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Demand Aggregation under Structural Stability

by

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Abstract

The goal of this paper is to model the mean (aggregate) consumption expenditure of a large and heterogeneous population of households. The aggregation process is based on assumptions of how the income distribution and the composition of the population evolves over time (structural stability). It is shown that the change in the aggregate consumption expenditure ratio can be decomposed into an effect of changing income dispersion, an effect of income growth, an effect of price-inflation and an effect of changing composition of the population.

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1 Introduction

1. The goal of this paper is to model the change over time of *mean consumption expenditure* C_t of a *large and heterogeneous population* H_t of households:

$$C_t = \frac{1}{\# H_t} \sum_{h \in H_t} c_t^h \tag{1}$$

where c_t^h denotes the consumption expenditure of household h in current prices during period t on all commodities that belong to a certain consumption category, such as food, housing or non-durables.

The starting point of any analysis of aggregation across households is a model of individual household behaviour.

To concentrate in this introduction on the essential we start directly from a *micro-relation* $c(x, \chi)$

$$c_t^h = c(x_t^h, \chi_t^h) \quad , h \in H_t \tag{2}$$

where x_t^h denotes disposable *income* in period t of household h and $\chi_t^h = (\chi_{t,1}^h, \chi_{t,2}^h, \ldots)$ denotes a vector of *household characteristics* that are used as explanatory variables in the underlying model of household behaviour (e.g. preferences). We do not explicitly mention prices and interest rates in the introduction; this amounts to assuming that they do not change over time.

The population of households in period t is described by the joint distribution μ_t of household income x^h and characteristics χ^h across the population H_t. Given the micro-relation (2), one obtains for mean consumption expenditure

$$C_t = \int c(x, \chi) d\mu_t \; . \tag{3}$$

Thus, given the micro-relation c, C_t is a function of μ_t; $C_t = C(\mu_t)$.

The distribution μ_t, however, is not a useful explanatory variable for mean consumption expenditure, because it is a far too detailed description of the population. The goal of aggregation theory[1] is to simplify the function $C(\mu_t)$ by reducing the entire distribution μ_t to certain relevant characteristics of μ_t, such as mean or dispersion. Obviously, such a simplification – even if one is satisfied with an approximation to $C(\mu_t)$ – is only possible if one restricts the way in which the distribution μ_t changes over time and/or if one appropriately specifies the micro-relation.

[1]There is a large literature on aggregation starting with Antonelli in 1886. For a general discussion of the various aspects of aggregation theory we recommend Malinvaud (1993).

In order to illustrate this point we give a simple example. If the Engel curve of the population

$$x \mapsto \int c(x, \chi) d\mu_t | x =: \bar{c}_t(x)$$

is time-invariant, i.e., $\bar{c}_t = \bar{c}$ (definitely, an unrealistic assumption), then (3) becomes

$$C_t = \int \bar{c}(x) \rho_t(x) dx$$

where ρ_t denotes the density of the income distribution in period t.

Thus, if $\bar{c}$ is linear, then mean consumption expenditure $C_t = \bar{c}(X_t)$, and hence depends only on mean income X_t without any restriction on the changes in the income distributions.

On the other hand, if changes in the income distributions are restricted to proportional changes in household income, hence the relative income distribution is time-invariant, say equal to ρ^*, then one obtains $C_t = \int \bar{c}(X_t \cdot x) \rho^*(x) dx$. Thus, mean consumption expenditure depends only on mean income X_t without any restriction on the Engel curve $\bar{c}$.

In this paper we want to avoid, as far as possible, any assumption on the micro-relation (other than being smooth in the relevant variables, e.g., income). Thus, the micro-relation is merely a notation; it just specifies the set of explanatory variables for consumption expenditure on the household level. To achieve the desired simplification of $C(\mu_t)$ we must therefore restrict the evolution over time of the distribution μ_t.

Since some of the household characteristics – that are explanatory variables in the micro-relation – are *unobservable*, we consider in addition to household characteristics also *observable* household *attributes*, such as age and employment status or household size. Household characteristics that are observable may be listed among attributes as well. Household income and attributes are used to stratify the population.

Then we obtain

$$C_t = \int \left[\int c(x, \chi) d\mu_t | (x, a) \right] d\nu_t$$

that is to say, we first consider mean consumption expenditure of the subpopulation consisting of all households with income x and attribute profile a and then we average over the subpopulations, i.e., we integrate with respect to the joint distribution ν_t of income and attributes.

2. In section 2 we model the changes over time of the conditional distribution $\mu_t | (x, a)$ of household characteristics (Hypothesis 1) and of the joint distribution ν_t of household income and attributes (Hypotheses 2 and 3).

It is our goal to "explain" the observed changes over time of C_t by changes in the observable income-attribute distribution ν_t. Such an "explanation" is only satisfactory if changes in C_t are not attributed to changes in the unobservable distributions $\mu_t|(x,a)$. Therefore we must somehow link changes in $\mu_t|(x,a)$ to changes in ν_t. This is achieved by Hypothesis 1, which is called *"structural stability*[2] *of household characteristics with respect to household attributes"*. In the special case where the conditional distribution $\mu_t|(x,a)$ does not depend on x, the hypothesis simply expresses that the distribution $\mu_t^{\chi}|a$ of household characteristics across all households with attribute profile a changes very slowly over time such that the distributions $\mu_s^{\chi}|a$ and $\mu_t^{\chi}|a$ can be considered as identical for periods s and t that are not too far apart from each other (local time-invariance).

Hypothesis 2 describes how the income distributions are allowed to change over time. Since we want to allow for changing income dispersion (e.g., changing Gini-coefficient) we can not rely on the simple assumption of time-invariance of the relative income distribution. Obviously, the actual evolution of household income is more complex than just a proportional change. No single assumption can exactly describe the complex evolution of income. We have chosen the simple hypothesis of local time-invariance of the standardized log income distribution (Hypothesis 2). Of course, this hypothesis should only be considered as an approximation to the complex actual changes of income distributions. The descriptive accuracy of Hypothesis 2 is illustrated in Figures 1 and 2.

Finally, we model how the attribute distributions are allowed to change over time. Hypothesis 3 expresses that the income-conditioned attribute distribution $\nu_s|x_s$ in period s is "approximately" equal to the income-conditioned attribute distribution $\nu_t|x_t$ in period t for two periods s and t that are close to each other, provided the income levels x_s and x_t are in the same percentile position (quantile) in the income distribution in period s and t, respectively.

As in the case of Hypothesis 2, this Hypothesis should be interpreted as an approximation, capturing the main tendency of the actual very complex change of attributes. The empirical content of Hypothesis 3 is illustrated in Figures 3, 4 and 5.

3. The propositions in Section 3 are based on a strong version of Hypothesis 3. It is assumed that the difference between the attribute distributions $\nu_s|x_s$ and $\nu_t|x_t$ can be neglected (Hypothesis 3^+). This requires, of course, that the periods s and t are close to each other.

[2]The idea of "structural stability" is borrowed from Malinvaud (1981), chapter 2.3 and (1993), section 10.

We then derive for the change of the mean consumption expenditure ratio a first-order approximation (Proposition 2):

$$C_t/X_t - C_s/X_s = \alpha_s \log \frac{\sigma_t}{\sigma_s} + \beta_s \log \frac{X_t}{X_s}$$
$$+ O\left(\max\left\{(\log \frac{\sigma_t}{\sigma_s})^2, (\log \frac{X_t}{X_s})^2\right\}\right)$$

where the coefficients α_s and β_s are determined by the micro-relation c and the distribution μ_s and σ_s is a measure of income dispersion.

Consequently, neglecting second-order terms, the change in the mean consumption expenditure ratio is the sum of two terms: the effect of the changing income dispersion, $\alpha_s \log \frac{\sigma_t}{\sigma_s}$, and the effect of mean income growth, $\beta_s \log \frac{X_t}{X_s}$.

The sign of the coefficients α_s and β_s are, of course, important. For example, a negative α_s implies that increasing income dispersion, (hence $\log \frac{\sigma_t}{\sigma_s}$ is positive), decreases the mean consumption expenditure ratio.

In Section 3 we show what kind of information on the micro-relation and the distribution μ_s is required to infer the sign or magnitude of the coefficients α_s and β_s. It turns out that no assumption on the micro-relation alone determines the sign of the coefficient α_s.

In certain circumstances, that are explained in Section 3, one can estimate the coefficients α_s and β_s from cross-section data in period s. In this case one does not need the micro-relation to determine α_s and β_s; only the actual household consumption expenditure in period s is needed.

4. In Section 4 we extend the approximation of Section 3 to the case where the difference of the attribute distributions $\nu_s|x_s$ and $\nu_t|x_t$ is not negligible.

2 Notation, Definitions and the Modelling Methodology

2.1

The goal of this analysis is to explain or more modestly to model the *change over time of aggregate consumption expenditure* C_t on a certain consumption category (such as Food, Housing and Non-Durables) in current prices of a *large and heterogeneous population.* What are the relevant explanatory variables for modelling the change in C_t?

Every analysis that accounts for aggregation over economic agents must begin with a model of individual behavior. Consequently, the starting point of our analysis is a *micro-relation* which relates household h's consumption expenditure c_t^h in period t to the price system p_t, the interest rate r_t, the disposable income x_t^h and certain theoretical household characteristics $\chi_t^h = (\chi_{1,t}^h, \chi_{2,t}^h, \dots)$:

$$c_t^h = c(p_t, r_t, x_t^h, \chi_t^h) \tag{1}$$

The nature of these household characteristics χ^h depends on the specification of the micro-model of individual behavior. Typically, some of the household characteristics are unobservable. If individual behavior is modelled as an intertemporal decision problem (foreward looking households) then, the micro-relation depends also on past information, for example, past income $x_{t-1}^h, \dots$ and past prices $p_{t-1}, \dots$ since this information is needed to predict future income and future prices. In the present paper, however, we do not treat this general case, since it greatly complicates the analysis. We start from a micro-relation (1) defined by a function c which is the same for all households; households differ, however, in income $x^h \in \mathbb{R}_+$ and in characteristics $\chi^h \in \mathcal{X}$. We assume that space of household characteristics $\mathcal{X}$ can always be considered as a metric space.

