solution* $\varphi^*(f)$ *to (4.24). The function* $f\mapsto\varphi^*(f)$ *is twice continuously differentiable with bounded derivatives and its first derivative is positive.*

Proof: Existence of a solution follows since for all $f\in\mathbb{R}_+$ the mapping $\varphi\mapsto\varphi-h'(\psi(T,f,\rho\cdot\varphi))$ is continuous. Condition (4.25) implies that this mapping is strictly increasing which implies uniqueness. Differentiability follows from the Implicit Function Theorem. By differentiating both sides of (4.24) we get

$$\frac{\partial}{\partial f}\varphi^*(f) = \frac{h''\left(\psi(T,f,\rho\cdot\varphi^*)\right)\cdot\psi_f(T,f,\rho\cdot\varphi^*)}{1-\rho h''\left(\psi(T,f,\rho\cdot\varphi^*)\right)\cdot\psi_\alpha(T,f,\rho\cdot\varphi^*)}$$

which is positive by (4.25) and the convexity of h. Boundedness of the derivatives of the function $\varphi^*(f)$ follows, since h'' has compact support by (A.3).

5 Analysis of the PDE for the Hedging-Strategy

In this section we prove existence and uniqueness of a solution to the terminal value problem given by the PDE (4.22) and the terminal condition

$$\phi(T,f) = g(f)\,. \tag{5.26}$$

We make the following regularity assumptions on the terminal values.

Assumption (A.4) *The function* $\tilde{g}:\mathbb{R}\to\mathbb{R}$, $x\mapsto g(\exp(x))$ *belongs to* $\mathcal{C}^2(\mathbb{R})$, *its derivatives are bounded and* $\tilde{g}''$ *is Hölder-continuous on* $\mathbb{R}$ *for some Hölder-exponent* $\beta\in(0,1)$. *Moreover* $\sup_{f\in\mathbb{R}_+}|g(f)|\le 1$ *and* $\frac{\partial}{\partial f}g(f) > 0$.

Note that by Lemma 4.5 (A.4) is satisfied for the function defined by the terminal condition (4.23). Before we can state the main result of this section we have to specify the regularity conditions we impose on our solutions.

Definition 5.1 *Let β be some number from the intervall $(0,1)$.*

1. A function $u \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ is said to be Hölder-continuous of class $H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$, if u and its derivatives are bounded on $[0,T]\times\mathbb{R}$, and if moreover the derivatives u_x, u_{xx} and u_t satisfy a Hölder condition in x with exponent β and a Hölder condition in t with exponent $\beta/2$.

2. A function $\phi \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R}_+)$ belongs to the space $H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R}_+)$ if the function u defined by $u(t,x) := \phi(t,\exp(x))$ belongs to $H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$.

A more formal definition of this and related Hölder spaces is given in (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, chapter 1). To guarantee existence of a solution to the terminal value problem we have to impose the following restrictions on the reaction function ψ.

Assumption (A.5) *For every compact set $K \subset I$ there are finite constants $K_1,\dots,K_5$ such that for all $t\in[0,T], f>0, \alpha\in K$*

$$\begin{array}{lll} |\psi_\alpha(t,f,\alpha)/\psi(t,f,\alpha)| < K_1, & |\psi_{\alpha\alpha}(t,f,\alpha)/\psi(t,f,\alpha)| < K_2, & |\psi_{f\alpha}(t,f,\alpha)| < K_3 \\ |f\cdot\psi_{ff}(t,f,\alpha)| < K_4, & |\psi_t(t,f,\alpha)/\psi(t,f,\alpha)| < K_5 & \end{array}$$

Remarks: The constants $K_1,\dots,K_3$ can be interpreted as measures of market liquidity. Assumption (A.5) is always satisfied if — as in the models of Jarrow (1994), Platen and Schweizer (1994) or Frey and Stremme (1995) — the reaction function is of the particular form $\psi(t,f,\alpha) = \widetilde{\psi}(t,\alpha)\cdot f$. It holds true for many other reaction functions with similar asymptotic properties for $f\to 0$ and $f\to\infty$, too.

Theorem 5.2 *Suppose that Assumptions (A.3), (A.5) hold and that (A.4) is satisfied for some Hölder exponent β. Then the following holds*

(i) There is some $0<\bar\rho\le 1$ such that for every $\rho\le\bar\rho$ the terminal value problem (4.22), (5.26) has a solution contained in $H^{2+\beta,1+\beta/2}$ $([0,T]\times\mathbb{R}_+)$.

(ii) Every solution $\phi\in H^{2+\beta,1+\beta/2}$ $([0,T]\times\mathbb{R}_+)$ of this terminal value problem has the following properties

- $\inf_{f\in\mathbb{R}_+} g(f) \le \phi(t,f) \le \inf_{f\in\mathbb{R}_+} g(f)$
- $\frac{\partial}{\partial f}\phi(t,f) > 0$

(iii) For every $\beta>0$ and every $\rho\in[0,1)$ there is at most one solution of the terminal value problem (4.22), (5.26) belonging to the Hölder space $H^{2+\beta,1+\beta/2}$ $([0,T]\times\mathbb{R}_+)$.

