# LOGNORMALITY OF RATES AND TERM STRUCTURE MODELS

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**ABSTRACT**

A term structure model with lognormal type volatility structure is proposed. The Heath, Jarrow and Morton (HJM) framework, coupled with the theory of stochastic evolution equations in infinite dimensions, is used to show that the resulting rates are well defined (they do not explode) and remain positive. They are bounded from below and above by lognormal processes. The model can be used to price and hedge caps, swaptions and other interest rate and currency derivatives including the Eurodollar futures contract, which requires integrability of one over zero coupon bond. This extends results obtained by Sandmann and Sondermann (1993), (1994) for Markovian lognormal short rates to (non-Markovian) lognormal forward rates.

## 1. INTRODUCTION

We work with a modified HJM framework in which $r(t, x)$ is the forward rate prevailing at time $t$ over the time interval $[t + x, t + x + dx]$. The price process of the savings account

$\beta(t)$ and the discount function $D(t,x)$ for the interval $[t, t+x]$ (i.e. the price at time $t$ of \$1 at time $t+x$) are given by

$$\beta(t) = \exp\left(\int_0^t r(s,0)ds\right),$$

$$D(t,x) = \exp\left(-\int_0^x r(t,u)du\right).$$

Uncertainty in the HJM framework is represented by the $\mathbf{R}^d$-valued volatility process $\{\tau(t,x) : t,x \geq 0\}$, indexed by the "time" variable $t$ and the "space" variable $x$ and defined on the probability space $(\Omega, \{\mathcal{F}_t : t \geq 0\}, \mathbf{P})$. The filtration $\{\mathcal{F}_t : t \geq 0\}$ is the $\mathbf{P}$-augmentation of the natural filtration generated by a standard $d$-dimensional Brownian motion $W = \{W(t) : t \geq 0\}$.

Assume that the process $\{r(t,x) : t,x \geq 0\}$ satisfies

$$dr(t,x) = \left(\frac{\partial}{\partial x} r(t,x) + \tau(t,x) \cdot \int_0^x \tau(t,u)du\right) dt + \tau(t,x) \cdot dW(t)$$

where $\cdot$ stands for the usual inner product in $\mathbf{R}^d$. This implies that discounted with the savings account $\beta(t)$ zero coupon bonds of all maturities are martingales under the measure $\mathbf{P}$ and leads to the following dynamics

$$dD(t,x) = D(t,x)\left(\left(r(t,0) - r(t,x)\right)dt - \sigma(t,x) \cdot dW(t)\right)$$

of the discount function, where $\sigma(t,x) = \int_0^x \tau(t,u)du$. The process $\{\sigma(t,x);\ t,x \geq 0\}$ can be interpreted as a stochastic price volatility process. Hence the case when $\tau(t,x)$ is deterministic corresponds to the parametrization in terms of the usual price volatility. Unfortunately this implies that the continuously compounded rates are normally distributed and therefore can assume negative values with positive probabilities. Lognormal type volatility structure of the form $\tau(t,x) = r(t,x)\gamma(t,x)$, where $\gamma(t,x)$ is deterministic, was expected to lead to a term structure model with positive rates. However, then, as shown by Morton (1988), the resulting rates explode (assume infinite value) with positive probability, implying zero prices for bonds and hence arbitrage opportunities. Fortunately, the volatility process $\tau(t,x) = \min(\lambda, r(t,x))\gamma(t,x)$ gives finite and positive forward rates (cf. Heath *et al* (1992)). On the one hand this result is reassuring. It shows that it is possible to build a model with positive rates under the arbitrage free measure. Also, because the volatility process is bounded the model can be used to price and hedge many derivatives including the Eurodollar futures contract which represents a challenge for any model with positive rates. On the other hand it is disappointing because the parameters $\gamma(t,x)$ loose their intuitive interpretation as "volatilities" of forward rates.

We follow Sandmann and Sondermann (1993), (1994) and shift the volatility specification from the rates $r(t,x)$ to the rates $j(t,x)$ which satisfy

$$e^{r(t,x)} = 1 + j(t,x).$$

We assume the lognormal type volatility structure on $j(t,x)$, that is

$$dj(t,x) = \cdots dt + j(t,x)\gamma(t,x) \cdot dW(t) \ ,$$

where $\gamma(t,x)$ is deterministic. The parameters $\gamma(t,x)$ define the term structure dynamics and hence define the volatilities and more generally the joint quadratic variations between the forward rates. Also they correspond to the yield volatilities. Because

$$dj(t,x) = d(e^{r(t,x)} - 1) = e^{r(t,x)} dr(t,x) + \frac{1}{2} e^{r(t,x)} d < r(\cdot,x) > (t)$$

$$= e^{r(t,x)} \left( \frac{\partial}{\partial x} r(t,x) + \tau(t,x) \cdot \int_0^x \tau(t,u)\, du + \frac{1}{2} |\tau(t,x)|^2 \right) dt + e^{r(t,x)} \tau(t,x) \cdot dW(t)$$

we get that

$$e^{r(t,x)} \tau(t,x) = j(t,x)\gamma(t,x) = (e^{r(t,x)} - 1)\gamma(t,x)$$

or equivalently that

$$\tau(t,x) = \left( 1 - e^{-r(t,x)} \right) \gamma(t,x) \ .$$

It follows that the process $\{j(t,x);\ t,x \geq 0\}$ must satisfy

$$dj(t,x) = \left( \frac{\partial}{\partial x} j(t,x) + j(t,x)\gamma(t,x) \cdot \int_0^x \frac{j(t,u)}{1 + j(t,u)} \gamma(t,u) du + \frac{1}{2} \frac{j^2(t,x)}{1 + j(t,x)} |\gamma(t,x)|^2 \right) dt + j(t,x)\gamma(t,x) \cdot dW(t) \ . \tag{1.1}$$

The family of rates $\{j(t,x); t,x \geq 0\}$ can be viewed as a stochastic process $\{j(t) = j(t,\cdot) : t \geq 0\}$ with values in a space of functions of $x$. The theory of stochastic evolution equations in infinite dimensions, as outlined in Da Prato and Zabczyk (1992), provides then a natural framework in which to analyse equation (1.1).