Given the micro-relation (1) the population of households in period t is described by the joint *distribution* μ_t *of income* $x \in \mathbb{R}_+$ *and household characteristics* $\chi \in \mathcal{X}$; thus μ_t is a distribution on $\mathbb{R}_+ \times \mathcal{X}$.

Mean consumption expenditure C_t in period t is then defined by

$$C_t := \int_{\mathbb{R}_+ \times \mathcal{X}} c(p_t, r_t, x, \chi) d\mu_t = C(p_t, r_t, \mu_t) \tag{2}$$

In addition to the *theoretical household characteristics* we shall consider *observable household attributes*, such as age, employment status, household size and composition etc. Such a profile of household attributes is denoted by $a = (a_1, a_2, \dots, a_n)$ which takes values in a set $\mathcal{A}$, a subset in $\mathbb{R}^n$.

Household characteristics which are *observable* belong also to the set of attributes (e.g. stock of financial assets). Typically, there are household attributes which are not household characteristics, that is to say, the micro-model is not formulated in terms of these attributes.

Notation

ν_t on $\mathbb{R}_+ \times \mathcal{A}$	the joint distribution of income x and household attributes a
$\nu_t \vert x$ on $\mathcal{A}$	the conditional distribution of household attributes given income x
ν_t^a on $\mathcal{A}$	the marginal distribution of household attributes
μ_t on $\mathbb{R}_+ \times \mathcal{X}$	the joint distribution of income x and household characteristics χ
$\mu_t \vert (x,a)$ on $\mathcal{X}$	the conditional distribution[3] of household characteristics given (x,a)
$\mu_t \vert x$ on $\mathcal{X}$	the conditional distribution of household characteristics given income x
μ_t^χ on $\mathcal{X}$	the marginal distribution of household characteristics

$\bar{c}_t(p_t, r_t, x, a) := \int_{\mathcal{X}} c(p_t, r_t, x, \chi) d\mu_t|(x,a)$
the *regression function* of consumption expenditure versus income x and household attribute a.

$\tilde{c}_t(p_t, r_t, x) := \int_{\mathcal{X}} c(p_t, r_t, x, \chi) d\mu_t|x$
the *regression function* of consumption expenditure versus income x; $\tilde{c}_t(p_t, r_t, \cdot)$ is called the *Engel curve* in period t.

Prototype example:

There are two distributions τ_1 *and* τ_2 *on* $\mathcal{X}$. *There is a function* π_t *from* $\mathbb{R}_+$ *into* $[0,1]$ *and* ρ_t *denotes the density of an income distribution. We consider one attribute which can take two values;* $\mathcal{A} = \{a^1, a^2\}$.

[3]this conditional distribution has to be derived from the joint distribution of (x, χ, a)

Define

$$\nu_t|x := \begin{cases} a^1 & \textit{with probability } \pi_t(x) \\ a^2 & \textit{with probability } (1-\pi_t(x)) \end{cases}$$

$$\nu_t^a := \begin{cases} a^1 & \textit{with probability } \int \pi_t(x)\rho_t(x)dx \\ a^2 & \textit{with probability } \int (1-\pi_t(x))\rho_t(x)dx \end{cases}$$

$$\mu_t|x := \pi_t(x)\tau_1 + (1-\pi_t(x))\tau_2$$

$$\mu_t|(x,a^1) := \tau_1, \;\; \mu_t|(x,a^2) := \tau_2$$

$$\begin{aligned} \bar{c}_t(p_t, r_t, x, a^i) &= \int_{\mathcal{X}} c(p_t, r_t, x, \chi) d\tau_i \,, \quad i = 1, 2 \\ \tilde{c}_t(p_t, r_t, x) &= \pi_t(x)\bar{c}_t(p_t, r_t, x, a^1) + (1-\pi_t(x))\bar{c}_t(p_t, r_t, x, a^2) \end{aligned}$$

Note that in this example, income x and household characteristics χ are independently distributed if one conditions on household attributes. Consequently

$$\partial_x \bar{c}_t(p_t, r_t, x, a^i) = \int_{\mathcal{X}} \left[\partial_x c(p_t, r_t, x, \chi)\right] d\mu_t|(x, a^i).$$

However,

$$\partial_x \tilde{c}_t(p_t, r_t, x) \neq \int_{\mathcal{X}} \partial_x c(p_t, r_t, x, \chi) d\mu_t|x.$$

The evolution over time of μ_t is determined by the evolution of ρ_t and π_t. Note that without specific assumptions on the evolution of ρ_t and π_t the marginal distribution μ_t^χ of household characteristics is not time-invariant.

In order to model the change over time of mean consumption expenditure

$$C_t = \int_{\mathbb{R}_+ \times \mathcal{A}} \left[\int_{\mathcal{X}} c(p_t, r_t, x, \chi) d\mu_t|(x, a)\right] d\nu_t \tag{3}$$

we have to formulate hypotheses on the *change over time* of

1) the conditional distribution $\mu_t|(x,a)$ of household characteristics given
   income x and attribute profile a, or alternatively,
   the regression function $\bar{c}_t(p_t, r_t, x, a)$

2) the joint distribution ν_t of income and attributes.

2.2 Structural stability of household characteristics

Hypotheses on the change over time of the conditional distribution $\mu_t|(x,a)$ are delicate. Any hypothesis on the change of the distribution $\mu_t|(x,a)$ is purely theoretical, that is to say, speculative, since $\mu_t|(x,a)$ describes a distribution of unobservable household characteristics.

It is our goal to "explain" the observed changes over time of C_t by changes in the observable distribution ν_t (thus, in particular, invariance of ν_t, p_t and r_t must imply invariance of C_t). Therefore we must link possible changes in $\mu_t|(x,a)$ with changes in ν_t. Otherwise ν_t cannot serve as a satisfactory explanatory variable since changes in C_t can then always be attributed to unobservable changes in $\mu_t|(x,a)$. Of course, it might turn out that the observed changes in C_t cannot be "explained" by the observed changes in ν_t since the set $\mathcal{A}$ of attributes is not sufficiently comprehensive.

The distribution μ_t^χ of household characteristics of the whole population may change over time, yet this change should be caused by a change in the distribution ν_t of income and attributes. The simplest way to achieve this is to *postulate that the distribution $\mu_t^\chi|a$ of characteristics of all households with attribute profile a is time-invariant* or, at least, changes very slowly over time (local time-invariance). This postulate is based on the idea that households typically keep their characteristics if their attribute profile does not change over time.

In the case where the conditional characteristic distribution $\mu_t|(x,a)$ *does not depend on x* (i.e., within the subpopulation of all households with attribute profile a, income x and household characteristics χ are independently distributed), the above postulate can serve as a definition of "structural stability" of household characteristics with respect to a set of household attributes. In this case the *regression function* $\bar{c}_t(\cdot,\cdot,\cdot,a)$, defined by

$$\bar{c}_t(p,r,x,a) = \int_{\mathcal{X}} c(p,r,x,\chi)d\mu_t|(x,a)$$

is time-invariant; a property which usually is assumed in applied micro-econometrics (e.g. Stoker (1993), Jorgenson et.al. (1982) and Blundell et. al. (1993)).

We would like to define "structural stability" also in the case where $\mu_t|(x,a)$ might depend on x. Then, with household income and prices changing over time, it does not seem to us meaningful to postulate time-invariance of $\mu_t|(x,a)$, since x denotes nominal income. Rather one should condition on "real" income or on quantiles in the income distribution.

Definition: The income levels x_s and x_t in period s and period t, respectively, *are in the same percentile position* in the income distribution in period s and t, respectively, if

$$\int_0^{x_s} \rho_s(x)dx = \int_0^{x_t} \rho_t(x)dx \, .$$

The following heuristic argument motivates the definition of structural stability that is basic for our analysis.

Consider two periods s and t and the income densities ρ_s and ρ_t. For periods that are close to each other one expects a *high positive association* between household's income in period s and the later period t. If this association were perfect then households would remain in the same percentile position in the income distributions ρ_s and ρ_t. Furthermore, if *households whose attribute profile does not change* in going from period s to period t *keep their characteristics* then the distributions of household characteristics $\mu_s|(x_s, a)$ and $\mu_t|(x_t, a)$ will approximately be the same if x_s and x_t are in the same percentile position in ρ_s and ρ_t, respectively.

Hypothesis 1: structural stability of household characteristics

Structural stability of household characteristics with respect to the set $\mathcal{A}$ of household attributes is defined by the following property:

> *if x_s and x_t are in the same percentile position in the income distribution of the whole population in period s and t, respectively, then, for every $a \in \mathcal{A}$,*
>
> $$\mu_s|(x_s, a) = \mu_t|(x_t, a).$$

Structural stability implies the above postulate that the *distribution of household characteristics of all households with attribute profile a is time-invariant.* This justifies the label "structural stability" (in the sense of time-invariance). This property, in turn, implies structural stability if $\mu_t|(x, a)$ does not depend on x, that is to say, conditioned on household attribute profile a, income x and household characteristics χ are independent. We remark that the prototype example is structurally stable.

2.3 The change over time of the distribution ν_t

In formulating hypotheses on the change over time of the observable distribution ν_t we have to face the following well-known empirical facts:
1) ν_t describes a multi-variate distribution whose *components are not independently distributed*, that is to say, the joint distribution ν_t of income and attributes

is not the product of its marginals (e.g. the distributions of income, age, household size, etc.). Typically there is a high correlation between income and the various household attributes.
2) There is *no* satisfactory *a priori* given *functional form* (determined up to some few parameters!) for the distribution ν_t and even for its marginals.

For example, the shape and the change over time of income distributions are the outcome of many different forces some of which are operating in different directions. Furthermore, the shape and the change over time of income distributions depend on the notion of income (e.g. "disposable income"), on the units over which the distribution is defined (e.g. "household") and on the population (e.g. "full-time employed household head").

By "modelling the change over time" of the distribution ν_t we do not mean "to predict the evolution of ν_t" but to suitably restrict the possible changes, that is, we want a "parametrization of the *transition*" from ν_s in period s to ν_t in period t.