Remarks: To guarantee existence of a solution to the terminal problem for the hedge ratio we have to restrict the market weight ρ of the large trader. This additional qualification is needed, since when dealing with nonlinear PDE's one has to impose certain restrictions on the character of the nonlinear occurrences of the solution and its first derivative in the coefficients of the equation, cf. (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, chapter 1.3). The constant $\bar\rho$ depends essentially on two factors, first on the "Gamma" of the terminal payoff, that is on $\sup\{g'(f)\cdot f,\ f\in \mathbb{R}_+\}$ and second on the liquidity of the market as measured by the K_i in (A.5). Statement (ii) implies that the qualitative properties of the hedge ratio are unaltered by tworking in finitely elastic markets. However, we will see in the simulations of section 6 that there are quantitative differences which may be quite large.

The rest of this section is devoted to the proof of theorem 5.2. Our main tool will be some results from Ladyzenskaja, Solonnikov, and Ural'ceva (1968). To apply these results we have to transform the terminal value problem on $\mathbb{R}_+$ into an initial value problem on $\mathbb{R}$. To this end we introduce the new time variable $\tau(t)=T-t$ and the new space variable $x(f)=\ln(f)$. We define a function $u:[0,T]\times\mathbb{R}_+\to\mathbb{R}$ by $\phi(t,f)=:u(\tau(t),x(f))$. Elementary calculations show that ϕ solves the terminal value problem (4.22), (5.26) if and only if the function u is a solution of the Cauchy problem

$$u_t = a(\rho,t,x,u,u_x)\cdot u_{xx} + b(\rho,t,x,u,u_x)\cdot u_x \tag{5.27}$$

$$u(0,x) = \tilde g(x)\,, \tag{5.28}$$

where the functions a and b are given by

$$a(\rho,t,x,u,q) = \frac{1}{2}\eta^2\cdot\left(1+2\cdot\rho\cdot q\cdot e^{-x}\cdot\frac{\psi_\alpha(T-t,e^x,\rho\cdot u)}{\psi_f(T-t,e^x,\rho\cdot u)}\right) \tag{5.29}$$

$$b(\rho,t,x,u,q) = \frac{\eta^2}{\psi_f}\cdot\left(\frac{1}{2}\psi_f+\frac{1}{2}\cdot e^x\psi_{ff}-e^{-x}\psi_t+\rho q\psi_{f\alpha}+\frac{\rho^2q^2}{2}e^{-x}\cdot\psi_{\alpha\alpha}\right)\,. \tag{5.30}$$

In (5.30) the arguments of ψ and its derivatives are given by $(T-t,e^x,\rho\cdot u)$, too. From now on we concentrate on the initial value problem (5.27) (5.28). For technical reasons we have to introduce truncated versions $\bar a$ and $\bar b$ of our coefficients. Their precise definition is given in the Appendix A.2. For an appropriate choice of $\bar a$ the PDE

$$u_t = \bar a(\rho,t,x,u,u_x)\cdot u_{xx} + \bar b(\rho,t,x,u,u_x)\cdot u_x \tag{5.31}$$

is *parabolic*, such that we can apply results from the theory of quasilinear parabolic PDE's to our problem. In the next proposition we establish existence and uniqueness of the Cauchy problem (5.31) (5.28). After that we will proof certain properties of the solutions, thereby showing that for ρ sufficiently small a solution to the PDE (5.31) solves also the original equation (5.27), which proves Theorem 5.2 (i) and (ii).

Proposition 5.3 *Suppose that the initial values g satisfy Assumption (A.4).*

(i) Then for all $\rho\in[0,1)$ there is at least one solution u to the Cauchy problem (5.31), (5.28) belonging to $H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$.

(ii) For every $0\le\rho_0<1$ there is some constant K depending only on the Hölder norm of the initial values $\tilde g$ and on the size of the constants in Assumption (A.5), such that $|u_x(t,x)|<K$ for all $0\le\rho\le\rho_0$, $t\in[0,T]$, $x\in\mathbb{R}$.

(iii) For any $\beta\in(0,1)$ and any $\rho\in[0,1]$ there is at most one solution of the Cauchy problem (5.31), (5.28) belonging to $H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$.

The proof consists of an application of (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, Theorem 5.8.1) and is given in Appendix A.2.

In the following proposition we prove the most important properties of solutions to the Cauchy problem (5.31), (5.28) and show that they carry over to the unrestricted Cauchy problem (5.27), (5.28).

Proposition 5.4 *Every solution of the Cauchy problem (5.31), (5.28) belonging to some Hölder space* $H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$ *for some* $\beta>0$ *satisfies* $\forall\,(t,x)\in[0,T]\times\mathbb{R}$:

$$\inf_{x\in\mathbb{R}}\tilde{g}(x)\le u(t,x)\le\sup_{x\in\mathbb{R}}\tilde{g}(x)\ \text{and}\ u_x(t,x)\ge 0$$

These properties carry over to solutions of the unrestricted Cauchy problem (5.27), (5.28) belonging to $H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$ *for some* $\beta>0$.

PROOF: The key of the proof is the following observation: Whenever u solves the quasilinear PDE (5.31), it is also a solution to the following *linear* parabolic equation

$$u_t=a^u(t,x)u_{xx}+b^u(t,x)u_x \tag{5.32}$$

where $a^u(t,x):=\bar{a}(\rho,t,x,u(t,x),u_x(t,x))$ and $b^u(t,x):=\bar{b}(\rho,t,x,u(t,x),u_x(t,x))$.[8] The bounds on u follow therefore directly from the maximum principle for linear parabolic PDE's, or they can be read from the Feynman-Kac representation of u.