Such volatility specification is analogous to the one introduced by Sandmann and Sondermann (1993) on the spot rate $r(t, 0)$ under an extra assumption that the process $\{r(t, 0) : t \geq 0\}$ is Markov. Similar HJM models were analysed by Miltersen (1994) and Musiela (1995), for example, the case of rates $q(t, x)$ defined by

$$(1 + \delta q(t, x))^{1/\delta} = e^{r(t,x)},$$

in which the HJM volatility process $\tau(t, x)$ takes the form

$$\tau(t, x) = \delta^{-1} \left(1 - e^{-\delta r(t,x)}\right) \gamma(t, x).$$

Note that, for $\delta = 0$ we obtain the volatility $\tau(t, x) = r(t, x)\gamma(t, x)$ which defines the HJM lognormal model with exploding forward rates (for $\delta > 0$ no explosion occurs).

Let us point out here that another class of closely related lognormal models was studied in parallel. The effective annual rates $f(t, x, \delta)$, defined by

$$(1 + f(t, x, \delta))^{\delta} = \exp\left(\int_x^{x+\delta} r(t, u)\, du\right),$$

were used by Sandmann *et al* (1995). A model based on the simple forward rates $f(t, T, \delta)$, defined by

$$1 + \delta f(t, T, \delta) = \exp\left(\int_{T-t}^{T+\delta-t} r(t, u)\, du\right),$$

was proposed in Miltersen *et al* (1995). An extensive study of the theoretical and implementation issues for a model based on the forward LIBOR rates $L(t, x)$ defined by

$$1 + \delta L(t, x) = \exp\left(\int_x^{x+\delta} r(t, u)\, du\right)$$

was carried out by Brace *et al* (1995). An alternative construction of an arbitrage free family of forward LIBOR processes was proposed by Musiela and Rutkowski (1995).

To compare the abovementioned classes of lognormal term structure models one can analyse the corresponding price volatilities $\sigma(t, x)$. In the first class

$$\sigma(t, x) = \int_0^x \delta^{-1} \left(1 - \exp\left(-\delta r(t, u)\right)\right) \gamma(t, u)\, du$$

(cf. Musiela (1995)). In this paper we consider the case when $\delta = 1$. In the second class, that is for models based on forward LIBOR rates, we have

$$\sigma(t,x) = \sum_{k=1}^{[\delta^{-1}x]} \left(1 - \exp\left(-\int_{x-k\delta}^{x+k\delta} r(t,u)\,du\right)\right)\gamma(t, x-k\delta)$$

if $\sigma(t,x) = 0$ for $0 \le x < \delta$. Clearly taking a piecewise constant approximation of the forward curve $r(t,\cdot)$ and the volatility curve $\gamma(t,\cdot)$ at maturities of the form $u = k\delta$, $k = 1, 2, \ldots$ we see that both volatilities are identical. Consequently one can expect that both models can be calibrated to the same market information about rates and volatilities with the same piecewise constant volatility function $\gamma(t,u)$ and will show very similar numerical behaviour. In particular the first approach does not lead to a closed form pricing formula for a caplet even if numerically prices of quaterly caplets ($\delta = .25$) will not differ from the market prices. Mathematically there are important differences. In the forward LIBOR model the volatility function $\sigma(t,x)$ is defined inductively. To start the inductive process one needs to assign values to zero coupon bond volatilities of maturities shorter than $\delta$. Therefore the price volatility is uniquely determined by this arbitrary initial specification and the requirement that the LIBOR volatilities are of lognormal type. External to the model interpolation type arguments serve here to define the model. In the first case we have only the requirement of lognormal type volatility function for the rates $j(t,x)$. There is a finite dimensional diffusion underlying the dynamics of the LIBOR model (cf. Jamshidian 1996), there are no state variables describing the evolution of $j(t,x)$. A continuum of zero coupon bonds is not necessary to derive the dynamics of the forward LIBOR process and evaluate associated LIBOR derivatives. Zero coupon bonds of all maturities are necessary to derive the arbitrage free dynamics of $j(t,\cdot)$ etc.

The papers is organized as follows. In Section 2 we show that there is a unique positive solution to (1.1). It is worth mentioning here that, working in the setting of the HJM model rather than in the setting of infinite dimensional diffusions, Miltersen (1994) also gives sufficient conditions to assure that the HJM forward rates exist and are positive. However his argument are based on Morton (1989) results which require globally Lipschitz volatility functions and hence cannot be applied in our case.

In Section 3, considering the time homogenous case when $\gamma(t,\cdot) = \gamma(\cdot)$, we show that the distribution of $j(t,\cdot)$ tends to an invariant distribution as $t$ tends to infinity. This property, known in the theory of stochastic processes as ergodicity, is much stronger than the well documented mean reversion of interest rates, and means, in practical terms, that on periods of constant volatility the market tends to generate yield curves from the same distribution.

The question how "different" is the process $j(t,x)$ from a lognormal process is analyzed in Section 4. One expects that the nonlinearity in the arbitrage free drift has a limited effect on prices of caps and floors given that the method currently used in the market for pricing caps assumes no drift. It turns out that the arbitrage free process is bounded from above and below by lognormal processes implying that caplet prices can be estimated from above and below by the appropriate Black-Scholes prices. The estimate of the process from above can be also used to show that the Eurodollar futures price is well defined. The Eurodollar futures contract is commonly related to the LIBOR rate (which is an add-on rather than a discount rate) and hence requires integrability of $D(T,\delta)^{-1}$. This may not hold for term structure models in which zero coupon rates have distributions with exponential tails, like interest rate models related to the square-root process.

Ratio of forward to future prices are deterministic under the normality of rates. Hence in a Gaussian HJM framework it is not possible to explain randomness between forward and future markets. In Section 5 we show that, under lognormality of the effective annual rates, ratios of forward to futures prices are not deterministic but are bounded from below and above by constants.

In the final Section 6 we show that the lognormal model (1.1) is naturally path dependent. In fact, we prove that, if the forward curve $q(t,\cdot)$ is a deterministic function of a 1-dimensional diffusion process, then the volatility function $\gamma$ must be equal to zero (i.e. $\gamma(t,x)=0$ for all $t,x>0$). This implies that the Markov property of the yield curve dynamics can be obtained only in an infinite dimensional space of functions. This points to another difference with a Gaussian HJM framework in which it is possible to explain the yield curve dynamics in terms of a finite number of Ornstein-Uhlenbeck processes under the appropriate volatility specifications.