To explain this "parametrization of the transition" we consider first the case of income distributions.

Example: Time-invariance of the relative income distribution

The *relative* income distribution in period t is obtained by dividing the income x_t^h of every household in the population by mean income X_t. If ρ_t and ρ_t^* denote the density of the income and relative income distribution, respectively, then one obtains

$$\rho_t^*(\xi) = X_t \rho_t(X_t \cdot \xi) \,. \tag{4}$$

Time-invariance of ρ_t^* then implies that for two periods s and t

$$\rho_t(x) = \frac{X_s}{X_t} \rho_s \left(\frac{X_s}{X_t} \cdot x \right) . \tag{5}$$

Thus, the *transition* from ρ_s to ρ_t is parametrized by mean income X_t, that is to say, if one knows ρ_s and X_t then ρ_t is determined.

Time-invariance of the relative income distribution cannot serve as a suitable hypothesis for our analysis. Indeed, it implies that the income dispersion, measured, for example, by the Gini-coefficient or the standard deviation of log income, remains constant over time. It is however a well-established empirical fact (e.g. Gottschalk and Smeeding (1997)) that for most countries the income dispersion changes over time, even though, for some countries, the change is very slow. To take into account a changing income dispersion we consider the following

Hypothesis 2: Time-invariance of the standardized log income distribution

Instead of income x consider the logarithm of income, $\log x$. Let m_t and σ_t denote the mean and standard deviation, respectively, of the distribution of $\log x$ in period t.

The *standardized log income distribution* is then defined as the distribution of

$$\frac{1}{\sigma_t}(\log(x) - m_t).$$

The density of this distribution is denoted by ρ_t^+. The relationship between the densities ρ_t and ρ_t^+ is given by

$$\rho_t^+(z) = y_t \sigma_t \exp(\sigma_t \cdot z) \rho_t(y_t \exp(\sigma_t \cdot z)) \tag{6}$$

where $y_t = \exp(m_t)$. Note, y_t is equal to the median of the income distribution ρ_t if $\log x$ is symmetrically distributed. Furthermore, if income were log-normally distributed (i.e., $\log x$ has a normal distribution) then ρ_t^+ is the normal distribution with mean 0 and variance 1.

Time-invariance of ρ_t^+ then implies for two periods s and t

$$\rho_t(x) = \frac{\sigma_s}{\sigma_t} \cdot \frac{y_s}{y_t^{\sigma_s/\sigma_t}} \cdot x^{(\sigma_s/\sigma_t)-1} \rho_s \left(\frac{y_s}{y_t^{\sigma_s/\sigma_t}} \cdot x^{(\sigma_s/\sigma_t)} \right) \tag{7}$$

It is not hard to show that the transition can also be parametrized by mean income X_t and σ_t;

$$\rho_t(x) = \sigma_s/\sigma_t \cdot \left(\frac{m_s(\sigma_t/\sigma_s)}{X_t} \right)^{\sigma_s/\sigma_t} \cdot x^{(\sigma_s/\sigma_t)-1} \cdot \rho_s \left(\left(\frac{m_s(\sigma_t/\sigma_s)}{X_t} \cdot x \right)^{\sigma_s/\sigma_t} \right) \tag{8}$$

where $m_s(\sigma) = \int x^\sigma \rho_s(x) dx$.

Thus, the *transition* from ρ_s to ρ_t is parametrized by (X_t, σ_t), that is to say, if one knows ρ_s, hence X_s and σ_s, and X_t, σ_t then ρ_t is determined.

It follows from (8) that x_s and x_t are in the same percentile position of ρ_s and ρ_t, respectively, if

$$x_s = \varphi(x_t) \tag{9}$$

where the function φ is defined by $\varphi(x) := \left(\frac{m_s(\sigma_t/\sigma_s)}{X_t} \cdot x \right)^{\sigma_s/\sigma_t}$.

Remark: Naturally, time-invariance of the standardized log income distribution will never hold exactly, even for periods s and t that are close to each other.

Hypothesis 2 should be considered as an *approximation* to the actual change in the short-run. It is important to remark that the income distributions can be estimated and therefore one can decide whether the hypothesis satisfactorily captures the main tendency of the actual change. Since we need the income distribution only to compute the mean (integral) it might be sufficient to know ρ_t approximately provided the regression function that we want to integrate, is sufficiently regular.

It might well be that alternative hypotheses will be found that yield a better approximation. The standardized log transformation is particularly simple, yet our approach can be adapted to any other transformation of income distributions that leads to time-invariance. Given ρ_s, (8) defines a parametrization of ρ_t in terms of X_t, σ_t. This parametrization of income distribution is "mean-scaled" in the sense of Lewbel (1990) and (1992).

Figures 1 and 2 show kernel density estimates of the standardized log income distribution based on data from the U.K. Family Expenditure Survey.

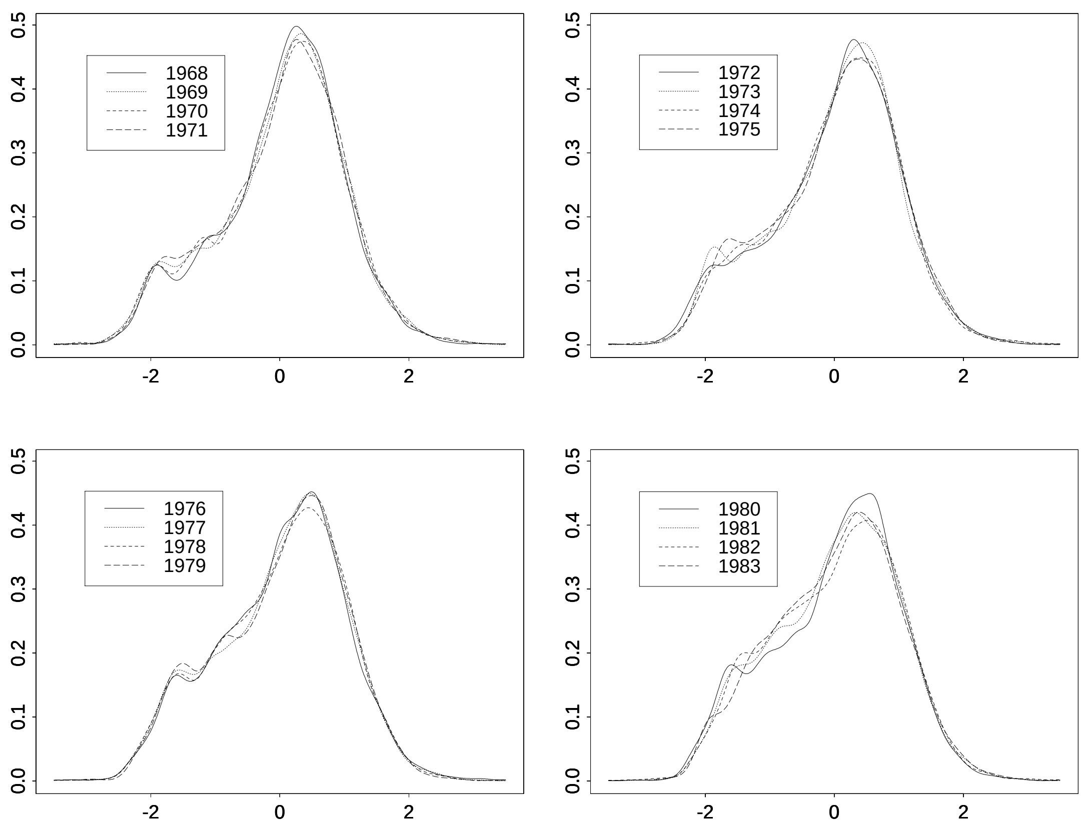

Figure 1: Kernel density estimates of the standardized log income distribution; U.K.-FES, total population.

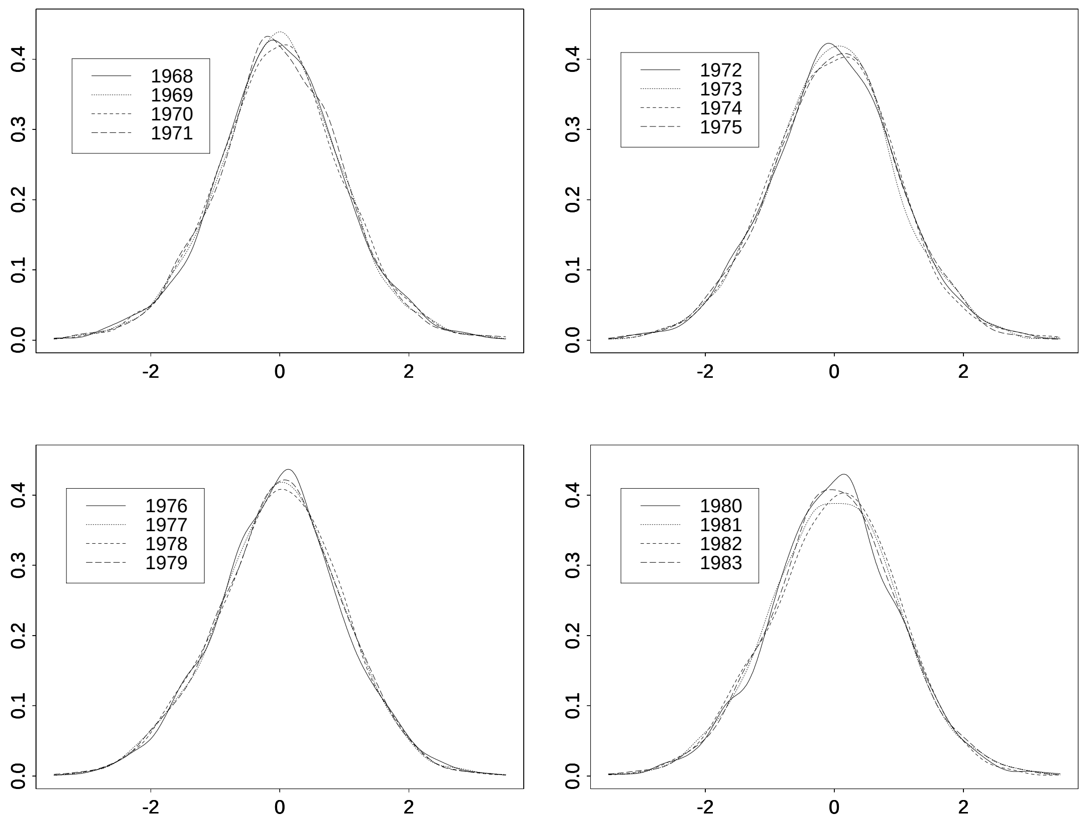

Figure 2: Kernel density estimates of the standardized log income distribution; U.K.-FES, subpopulation of full-time employed head of household.