To prove the positivity of u_x we first note that (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, Theorem 3.12.2) can be applied to the PDE (5.32), yielding $u\in H^{3+\beta,(3+\beta)/2}((0,t)\times\Omega)$ for every $\Omega\subset\subset\mathbb{R}$. In particular the derivatives $\frac{\partial}{\partial t}\frac{\partial}{\partial x}u$ and $\partial^3u/\partial x^3$ are well-defined. Hence by differentiating (5.32) we obtain the following linear parabolic PDE for $v(t,x):=u_x(t,x)$

$$v_t=a^u(t,x)v_{xx}+v_x\cdot\left(\frac{\partial}{\partial x}a^u(t,x)+b^u(t,x)\right)+v\cdot\frac{\partial}{\partial x}b^u(t,x)$$

Of course v is continuous on $[0,T]\times\mathbb{R}$ and has initial values $v(0,x):=\frac{\partial}{\partial x}g(\exp(x))$. By Assumption (A.4) we hence get $v(0,x)\ge 0$. It can be checked that the regularity conditions for the Feynman-Kac theorem are fulfilled. Therefore we obtain the following stochastic representation for v:

$$v(t,x)=E^{(T-t,x)}\left[v(0,Y_T)\cdot\exp\left(\int_{T-t}^{T}\frac{\partial}{\partial x}b^u(s,Y_s)ds\right)\right] \tag{5.33}$$

where Y solves the SDE

$$dY_t=\sqrt{a^u(t,Y_t)}dW_t+\left(b^u(t,Y_t)+\frac{\partial}{\partial x}a^u(t,Y_t)\right)dt$$

Since all the terms in (5.33) are nonnegative it follows immediately that $v(t,x)\ge 0$.

Now let us turn to the second claim. Suppose we are given a function $u\in H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$ solving the PDE (5.27) with initial values (5.28), for which there exists some (t_0,x_0) with $u_x(t_0,x_0)<0$. Since $u_x(0,x)\ge 0$, and since by definition of the Hölder space $H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$ u_x is Hölder continuous in t uniformly in x, there is then for every $\varepsilon>0$ some pair (t^*,x^*) with

[8] We omit ρ from the definition of a^u and b^u since in the proof this parameter is kept constant

- $\forall\, t \le t^*,\ \forall\, x \in \mathbb{R} \quad u_x(t,x) > -\varepsilon/2.$
- $u_x(t^*, x^*) < 0$

On the other hand for appropriate truncation functions u is also a solution to the restricted Cauchy problem (5.31), (5.28) on $[0,t^*] \times \mathbb{R}$. Hence $u_x(t^*, x^*) \ge 0$, a contradiction. □

REMARK: The proof of the positivity of u_x shows that option prices are convex functions of the price of the underlying security in any Markovian model where the stock price follows a diffusion equation of the form $dX_t = \sigma(t, X_t)X_t dW_t$ for a sufficiently smooth function σ. This result has independently been proven by Bergman, Grundy, and Wiener (1995) and, using a probabilistic argument, by El Karoui, Jeanblanc-Picqué, and Shreve (1995).

It is now easy to proof Theorem 5.2 using Propositions 5.3 and 5.4; see Appendix A.2.

6 Results from Simulations

In case there are no feedback effects from the large trader's position into equilibrium prices our theory for option pricing and hedging boils down to the standard theory as developed for instance by Black and Scholes (1973) and Harrison and Pliska (1981). Using explicit numerical computations we now want to compare the hedge ratio and the value of the hedge portfolio for an option in a finitely elastic market to option prices and hedge ratios in the Black-Scholes model. In all simulations we work with the reaction function

$$\psi(t, f, \alpha) = f/(1-\alpha)$$

introduced in section 2 and with a terminal payoff given by

$$h(x) = \frac{1}{2}\left(x - K + \sqrt{(x-K)^2 + \alpha}\,\right)$$

for some small $\alpha > 0$, i.e. we are considering Call options with "smoothed kinks". For these data there exists a solution to the option replication problem (3.8) by Theorems 4.4 and 5.2. To numerically solve the PDE (4.22) we used the method of *implicit finite differences* as explained for instance in (Willmott, Dewynne, and Howison 1993, Chapter 19). By Theorem 5.2 (ii) we know that $\frac{\partial}{\partial f}\phi(t,f) > 0$. Therefore we get $\frac{\partial}{\partial f}X^\phi(t,f) > 0$. Hence for every fixed t the fundamental f can be expressed as $(X^\phi)^{-1}(t,x)$. This allows us to represent also the hedging strategy as a function $\varphi(t,x)$ of the equilibrium price:

$$\phi(t,f) = \phi\left(t, (X^\phi)^{-1}(t, X^\phi(t,f))\right) =: \varphi(t, X^\phi(t,f)) \tag{6.34}$$

The first simulation we have run illustrates that the qualitative properties of the hedge ratio remain unaltered by assuming that markets are only finitely elastic, a fact we have proven already in point (ii) of Theorem 5.2. In figure 1 we have graphed the solution of our hedge problem as function $\varphi(t,x)$ of time t and price of the underlying asset x for $\rho = 0.2$. The plot looks very similar to usual pictures of the "Delta" in the Black-Scholes model. However, there are quantitative differences, as it is shown by figure 2 where we have plotted the hedge ratio at $t = 1$ as a function of x for different values of ρ.