## 2. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Our first aim is to show that (1.1) has a unique and strictly positive solution in an appropriately chosen function space. Let $L^2_\alpha(0,\infty)$ be the space of functions with the finite norm

$$\|f\|^2_\alpha = \int_0^\infty f^2(x)e^{\alpha x}\,dx,$$

where $\alpha$ is an arbitrary real number. The space $L^2_\alpha(0,\infty)$ is a Hilbert space with the inner product

$$\langle f,g\rangle_\alpha = \int_0^\infty f(x)g(x)e^{\alpha x}\,dx.$$

Roughly speaking, we shall show that if (A) below holds for a certain $\alpha$ then (1.1) has a unique solution in $L^2_\alpha(0,\infty)$, that is the integrability properties of the initial yield curve propagate through time. The abstract analogue of (1.1) in the space $L^2_\alpha(0,\infty)$ is given by the equation

$$dX = (AX + F(t,X))dt + B(t)X\,dW \tag{2.1}$$

with the initial condition $X(0) = \varphi \geq 0$. The operator $A = \frac{\partial}{\partial x}$ with the domain

$$D(A) = H^1_\alpha(0,\infty) = \left\{ f \in L^2_\alpha(0,\infty) : \int_0^\infty \left| \frac{\partial f}{\partial x}(x) \right|^2 \, dx < \infty \right\}$$

generates a strongly continuous semigroup of left shifts in $L^2_\alpha(0,\infty)$ i.e., for every $t \geq 0$ and $\varphi \in L^2_\alpha(0,\infty)$

$$S(t)\varphi(x) = \varphi(t+x)$$

$x$ a.s. For nonnegative $\varphi \in L^2_\alpha(0,\infty)$ we define $F(t,\varphi)$ as follows:

$$F(t,\varphi)(x) = \gamma(t,x)\varphi(x) \cdot \int_0^x \frac{\varphi(u)}{1+\varphi(u)} \gamma(t,u)du + \frac{1}{2}\frac{\varphi^2(x)}{1+\varphi(x)} |\gamma(t,x)|^2 \; .$$

If $\gamma$ is a locally integrable function and $\phi \geq 0$ then

$$0 \leq F(t,\varphi)(x) \leq \varphi(x) \int_0^x |\gamma(t,x) \cdot \gamma(t,u)| du + \frac{1}{2} |\gamma(t,x)|^2 \varphi(x) \tag{2.2}$$

and therefore the function $F(t,\varphi)$ is well defined and finite $x$-a.s. The operator $B(t)\varphi : R^d \to L^2_\alpha(0,\infty)$ is given by the formula

$$B(t)\varphi(x)h = \varphi(x)\gamma(t,x) \cdot h$$

for $h \in \mathbf{R}^d$.

To show the existence and uniqueness of solution to (2.1) in $L^2_\alpha(0,\infty)$ we need to impose some restrictions on the volatility function $\gamma$.

**Assumption A.** *For a certain $\alpha \in \mathbf{R}$*

$$\sup_{t,x \geq 0} |\gamma(t,x)| + \sup_{t \geq 0} \int_0^\infty |\gamma(t,x)| \left(1 + e^{-\alpha x}\right) \, dx = C < \infty. \tag{A}$$

We say that an $L^2_\alpha(0,\infty)$-valued predictable process $X$ is a solution to (2.1) if for every $t \geq 0$

$$X(t) = S(t)\varphi + \int_0^t S(t-s)F(s,X(s))ds + \int_0^t S(t-s)B(s)X(s)dW(s)\ . \tag{2.3}$$

Clearly, if $X$ is a solution to (2.1) then for every $t \geq 0$

$$X(t,x) = \varphi(x+t) + \int_0^t \Bigg( X(s,x+t-s)\gamma(s,x+t-s)\cdot \int_0^{x+t-s} \frac{X(s,u)}{1+X(s,u)}\gamma(s,u)du$$

$$+\frac{1}{2}\,\frac{X^2(s,x+t-s)}{1+X(s,x+t-s)}|\gamma(s,x+t-s)|^2\Bigg)ds + \int_0^t X(s,x+t-s)\gamma(s,x+t-s)dW(s) \tag{2.4}$$

$x$-a.s.

If (A) holds then

$$|B(t)\varphi(x)| \leq C\varphi(x)$$

and by (2.2)

$$F(t,\varphi)(x) \leq \frac{3}{2}C^2\varphi(x)$$

for $\varphi \geq 0$. It follows that $B(t)\varphi : R^d \to L^2_\alpha(0,\infty)$ is a bounded linear operator and

$$\|F(t,\varphi)\|_\alpha \leq \frac{3}{2}C^2\|\varphi\|_\alpha$$

for $\varphi \geq 0$. Moreover, the definition of the mapping $F(t,\phi)$ yields for $\phi,\psi \geq 0$

$$F(t,\varphi)(x) - F(t,\psi) = \gamma(t,x)(\varphi(x)-\psi(x))\cdot \int_0^x \frac{\varphi(u)}{1+\varphi(u)}\gamma(t,u)du$$

$$+\gamma(t,x)\psi(x)\cdot \int_0^x \left(\frac{\varphi(u)}{1+\varphi(u)} - \frac{\psi(u)}{1+\psi(u)}\right)\gamma(t,u)du$$

$$+\frac{1}{2}\left(\frac{\varphi^2(x)}{1+\varphi(x)} - \frac{\psi^2(x)}{1+\psi(x)}\right)|\gamma(t,x)|^2.$$

Taking into account the inequality

$$\left|\frac{x^i}{1+x} - \frac{y^i}{1+y}\right| \leq |x-y|$$

for $x, y \geq 0$ and $i = 1, 2$ we obtain

$$|F(t,\phi)(x) - F(t,\psi)(x)| \leq |\phi(x) - \psi(x)||\gamma(t,x)| \int_0^\infty |\gamma(t,u)|\, du$$

$$+|\gamma(t,x)|\psi(x) \int_0^\infty |\gamma(t,u)||\phi(u) - \psi(u)|\, du + \frac{1}{2}\, |\gamma(t,x)|^2 |\varphi(x) - \psi(x)|$$

and hence

$$\|F(t,\phi) - F(t,\psi)\|_\alpha \leq \frac{3}{2} C^2 \|\phi - \psi\|_\alpha + C^2 \|\psi\|_\alpha \cdot \|\phi - \psi\|_\alpha .$$

Therefore the function $F(t)$ is locally Lipschitz with respect to the $L^2_\alpha(0,\infty)$-norm in its domain for all $\alpha \in \mathbf{R}$.