Next we have to model the change over time of the distribution of household attributes. The modelling approach is formally analogous to the one in section 2.2 in the case of distributions of household characteristics. There is, however, an important difference; distributions of attributes are observable. For example, it is a well-established empirical fact that the attribute distribution $\nu_t|x$ of all households with income x *depends quite strongly on the income level* x and the distribution $\nu_t|x$ is *not time-invariant* (e.g. see Figures 3, 4 and 5).

The heuristic argument preceding the definition of structural stability in section 2.2 suggests the following

Hypothesis 3: *For two periods s and t that are close to each other the income-conditioned attribute distribution $\nu_s|x_s$ is "approximately" equal to the income-conditioned attribute distribution $\nu_t|x_t$ if x_s and x_t are in the same percentile position in ρ_s and ρ_t, respectively.*

This hypothesis is based on the view that household attributes change relatively slowly as compared with income. We shall consider two versions of Hypothesis 3; we shall assume in section 3 that the difference $\nu_s|x_s - \nu_t|x_t$ is negligible and in section 4 that the difference is "small", in a sense to be explained later.

To illustrate the empirical content of Hypothesis 3 we show in Figures 3, 4 and 5 estimates of the age distribution of head of household, the distribution of household size and the distribution of employment status of head of household, respectively, conditioned on four percentile positions: 'poor', 'lower middle-class', 'upper middle-class', and 'rich'.

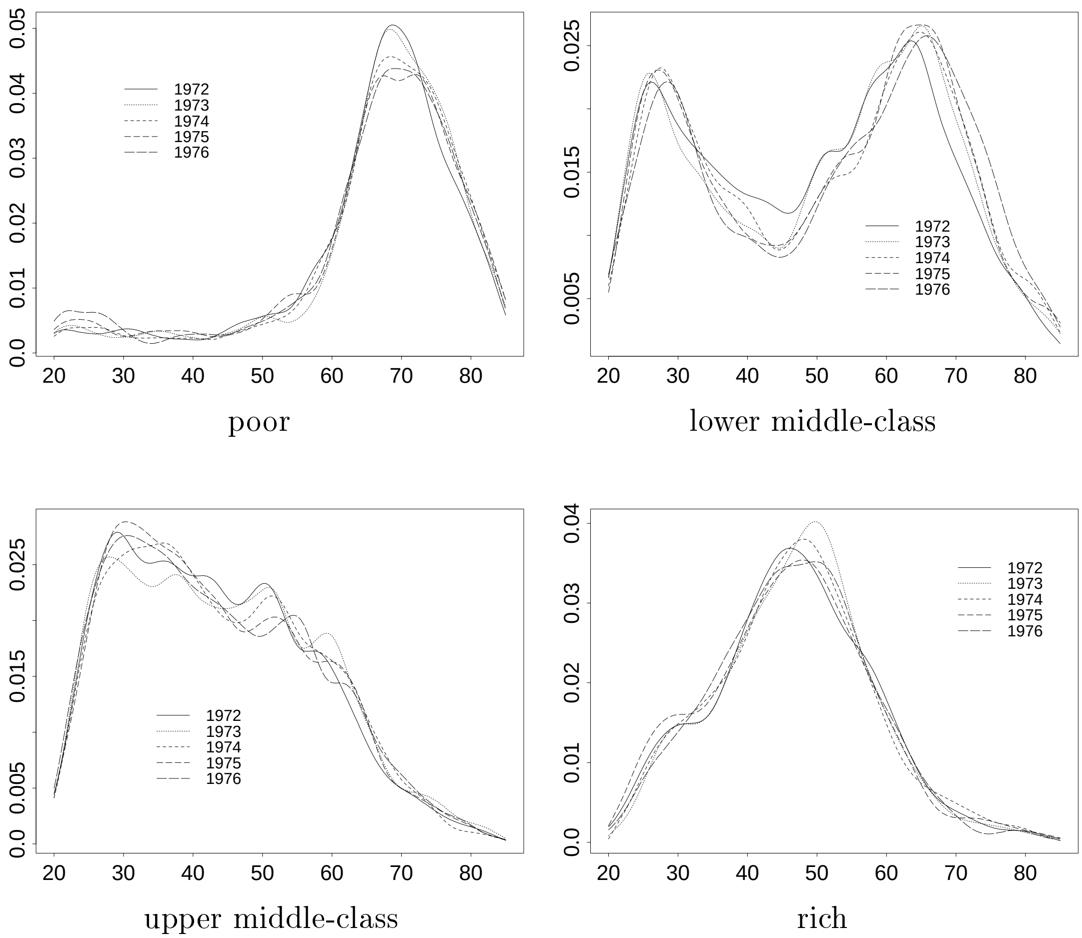

Figure 3: Estimates of age distributions of head of household conditioned on four percentile interval positions: “poor” (0 - 16 %), “lower middle-class” (17 - 50 %), “upper middle-class” (50 - 84 %), and “rich” (85 - 100 %). U.K.-FES, total population.

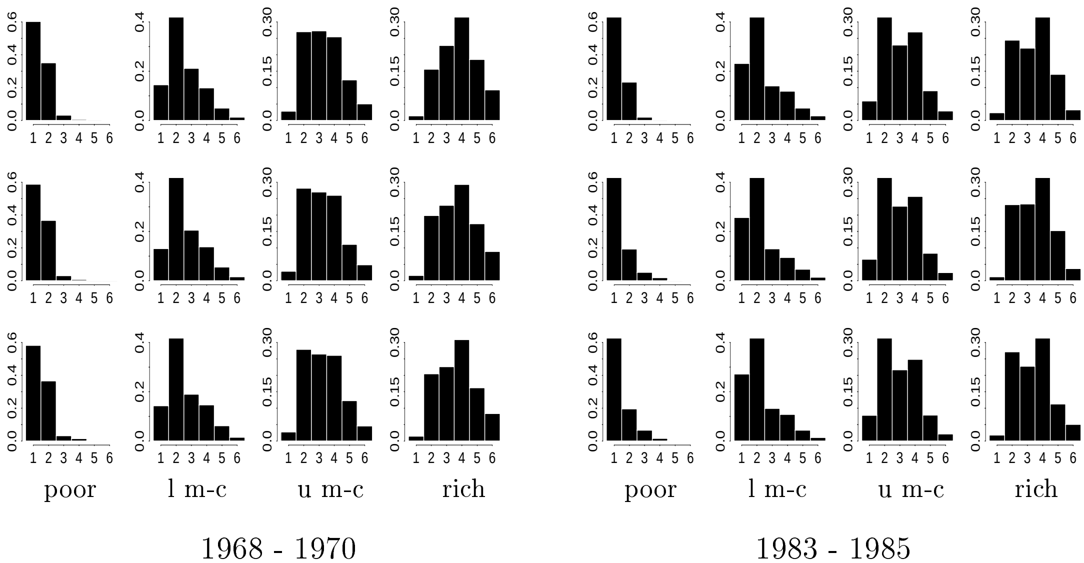

Figure 4: Estimates of the distribution of number of persons in household conditioned on four percentile interval positions: "poor", "lower middle-class", "upper middle-class", and "rich".

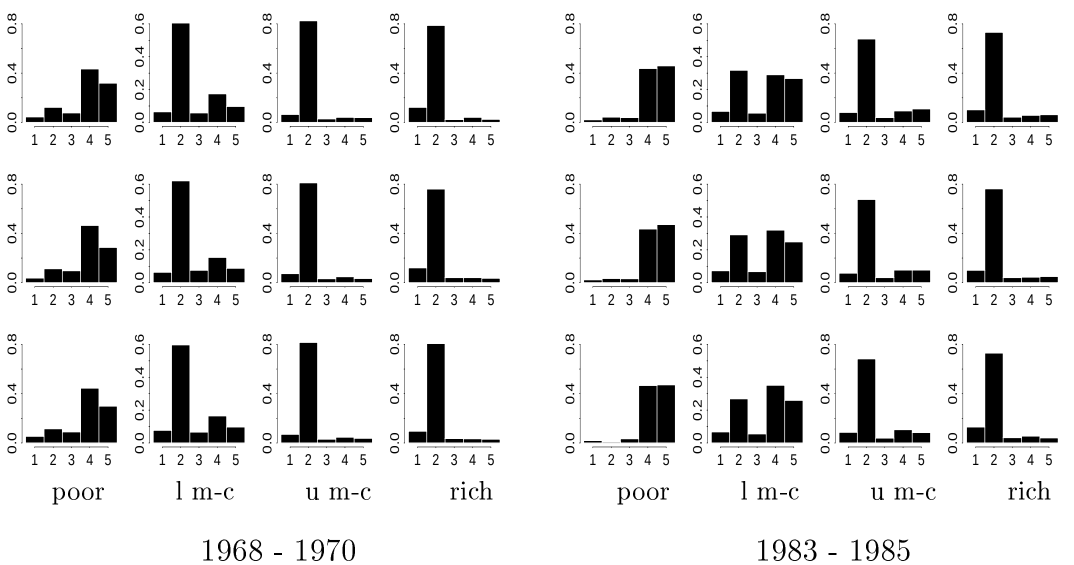

Figure 5: Estimates of the distribution of employment status of head of household conditioned on four percentile interval positions: "poor", "lower middle-class", "upper middle-class", and "rich". Self employed 1, full-time employed 2.