We have also computed H_0, the hedge costs per contract as defined in Proposition 4.1, for different values of ρ. The results of the simulations are plotted in figure 3 and figure 4. It is obvious that the initial value of the hedge portfolio increases with increasing ρ. Comparing figure 3 where we have plotted H_0 against the fundamental f to figure 4 where we have varied the underlying's price x we see that in figure 3 the increase in the value of the hedge portfolio caused by a rise in ρ is much more pronounced. This comparison reveals two reasons for the increase in the hedge costs. First an increasing ρ implies an increase in the large trader's stock position and hence increasing asset prices. Second the rise in ρ causes a rise in stock price volatility, which explains why the hedge costs increase even if the asset price is kept constant.

This dependence of the hedge costs on the overall amount of hedging is genuine to our model with finitely elastic markets. It also shows that — in contrast to standard option pricing theory — in our framework derivative prices do depend on the large trader's derivative position. As indicated already by Jarrow (1994) this could possibly explain some anomalies on real options markets such as the smile pattern of implied volatilities first observed by Rubinstein (1985).

7 Conclusion

The paper studies the pricing and hedging of derivatives in an economy with a large trader whose trades move prices. It extends previous results of Jarrow (1994) on this issue to a continuous time setting. First a theory for option pricing in this framework is developed. It turns out that in finitely elastic markets we need an additional assumption, the synchronous market condition, to arrive at a fully specified pricing theory. We then go on and study the problem of option replication in our framework. We characterize the strategy by which the large trader can synthesize the payoff of a derivative contract as solution of a nonlinear PDE. Additional qualifications on market liquidity and the second derivative of the payoff not needed in the standard theory are necessary to guarantee existence of a solution to this PDE. We show that the qualitative shape of the hedge portfolio is unaltered by working in finitely elastic markets. However, simulations reveal that there are qualitative differences which may be quite large. Moreover, we find that hedge costs and hence option prices do depend on the large trader's position in stock and derivatives. This observation might help to explain some "anomalies" on real options markets. However, this awaits future research.[9]

[9] A first attempt into this direction has already been made in the paper (Platen and Schweizer 1994) which seeks to explain volatility smiles by feedback effects from dynamic hedging.