**Theorem 2.1.** *Let (A) hold for a certain $\alpha \in \mathbf{R}$. If $\varphi \geq 0$ then there exists a unique $L^2_\alpha(0,\infty)$-valued nonnegative Markov process which is a solution of (2.1). Moreover, if $\varphi > 0$ $\lambda$-a.s. then $X(t) > 0$ $\lambda$-a.s. for every $t \geq 0$.*

*Proof.* The proof is divided into a sequence of steps.

*Step 1.* Consider the equation

$$dX_n = (AX_n + F_n(t, X_n))\, dt + B(t) X_n\, dW \tag{2.5}$$

with $X(0) = \varphi$, where

$$F_n(t,\varphi) = \begin{cases} F(t, |\varphi|) & \|\varphi\|_\alpha \leq n \\ F\left(t, n\frac{|\varphi|}{\|\varphi\|_\alpha}\right) & \|\varphi\|_\alpha \geq n\ . \end{cases}$$

It is easy to check that $F_n(t) : H \to H$ is globally Lipschitz with Lipschitz constant independent of $t$. Therefore Theorem 7.4 of DaPrato and Zabczyk (1992) assures existence of a unique continuous solution $X_n$ to (2.5). Moreover

$$\sup_{t \leq T} E\|X_n(t)\|^p_\alpha \leq C_{p,T}(1 + \|\varphi\|^p_\alpha)$$

for every $p \geq 2$ with a constant $C_{p,T}$ independent of $n$.

*Step 2.* For all integers $n > \|\varphi\|_\alpha$ define stopping times

$$\theta_n = \inf\{t \leq T :\ \|X_n(t)\|_\alpha \geq n\}$$

where as usual $\inf \emptyset = T$. Since the process $X_n$ is continuous we have $P(\theta_n > 0) = 1$. On the other hand the uniqueness of solutions to (2.5) implies that $X_n(t) = X_{n+1}(t)$ for $t < \theta_n$. Hence $\theta_n \le \theta_{n+1}$ and the process

$$X(t) = \sum_n X_n(t) I_{[\theta_n, \theta_{n+1})}$$

is a unique solution to the equation

$$dY = (AY + F(t, |Y|))dt + B(t)Y dW(t)$$

with $Y(0) = \varphi$.

*Step 3.* We show now that $Y(t,x) \ge 0$ $x$-a.s. for every $t \ge 0$ provided $\varphi \ge 0$ and therefore $Y$ is a solution to (2.3) or, equivalently, to (1.1). Let the process $k(t,T)$, $t \le T$, be defined as

$$k(t,T) = Y(t, T-t) \ .$$

It follows from (2.4) that

$$k(t,T) = k(0,T) + \int_0^t \Bigg( |k(s,T)| \gamma(s, T-s) \cdot \int_s^T \frac{|k(s,u)|}{1+|k(s,u)|} \gamma(s, u-s) du$$

$$+ \frac{1}{2} \frac{k^2(s,T)}{1+|k(s,T)|} |\gamma(s, T-s)|^2 \Bigg) ds + \int_0^t k(s, T-s) \gamma(s, T-s) \cdot dW(s)$$

and hence $T$-a.s. the process $\{k(t,T) : \ 0 \le t \le T\}$ is a semimartingale and satisfies

$$dk(t,T) = k(t,T)\gamma(t, T-t) \cdot \Bigg( \Bigg( \int_t^T \frac{|k(t,u)|}{1+|k(t,u)|} \gamma(t, u-t) du$$

$$+ \frac{1}{2} \frac{|k(t,T)|}{1+|k(t,T)|} \gamma(t, T-t) \Bigg) \operatorname{sign} k(t,T) dt + dW(t) \Bigg) \tag{2.6}$$

with $k(0,T) = \varphi(T) > 0$. Let $\underline{k}$ be a solution to the equation

$$d\underline{k}(t,T) = \underline{k}(t,T)\gamma(t, T-t) \cdot dW(t) \ . \tag{2.7}$$

It is easy to see that $\underline{k}(t,T) > 0$ for every $t \le T$. Moreover, because

$$E \exp \left( \frac{1}{2} \int_0^T \left| \int_t^T \frac{\underline{k}(t,u)}{1+\underline{k}(t,u)} \gamma(t, u-t) du + \frac{1}{2} \frac{\underline{k}(t,T)}{1+\underline{k}(t,T)} \gamma(t, T-t) \right|^2 dt \right) < \infty \ ,$$

the Girsanov theorem implies that the processes $k(\cdot, T)$ and $\underline{k}(\cdot, T)$ are mutually absolutely continuous and consequently we find that $k(t, T) > 0$ for every $t \leq T$, $T$ a.s. Finally, $X(t, x) > 0$ for every $t$, $x$-a.s. and this fact together with *Step 2* gives existence of a unique positive solution to equation (2.1). A simple modification of Theorem 9.8 of DaPrato and Zabczyk (1992) implies that this solution is a Markov process. □

## 3. INVARIANT MEASURES

In this section we discuss ergodic properties of the solution to equation (2.1) in the time homogenous case when $\gamma(t, x) = \gamma(x)$ under the assumption (A1) below.

**Assumption A1.** *For a certain* $\alpha > 0$

$$\|\gamma\|_\infty + \|\gamma\|_\alpha = C < \infty, \tag{A1}$$

*where* $\|\gamma\|_\infty = \sup_{x \geq 0} |\gamma(x)|$.

Ergodicity of the term structure dynamics may be viewed as a stronger property than the well documented mean reversion of interest rates. It implies that the yield curve distribution converges to a steady state and means in practical terms that locally in time, on periods between significant shifts in the volatility levels, the market generates yield curves from a distribution which is not far from a steady state. The limiting distribution depends only on the volatility function $\gamma$ which in turn can be estimated from historical data.

Note that the process $X = 0$ is a solution to (2.1) or, equivalently the Dirac measure $\delta_0$ concentrated at zero is invariant for (2.1). Below we show that there exist other invariant measures for this equation. In order to eliminate the trivial solution $X = 0$ we consider first the process $Y(t, x) = \log X(t, x)$. For $\alpha > 0$ we consider the equation

$$dY = (AY + G(Y))dt + \gamma dW \tag{3.1}$$

in the space $L^2_\alpha(0, \infty)$, where

$$G(\varphi)(x) = -\gamma(x) \cdot \int_0^x \frac{1}{1 + e^{\varphi(u)}} \gamma(u) du - \frac{1}{2}|\gamma(x)|^2 \frac{1}{1 + e^{\varphi(x)}} + \gamma(x) \cdot \int_0^x \gamma(u)\, du.$$

If $Y$ is a solution to (3.1) then for every $t \geq 0$

$$Y(t, x) = \phi(t + x) + \int_0^t G(Y)(x + t - s)\, ds + \int_0^t \gamma(x + t - s) \cdot dW(s) \tag{3.2}$$

$x$-a.s.