3 Aggregation under structural stability of household attributes

3.1

In this section we explore the implications of Hypothesis 1, structural stability of household characteristics, Hypothesis 2, time-invariance of the standardized log income distributions, and a strong version of Hypothesis 3, which is

Hypothesis 3^+: Structural stability of household attributes
If x_s and x_t are in the same percentile position in the income distributions of period s and t, respectively, then

$$\nu_s|x_s = \nu_t|x_t \,.$$

Hypothesis 3^+ is very restrictive; it will be modified later. The hypothesis implies that the distribution of household attributes is time-invariant. Yet the distributions of age, household size etc. that are estimated from time series of cross-section data are not time-invariant, even though they change over time very slowly (see Hildenbrand, Kneip, and Utikal (1997)).

Proposition 1: *Hypothesis 1, 2 and 3^+ imply*

$$C_t = \int_{\mathrm{I\!R}_+ \times \mathcal{X}} c(p_t, r_t, \frac{X_t}{m_s(\sigma_t/\sigma_s)} \cdot x^{\sigma_t/\sigma_s}, \chi) d\mu_s =: K_{\mu_s}(p_t, r_t, X_t, \sigma_t) \tag{10}$$

that is to say, given the micro-relation c and the distribution μ_s in period s, then mean consumption expenditure C_t in period t is a function in p_t, r_t, X_t and σ_t.

Proof: By Hypothesis 2 we obtain

$$\rho_t(x) = \sigma_s/\sigma_t \cdot \left(\frac{m_s(\sigma_t/\sigma_s)}{X_t}\right)^{\sigma_s/\sigma_t} \cdot x^{(\sigma_s/\sigma_t)-1} \rho_s \left(\left(\frac{m_s(\sigma_t/\sigma_s)}{X_t} \cdot x\right)^{\sigma_s/\sigma_t}\right) \tag{11}$$

Hypothesis 1, structural stability of household characteristics, and Hypothesis 3^+, structural stability of household attributes, imply

$$\mu_t|x = \mu_s| \left(\frac{m_s(\sigma_t/\sigma_s)}{X_t} \cdot x\right)^{\sigma_s/\sigma_t} \tag{12}$$

We now substitute (11) and (12) into the definition of C_t and obtain with the notation $\sigma = \sigma_t/\sigma_s$

$$C_t = \int_{\mathbb{R}_+} \left[\int_{\mathcal{X}} c(p_t, r_t, x, \chi) d\mu_s | \left(\frac{m_s(\sigma)}{X_t} \cdot x \right)^{1/\sigma} \right] \frac{1}{\sigma} \left(\frac{m_s(\sigma)}{X_t} \cdot x \right)^{1/\sigma} \frac{1}{x} \rho_s \left(\left(\frac{m_s(\sigma)}{X_t} \cdot x \right)^{1/\sigma} \right) dx$$

The substitution $\xi = \left(\frac{m_s(\sigma)}{X_t} \cdot x\right)^{1/\sigma}$, hence $x = \frac{X_t}{m_s(\sigma)} \xi^\sigma$ and $\frac{dx}{d\xi} = \frac{X_t}{m_s(\sigma)} \cdot \sigma \cdot \xi^{\sigma - 1}$, leads to

$$C_t = \int_{\mathbb{R}_+} \left[\int_{\mathcal{X}} c(p_t, r_t, \frac{X_t}{m_s(\sigma)} \cdot x^\sigma, \chi) d\mu_s | x \right] \rho_s(x) dx.$$

Q.E.D.

Proposition 1 shows that $C_t = K_{\mu_s}(p_t, r_t, X_t, \sigma_t)$ is determined by the micro-relation c – which we did not specify up to now – and the distribution μ_s – which is only partially observable. In the following we want to approximate the integral in (10) by a simple expression in the variables p_t, r_t, X_t and σ_t.

The simplest approximation of $K_{\mu_s}(p_t, r_t, X_t, \sigma_t)$ which comes into mind is a first-order Taylor expansion in the variables p, r, X and σ at the values p_s, r_s, X_s and σ_s. However, even for periods s and t which are close to each other, say $t = s + 1$, the change $X_t - X_s$ of mean income (or the price change $p_t - p_s$) will typically not be *very* small, for example, mean income might well increase by 10% or even more. Consequently, to obtain a satisfactory approximation of C_t we have either to take a higher-order Taylor expansion – which will complicate the analysis – or we look for a suitable non-linear first-order approximation which is a satisfactory approximation even for values of the variables p, r, X, σ that are not very near to p_s, r_s, X_s, σ_s. The choice of such an approximation requires, of course, some knowledge of the shape of the function that we want to approximate.

The particular approximation that is given in Proposition 2 is based on the assumption that the regression

$$x \mapsto w_s(p_s, r_s, x, a) = \int_{\mathcal{X}} \frac{1}{x} c(p_s, r_s, x, \chi) d\mu_s | (x, a), \quad a \in \mathcal{A}$$

has the following property (that is a well-known empirical fact):

> $\partial_\lambda w_s(p_s, r_s, \lambda x, a)|_{\lambda=1} = x \cdot \partial_x w_s(p_s, r_s, x, a)$ does not depend very sensitively on the income level x.

To simplify the presentation we replace the comprehensive price system p_t by two price indices π_t^1 and π_t^2. We shall write $p_t = (p_t^1, p_t^2)$, where p_t^1 is the price system of all commodities for which we consider consumption expenditure and p_t^2 is the price system of all other commodities. Let π_t^1 and π_t^2 denote a price index for p_t^1 and p_t^2, respectively.

We now define the function G by

$$G(\pi_t^1, \pi_t^2, r_t, X_t, \sigma_t) := K_{\mu_s}(\frac{\pi_t^1}{\pi_s^1} p_s^1, \frac{\pi_t^2}{\pi_s^2} p_s^2, r_t, X_t, \sigma_t)$$

Proposition 2: *Hypothesis 1, 2 and 3^+ imply for a smooth micro-relation c*

$$\begin{aligned} & G(\pi_t^1, \pi_t^2, r_t, X_t, \sigma_t) \\ & \quad = X_t \left[\frac{C_s}{X_s} + \alpha_s \log \frac{\sigma_t}{\sigma_s} + \beta_s \log \frac{X_t}{X_s} + \gamma_s \log \frac{\pi_t^1}{\pi_s^1} + \delta_s \log \frac{\pi_t^2}{\pi_s^2} + \eta_s \log \frac{r_t}{r_s} \right] \\ & \qquad + O\left(\max \left\{ (\log \frac{\sigma_t}{\sigma_s})^2, (\log \frac{X_t}{X_s})^2, (\log \frac{\pi_t^1}{\pi_s^1})^2, (\log \frac{\pi_t^1}{\pi_s^2})^2, (\log \frac{r_t}{r_s})^2 \right\} \right); \end{aligned} \tag{13}$$

the coefficients $\alpha_s, \beta_s, \gamma_s, \delta_s$ and η_s are defined by

$$\begin{aligned} \alpha_s &= \partial_\sigma \left[\frac{1}{m_s(\sigma)} \int_{\mathbb{R}_+} x^\sigma \left[\int_{\mathcal{X}} \partial_x c(p_s, r_s, x, \chi) d\mu_s | x \right] \rho_s(x) dx \right]_{\sigma=1} \\ \beta_s &= \frac{1}{X_s} \int_{\mathbb{R}_+} x \left[\int_{\mathcal{X}} \partial_\lambda [w(p_s, r_s, \lambda x, \chi)]_{\lambda=1} d\mu_s | x \right] \rho_s(x) dx \\ \gamma_s &= \frac{1}{X_s} \int_{\mathbb{R}_+ \times \mathcal{X}} \partial_\pi \left[c(\pi \cdot p_s^1, p_s^2, r_s, x, \chi) \right]_{\pi=1} d\mu_s \\ \delta_s &= \frac{1}{X_s} \int_{\mathbb{R}_+ \times \mathcal{X}} \partial_\pi \left[c(p_s^1, \pi \cdot p_s^2, r_s, x, \chi) \right]_{\pi=1} d\mu_s \\ \eta_s &= \frac{1}{X_s} \int_{\mathbb{R}_+ \times \mathcal{X}} \partial_r \left[c(p_s, r \cdot r_s, x, \chi) \right]_{r=1} d\mu_s \end{aligned}$$

We emphasize that the coefficients are defined by the micro-relation and the distribution μ_s in period s; hence they can be interpreted as behavioral parameters that depend on the composition of the population (see section 3.2 for details).

Proof: With the notation $\pi_1 := \pi_t^1/\pi_s^1, \pi_2 := \pi_t^2/\pi_s^2, \sigma := \sigma_t/\sigma_s$ and $w(p_t, r_t, \xi, \chi) := \frac{1}{\xi} c(p_t, r_t, \xi, \chi)$ we obtain from Proposition 1 that

$$G(\pi_t^1, \pi_t^2, r_t, X_t, \sigma_t)/X_t = \int_{\mathbb{R}_+} \frac{x^\sigma}{m_s(\sigma)} \left[\int_{\mathcal{X}} w(\pi_1 \cdot p_s^1, \pi_2 \cdot p_s^2, r_t, \frac{X_t}{m_s(\sigma)} \cdot x^\sigma, \chi) d\mu_s | x \right] \rho_s(x) dx.$$

Let $X := X_t/X_s$ and $r := r_t/r_s$. The integral then becomes

$$\int_{\mathbb{R}_+\times\mathcal{X}} \frac{x^\sigma}{m_s(\sigma)} w(\pi_1\cdot p_s^1, \pi_2\cdot p_s^2, r\cdot r_s, \frac{X_s}{m_s(\sigma)}\cdot x^\sigma\cdot X, \chi) d\mu_s \tag{14}$$

which defines a function f_s in $(\pi_1, \pi_2, r, X, \sigma)$;

$f_s(1,\dots,1) = G(\pi_s^1, \pi_s^2, r_s, X_s, \sigma_s)/X_s = C_s/X_s$ and

$f_s(\pi_1,\pi_2,r,X,\sigma) = G(\pi_t^1, \pi_t^2, r_t, X_t, \sigma_t)/X_t$.