8 Results from Simulations

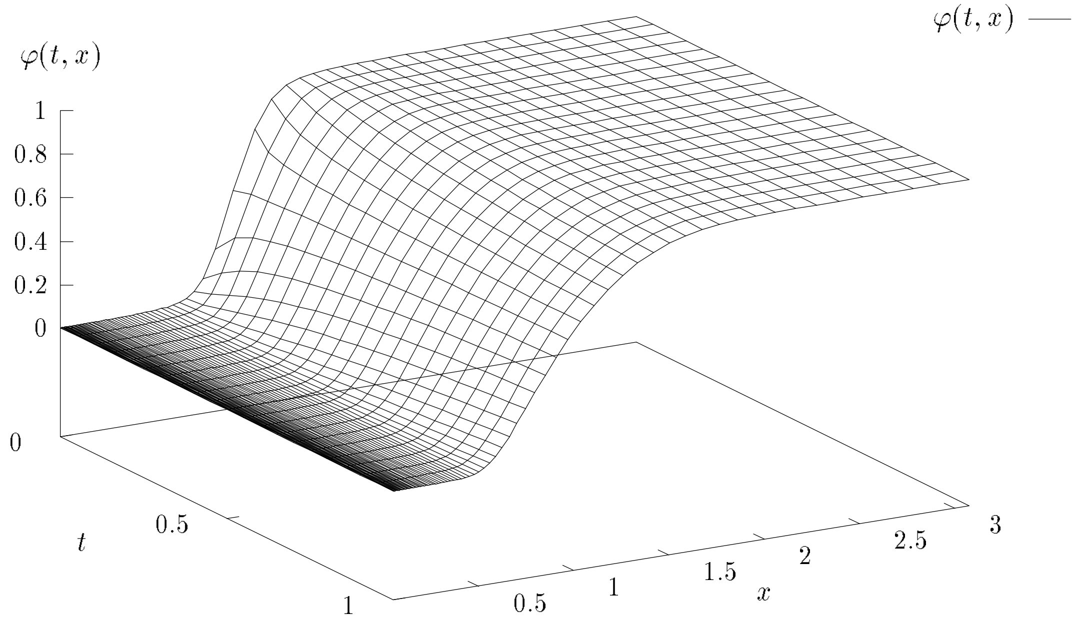

Figure 1: Hedge ratio as function of price and time for a value $\rho = 0.2$

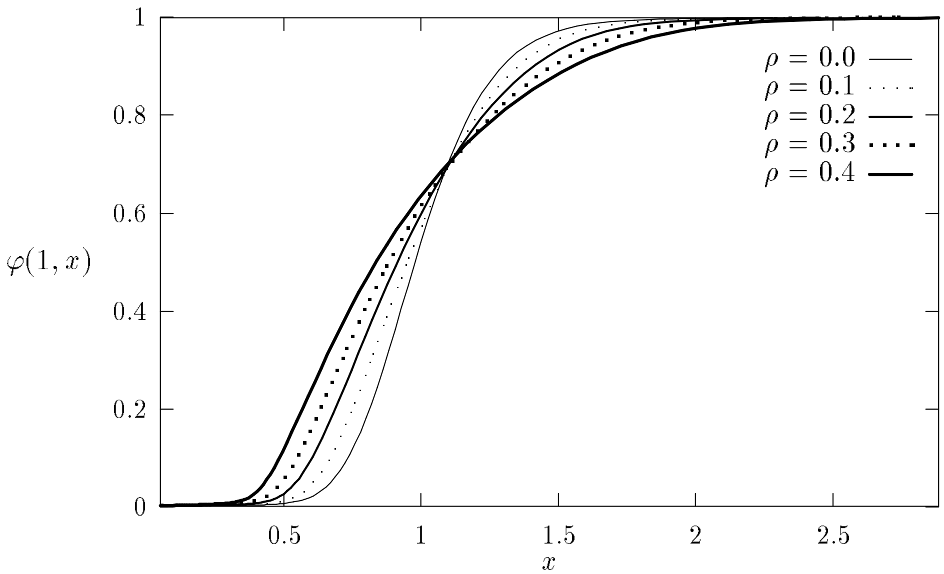

Figure 2: Hedge ratio $\varphi(1,x)$ at $t=1$ for different values of ρ

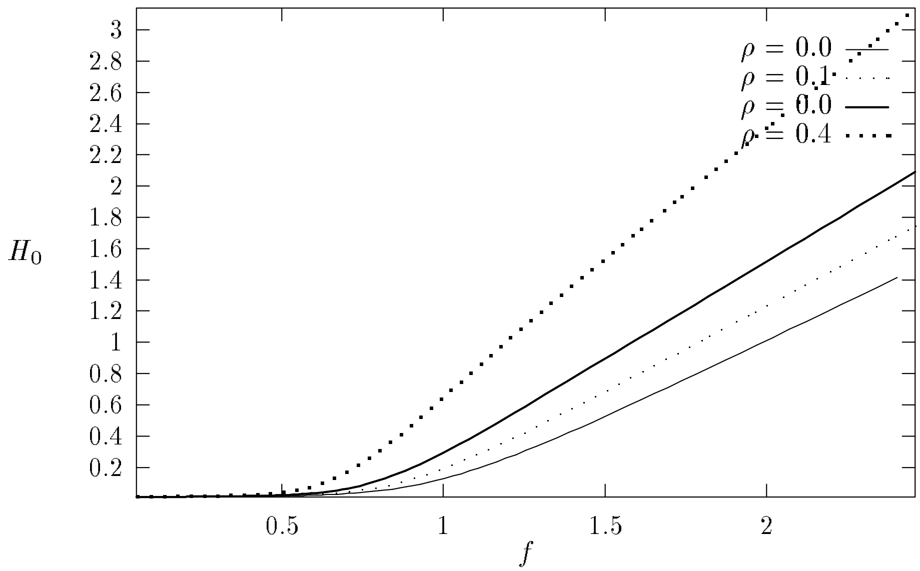

Figure 3: Cost of hedging per contract H_0 as a function of the fundamental f for different values of ρ

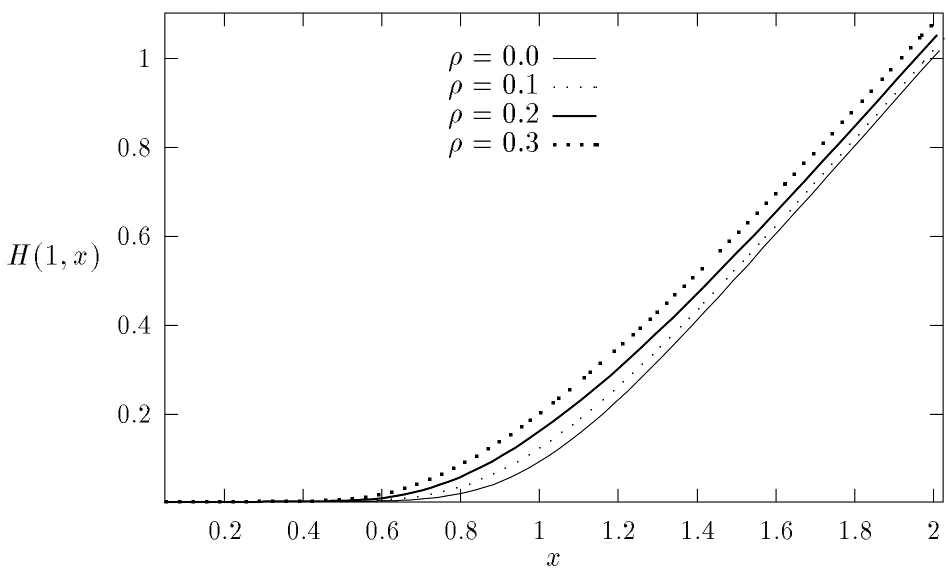

Figure 4: Cost of hedging per contract H_0 as a function of the current price x for different values of ρ

A Mathematical Appendix

A.1 Proof of Lemma 4.3

To shorten the notation we will always omit the arguments (t,f). We start by computing $\frac{\partial}{\partial t}(\phi\cdot\frac{\partial}{\partial f}X^{\phi})$ and get

$$\frac{\partial}{\partial t}(\phi\cdot\frac{\partial}{\partial f}X^{\phi}) = \frac{\partial}{\partial t}\phi\cdot\frac{\partial}{\partial f}X^{\phi} + \phi\cdot\frac{\partial^2}{\partial f\partial t}X^{\phi} \tag{A.35}$$