**Lemma 3.1.** *Under (A1) there exists a unique solution to equation (3.1) in the space* $L^2_\alpha(0,\infty)$.

*Proof.* For all $\phi \in L^2_\alpha(0,\infty)$

$$|G(\varphi)(x)| \le C|\gamma(x)|$$

and therefore it follows immediately from (A1) that $G$ maps $L^2_\alpha(0,\infty)$ into $L^2_\alpha(0,\infty)$. Similarly, taking into account that

$$\left|(1+e^x)^{-1} - (1+e^y)^{-1}\right| \le |x-y|$$

we find that for all $\phi,\psi \in L^2_\alpha(0,\infty)$

$$|G(\varphi)(x) - G(\psi)(x)|$$

$$\le |\gamma(x)| \int_0^x |\varphi(y)-\psi(y)||\gamma(y)|dy + \frac{1}{2}|\gamma(x)|^2|\varphi(x)-\psi(x)|$$

$$\le |\gamma(x)| \left(\int_0^\infty |\varphi(y)-\psi(y)|^2 dy\right)^{1/2} \left(\int_0^\infty |\gamma(y)|^2 dy\right)^{1/2} + \frac{1}{2}|\gamma(x)|^2|\varphi(x)-\psi(x)|.$$

Because $\alpha > 0$ the condition (A1) yields

$$\|G(\varphi) - G(\psi)\|_\alpha \le \frac{3}{2}C^2\|\varphi-\psi\|_\alpha.$$

By Theorem 7.4 of DaPrato and Zabczyk (1992) equation (3.1) has a unique solution.

Approximating the process $X$ by a sequence of strong solutions and applying the Ito formula one can show that the process

$$X(t,x) = e^{Y(t,x)}$$

is a solution to equation (2.1).

**Proposition 3.2.** *If*

$$\frac{1}{2}\|\gamma\|^2_\infty + \|\gamma\|_\alpha \cdot \|\gamma\|_{-\alpha} < \alpha \tag{3.3}$$

*then there exists a unique invariant measure for equation (3.1) in the space* $L^2_\alpha(0,\infty)$.

*Proof.* In view of Theorem 11.22 in DaPrato and Zabczyk (1992) it is enough to check that

$$\langle A(\phi-\psi), \phi-\psi\rangle_\alpha + \langle G(\phi)-G(\psi), \phi-\psi\rangle_\alpha \le -\omega\|\phi-\psi\|^2_\alpha \tag{3.4}$$

for certain $\omega > 0$. Note first that for $\alpha > 0$

$$\langle A\phi, \phi\rangle_\alpha \le -\frac{1}{2}\alpha\|\phi\|_\alpha^2.$$

Let

$$G(\phi)(x) = -\gamma(x)\cdot\int_0^x \frac{1}{1+e^{\varphi(u)}}\gamma(u)du - \frac{1}{2}|\gamma(x)|^2\frac{1}{1+e^{\varphi(x)}} + \gamma(x)\int_0^x \gamma(u)\,du$$

$$= G_1(\phi)(x) + G_2(\phi)(x) + G_3(x).$$

Then

$$\langle G_1(\phi) - G_1(\psi), \phi - \psi\rangle_\alpha$$

$$= \int_0^\infty \left(\int_0^x \frac{e^{-\psi(u)} - e^{-\phi(u)}}{\left(1+e^{-\phi(u)}\right)\left(1+e^{-\psi(u)}\right)}\gamma(u)\,du\right)(\phi(x) - \psi(x))\gamma(x)e^{\alpha x}\,dx.$$

As a consequence we find that

$$|\langle G_1(\phi) - G_1(\psi), \phi - \psi\rangle_\alpha|$$

$$\le \int_0^\infty |\phi(u) - \psi(u)|\gamma(u)\,du\int_0^\infty |\phi(x) - \psi(x)|\gamma(x)e^{\alpha x}\,dx$$

$$\le \left(\int_0^\infty \gamma^2(x)e^{-\alpha x}\,dx\right)^{1/2}\left(\int_0^\infty \gamma^2(x)e^{\alpha x}\,dx\right)^{1/2}\|\phi - \psi\|_\alpha^2$$

$$= \|\gamma\|_\alpha\|\gamma\|_{-\alpha}\|\phi - \psi\|_\alpha^2. \tag{3.5}$$

For $G_2$ we obtain

$$\langle G_2(\phi) - G_2(\psi), \phi - \psi\rangle_\alpha$$

$$= \frac{1}{2}\int_0^\infty \frac{e^{-\psi(x)} - e^{-\phi(x)}}{\left(1+e^{-\phi(x)}\right)\left(1+e^{-\psi(x)}\right)}(\phi(x) - \psi(x))|\gamma(x)|^2e^{\alpha x}\,dx$$

and therefore

$$|\langle G_2(\phi) - G_2(\psi), \phi - \psi\rangle_\alpha| \le \frac{1}{2}\|\gamma\|_\infty^2\|\phi - \psi\|_\alpha^2. \tag{3.6}$$

Finally, taking into account (3.5) and (3.6) we obtain

$$|\langle G(\phi) - G(\psi), \phi - \psi\rangle_\alpha| \le \left(\frac{1}{2}\|\gamma\|_\infty^2 + \|\gamma\|_\alpha\|\gamma\|_{-\alpha}\right)\|\phi - \psi\|_\alpha^2$$

and hence (3.3) yields (3.4) for a certain $\omega > 0$.

**Remark 3.1.** Let

$$|\gamma(x)| \leq a e^{-bx}$$

for some $a, b > 0$. Then (A1) holds for any $\alpha \in [0, 2b)$. Moreover, if

$$a^2 \left(1 + \frac{2}{\sqrt{4b^2 - \alpha^2}}\right) < \alpha$$

then the assumption of Proposition 3.2 holds and $Y$ is a well defined Markov process in $L^2_\alpha(0, \infty)$ with a unique invariant measure.