We now take a first-order Taylor expansion of the function f_s in $\log\pi_1 =: \tilde\pi_1, \log\pi_2 =: \tilde\pi_2, \log r =: \tilde r, \log X =: \tilde X$ and $\log\sigma =: \tilde\sigma$ at $(\tilde\pi_1,\tilde\pi_2,\tilde r,\tilde X,\tilde\sigma) = (0,\dots,0)$, that is to say, we take a usual first order Taylor expansion of the function

$$(\tilde\pi_1,\tilde\pi_2,\tilde r,\tilde X,\tilde\sigma) \mapsto f_s(\exp\tilde\pi_1, \exp\tilde\pi_2, \exp\tilde r, \exp\tilde X, \exp\tilde\sigma) \tag{15}$$

at $(\tilde\pi_1,\dots,\tilde\sigma) = (0,\dots,0)$. Thus we obtain

$$\begin{aligned} f_s(\pi_1,\pi_2,r,X,\sigma) \approx{}& f_s(1,\dots,1) + \alpha_s\log\sigma + \beta_s\log X \\ &+ \gamma_s\log\pi_1 + \delta_s\log\pi_2 + \eta_s\log r \end{aligned}$$

which is the claimed approximation. The coefficients $\alpha_s,\dots,\eta_s$ are defined as partial derivatives of the above function (15) at $(0,\dots,0,0)$.

The coefficient α_s requires a comment. By definition

$$\begin{aligned} \alpha_s &:= \partial_{\tilde\sigma}\left[f_s(1,1,1,1,\exp\tilde\sigma)\right]_{\tilde\sigma=0} = \partial_\sigma\left[f_s(1,1,1,1,\sigma)\right]_{\sigma=1} \\ &= \partial_\sigma\left[\int_{\mathbb{R}_+\times\mathcal{X}} \frac{x^\sigma}{m_s(\sigma)} w(p_s, r_s, \frac{X_s}{m_s(\sigma)}\cdot x^\sigma, \chi) d\mu_s\right]_{\sigma=1} \end{aligned}$$

Define $g(\sigma) = \frac{x^\sigma}{m_s(\sigma)}$. Then we obtain

$$\alpha_s = \int_{\mathbb{R}_+\times\mathcal{X}} \partial_\sigma\left[g(\sigma)w(p_s, r_s, g(\sigma)X_s, \chi)\right]_{\sigma=1} d\mu_s$$

We now compute the derivative under the integral.

$$\begin{aligned} \partial_\sigma\left[g(\sigma)w(p_s,r_s,g(\sigma)X_s,\chi)\right]_{\sigma=1} &= g'(\sigma)|_{\sigma=1}\cdot\left(w(p_s,r_s,x,\chi) + x\partial_x w(p_s,r_s,x,\chi)\right) \\ &= g'(\sigma)|_{\sigma=1}\cdot\partial_x c(p_s,r_s,x,\chi). \end{aligned}$$

Consequently, $\alpha_s = \int \partial_\sigma\left[\frac{x^\sigma}{m_s(\sigma)}\cdot\partial_x c(p_s,r_s,x,\chi)\right]_{\sigma=1} d\mu_s$ which is equal to the expression claimed in Proposition 2.

By definition

$$\beta_s := \partial_{\tilde X}\left[f_s(1,1,1,\exp\tilde X, 1)\right]_{\tilde X=0}.$$

Since

$$\partial_{\tilde{X}} f_s(1,1,1,\exp \tilde{X},1) = \partial_{\tilde{X}} \int \frac{x}{X_s} w(p_s, r_s, x \exp \tilde{X}, \chi) d\mu_s$$
$$= \frac{1}{X_s} \int x \partial_{\text{inc}} w(p_s, r_s, x \exp \tilde{X}, \chi) \cdot x \exp \tilde{X} d\mu_s,$$

where ∂_{inc} denotes the partial derivative with respect to income, we obtain for $\tilde{X} = 0$ the expression for β_s as claimed in Proposition 2. Furthermore, the second derivative

$$\partial^2_{\tilde{X}} f_s(1,1,1,\exp \tilde{X},1)$$

can be expected to be quite small for $\tilde{X}$ in a large neighbourhood around 0 if on the micro-level for every x the expression

$$\lambda x \partial_{\text{inc}} w(p_s, r_s, \lambda x, \chi) \tag{16}$$

is approximately constant for λ in a large neighborhood of 1. Consequently, if one assumes property (16) then it is justified to take a Taylor expansion in $\log X$ instead of X.

We remark that property (16) is well supported by empirical evidence if it is applied to the regression function $w_s(p_s, r_s, x, a) = \int w(p_s, r_s, x, \chi) d\mu_s |(x, a)$.

By definition

$$\gamma_s := \partial_{\tilde{\pi}_1} \left[f_s(\exp \tilde{\pi}_1, 1, 1, 1, 1) \right]_{\tilde{\pi}_1 = 0}$$
$$= \partial_{\tilde{\pi}_1} \left[\frac{1}{X_s} \int x w(\exp \tilde{\pi}_1 \cdot p_s^1, p_s^2, r_s, x, \chi) d\mu_s \right]_{\tilde{\pi}_1 = 0}$$
$$= \frac{1}{X_s} \int \partial_\pi \left[c(\pi \cdot p_s^1, p_s^2, r_s, x, \chi) \right]_{\pi=1} d\mu_s$$

Analogously for the coefficients δ_s and η_s.

Q.E.D.

3.2 Discussion of the coefficients

The coefficient α_s:

By definition of α_s in Proposition 2 we obtain

$$\alpha_s = \partial_\sigma \left[\frac{1}{m_s(\sigma)} \int_{\mathbb{R}_+} x^\sigma \cdot MMP_s(x) \rho_s(x) dx \right]_{\sigma=1}$$

where $MMP_s(x) := \int_{\mathcal{X}} \partial_x c(p_s, r_s, x, \chi) d\mu_s | x$, i.e., the mean marginal propensity to consume of all households with income x.

The sign of α_s depends on the form of the function $MMP_s(x)$. Indeed, it follows from the Lemma in the Appendix that for every density ρ_s the coefficient

$\alpha_s \leq 0$ (≥ 0) *if $MMP_s(x)$ is a decreasing (increasing) function in x*

More generally, $\alpha_s \leq 0$ if for all z in a neighborhood of 1,

$$\min_{0 \leq x \leq z} MMP_s(x) \geq \max_{z \leq x} MMP_s(x).$$

If the individual household propensity to consume $\partial_x c(p_s, r_s, x, \chi)$ is decreasing in x, i.e. $c(p_s, r_s, x, \chi)$ is a concave function in x – which is frequently postulated for the micro-model – then it does not necessarily follow that the $MMP_s(x)$ is also decreasing in x since the conditional distribution $\mu_s|x$ depends on x.

For example, let $\mu_s|x$ be equal to χ^1 with probability $\pi(x)$ and χ^2 with probability $1 - \pi(x)$. Let $c(p_s, r_s, x, \chi^i)$ be linear in x and $\partial_x c(p_s, r_s, x, \chi^1) \neq \partial_x c(p_s, r_s, x, \chi^2)$. Then the $MMP(x)$ can be decreasing, increasing or not be monotone at all depending on the function $\pi(x)$.

Figure 6 shows three examples of the function $\pi(x)$.

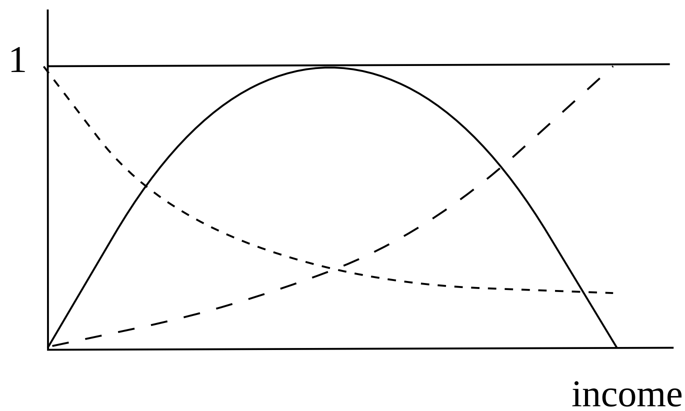

Figure 6

Figure 7 shows for each function π the mean marginal propensity to consume $MMP(x)$.

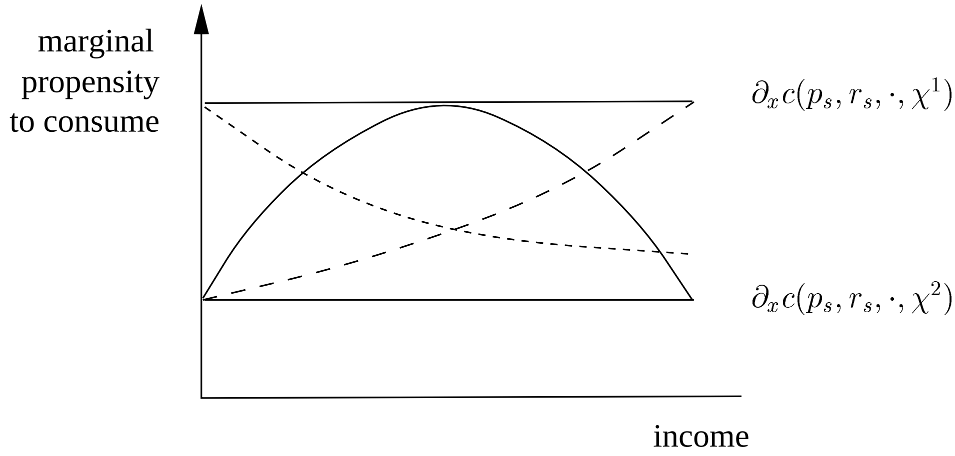

Figure 7

Thus we have shown that the sign of α_s does not only depend on suitable properties of the micro-model but also on the composition of the population. In particular, *linearity in income on the household level does not necessarily imply that the distribution effect of income is positive or negative nor that it can be neglected.*

The magnitude of the coefficient α_s depends not only on the form of $MMP_s(x)$, but also on the income distribution ρ_s; in particular, the magnitude of α_s depends on the dispersion σ_s. This is best illustrated by an example: let $MMP_s(x) = d_s + b_s \log x$ and ρ_s a log normal density with parameters (m_s, σ_s). Then one can explicitly compute the coefficient α_s and obtains $\alpha_s = 2\sigma_s^2 \cdot b_s$.