Now we turn to calculating the right hand side of (4.21). We get

$$\begin{aligned}\phi\left(\frac{\partial}{\partial t}X^{\phi}+\frac{1}{2}\eta^2 f^2\frac{\partial^2}{\partial f^2}X^{\phi}\right) &- \frac{1}{2}\eta^2 f^2\frac{\partial}{\partial f}\left(\phi\cdot\frac{\partial}{\partial f}X^{\phi}\right) = \\ &= \phi\cdot\frac{\partial}{\partial t}X^{\phi}-\frac{1}{2}\eta^2 f^2\cdot\frac{\partial}{\partial f}\phi\cdot\frac{\partial}{\partial f}X^{\phi}\end{aligned} \tag{A.36}$$

Computation of the derivative of (A.36) wrt. f now yields the right hand side of (4.21). Here we get

$$\begin{aligned}\phi\cdot\frac{\partial^2}{\partial f\partial t}X^{\phi} &+ \frac{\partial}{\partial f}\phi\cdot\frac{\partial}{\partial t}X^{\phi}-\eta^2 f\cdot\frac{\partial}{\partial f}\phi\cdot\frac{\partial}{\partial f}X^{\phi}- \\ &-\frac{1}{2}\eta^2 f^2\cdot\frac{\partial^2}{\partial f^2}\phi\cdot\frac{\partial}{\partial f}X^{\phi}-\frac{1}{2}\eta^2 f^2\cdot\frac{\partial}{\partial f}\phi\cdot\frac{\partial^2}{\partial f^2}X^{\phi}\end{aligned} \tag{A.37}$$

Equating both sides of (4.21), i.e. (A.35) and (A.37) now yields the following PDE for ϕ:

$$\begin{aligned}\frac{\partial}{\partial t}\phi\cdot\frac{\partial}{\partial f}X^{\phi}+\phi\cdot\frac{\partial^2}{\partial f\partial t}X^{\phi} &= \phi\cdot\frac{\partial^2}{\partial f\partial t}X^{\phi}+\frac{\partial}{\partial f}\phi\cdot\frac{\partial}{\partial t}X^{\phi}-\eta^2 f\cdot\frac{\partial}{\partial f}\phi\cdot\frac{\partial}{\partial f}X^{\phi} \\ &- \frac{1}{2}\eta^2 f^2\cdot\frac{\partial^2}{\partial f^2}\phi\frac{\partial}{\partial f}X^{\phi}-\frac{1}{2}\eta^2 f^2\cdot\frac{\partial}{\partial f}\phi\cdot\frac{\partial^2}{\partial f^2}X^{\phi}.\end{aligned} \tag{A.38}$$

Since

$$\begin{aligned}\frac{\partial}{\partial t}\phi\cdot\frac{\partial}{\partial f}X^{\phi} &= \frac{\partial}{\partial t}\phi\cdot\left(\psi_f+\rho\cdot\psi_\alpha\cdot\frac{\partial}{\partial f}\phi\right) \text{ and} \\ \frac{\partial}{\partial f}\phi\cdot\frac{\partial}{\partial t}X^{\phi} &= \frac{\partial}{\partial f}\phi\cdot\left(\psi_t+\rho\cdot\psi_\alpha\cdot\frac{\partial}{\partial t}\phi\right),\end{aligned}$$

cancelling terms on both sides yields the following version of the PDE (A.38).

$$\begin{aligned}\frac{\partial}{\partial t}\phi\cdot\psi_f &= \frac{\partial}{\partial f}\phi\cdot\psi_t-\eta^2\cdot f\cdot\frac{\partial}{\partial f}\phi\cdot\frac{\partial}{\partial f}X^{\phi} \\ &-\frac{1}{2}\eta^2 f^2\cdot\frac{\partial^2}{\partial f^2}\phi\cdot\frac{\partial}{\partial f}X^{\phi}-\frac{1}{2}\eta^2 f^2\cdot\frac{\partial}{\partial f}\phi\cdot\frac{\partial^2}{\partial f^2}X^{\phi}\end{aligned}$$

Now we use that

$$\begin{aligned}\frac{\partial}{\partial f}X^{\phi} &= \psi_f+\rho\cdot\psi_\alpha\cdot\frac{\partial}{\partial f}\phi \\ \frac{\partial^2}{\partial f^2}X^{\phi} &= \psi_{ff}+\psi_{\alpha f}\cdot\rho\cdot\frac{\partial}{\partial f}\phi+\rho\cdot\frac{\partial}{\partial f}\phi\cdot\left(\psi_{\alpha f}+\psi_{\alpha\alpha}\cdot\rho\cdot\frac{\partial}{\partial f}\phi\right)+\rho\cdot\psi_\alpha\cdot\frac{\partial^2}{\partial f^2}\phi \\ &= \psi_{ff}+2\psi_{\alpha f}\cdot\rho\cdot\frac{\partial}{\partial f}\phi+\rho^2\psi_{\alpha\alpha}\left(\frac{\partial}{\partial f}\phi\right)^2+\rho\psi_\alpha\frac{\partial^2}{\partial f^2}\phi\end{aligned}$$

and obtain the following version of (A.38)

$$\begin{aligned}
\psi_f \cdot \frac{\partial}{\partial t}\phi &= -\frac{1}{2}\eta^2 f^2 \cdot \frac{\partial^2}{\partial f^2}\phi \cdot \left(\psi_f + \rho \cdot \psi_\alpha \cdot \frac{\partial}{\partial f}\phi + \rho \cdot \psi_\alpha \frac{\partial}{\partial f}\phi\right) \\
&- \frac{\partial}{\partial f}\phi \cdot \left(-\psi_t + \eta^2 \cdot f \cdot \psi_f + \frac{1}{2}\eta^2 f^2 \psi_{ff}\right) \\
&- \left(\frac{\partial}{\partial f}\phi\right)^2 \cdot (\eta^2 \cdot f^2 \cdot \rho \cdot \psi_{\alpha f} + \eta^2 \cdot f \cdot \rho \cdot \psi_\alpha) - \left(\frac{\partial}{\partial f}\phi\right)^3 \cdot \left(\frac{1}{2}\eta^2 \cdot \rho^2 \cdot \psi_{\alpha\alpha}\right)
\end{aligned}$$