**Theorem 3.3.** *Assume (3.3). Then there exists an invariant measure for equation (2.1) in the space $L^2_{-\alpha}(0, \infty)$.*

*Proof.* Choose $\phi \in L^2_\alpha(0, \infty)$ such that $e^\phi \in L^2_{-\alpha}(0, \infty)$. If $Y$ is a solution to (3.1) with $Y(0) = \phi$ then $X(t) = e^{Y(t)} \in L^2_{-\alpha}(0, \infty)$ and $X$ is a solution to (2.1). By Proposition 3.2 $Y$ has a unique invariant measure and therefore there exists an invariant measure for the process $X$.

## 4. EURODOLLAR FUTURES

A Eurodollar deposit is any U.S. dollar deposit with a bank outside the U.S. or with an international banking facility within the U.S. The Eurodollar futures contract is related to the Eurodollar deposit. The most common contract relates to the LIBOR rate $L(T)$ which is an add-on rather than a discount rate but can be expressed in terms of the discount function $D(T, \delta)$, namely,

$$1 + \delta L(T) = D(T, \delta)^{-1},$$

where $\delta = .25$ in case of three month LIBOR. The futures payoff is $\delta L(T)$ and the futures price is the expected value of this payoff. Thus the finiteness of the Eurodollar futures price is equivalent to the finiteness of the expected value of $D(T, \delta)^{-1}$.

In this section we show that the Eurodollar futures price is well defined. First, however, we show that the rates $j(t, x)$ are bounded from above and below by lognormal processes. Let for $0 \leq t \leq T$, $T > 0$

$$k(t, T) = j(t, T - t).$$

It follows from *Step 3* in Theorem 2.1 that the process $\{k(t,T);\ 0 \le t \le T\}$ is a semimartingale and satisfies

$$dk(t,T) = k(t,T)\gamma(t,T-t)\cdot\Bigg(\bigg(\int_t^T \frac{k(t,u)}{1+k(t,u)}\gamma(t,u-t)du$$

$$+\frac{1}{2}\frac{k(t,T)}{1+k(t,T)}\gamma(t,T-t)\bigg)dt + dW(t)\Bigg). \tag{4.1}$$

Let $\underline{k}(t,T)$ and $\overline{k}(t,T)$ be, respectively, the solutions to

$$d\underline{k}(t,T) = \underline{k}(t,T)\gamma(t,T-t)\cdot dW(t), \underline{k}(0,T) = k(0,T)\ , \tag{4.2}$$

and

$$d\overline{k}(t,T) = \overline{k}(t,T)\gamma(t,T-t)\cdot\Bigg(\bigg(\int_t^T \gamma(t,u-t)du + \frac{1}{2}\gamma(t,T-t)\bigg)dt + dW(t)\Bigg),$$

$$\overline{k}(0,T) = k(0,T)\ . \tag{4.3}$$

The main assumption in the proposition below is that all rates $j(t,x)$ are positively correlated.

**Proposition 4.1.** *If* (A) *holds and for all* $t,x,y \ge 0$ $\gamma(t,x)\cdot\gamma(t,y) \ge 0$ *then for all* $0 \le t \le T$

$$\underline{k}(t,T) \le k(t,T) \le \overline{k}(t,T)\ ,\ T = a.s.,$$

*where* $k, \underline{k}$ *and* $\overline{k}$ *are given by* (4.1), (4.2) *and* (4.3), *respectively.*

*Proof.* Define the semimartingale

$$Z_t = k(t,T) - \overline{k}(t,T)\ .$$

Then

$$d<Z>_t = Z_t^2|\gamma(t,T-t)|^2dt$$

and

$$\int_0^t \frac{1_{]0,\infty[}(Z_s)}{Z_s^2}d<Z>_s = \int_0^t 1_{]0,\infty[}(Z_s)|\gamma(s,T-s)|^2ds \le C^2t$$

for all $0 \le t \le T$. Consequently, the local time of $Z$ at zero $L_t^0(Z) = 0$ (see Le Gall (1983) for example) and from the Tanaka formula it follows that

$$Z_t^+ = \int_0^t 1_{]0,\infty[}(Z_s) dZ_s \ .$$

Let $\phi(t) = EZ_t^+$, then following the proof of Theorem 3.7 of Chapter 9 of Revuz and Yor (1991) we can show that

$$\phi(t) \le K_t \int_0^t \phi(s)\, ds \ .$$

Finally, using the Gronwall lemma we get that $\phi(t) = 0$, or

$$k(t,T) \le \overline{k}(t,T)$$

for all $0 \le t \le T$. To obtain the second inequality we define the process $U_t = \underline{k}(t,T) - k(t,T)$ and the function $\psi(t) = EU_t^-$. Using the sama arguments as in the first part of the proof we find that $L_t^0(U) = 0$ and because $\gamma(t,x) \cdot \gamma(t,y) \ge 0$ we can check easily that $\psi = 0$ hence the second inequlaity. □

**Corollary 4.1.** *Assume that (A) holds and the initial condition $j(0,\cdot)$ is a locally bounded function. Then $ED(T,\delta)^{-1} < \infty$ and hence the Eurodollar futures price is well defined.*

*Proof.* Note first that for all $\delta > 0$

$$\int_0^\delta E\left(\overline{j}(T,u)\right)^\delta \, du < \infty,$$

where $\overline{j}(T,u) = \overline{k}(T, T+u)$. Indeed, by (4.3)

$$\overline{j}(T,u) = \overline{k}(0, T+u) \exp\left( A(T, T+u) + \int_0^T B(s, T+u)\, dW(s) \right)$$

with $A$ and $B$ deterministic and by (A) locally bounded in two variables. Because $\overline{k}(0, T+\cdot)$ is locally bounded the above estimate follows by standard argument for all $\delta > 0$. For all $T, \delta > 0$

$$ED(T,\delta)^{-1} = E\exp\left(\int_0^\delta r(T,u)\, du\right) = E\exp\left(\int_0^\delta \log(1 + j(T,u))\, du\right)$$

$$\leq E \exp\left(\int_0^\delta \log(1+\bar{j}(T,u))\,du\right) \leq \frac{1}{\delta}\int_0^\delta E(1+\bar{j}(T,u))^\delta\,du < \infty$$

and these estimates conclude the proof.

## 5. RATIO OF FORWARD TO FUTURES PRICES

It is well known (cf. Jamshidian 1993) that in a Gaussian HJM framework (i.e., with deterministic volatilities) the ratio of forward to futures prices on the same underlying is deterministic.

In this section we show that, under the lognormal volatility structure on $j(t,x)$ the ratio of forward to futures prices on a zero coupon bond is not deterministic but it is bounded from below and above by deterministic constants.