How can one estimate the coefficient α_s? If one conditions on household attributes one might expect (or assume) that the conditional distribution $\mu_s|(x,a)$ does not depend very sensitively on x, more precisely,

$$\begin{aligned}\int_{\mathcal{X}} \partial_x c(p_s, r_s, x, \chi) d\mu_s|(x,a) &\approx \partial_x \int_{\mathcal{X}} c(p_s, r_s, x, \chi) d\mu_s|(x,a) \qquad (\star)\\ &= \partial_x \bar{c}_s(p_s, r_s, x, a)\end{aligned}$$

If this condition is satisfied one can define the following proxi for α_s:

$$\bar{\alpha}_s := \partial_\sigma \left[\frac{1}{m_s(\sigma)} \int_{\mathbb{R}_+} x^\sigma \left[\int_{\mathcal{A}} \partial_x \bar{c}_s(p_s, r_s, x, a) d\nu_s|x \right] \rho_s(x) dx \right]_{\sigma=1}$$

Indeed, if condition $(\star)$ holds with equality, then $\alpha_s = \bar{\alpha}_s$. The important point now is that $\bar{\alpha}_s$ *can be estimated from cross-section data in period* s. For estimating $\bar{\alpha}_s$ one does not need a micro-model c, only the actual consumption decisions in period s are needed.

We have estimated the coefficient $\bar{\alpha}_t$ (stratification with respect to household size and age in 5×8 disjoint groups) using the data from the U.K. Family Expenditure Survey. The mean value of $\bar{\alpha}_t$ for the years 1968-1986 is -0.036 for consumption expenditure on non-durables; -0.009 for food and $+0.006$ for services. The method of estimation and further details can be found in Hildenbrand and Kneip (1997).

The coefficient β_s:

Since

$$\partial_\lambda \left[w(p_s, r_s, \lambda x, \chi) \right]_{\lambda=1} = \partial_x c(p_s, r_s, x, \chi) - \frac{1}{x} c(p_s, r_s, x, \chi)$$

one obtains from the definition of β_s that

$$\begin{aligned} \beta_s &= \frac{1}{X_s} \int_{\mathbb{R}_+} \left[\int_{\mathcal{X}} (x \partial_x c(p_s, r_s, x, \chi) - c(p_s, r_s, x, \chi)) d\mu_s | x \right] \rho_s(x) dx \\ &= \frac{1}{X_s} \int x \partial_x c(p_s, r_s, x, \chi) d\mu_s - C_s / X_s \end{aligned}$$

Hence, *the coefficient β_s is negative (positive) if in average*

$$\int_{\mathcal{X}} \left(x \partial_x c(p_s, r_s, x, \chi) - c(p_s, r_s, x, \chi) \right) d\mu_s | x$$

is negative (positive).

Note that *the smaller the individual household marginal propensity to consume, $\partial_x c(p_s, r_s, x, \chi)$, the larger is $-\beta_s$.* In the extreme case, where $\partial_x c(p_s, r_s, x, \chi) \approx 0$ one obtains $\beta_s \approx -\frac{C_s}{X_s}$.

As in the case of the coefficient α_s one can define a proxi $\bar{\beta}_s$ for the coefficient β_s which can be estimated from cross-section data in period s;

$$\begin{aligned} \bar{\beta}_s &:= \frac{1}{X_s} \int_{\mathbb{R}_+} x \left[\int_{\mathcal{A}} \partial_x \left[\bar{w}_s(p_s, r_s, \lambda x, a \right]_{\lambda=1} d\nu_s | x \right] \rho_s(x) dx \\ &= \frac{1}{X_s} \int_{\mathbb{R}_+} x \left[\int_{\mathcal{A}} \partial_x \bar{c}_s(p_s, r_s, x, a) d\nu_s | x \right] \rho_s(x) dx - \frac{C_s}{X_s} \end{aligned}$$

As before, $\bar{\beta}_s$ is a proxi for β_s provided condition $(\star)$ is satisfied. For estimating $\bar{\beta}_s$ one does not need a micro-relation.

Estimates of the coefficient $\bar{\beta}_s$ (stratification with respect to household size and age in 5×8 disjoint groups) for the years 1968-1986 yield a mean value of -0.242 for consumption expenditure on non-durables; -0.152 for food and $+0.022$ for services. For details see Hildenbrand and Kneip (1997).

The coefficients γ_s, δ_s and η_s

The interpretation of the coefficient γ_s, δ_s and η_s can be short. By definition they depend on how, on average, households react to changes in the price level or interest rate.

The coefficient $\gamma_s = \frac{1}{X_s}\partial_\pi \left[C_s(\pi \cdot p_s^1, p_s^2, r_s, \mu_s)\right]_{\pi=1}$ depends on the price-elasticity of demand. For example, if, for every income level, demand is zero-elastic – an implausible situation – then one obtains $\gamma_s = C_s/X_s$. On the other hand, if an increase in the price level leads, on average, to a reduction of demand, yet to an unchanged expenditure, i.e. unit-elastic demand, then one obtains $\gamma_s = 0$.

The coefficient $\delta_s = \frac{1}{X_s}\partial_\pi \left[C_s(p_s^1, \pi \cdot p_s^2, r_s, \mu_s)\right]_{\pi=1}$ depends on the cross-elasticity of demand with respect to an increase in the price-level for all commodities other than those that are included in expenditure.

The coefficient $\eta_s = \frac{1}{X_s}\partial_r \left[C_s(p_s^1, p_s^2, r \cdot r_s, \mu_s)\right]_{r=1}$ is often considered as negligible. Yet at the level of an individual household a change in the current interest rate might well have a non-negligible effect. For example, analyzing the standard intertemporal decision problem of a household shows that a change in the interest rate matters; yet *the effect is indeterminate*. An increase in the current interest rate can lead to either an increase or a decrease in current consumption expenditure depending on the intertemporal utility function, the expectation function for future interest rates and asset holdings. Therefore, it might well happen that on average over the population the effect of a change in the current interest rate is negligible, hence $\eta_s = 0$ even though at the individual level it matters.

4 Aggregation under slowly changing attribute distributions

The last section was based on the hypothesis of structural stability not only of household characteristics (that is, Hypothesis 1) but also of household attributes (that is, Hypothesis 3^+). As mentioned already in section 3 this last hypothesis is in contradiction with empirical facts: indeed, the distributions of household attributes typically change over time; this change, however, is quite slow (see H-K-U (1997)). Therefore we base this section on Hypothesis 3 in a less restrictive version than in section 3.

The distribution μ_t in period t has been decomposed in the conditional distribution of household characteristics

$$\mu_t|(x,a)\,,$$

the conditional distribution of household attributes

$$\nu_t|x$$

and the income density

$$\rho_t(x)\,.$$

Hypothesis 2, i.e., the time-invariance of the standardized log income distribution, implies (see (8) and (9) of section 2)

$$\rho_t(x) = \varphi'(x)\rho_s(\varphi(x))\,,$$

where $\varphi(x) := \left(\frac{m_s(\sigma_t/\sigma_s)}{X_t}\cdot x\right)^{\sigma_s/\sigma_t}$.

Hypothesis 1, i.e., structural stability of household characteristics, expresses that

$$\mu_t|(x,a) = \mu_s|(\varphi(x),a)\,.$$

Instead of assuming $\nu_t|x = \nu_s|\varphi(x)$ as in section 3, we now assume Hypothesis 3, that is, for periods s and t that are close to each other the difference

$$\nu_t|x - \nu_s|\varphi(x)$$

is "small", but not necessarily negligible. We shall explain in the sequel (see the approximation (18)) in which sense the difference $\nu_t|x - \nu_s|\varphi(x)$ should be small.

By definition

$$C_t = \int_{\mathrm{I\!R}_+}\left[\int_{\mathcal{A}}\left\{\int_{\mathcal{X}} c(p_t,r_t,x,\chi)d\mu_t|(x,a)\right\}d\nu_t|x\right]\rho_t(x)dx\,.$$

Since by assumption $\mu_t|(x,a) = \mu_s|(\varphi(x),a)$ and $\rho_t(x) = \varphi'(x)\rho_s(\varphi(x))$ we obtain

$$C_t = \int_{\mathbb{R}_+} \left[\int_{\mathcal{A}} \left\{ \int_{\mathcal{X}} c(p_t, r_t, x, \chi) d\mu_s|(\varphi(x), a) \right\} d\nu_t|x \right] \varphi'(x)\rho_s(\varphi(x))dx \, .$$

Substituting $\nu_t|x = \nu_s|\varphi(x) + (\nu_t|x - \nu_s|\varphi(x))$ yields by Proposition 1

$$C_t = \int_{\mathbb{R}_+ \times \mathcal{X}} c(p_t, r_t, \varphi^{-1}(x), \chi) d\mu_s + A \tag{17}$$

where

$$A := \int_{\mathbb{R}_+} \left[\int_{\mathcal{A}} \left\{ \int_{\mathcal{X}} c(p_t, r_t, x, \chi) d\mu_s|(\varphi(x), a) \right\} d(\nu_t|x - \nu_s|\varphi(x)) \right] \varphi'(x)\rho_s(\varphi(x))dx \, .$$

For the first term on the right hand side of (17) we have given an approximation in Proposition 2 of section 3. We shall now develop an approximation for the second term, that is to say, for the integral which is denoted by A.

Substituting $\xi = \varphi(x)$ yields

$$A = \int c(p_t, r_t, \varphi^{-1}(\xi), \chi) d\mu_s|(\xi, a) \left\{ \nu_t|\varphi^{-1}(\xi) - \nu_s|\xi \right\} \rho_s(\xi) d\xi$$

Since $\varphi^{-1}(\xi) = \frac{X_t}{m_s(\sigma_t/\sigma_s)} \xi^{\sigma_t/\sigma_s}$ we obtain

$$A = \int_{\mathbb{R}_+} \left[\int_{\mathcal{A}} \left\{ \frac{X_t}{m_s(\sigma)} \int_{\mathcal{X}} \xi^{\sigma} w(p_t, r_t, \frac{X_t}{m_s(\sigma)} \cdot \xi^{\sigma}, \chi) d\mu_s|(\xi, a) \right\} d\left(\nu_t| \frac{X_t}{m_s(\sigma)} \cdot \xi^{\sigma} - \nu_s|\xi \right) \right] \rho_s(\xi) d\xi$$

where $\sigma = \sigma_t/\sigma_s$ and $w(p_t, r_t, x, \chi) = \frac{1}{x} c(p_t, r_t, x, \chi)$.