Rearranging terms we see that this is the PDE from Lemma 4.3. □

A.2 Complements to the Proof of Theorem 5.2

Definition of the truncated coefficients in the PDE (5.31)

We define the truncated versions $\bar{a}$ and $\bar{b}$ of the coefficient functions defined in (5.29) and (5.30) as follows. We set

$$\bar{a}(\rho,t,x,u,q) := a(\rho,t,x,u,\frac{1}{\rho}\cdot c_1(\rho q))\,,\ \bar{b}(\rho,t,x,u,q) := b(\rho,t,x,u,\frac{1}{\rho}\cdot c_2(\rho q))$$

Here $c_1,\ c_2 :\ \mathbb{R} \to \mathbb{R}$ are smooth functions with

$$\text{(A.39)} \qquad \begin{array}{lll} 1 \geq c_1' \geq 0, & c_1(y) = y \text{ on } [-\varepsilon/2, M - \varepsilon/2]\,, & c_1(y) \in [-\varepsilon, M] \\ 1 \geq c_2' \geq 0, & c_2(y) = y \text{ on } [-M + \varepsilon/2, M - \varepsilon/2], & c_2(y) \in [-M, M], \end{array}$$

where $\varepsilon > 0$ is small and M is some large positive constant. As the solutions to our Cauchy problems are bounded (see Proposition 5.4), by Assumption (A.5) we are able to find for every $0 \leq \rho_0 < 1$ some ε in (A.39) which is small enough to ensure that there is some constant $\tilde{K}_0 > 0$ such that

$$\text{(A.40)} \qquad \inf\{\bar{a}(\rho,t,x,u,q),\ \rho \in [0,\rho_0], t \in (0,T], x \in \mathbb{R}, u \in I^g, q \in \mathbb{R}\} > \tilde{K}_0$$

Hence for this choice of c_1 the PDE (5.31) is parabolic.

Proof of Proposition 5.3

To prove the proposition we have to show that (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, Theorem 5.8.1) can be applied to the Cauchy problem (5.31), (5.28). This theorem is on equations in *divergence form* that is on PDEs of the form

$$u_t - \frac{\partial}{\partial x}[a^{\text{div}}(t,x,u(t,x),u_x(t,x))] + b^{\text{div}}(t,x,u(t,x),u_x(t,x)) = 0$$

To write the PDE (5.31) in divergence form we have to choose

$$\begin{aligned}
a^{\text{div}}(\rho,t,x,u,q) &:= \int_0^q \bar{a}(\rho,t,x,u,\tau)d\tau \\
b^{\text{div}}(\rho,t,x,u,q) &:= -\bar{b}(\rho,t,x,u,q)\cdot q + \frac{\partial a^{\text{div}}}{\partial x}(\rho,t,x,u,q) + \frac{\partial a^{\text{div}}}{\partial u}(\rho,t,x,u,q)\cdot q\,.
\end{aligned}$$

To prove the statement on existence of solutions we now check that the hypothesis of (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, Theorem 5.8.1) are satisfied.

ad a): This hypothesis is directly implied by Assumption (A.4).

ad b): Here we have for $A(\rho, t, x, u, q)$ defined in (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, Chapter 5, equation (8.5))

$$A(\rho,t,x,u,q) := b^{\mathrm{div}}(\rho,t,x,u,q) - \frac{\partial a^{\mathrm{div}}}{\partial x}(\rho,t,x,u,q) - \frac{\partial a^{\mathrm{div}}}{\partial u}(\rho,t,x,u,q)\cdot q = -\bar{b}(\rho,t,x,u,q)\cdot q$$

and hence $A(\rho, t, x, u, 0) = 0$.

ad c) To verify this hypothesis we have to work a little harder. We have to check condition b) of (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, Theorem 5.6.1). Suppose we want to prove existence for some $\rho \in [0, 1)$. Let us first fix some $1 > \rho_0 \geq \rho$. Defining $\tilde{K}_1$ by

$$\text{(A.41)} \qquad \tilde{K}_1 := \sup\{|\bar{a}(\rho,t,x,u,q)|,\ \rho\in[0,\rho_0], t\in[0,T], x\in\mathbb{R},\ u\in I^g,\ q\in\mathbb{R}\}$$

we get immediately from the first inequality in Assumption (A.5) that $\tilde{K}_1 < \infty$. Here I^g denotes the interval $[\min_{f>0} g(f), \max_{f>0} g(f)]$. Moreover, we have

$$\tilde{K}_0 \leq \bar{a}(\rho,t,u,q) \leq \tilde{K}_1\,,$$

where $\tilde{K}_0$ is defined in (A.40). This proves the first part of the condition. The following estimates show that the second half is fulfilled, too:

$$|a^{\mathrm{div}}(\rho,t,x,u,q)|\cdot(1+|q|) \leq |q|\cdot\tilde{K}_1\cdot(1+|q|) \leq \tilde{K}_1\cdot(1+|q|)^2$$