The HJM forward rate $f(t,T)$ at time $t$ for time $T(t \leq T)$ corresponding to the rate $k(t,T)$ satisfies

$$f(t,T) = \log(1+k(t,T)) \ .$$

It follows from the Ito formula and (3.1) that

$$df(t,T) = \sigma_{HJM}(t,T)\cdot\int_t^T \sigma_{HJM}(t,u)du dt + \sigma_{HJM}(t,T)\cdot dW(t) \ ,$$

where

$$\sigma_{HJM}(t,T) = (1-e^{-f(t,T)})\gamma(t,T-t) \ .$$

Let $P(t,T) = D(t,T-t)$ denote the time $t$ price of a $T$ maturity zero coupon bond $(t \leq T)$ and let for $T \leq T_1$

$$F_T(t,T_1) = \frac{P(t,T_1)}{P(t,T)}$$

denote the forward price at time $t$ for settlement at time $T$, in a contract on a $T_1$ maturity zero coupon bond. Because

$$dP(t,T) = P(t,T)\left(r(t,0)dt - \int_t^T \sigma_{HJM}(t,u)\,du\cdot dW(t)\right)$$

we also have

$$dF_T(t,T_1) = -F_T(t,T_1)\int_T^{T_1}\sigma_{HJM}(t,u)\,du\cdot\left(\int_t^T \sigma_{HJM}(t,v)\,dv dt + dW(t)\right)$$

and

$$d\left\langle F_T\left(\cdot, T_1\right),\, P(\cdot, T)\right\rangle(t)=F_T\left(t, T_1\right) P(t, T) \int_T^{T_1} \sigma_{HJM}(t, u)\, du \cdot \int_t^T \sigma_{HJM}(t, v)\, dv dt\ .$$

This implies (cf. Jamshidian 1993) that

$$F_T(t, T_1)=E\left(P\left(T, T_1\right) \exp \left(\int_t^T \int_T^{T_1} \sigma_{HJM}(s, u)\, du \cdot \int_s^T \sigma_{HJM}(s, v)\, dv ds\right) \middle| \mathcal{F}_t\right) .$$

The futures price at time $t$ in the contract with expiry date $T(t \leq T)$ on a zero coupon bond with maturity $T_1(T \leq T_1)$ is

$$G_T(t, T_1)=E(P(T, T_1) | \mathcal{F}_t)\ .$$

**Proposition 5.1.** *If* $(A)$ *holds and for all* $t, x, y \geq 0, \gamma(t, x) \cdot \gamma(t, y) \geq 0$ *then for all* $0 \leq t \leq T \leq T_1$

$$1 \leq \frac{F_T(t, T_1)}{G_T(t, T_1)} \leq \exp \left(\int_t^T \int_T^{T_1} \gamma(s, u-s)\, du \cdot \int_s^T \gamma(s, v-s)\, dv ds\right) .$$

*Proof.* Note that

$$\begin{aligned} 0 & \leq \int_t^T \int_T^{T_1} \sigma_{HJM}(s, u) du \cdot \int_s^T \sigma_{HJM}(s, v)\, dv ds \\ & \leq \int_t^T \int_T^{T_1} \gamma(s, u-s) du \cdot \int_s^T \gamma(s, v-s)\, dv ds. \end{aligned}$$

□

## 6. PATH DEPENDENCE

Many term structure models are defined in terms of a finite number of state variables which follow a Markov process. For example, Vasicek (1977) model is constructed on an Ornstein-Uhlenbeck process , CIR model (see Cox *et al* (1985)) on a square root process, BDT model (see Black *et al* (1990)) on a lognormal process. The role of state variables in interest rates models is discussed in El Karoui *et al* ((1995). Such approach has obvious advantages. In particular, if the process of short rate is Markov then it is quite easy to build path independent trees and use them to value arbitrary European or American interest rate options.

The HJM models provide a general framework to analyse almost any interest rate model. Vasicek model, for example, is a particular case of a Gaussian HJM model with an exponential volatility function. Therefore even if the HJM models are typically path dependent sometimes it is possible to choose volatility functions which lead to a small number of Markovian state variables defining the term structure dynamics.

The question then arises whether it is possible to find a function $\gamma$ such that the process $Y(t,x)$, solution to (3.1), is a deterministic function of a finite dimensional Markov process. The aim of this section is to show that the process $Y$ cannot be obtained as a nonlinear transformation of a one-dimensional diffusion process.

We assume that the Brownian motion $W$ is one-dimensional and (A) holds for certain $\alpha \geq 0$. By Lemma 3.1 there exists a unique solution $Y^\phi$ to (3.1) with the initial condition $Y(0) = \phi$. Moreover, if the functions $\gamma$ and $\phi$ are of class $C^1$ with the derivatives $\gamma', \phi' \in L^2_\alpha(0,\infty)$ then it follows by standard arguments that the process $Y^\phi$ is a classical solution of the following stochastic partial differential equation

$$dY(t,x) = \left( \frac{\partial}{\partial x} Y(t,x) + G(Y(t))(x) \right) dt + \gamma(x) dW(t) \tag{6.1}$$

with

$$G(\psi)(x) = -\gamma(x) \int_0^x \frac{1}{1+e^{\psi(u)}} \gamma(u) du - \frac{1}{2}\gamma^2(x) \frac{1}{1+e^{\psi(x)}} + \gamma(x) \int_0^x \gamma(u)\, du.$$

Let $\{Z^z(t) : t \geq 0\}$ be a unique solution of the stochastic differential equation

$$\begin{aligned} dZ &= b(Z(t))dt + \sigma(Z(t))dW(t), \\ Z(0) &= z \end{aligned}$$

with continuous coefficients $b$ and $\sigma$.