We now want to develop an approximation for A.

If $\sigma_t/\sigma_s, X_t/X_s, p_t/p_s$ and r_t/r_s tend to 1 then

$$\left\{ \int_{\mathcal{X}} \xi^{\sigma} w(p_t, r_t, \frac{X_t}{m_s(\sigma)} \cdot \xi^{\sigma}, \chi) d\mu_s|(\xi, a) \right\}$$

tends to

$$\int_{\mathcal{X}} \xi \, w(p_s, r_s, \xi, \chi) d\mu_s|(\xi, a) = \bar{c}_s(p_s, r_s, \xi, a)$$

Note, however, that $\{\nu_t|\frac{X_t}{m_s(\sigma)} \cdot \xi^{\sigma} - \nu_s|\xi\}$ does not necessarily tend to zero. At this point we make use of Hypothesis 3: for periods s and t that are close to each other, $\{\nu_t|\frac{X_t}{m_s(\sigma)} \cdot \xi^{\sigma} - \nu_s|\xi\}$ will be sufficiently small in order to justify the following approximation:

$$A \approx \bar{A} := \int_{\mathbb{R}_+} \left[\frac{X_t}{X_s} \int_{\mathcal{A}} \bar{c}_s(p_s, r_s, x, a) \, d\left(\nu_t| \frac{X_t}{m_s(\sigma)} \cdot x^{\sigma} - \nu_s|x \right) \right] \rho_s(x) dx \tag{18}$$

Thus, in this section the term "approximate" in Hypothesis 3 has to be interpreted as implying $A \approx \bar{A}$.

To evaluate $\bar{A}$ we assume that the set of household attributes is finite $\mathcal{A} = \{a^1, \dots, a^m\}$ and use the following notation:

$$\nu_s|x\{a^i\} =: \nu_s^i(x) \text{ and } \nu_s\{\mathbb{R}_+ \times a^i\} =: \nu_s^i.$$

Furthermore we make the following

Hypothesis 4: *For every x_s and x_t that are in the same percentile position in period s and t, respectively,*

$$\frac{\nu_t^i(x_t)}{\nu_s^i(x_s)} \approx \frac{\nu_t^i}{\nu_s^i}$$

Let us recall, Hypothesis 3 is used to justify the approximation $A \approx \bar{A}$ and Hypothesis 4 is a technical assumption which allows to evaluate $\bar{A}$.

Indeed,

$$\bar{A} = \frac{X_t}{X_s} \int_{\mathbb{R}_+} \left[\sum_{i=1}^{m} \bar{c}_s(p_s, r_s, x, a^i) \left(\nu_t^i(\frac{X_t}{m_s(\sigma)} \cdot x^\sigma) - \nu_s^i(x) \right) \right] \rho_s(x) dx.$$

Since $\sum_{i=1}^m \nu_t^i(\xi) = 1$ and $\sum_{i=1}^m \nu_s^i(x) = 1$ we obtain

$$\sum_{i=1}^{m} \tilde{c}_s(p_s, r_s, x) \left(\nu_t^i(\frac{X_t}{m_s(\sigma)} \cdot x^\sigma) - \nu_s^i(x) \right) = 0$$

Hence

$$\bar{A} = \frac{X_t}{X_s} \int_{\mathbb{R}_+} \left[\sum_{i=1}^{m} [\bar{c}_s(p_s, r_s, x, a^i) - \tilde{c}_s(p_s, r_s, x)][\nu_t^i(\frac{X_t}{m_s(\sigma)} \cdot x^\sigma) - \nu_s^i(x)] \right] \rho_s(x) dx$$

Now

$$\begin{aligned}
\left[\nu_t^i(\frac{X_t}{m_s(\sigma)} \cdot x^\sigma) - \nu_s^i(x) \right] &= \left[\frac{\nu_t^i(\frac{X_t}{m_s(\sigma)} \cdot x^\sigma)}{\nu_t^i} \cdot \nu_t^i - \frac{\nu_s^i(x)}{\nu_s^i} \cdot \nu_s^i \right] \\
&= \frac{\nu_s^i(x)}{\nu_s^i} \left(\nu_t^i - \nu_s^i \right) \\
&= \nu_s^i(x) \left(\frac{\nu_t^i}{\nu_s^i} - 1 \right).
\end{aligned}$$

Thus,

$$\bar{A} = X_t \cdot \sum_{i=1}^{m} \lambda_s^i \left(\frac{\nu_t^i}{\nu_s^i} - 1 \right)$$

where

$$\lambda_s^i = \frac{1}{X_s} \int_{\mathbb{R}_+} [\bar{c}_s(p_s, r_s, x, a^i) - \tilde{c}_s(p_s, r_s, x)] \nu_s^i(x) \rho_s(x) dx.$$

Note that the coefficient λ_s^i can be estimated from cross-section data.

The coefficient λ_s^i measures to what extent the Engel curve of the subpopulation consisting of all households with attribute a^i, i.e., $\bar{c}_s(p_s, r_s, \cdot, a^i)$ differs (on average) from the Engel curve of the whole population, i.e., $\tilde{c}_s(p_s, r_s, \cdot)$. Consequently,

$$\lambda_s^i \left(\frac{\nu_t^i}{\nu_s^i} - 1 \right)$$

describes the effect (on C_t/X_t) of the changing composition of the population with respect to the attribute a^i.

In summary, in this section we extended Proposition 2 and derived the following approximation:

$C_t/X_t - C_s/X_s$	*change in the aggregate consumption ratio*
$\approx \quad \alpha_s \log \frac{\sigma_t}{\sigma_s}$	*effect of the changing income dispersion*
$+\beta_s \log \frac{X_t}{X_s}$	*effect of mean (nominal) income growth*
$+\gamma_s \log \frac{\pi_t^1}{\pi_s^1}$	*effects of price-inflation*
$+\delta_s \log \frac{\pi_t^2}{\pi_s^2}$	
$+\eta_s \log \frac{r_t}{r_s}$	*effect of interest rate changes*
$+ \sum_{i=1}^m \lambda_s^i \left(\frac{\nu_t^i}{\nu_s^i} - 1 \right)$	*effect of the changing distribution of attributes*

Appendix

Lemma: *For every continuous and decreasing function v of $\mathbb{R}_+$ into $\mathbb{R}$ and every density ρ on $\mathbb{R}_+$ such that*

$$A(\sigma) := \frac{\int x^\sigma v(x)\rho(x)dx}{\int x^\sigma \rho(x)dx}$$

is defined on an open interval around $\sigma = 1$ if follows that

$$\partial_\lambda \left[A(\sigma)\right]_{\sigma=1} \leq 0$$

Proof: In order to prove the assertion of the lemma it suffices to show that for all σ with $0 < \sigma \leq 1$,

$$\frac{\int x^\sigma v(x)\rho(x)dx}{\int x^\sigma \rho(x)dx} \geq \frac{\int x\ v(x)\rho(x)dx}{\int x\ \rho(x)dx} \tag{1}$$

Let $m(\sigma) := \int x^\sigma \rho(x)dx$ and $z(\sigma) := \left(\frac{m(1)}{m(\sigma)}\right)^{\frac{1}{1-\sigma}}$. Then, since $\sigma < 1$,

$$\frac{x^\sigma}{m(\sigma)} \geq \frac{x}{m(1)} \qquad \text{if } 0 \leq x \leq z(\sigma) \tag{2}$$

and

$$\frac{x^\sigma}{m(\sigma)} \leq \frac{x}{m(1)} \qquad \text{if } z(\sigma) < x;. \tag{3}$$

Since $\int \frac{x^\sigma}{m(\sigma)}\rho(x)dx = \int \frac{x}{m(1)}\rho(x)dx = 1$, one obtains

$$\int_0^{z(\sigma)} \left[\frac{x^\sigma}{m(\sigma)} - \frac{x}{m(1)}\right] \rho(x)dx = -\int_{z(\sigma)}^{\infty} \left[\frac{x^\sigma}{m(\sigma)} - \frac{x}{m(1)}\right] \rho(x)dx;,$$

and by relations (2) and (3),

$$\int_0^{z(\sigma)} \left|\frac{x^\sigma}{m(\sigma)} - \frac{x}{m(1)}\right| \rho(x)dx = -\int_{z(\sigma)}^{\infty} \left|\frac{x^\sigma}{m(\sigma)} - \frac{x}{m(1)}\right| \rho(x)dx\ . \tag{4}$$

Relation (1) holds, if and only if for all σ with $0 < \sigma \leq 1$ one has

$$R(\sigma) := \int \left(\frac{x^\sigma}{m(\sigma)} - \frac{x}{m(1)}\right) v(x)\rho(x)dx \geq 0. \tag{5}$$

However, (5) is an immediate consequence of (4) and the assumption on v. Indeed,

$$\begin{aligned} R(\sigma) = & \int_0^{z(\sigma)} \left(\frac{x^\sigma}{m(\sigma)} - \frac{x}{m(1)}\right) v(x)\rho(x)dx + \int_{z(\sigma)}^{\infty} \left(\frac{x^\sigma}{m(\sigma)} - \frac{x}{m(1)}\right) v(x)\rho(x)dx \\ & \geq \left[\min_{0\leq x\leq z(\sigma)} v(x)\right] \cdot \int_0^{z(\sigma)} \left|\frac{x^\sigma}{m(\sigma)} - \frac{x}{m(1)}\right| \rho(x)dx \\ & \qquad - \left[\max_{z(\sigma)\leq x<\infty} v(x)\right] \cdot \int_{z(\sigma)}^{\infty} \left|\frac{x^\sigma}{m(\sigma)} - \frac{x}{m(1)}\right| \rho(x)dx = 0 \end{aligned}$$

Q.E.D.

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