Now, using Assumption (A.5) it is easily shown that the following constants are finite:

$$\begin{aligned}
\text{(A.42)}\ \tilde{K}_2 &:= \sup\left\{\left|\frac{\partial a^{\mathrm{div}}}{\partial u}(\rho,t,x,u,q)\right|,\ \rho\in[0,\rho_0], t\in[0,T], x\in\mathbb{R}, u\in I^g,\ q\in\mathbb{R}\right\}\\
\text{(A.43)}\ \tilde{K}_3 &:= \sup\left\{\left|\frac{\partial a^{\mathrm{div}}}{\partial u}(\rho,t,x,u,q)\right|,\ \rho\in[0,\rho_0], t\in[0,T], x\in\mathbb{R}, u\in I^g,\ q\in\mathbb{R}\right\}\\
\text{(A.44)}\ \tilde{K}_4 &:= \sup\{|\bar{b}(\rho,t,x,u,q)|,\ \rho\in[0,\rho_0], t\in[0,T], x\in\mathbb{R}, u\in I^g,\ q\in\mathbb{R}\}
\end{aligned}$$

These constants depend of course on ρ_0, on the "cutoff-level" M in (A.39) and on the value of the constants in Assumption (A.5). Now we get

$$\begin{aligned}
\left|\frac{\partial}{\partial u}a^{\mathrm{div}}(\rho,t,x,u,q)\cdot(1+|q|)\right| &\leq |q|\cdot\tilde{K}_2\cdot(1+|q|) \leq \tilde{K}_2\cdot(1+|q|)^2\\
\left|\frac{\partial}{\partial x}a^{\mathrm{div}}(\rho,t,x,u,q)\right| &\leq |q|\cdot\tilde{K}_2 \leq \tilde{K}_2\cdot(1+|q|) \text{ and}\\
|b^{\mathrm{div}}(\rho,t,x,u,q)| &\leq \left(|\bar{b}(\rho,t,x,u,q)| + |\frac{\partial a^{\mathrm{div}}}{\partial u}(\rho,t,x,u,q)|\right)\cdot|q| + |\frac{\partial a^{\mathrm{div}}}{\partial x}(\rho,t,x,u,q)|\\
&\leq (\tilde{K}_2+\tilde{K}_3+\tilde{K}_4)\cdot(1+|q|)^2
\end{aligned}$$

Since the above estimates are valid for all $x \in \mathbb{R}$ we get $u \in H^{2+\beta,1+\beta/2}([0,T]\times\mathbb{R})$ which proves the first statement of the proposition. As explained in (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, p 451) the norm

$$\|u_x\|_T := \sup\{|u_x(t,x)|, \quad x\in\mathbb{R}, t\in[0,T]\}$$

is bounded by some constant K depending only on the Hölder norm of the initial values $\tilde{g}$ and the constants $\tilde{K}_0, \dots, \tilde{K}_4$ from the above estimates. As these are valid for all $\rho \in [0, \rho_0]$ the second claim follows.

Uniqueness follows immediately from (Ladyzenskaja, Solonnikov, and Ural'ceva 1968, Theorem 5.6.1), since the coefficients of the PDE (5.31) are smooth functions. Hence for every $\rho \in [0, 1)$ these functions and their derivatives are bounded on every the compact set of the form $\{(t, x, u, q),\ t \in [0, T], |x| < N, u \in I^g, |q| \le K\}$, where K is the bound on u_x established in (ii). □

Proof of Theorem 5.2

ad (i): It is enough to prove existence of a solution to the initial value problem (5.27), (5.28). By Proposition 5.3 for every $\rho \in [0, 1)$ the restricted Cauchy problem (5.31), (5.28) has a solution from $H^{2+\beta, 1+\beta/2}([0, T] \times \mathbb{R})$. Moreover for every $\rho_0 \in [0, 1)$ there is a constant $K = K(\rho_0, M)$ depending[10] only on ρ_0 and the "cutoff level" M from (A.39) such that for all $\rho \in [0, \rho_0]$ the norm of the derivative u_x of the solution u to (5.31), (5.28) is bounded by K. Moreover by Proposition 5.4 we have $u_x \ge 0$. Hence whenever

$$\rho \le \bar{\rho} := \sup \left\{ \min \left\{ \rho_0, \frac{M}{K(M, \rho_0)} \right\}, \quad M > 0, \rho_0 \in [0, 1] \right\}$$

the "constraints" of equation (A.39) are not binding such that u is also a solution of the unrestricted PDE (5.27).

ad (ii) This statement follows directly from Proposition 5.4.

ad (iii): Uniqueness for the terminal value problem for $\phi(t, f)$ is equivalent to uniqueness for the initial value problem for $u(t, x)$. Now suppose that u^1 and u^2 are both solutions belonging to $H^{2+\beta, 1+\beta/2}([0, T] \times \mathbb{R})$ for some $\beta > 0$. By Proposition 5.4 we know that u_x^1 and u_x^2 are both nonnegative such that for M large enough both functions solve also the restricted PDE (5.31). Hence the claim follows from Proposition 5.3. □

[10] K depends on M and ρ_0 via the constants $\tilde{K}_i$ defined in the proof of Proposition 5.3.

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