**Theorem 6.1.** *Let $f \in C^1(\mathbf{R}_+ \times \mathbf{R})$, $\gamma' \in H^1_\alpha$ and $\gamma \geq 0$. Assume that for every initial condition $Y(0) = \phi$ there exists $z = z(\phi)$ such that*

$$Y^\phi(t,x) = f\left(x, Z^{z(\phi)}(t)\right)$$

*for all $t, x \geq 0$. Moreover, assume that $\sigma > 0$ on $\mathbf{R}$. Then $\gamma = 0$.*

*Proof.* Without loss of generality we can assume that $\sigma = 1$. Then applying the Ito formula to the process $Y^\phi(t,x) = f\left(x, Z^{z(\phi)}(t)\right)$ we obtain

$$dY^\phi(t,x) \tag{6.2}$$

$$= \left( b\left(Z^{z(\phi)}(t)\right) \frac{\partial f}{\partial z}\left(x, Z^{z(\phi)}(t)\right) + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}\left(x, Z^{z(\phi)}(t)\right)\right) dt + \frac{\partial f}{\partial z}\left(x, Z^{z(\phi)}(t)\right) dW(t).$$

Hence comparing (for each $x$) the semimartingale representations (6.1) and (6.2) we find that

$$\frac{\partial f}{\partial z}\left(x, Z^{z(\phi)}(t)\right) = \gamma(x)$$

**P**-a.s. Since the diffusion $Z$ is nondegenerate and $f$ is continuous we find that

$$\frac{\partial f}{\partial z}(x, z) = \gamma(x) \tag{6.3}$$

and similarly

$$\frac{\partial f}{\partial x}(x, z) + G(f)(x) = b(z)\frac{\partial f}{\partial z}(x, z) + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}(x, z) \tag{6.4}$$

for all $x \geq 0$. It follows from (6.3) that

$$f(x, z) = (z - z(\phi))\gamma(x) + \phi(x)$$

and

$$\phi(x) = z(\phi)\gamma(x)$$

for all $x \geq 0$. Hence

$$f(x, z) = z\gamma(x).$$

which together with (6.4) yields

$$b(z)\gamma(x) = z\gamma'(x) - \gamma(x)\int_0^x \frac{1}{1 + e^{z\gamma(u)}}\gamma(u)\, du$$

$$-\frac{1}{2}\gamma^2(x)\frac{1}{1 + e^{z\gamma(x)}} + \gamma(x)\int_0^x \gamma(u)\, du \tag{6.5}$$

for all $x \geq 0$. Therefore putting $x = 0$ we obtain

$$b(z) = \frac{\gamma'(0)}{\gamma(0)}z - \frac{1}{2}\frac{\gamma(0)}{1 + e^{\gamma(0)z}}$$

and in particular $b(0) = -\frac{1}{4}\gamma(0)$. Putting $z = 0$ in (6.5) we get

$$b(0) = \frac{1}{2}\int_0^x \gamma(u)\, du - \frac{1}{4}\gamma(x).$$

Then denoting

$$U(x) = \int_0^x \gamma(u)\,du$$

and solving the equation

$$U' = 2U - 4b(0)$$

we obtain

$$\gamma(x) = \gamma(0)e^{2x}.$$

Note that such a function $\gamma$ does not satisfy assumption (A1). We shall show that even if we consider the process $Y$ as a solution to the stochastic partial differential equation (6.1) with the above $\gamma$ then assumptions of Theorem 6.1 imply that $\gamma = 0$. In fact, it follows from (6.5) that

$$-\int_0^x \frac{1}{1+e^{z\gamma(u)}}\gamma(u)\,du - \frac{1}{2}\gamma(x)\frac{1}{1+e^{z\gamma(x)}} + U(x) = b(z) - 2z = -\frac{1}{2}\frac{\gamma(0)}{1+e^{z\gamma(0)}}.$$

If $z$ tends to infinity then the right hand side of this equation tends to zero and the left hand side to $U(x)$, therefore $\gamma(0) = 0$ and the proof is finished.

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## References

Black, F., Derman E. and Toy, W. (1990): "A one factor model of interest rates and its application to treasury bond options", *Financial Analyst J.*, I-II, 33-39

Brace ,A., Gątarek ,D. and Musiela ,M. (1995): "The market model of interest rate dynamics", UNSW Department of Statistics report No. S95-2

Cox, J.C., Ingersoll, J.E. and Ross, S.A. (1985): "A theory of term structure of interest rates", *Econometrica*, 53, 385-407

Da Prato, G. and Zabczyk, J. (1992): Stochastic equations in infinite dimensions, Cambridge University Press.

El Karoui, N., Gemann, H. and Lacoste, V. (1995): "On the role of state variables in interest rate models", working paper

Heath, D. Jarrow, R. and Morton, A. (1992): ‘Bond pricing and the term structure of interest rates: a new methodology’. *Econometrica*, **61(1)**, 77-105.

Jamshidian, F. (1993): ’Option and futures evaluation with deterministic volatilities’. *Mathematical Finance*, **3**, 149-156

Jamshidian, F. (1996): ’Libor and swap market models and measures’, working paper

Le Gall, J.F. (1983): ‘Applications du temps local aux équations différentilles stochastiques unidimensionnelles’, Sém. Prob. XVII. Lecture Notes in Mathematics, Vol. 986. Springer, Berlin Heidelberg New York, 15-31.

Miltersen, K.R. (1994): An arbitrage theory of the term structure of interest rates, *Ann. Appl. Probab.* 4, 953-967

Miltersen, K.R., Sandmann, K. and Sondermann, D. (1995): ”Closed form solutions for term structure derivatives with log-normal interest rates”, University of Bonn Discussion Paper B-308.

Morton, A. (1988): ”A class of stochastic differential equations arising in models for the evaluation of bond prices”, Technical Report, School of Operations Research and Industrial Engineering, Cornell University.

Morton, A. (1989): ”Arbitrage and martingales”, PhD dissertation, Cornell University.

Musiela, M. (1995): ’Nominal annual rates and lognormal volatility structure’, in Platen, E. (ed.): ’Workshop on Stochastics and Finance’, part II, ANU Research Report FMRR 005-95, 241-248

Musiela, M. and Rutkowski, M. (1995): ”Continuous-time term structure models”, preprint S95-14 of the School of Mathematics, The University of New South Wales

Revuz, D. and Yor, M. (1991): Continuous martingales and Brownian motion, Springer, Berlin Heidelberg New York.

Sandmann, K. and Sondermann, D. (1993): ‘A term structure model and the pricing of interest rate derivatives’. *The Review of Futures Markets,* **12**(2), 392–423.

Sandmann, K. and Sondermann, D. (1994): ‘On the stability of lognormal interest rate models and the pricing of Eurodollar futures’, University of Bonn Discussion Paper B-263.

Sandmann, K., Sondermann, D. and Miltersen, K.R. (1995): ‘Closed form term structure derivatives in a Heath-Jarrow-Morton model with log-normal annually compounded interest rates’. *Proceedings of the Seventh Annual European Futures Research Symposium, Bonn, September 1994.* Chicago Board of Trade, 145-165.

Vasicek, O. ((1977): "An equilibrium characterization of the term structure", *Journal of Financial Economics*, 5, 177-188

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